Minimal weakly isotropic forms

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Abstract

In this article weakly isotropic quadratic forms over a (formally) real field are studied. Conditions on the field are given which imply that every weakly isotropic form over that field has a weakly isotropic subform of small dimension. Fields over which every quadratic form can be decomposed into an orthogonal sum of a strongly anisotropic form and a torsion form are characterized in different ways.

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1 Introduction

The concept of weak isotropy was introduced by Prestel in [11] and studied in the 1970ies, especially by Prestel and Bröcker. The main motivation for this concept is the failure of any kind of local-global principle for isotropy of quadratic forms in a general situation. It turned out that, over any (formally) real field, weakly isotropic forms can be characterized, on the one hand in terms of the semi-orderings (cf. [11]), on the other hand in terms of the real places of the field (cf. [11], [2]). Applications of weakly isotropic forms occur in real algebra and geometry (cf. [13]).

The present article investigates weakly isotropic forms in view of the question what kind of subforms and orthogonal decompositions they admit. This leads to consider in particular those weakly isotropic forms which are minimal in the sense that they do not contain any proper subform which is weakly isotropic. The question of how large the dimension of such a form may be gives rise to a new field invariant. This invariant will be compared with other invariants and properties of fields. This work is mainly inspired by ideas and results due to Arason and Pfister, found in [1].

The reader is assumed to be familiar with the general theory of quadratic forms over fields as it is presented for example in the books by Lam and Scharlau (cf. [5] or [14]). The terms 'form' and 'quadratic form' shall always refer to a regular, finite dimensional quadratic form.

Let F always be a field of characteristic different from 2. A quadratic form over F may be given by a diagonalization $\langle a_1, \ldots, a_n \rangle$ with entries $a_1, \ldots, a_n \in F^\times$ which determines the form up to isometry. We denote by WF the Witt ring of (Witt equivalence classes of) quadratic forms over F and by IF the fundamental ideal in WF, given by the even-dimensional forms. For $n \geq 1$ we write I^nF for the nth power $(IF)^n$ of the ideal IF.

A quadratic form φ over F is said to be *weakly isotropic* if there is an integer $m \geq 1$ such that $m \times \varphi = \varphi \perp \ldots \perp \varphi$ is isotropic; the lowest such m shall then be called the *weak isotropy index* of φ and denoted by $wi(\varphi)$. Note that $wi(\varphi) = 1$ means that φ is isotropic. We say that φ is *strongly anisotropic* if it is not weakly isotropic an write $wi(\varphi) = \infty$ in this case.

In order to study weakly isotropic forms one needs also to consider torsion forms. A quadratic form φ over F is *torsion* if $m \times \varphi$ is hyperbolic for a certain $m \geq 1$. The least such m then is a power of 2 (cf. [14, p. 53]) and equal to the order of the class of φ in WF. By Pfister's local-global principle (cf. [9, Satz 22]), a quadratic form over F is torsion if and only if its signature at every ordering of F is zero.

We denote by $F^\times$ the multiplicative group of the field F and by $\sum F^{\times 2}$ the subgroup consisting of the non-zero sums of squares in F. The elements of $\sum F^{\times 2}$ are said to be *totally positive*. If $-1 \in \sum F^{\times 2}$, then the field F is said to be *nonreal*, otherwise F is *real*. If F is a nonreal field, then every quadratic form over F is torsion and in particular weakly isotropic. The notion of weak isotropy is therefore only interesting over real fields.

Let β be a binary (i.e. 2-dimensional) quadratic form over F. Writing $\beta = \langle a, -ad \rangle$ with $a, d \in F^\times$, we have that β is weakly isotropic if and only if β is torsion, if and only if $d \in \sum F^{\times 2}$. If a quadratic form contains a torsion binary form, then it is clearly weakly isotropic. The question whether the converse implication also holds is the starting point for the present investigation. Given a weakly isotropic quadratic form φ over a real field F, does φ contain a torsion binary form?

If the answer to this question is positive for every weakly isotropic form over F, then it follows by an induction argument that every quadratic form over F has a 'weak decomposition' in the following sense. If a quadratic form φ over a real field F can be decomposed into an orthogonal sum

$$\gamma \perp \beta_1 \perp \ldots \perp \beta_r$$

where γ is a strongly anisotropic form and $\beta_1, \ldots, \beta_r$ $(r \geq 0)$ are torsion

binary forms over F, then this is called a *weak decomposition* of φ. Note that the number r then is uniquely determined by φ, and so is the isometry class of $\gamma_{F_{pyth}}$ over the pythagorean closure F_{pyth} of F.

We say that F has the *weak decomposition property* or that F is a *WD-field*, if either F is nonreal, or if every quadratic form over F has a weak decomposition in the above sense.

In (4.2) we will characterize real *WD*-fields as those fields having the Property C investigated by Arason and Pfister in [1]. Property C means that torsion forms are 'strongly balanced', i.e. decomposable into a sum of binary torsion forms. Easily accessible examples of *WD*-fields are number fields and extensions of transcendence degree at most two of real closed fields.

Arason and Pfister have also obtained examples of real fields not satisfying Property C. In [1, Satz 2 & Satz 3] they show that if k is a real field with two totally positive elements u, v such that v is not represented by $\langle 1, u\rangle$, then the 3-dimensional form $\langle 1, X, -(u+Y^2+vX)\rangle$ over the field $k(X,Y)$ is weakly isotropic but contains no torsion binary form. In particular the form $\langle 1, X, -(1+Y^2+3X)\rangle$ over the field $\mathbb{Q}(X,Y)$ gives evidence that this is not a *WD*-field.

This example suggests to study the following concepts. A weakly isotropic quadratic form is said to be *minimal weakly isotropic* (*m.w.i.*) if none of its proper subforms is weakly isotropic. Since it is desirable to have an upper bound for the dimensions of minimal weakly isotropic forms over F, we define

$$w(F) \;=\; \sup\{\dim(\varphi) \mid \varphi \text{ m.w.i. form over } F\} \;\in\; \mathbb{N}\cup\{\infty\}\,.$$

We consider $w(F)$ as an invariant of the field F and want to compare it with other field invariants related to quadratic form theory. For a nonreal field F it seems natural to agree that $w(F) = 1$. If F is real, then the hyperbolic plane $\mathbb{H} = \langle 1, -1\rangle$ should be considered as a minimal weakly isotropic form so that we obtain $w(F) \geq 2$ in this case. With these definitions, F is a *WD*-field if and only if $w(F) \leq 2$.

The *Pythagoras number* of F is denoted by $p(F)$; this invariant is defined as the least integer $m \geq 1$ such that every totally positive element in F is equal to a sum of m squares if such an integer exists, otherwise $p(F) = \infty$.

The following section provides some preliminary results, in particular on products of two quadratic forms where one of them is multiplicative or represents only elements which are sums of squares in the field.

In Section 3 we obtain two upper bounds on the invariant $w(F)$. On the one hand, $w(F) \leq p(F)$ holds for any field F which is not real pythagorean (3.3). On the other hand, if $n \geq 3$ is such that I^nF is torsion-free, then

$w(F) \leq 2^{n-2}$ (3.5). The latter also allows us to compare $w(F)$ with the u-invariant of the field F (3.6).

In Section 4 we show that for $m \geq 3$ it suffices that there exists no minimal weakly isotropic form of dimension m over F in order to have $w(F) < m$ (4.1). This criterion allows us in (4.2) to characterize *WD*-fields in a way that extends [1, Satz 4]. A list of examples of *WD*-fields is given in (4.3).

Motivated by Property B considered in [1] we introduce in Section 5 three further field invariants b, b', and pu, which encode properties of (real) fields related to sums of squares. For a non-pythagorean field F, we refine the previously shown inequality $w(F) \leq p(F)$ to a chain of inequalities $w(F) \leq b(F) \leq b'(F) \leq pu(F) \leq p(F)$. Furthermore, several open problems are formulated in this final section.

2 Totally positive quadratic forms

For any quadratic form φ over F we denote by $D_F(\varphi)$ the set of non-zero elements of F which are represented by φ. Recall that the form φ over F is said to be *multiplicative* if either φ is hyperbolic or if φ is anisotropic and $a\varphi \cong \varphi$ holds for any $a \in D_F(\varphi)$. For $a_1, \ldots, a_n \in F^\times$ the form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ over F is called an *n-fold Pfister form* and denoted by $\langle\langle a_1, \ldots, a_n\rangle\rangle$. Pfister forms are multiplicative. In particular any Pfister form is either hyperbolic or anisotropic.

2.1 Proposition. *Let ρ be a multiplicative form and φ an arbitrary quadratic form over F.*

(*a*) *Assume that* $\dim(\varphi) \geq 2$ *and let* $a_1, \ldots, a_n \in F^\times$ *be such that* $\varphi \cong \langle a_1, \ldots, a_n\rangle$. *The form* $\rho \otimes \varphi$ *is isotropic (resp. hyperbolic) over* F *if and only if there exist* $t_1, \ldots, t_n \in D_F(\rho)$ *such that* $\langle a_1t_1, \ldots, a_nt_n\rangle$ *is isotropic (resp. hyperbolic). Moreover, the choice of* $t_1, \ldots, t_n$ *can be made such that* $t_1 = 1$.

(*b*) *If* $\rho \otimes \varphi$ *is hyperbolic and* ψ *is a subform of* φ *with* $\dim(\psi) > \frac{1}{2}\dim(\varphi)$, *then* $\rho \otimes \psi$ *is isotropic.*

(*c*) *If* $\dim(\varphi) \geq 3$ *and if* $\rho \otimes \varphi$ *is isotropic, then there exists a form* $\vartheta \in I^2F$ *with* $\dim(\vartheta) = 2\dim(\varphi) - 2$ *such that* φ *is a subform of* ϑ *and such that* $\rho \otimes \vartheta$ *is hyperbolic.*

Proof: If ρ is hyperbolic then all statements are trivial. We may thus assume that the multiplicative form ρ is anisotropic. Statement (a) is well-known (cf. [6, Chap. 1]). Statement (b) is immediately reduced to the case where

$\rho = \langle 1 \rangle$ for which it is very well-known. For the proof of (c), suppose that $\rho \otimes \varphi$ is isotropic. Let $a_1, \ldots, a_n \in F^\times$ be such that $\varphi \cong \langle a_1, \ldots, a_n \rangle$. We assume that $n = \dim(\varphi) \geq 3$. By (a), there exist $t_1, \ldots, t_n \in D_F(\rho)$ such that $\langle a_1 t_1, \ldots, a_n t_n \rangle$ is isotropic, hence of the form $\psi' \perp \langle c, 1, -1 \rangle$ for a certain $c \in F^\times$ and some form ψ' over F, where $\dim(\psi') = (n-3) \geq 0$. Let $t = t_1 \cdots t_n$. Since ρ is multiplicative we have $t \in D_F(\rho)$. We put $\psi = \psi' \perp \langle ct \rangle$ and $\vartheta = \varphi \perp -\psi$. Note that $\dim(\vartheta) = 2\dim(\varphi) - 2$. Now ϑ is Witt equivalent to the form $a_1\langle 1, -t_1 \rangle \perp \cdots \perp a_n \langle 1, -t_n \rangle \perp c\langle 1, -t \rangle$. Therefore $\vartheta \in I^2F$ and $\rho \otimes \vartheta$ is hyperbolic, as ρ is multiplicative and represents $t_1, \ldots, t_n$ and t. $\square$

Note that it might be useful to know more about the freedom of the choice of the form ϑ in statement (c) of the proposition.

We are now going to apply the proposition to link weakly isotropic forms and torsion forms with each other.

2.2 Corollary. *Let φ be a quadratic form over F.*

(a) *Assume that $\dim(\varphi) \geq 2$ and let $a_1, \ldots, a_n \in F^\times$ be such that $\varphi \cong \langle a_1, \ldots, a_n \rangle$. Then φ is weakly isotropic (resp. torsion) over F if and only if there exist $t_1, \ldots, t_n \in \sum F^{\times 2}$ such that the form $\langle a_1 t_1, \ldots, a_n t_n \rangle$ is isotropic (resp. hyperbolic). Moreover, the choice of $t_1, \ldots, t_n$ can be made such that $t_1 = 1$.*

(b) *If φ is torsion (of order 2^m, with $m \geq 0$) and if ψ is a subform of φ such that $\dim(\psi) > \frac{1}{2}\dim(\varphi)$, then ψ is weakly isotropic (of index $wi(\varphi) \leq 2^m$).*

(c) *If $\dim(\varphi) \geq 3$ and if φ is weakly isotropic, then there exists a torsion form $\vartheta \in I^2F$ with $\dim(\vartheta) = 2\dim(\varphi) - 2$ and such that φ is a subform of ϑ.*

Proof: For $m \geq 0$ we put $\rho_m = 2^m \times \langle 1 \rangle = \langle\langle -1, \ldots, -1 \rangle\rangle$ and observe that this form is multiplicative. A form φ is weakly isotropic (resp. torsion) if and only if $\rho_m \otimes \varphi$ is isotropic (resp. hyperbolic) for m sufficiently large. Now the statements follow immediately from the proposition. $\square$

2.3 Remark. Let T be a preordering of F, that is, a subset of F which is closed under addition and under multiplication, and which contains all squares in F. A quadratic form φ over F is said to be *T-isotropic* (resp. *T-hyperbolic*) if there exist $t_1, \ldots, t_m \in T \setminus \{0\}$ such that $\langle t_1, \ldots, t_m \rangle \otimes \varphi$ is isotropic (resp. hyperbolic). It is easily observed that all statements of the corollary remain valid if 'weakly isotropic' and 'torsion' are replaced by 'T-isotropic' and 'T-hyperbolic', respectively, and if $\sum F^{\times 2}$ is replaced by

$T^\times = T \setminus \{0\}$ in (a). The proof then uses the fact that a form φ over F is T-isotropic (resp. T-hyperbolic) if and only if $\langle\langle -t_1, \ldots, -t_m \rangle\rangle \otimes \varphi$ is isotropic for some non-zero elements $t_1, \ldots, t_n \in T$. For our purposes, we only need the special case where $T = \sum F^{\times 2} \cup \{0\}$, which is formulated in the corollary.

We say that a form ρ over F is *totally positive* if $D_F(\rho) \subset \sum F^{\times 2}$. If we have equality $D_F(\rho) = \sum F^{\times 2}$, then we say that ρ is *positive-universal*. A typical example of a positive-universal form over a real field F is obviously the form $p \times \langle 1 \rangle$ where $p = p(F) < \infty$.

2.4 Proposition. *Assume that F is real and ρ a positive-universal form over F. Then any quadratic form φ over F is weakly isotropic if and only if $\rho \otimes \varphi$ is isotropic.*

Proof: For m sufficiently large ρ is a subform of $m \times \langle 1 \rangle$. Therefore, if $\rho \otimes \varphi$ isotropic then φ is weakly isotropic. To prove the converse, assume that $\varphi = \langle a_1, \ldots, a_n \rangle$ is weakly isotropic. Then there exist $t_1, \ldots, t_n \in \sum F^{\times 2}$ such that $\widetilde{\varphi} = \langle t_1 a_1, \ldots, t_n a_n \rangle$ is isotropic. As the form ρ is positive-universal it represents $t_1, \ldots, t_n$. Then $\rho \otimes \varphi$ contains $\widetilde{\varphi}$ as a subform and is thus isotropic as well. $\square$

The statement of the proposition holds trivially also for nonreal fields except in the degenerate case where F is quadratically closed and ρ and φ are both 1-dimensional.

2.5 Corollary. *Let φ be a quadratic form of dimension at least 2 over F. If φ is weakly isotropic, then $wi(\varphi) \leq p(F)$.*

Proof: We may assume that F is real and $p(F) < \infty$. The statement follows by applying the last proposition to the positive-universal form $p(F) \times \langle 1 \rangle$. $\square$

The last statement is well-known at least in the case where F is real pythagorean, when it shows that weakly isotropic forms over F are already isotropic and therefore allows us to conclude that the Witt ring WF is torsion-free.

2.6 Remark. We may say that an element $t \in \sum F^{\times 2}$ is *positive-universal* if the binary form $\langle 1, t \rangle$ has this property, that is, if $D_F(\langle 1, t \rangle) = \sum F^{\times 2}$. If F is nonreal, then -1 is a positive-universal element. Over a real field there may exist no positive-universal element. However, if F happens to be a field with finitely many square classes and whose Witt ring WF is of elementary type (cf. [8]), then F has a positive-universal element; this is easily shown by induction on the construction of WF starting from basic Witt rings.

3 Weakly isotropic subforms

In this section we develop criteria for a form over F to contain a weakly isotropic subform of dimension at most m. In particular, this will give us upper bounds for the invariant $w(F)$.

The following proposition will be the crucial ingredient in many of the subsequent proofs. It generalizes an observation due to Elman and Lam; in [3, p. 289], they treated the case where the form ρ below is a binary form and obtained in this way their so-called '*β-decomposition*'.

3.1 Proposition. *Let φ and ρ be quadratic forms over F where φ is anisotropic. If $\rho \otimes \varphi$ is isotropic, then φ contains a subform ψ such that $\rho \otimes \psi$ is isotropic and* $\dim(\psi) \leq \dim(\rho)$.

Proof: Suppose that $\rho \otimes \varphi$ is isotropic. We write $\rho \cong \langle b_1, \ldots, b_m \rangle$ with $b_1, \ldots b_m \in F^\times$, $m = \dim(\rho)$. We consider φ as a quadratic map defined on a vector space V. Since $\rho \otimes \varphi \cong b_1\varphi \perp \cdots \perp b_m\varphi$ is isotropic, there exist $x_1, \ldots x_m \in V$, not all equal to zero, such that $b_1\varphi(x_1) + \cdots + b_m\varphi(x_m) = 0$. Let W we the subspace of V spanned by the vectors $x_1, \ldots, x_m$. Hence $1 \leq \dim(W) \leq m$. Let ψ denote the restriction of the quadratic map φ to this subspace. Since φ is anisotropic, so is ψ. Hence ψ is a regular quadratic form over F and a subform of φ. We further have $\dim(\psi) \leq m = \dim(\rho)$ and $\rho \otimes \psi$ is isotropic. $\square$

3.2 Corollary. *Let φ be a quadratic form over F and $m \geq 2$. If φ is weakly isotropic of index $wi(\varphi) \leq m$, then there exists a subform ψ of φ with* $\dim(\psi) \leq m$ *such that ψ is weakly isotropic of index $wi(\psi) \leq m$.*

Proof: If φ is isotropic then we may take for ψ the hyperbolic plane $\langle 1, -1 \rangle$. If φ is anisotropic, then we apply the proposition with $\rho = m \times \langle 1 \rangle$ to obtain the statement. $\square$

We are ready to give a first bound on $w(F)$.

3.3 Theorem. *If the field F is real pythagorean, then $w(F) = 2$, otherwise $w(F) \leq p(F)$.*

Proof: If F is nonreal, then $w(F) = 1 \leq p(F)$. Assume for the rest that F is real. If $p(F) = 1$, then F is real pythagorean and $\langle 1, -1 \rangle$ is the only minimal weakly isotropic form over F, in particular $w(F) = 2$.

Assume now that $p(F) \geq 2$. Let φ be an arbitrary weakly isotropic form over F. Then $\dim(\varphi) \geq 2$ as F is real and by (2.5), φ has index $wi(\varphi) \leq p(F)$. From (3.2) we obtain that φ contains a weakly isotropic

subform ψ with $\dim(\psi) \leq p(F)$. This shows that weakly isotropic forms over F of dimension greater than $p(F)$ are not minimal. Hence $w(F) \leq p(F)$. $\square$

We now obtain a criterion for the existence of weakly isotropic subforms of dimension bounded by a given number.

3.4 Proposition. *Let φ be an anisotropic quadratic form over F and $m \geq 2$. The following conditions are equivalent:*

(1) *φ contains a weakly isotropic subform ψ of dimension at most m.*

(2) *There exists an m-dimensional totally positive form ϑ over F such that $\vartheta \otimes \varphi$ is isotropic.*

Proof: Assume that φ has a weakly isotropic subform ψ with $\dim(\psi) \leq m$. Then taking $\tau = r \times \langle 1 \rangle$ for r large enough, we have that $\tau \otimes \psi$ is isotropic. By (3.1), τ then contains a subform ϑ' with $\dim(\vartheta') \leq \dim(\psi) \leq m$ such that $\vartheta' \otimes \psi$ is isotropic. Then ϑ' is totally positive and $\vartheta' \otimes \varphi$ is isotropic. We may further replace ϑ' by $\vartheta = \vartheta' \perp \langle 1, \ldots, 1 \rangle$ such that $\dim(\vartheta) = m$. This shows that (1) implies (2).

To prove the converse, assume now that ϑ is a totally positive form over F of dimension m and such that $\vartheta \otimes \varphi$ is isotropic. Then (3.1) says that there exists a subform ψ of φ such that $\dim(\psi) \leq \dim(\vartheta) = m$ and such that $\vartheta \otimes \psi$ is isotropic. Since ϑ is totally positive, it follows that ψ is weakly isotropic. $\square$

3.5 Theorem. *Assume that $n \geq 2$ is such that $I^n F$ is torsion-free. Then $wi(\varphi) \leq 2^{n-2}$ for any weakly isotropic form φ of dimension at least 3 over F. In particular, if $n \geq 3$ then $w(F) \leq 2^{n-2}$.*

Proof: Let φ be a quadratic form over F with $\dim(\varphi) \geq 3$. Suppose that φ is weakly isotropic. By (2.2, c) there exists a torsion form $\vartheta \in I^2 F$ which contains φ and with $\dim(\vartheta) = 2\dim(\varphi) - 2$. Since $I^n F$ is torsion-free, $2^{n-2} \times \vartheta$ is hyperbolic. Therefore φ has index $wi(\varphi) \leq 2^{n-2}$, by (2.2, b). If further $n \geq 3$, then $2^{n-2} \geq 2$ and (3.2) yields that φ contains a weakly isotropic subform of dimension at most 2^{n-2}. $\square$

Let $u(F)$ denote the u-invariant of F as introduced in [3], that is

$$u(F) = \sup \{\dim(\varphi) \mid \varphi \text{ anisotropic torsion form over } F\}.$$

3.6 Corollary. *Assume that F is a real field with $4 \leq u(F) < \infty$ and let $m \geq 1$ be the integer such that $2^m \leq u(F) < 2^{m+1}$. Then $w(F) \leq 2^{m-1}$, in particular $w(F) \leq \frac{1}{2} u(F)$.*

Proof: If $u(F) < 2^{m+1}$, then $I^{m+1} F$ is torsion-free, thus the statement follows from (3.5). $\square$

4 Fields with weak decomposition property

Next we obtain a criterion to decide whether the invariant $w(F)$ is smaller than a given number.

4.1 Theorem. *Let $m \geq 3$. If there exists no minimal weakly isotropic form of dimension m over F, then $w(F) < m$.*

Proof: We may suppose that F is real, since otherwise $w(F) = 1$. Assuming that there exists no minimal weakly isotropic form of dimension m, we have to show that every weakly isotropic form over F contains a weakly isotropic subform of dimension strictly smaller than m.

Let φ be a quadratic form over F. If φ is isotropic, then φ contains a (weakly) isotropic subform of dimension $2 < m$. Assume now that φ contains a weakly isotropic subform of dimension strictly smaller than m. We claim that if $\varphi = \psi \perp \langle a \rangle$, where $a \in F^\times$ and ψ is a form over F, then for any $t \in \sum F^{\times 2}$ the modified form $\widetilde{\varphi} = \psi \perp \langle at \rangle$ also contains a weakly isotropic subform of dimension strictly smaller than m. Once this claim is established, in view of (2.1, a) the general statement follows immediately by induction.

So let φ and $\widetilde{\varphi}$ be as above. From the assumption that $\varphi = \psi \perp \langle a \rangle$ has a weakly isotropic subform of dimension strictly smaller than m we conclude that ψ has a subform γ with $\dim(\gamma) < m$ such that $\gamma \perp \langle a \rangle$ is weakly isotropic. Then $\vartheta = \gamma \perp \langle at \rangle$ is also weakly isotropic, by (2.1, a). Moreover, ϑ is contained in $\widetilde{\varphi} = \psi \perp \langle at \rangle$ and $\dim(\vartheta) \leq m$. Furthermore, if $\dim(\vartheta) = m$ then by the assumption on F the weakly isotropic form ϑ is not minimal, whence it contains a weakly isotropic subform of dimension smaller than m. In any case we have established the claim. $\square$

The following corollary is an extension of [1, Satz 3].

4.2 Corollary. *For a real field F, the following conditions are equivalent:*

(1) $w(F) = 2$.

(2) *Every form φ over F can be decomposed into $\varphi \cong \gamma \perp \beta_1 \perp \cdots \perp \beta_r$ where γ is a strongly anisotropic form and $\beta_1, \ldots, \beta_r$ $(r \geq 0)$ are torsion binary forms.*

(3) *Every torsion form φ over F can be decomposed into an orthogonal sum of torsion binary forms.*

(4) *Every torsion 2-fold Pfister form over F contains a torsion binary form.*

(5) *Every 3-dimensional weakly isotropic form over F contains a torsion binary form.*

(6) *Every F-quaternion division algebra which is split over every real closure of F contains a totally positive quadratic extension of F.*

Proof: The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. The equivalence of (4) and (5) follows from the facts, that any 3-dimensional form φ is up to a scalar factor a subform of a unique 2-fold Pfister form π, that φ is weakly isotropic if and only if π is torsion, and that φ contains a torsion binary form if and only if π does. The equivalence of (4) and (6) follows from Pfister's local-global principle together with the one-to-one correspondence between F-quaternion (division) algebras and (anisotropic) 2-fold Pfister forms. Finally, (5) implies (1), by the above theorem. $\square$

In view of the last corollary, especially of condition (2), we say that a real field F has the *weak decomposition property* or is a *WD-field*, if $w(F) \leq 2$.

4.3 Examples. (1) If I^3F is torsion-free, then F is a *WD*-field, by (3.5). This shows that any number field as well as any extension of transcendence degree at most two of a real closed field has the weak decomposition property.

(2) If F is an *ED*-field (cf. [11]), then it is (obviously) a *WD*-field.

(3) By [1, Satz 5], if F is the rational function field $k(X)$ over a number field k, then F is a *WD*-field.

(4) If $p(F) \leq 2$, then F is a *WD*-field. This shows for example that the rational function field $F = k(X)$ over any hereditarily pythagorean field k is a *WD*-field, since $p(F) = 2$ (by Milnor's exact sequence).

4.4 Conjecture. *Let k be a hereditarily pythagorean field. If F/k is a field extension of transcendence degree one then F is a WD-field.*

In view of condition (5) in the corollary above we want to give a characterization of minimality for weakly isotropic forms in dimension three.

Note that in order to study questions about (weak) isotropy we need to know quadratic forms only up to a scalar factor. In particular, when dealing with odd-dimensional forms we may assume them to have determinant equal to some given non-zero element of the field.

4.5 Proposition. *Let φ be a 3-dimensional quadratic form of trivial determinant over F. Let V denote the underlying vector space of φ.*

(*a*) *The form φ is weakly isotropic if and only if it is totally indefinite, if and only if for any ordering P of F there exists a non-zero vector $v \in V$ such that $\varphi(v)$ is negative at P.*

(*b*) *If φ is weakly isotropic, then φ is minimal weakly isotropic if and only if for any non-zero vector $v \in V$ there exists an ordering P of F such that $\varphi(v)$ is positive at P.*

Proof: The first equivalence in (a) is well-known and can be checked, for example, by applying Pfister's local-global principle to the form $\langle 1\rangle \perp \varphi$ (which has signature 0 or 4 at any ordering). The second equivalence in (a) follows from the fact that the 3-dimensional form φ cannot be negative definite at any ordering of F, as its determinant is totally positive.

Assume now that φ is weakly isotropic but not minimal. Then it contains a binary subform β which is weakly isotropic and therefore of the form $\langle a, -ad\rangle$ where $a \in F^\times$ and $d \in \sum F^{\times 2}$. Comparing determinants we obtain $\varphi \cong \beta \perp \langle -d\rangle$. Hence there exists $v \in V$ such that $\varphi(v) = -d$, which obviously is negative at any ordering of F. This shows one implication claimed in (b).

To prove the converse implication, assume that φ is weakly isotropic and that there exists $v \in V$ for which $\varphi(v)$ is negative at every ordering of F. By the Artin-Schreier Theorem $d = -\varphi(v)$ is a sum of squares in F. Decomposing φ into $\langle -d\rangle \perp \beta$ and comparing determinants we find a torsion binary form β contained in φ. Hence φ is not minimal. $\square$

5 Further results and open problems

We are going to define three more field invariants. This will allow us to refine the earlier observation that $w(F) \leq p(F)$ holds if F is not real pythagorean.

We denote by $b(F)$ (resp. by $b'(F)$) the least $m \in \mathbb{N} \cup \{\infty\}$ such that, given any set $S \subset \sum F^{\times 2}$ with $|S| = m + 1$ (resp. with $|S| < \infty$), there exists an m-dimensional totally positive form ρ over F which represents all elements of S. The property B considered in [1] for a real field F corresponds to the condition that $b(F) \leq 2$. Let further $pu(F)$ denote the least integer $m \geq 1$ such that there exists a positive-universal form of dimension m over F; if there is no positive-universal form over F, then let $pu(F) = \infty$.

5.1 Proposition. *For any F one has the inequalities*

$$b(F) \leq b'(F) \leq pu(F) \leq p(F)\,.$$

Moreover, if any of these numbers is equal to 1*, then F is pythagorean and all four numbers are equal to* 1*.*

Proof: These facts are obvious from the definitions. $\square$

5.2 Examples. (1) If F is a real number field, then $b(F) = b'(F) = 2$. This can be shown by an approximation argument, as it was explained to the author by David Leep.

(2) If F is a real field with finite square class group and such that WF is of elementary type, then by (2.6) we have $b(F) = b'(F) = pu(F) \leq 2$, and therefore also $w(F) = 2$, by the next proposition.

We obtain the following refinement of (3.3).

5.3 Proposition. *If F is not real pythagorean then $w(F) \leq b(F)$.*

Proof: By (4.1), it suffices to show that there exists no minimal weakly isotropic form of dimension $n = b(F) + 1$. Let $\varphi = \langle a_1, \ldots, a_n \rangle$ be a weakly isotropic form over F of dimension n. By (2.1, a), there exist elements $t_1, \ldots, t_n \in \sum F^{\times 2}$ such that $\widetilde{\varphi} = \langle a_1 t_1, \ldots, a_n t_n \rangle$ is isotropic. By the definition of $b(F)$, there exists an $(n-1)$-dimensional totally positive form ρ over F representing all the elements $t_1, \ldots, t_n$. Now, $\rho \otimes \varphi$ contains $\widetilde{\varphi}$ and is therefore isotropic. Hence by (3.1), φ contains a weakly isotropic subform of dimension $n - 1$. $\square$

5.4 Proposition. *Assume that F carries a 2-henselian discrete valuation v and let F_v be its residue field. Then $w(F) = w(F_v)$.*

Proof: Let t be a uniformizer for v. Let φ be a quadratic form over F. We may write $\varphi \cong \varphi_0 \perp t\varphi_1$ where φ_0 and φ_1 have diagonalizations whose entries are units with respect to v. It is well-known that φ is isotropic (resp. weakly isotropic) over F if and only if at least one of the residue forms $\overline{\varphi_0}$ and $\overline{\varphi_1}$ is isotropic (resp. weakly isotropic) over F_v. Therefore φ is a minimal weakly isotropic form over F if and only if one of the residue forms is trivial and the other one is a minimal weakly isotropic form over F_v. This implies the statement. $\square$

5.5 Remark. The analogous statement holds for each of the invariants b, b', pu introduced here, as well as for the Pythagoras number. That is, under the hypothesis of the proposition we have $b(F) = b(F_v)$, $b'(F) = b'(F_v)$, $pu(F) = pu(F_v)$, and $p(F) = p(F_v)$. This follows just from the one-to-one correspondence between the square classes of totally positive elements in F and in F_v. Note that, in contrast to these invariants, the w-invariant is not determined only by the interaction between the totally positive elements of the given field.

By the last proposition, there can be no lower bound on the w-invariant in terms of the stability index. The latter field invariant was introduced in [2]. Fields of stability index 0 or 1 are *SAP*-fields (see [12] or [13] for this notion).

5.6 Question. *Is there an upper bound on $w(F)$ when F is a SAP-field?*

Recall that $w(F) \leq 2$ if F is an *ED*-field. This fact motivates to ask:

5.7 Question. *What values can $b(F)$ take when F is an ED-field?*

We have not seen examples of minimal weakly isotropic forms of dimension greater than 3 and thus have no evidence so far that the w-invariant can have values larger than 3. Needless to say, the author expects the following.

5.8 Conjecture. *Given any number $m \geq 2$, there exists a real field F with a minimal weakly isotropic form of dimension m.*

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