THE J-INVARIANT AND THE TITS ALGEBRAS OF A LINEAR ALGEBRAIC GROUP.

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Abstract. In the present paper we set up a connection between the indices of the Tits algebras of a simple linear algebraic group G and the degree one parameters of its J-invariant. Our main technical tool is the second Chern class map in the Riemann-Roch theorem without denominators.

As an application we recover some known results on the J-invariant of quadratic forms of small dimension; we describe all possible values of the J-invariant of an algebra with involution up to degree 8 and give explicit examples; we establish several relations between the J-invariant of an algebra A with involution and the J-invariant (of the quadratic form) over the function field of the Severi-Brauer variety of A.

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Introduction

The notion of a *Tits algebra* was introduced by J. Tits in his celebrated paper on irreducible representations [40]. This invariant of a linear algebraic group G plays a crucial role in the computation of the K-theory of twisted flag varieties by Panin [29] and in the statements and proofs of the index reduction formulas by Merkurjev, Panin and Wardsworth [27]. It has important applications to the classification of linear algebraic groups, and to the study of the associated homogeneous varieties.

Another invariant of a linear algebraic group, the *J-invariant*, has been recently defined in [34]. It can be viewed as an extension to arbitrary groups of the discrete motivic invariant of a quadratic form which was studied during the last decade, notably by Karpenko, Merkurjev, Rost and Vishik. This invariant describes the motivic behavior of the variety of Borel subgroups of G, and it is given by an r-tuple of integers $J = (j_1, j_2, \dots, j_r)$.

The goal of the present paper is to set up a connection between the indices of the Tits algebras of a group G and the degree one parameters of its J-invariant (see Cor. 3.3 and Thm. 3.8 below). The paper is organized as follows. In § 1, we recall the definitions of the characteristic maps, both for Chow groups and K-theory, and their relation with the restriction maps. The definition of the J-invariant of a twisted form $G = {}_\xi G_0$ of a split group G_0 is recalled in § 2,where we also describe precisely how it depends on the choice of the cocycle ξ. We then state and prove the main results in § 3. As a crucial ingredient, we use Panin's computation of $K_0(\mathfrak{X})$, where $\mathfrak{X}$ is the variety of Borel subgroups of G [29]. The result is obtained using the so-called γ-filtration on $K_0(\mathfrak{X})$, and relies on Lemma 3.11, which describes Chern classes of rational bundles of the first two layers of the γ-filtered group $K_0(\mathfrak{X})$.

The rest of the paper is devoted to applications. First, we recover very easily some known results on the J-invariant of quadratic forms of small dimension using our main theorem. We then explain how one can compute the J-invariant of an algebra with involution up to degree 8. We describe the possible values and give explicit examples. As opposed to what happens for quadratic forms, we also show that some values, which were not excluded before, actually are impossible (see 6.1). Finally, we study the relations between the J-invariant of an algebra A with involution and the J-invariant (of the quadratic form) over the function field of the Severi-Brauer variety of A.

Notations.

Algebraic groups and Borel varieties. We work over a base field k of characteristic different from 2. Let G_0 be a split simple linear algebraic group of rank n over k. We fix a split maximal torus $T_0 \subset G_0$, and a Borel subgroup $B_0 \supset T_0$, and we let $\hat{T}_0$ be the character group of T_0. We let $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a set of simple roots with respect to B_0, and $\{\omega_1, \omega_2, \dots, \omega_n\}$ the respective set of fundamental weights, so that $\alpha_i^\vee(\omega_j) = \delta_{ij}$. The roots and weights are always numbered as in Bourbaki [1].

Recall that $\Lambda_r \subset \hat{T}_0 \subset \Lambda_\omega$, where Λ_r and Λ_ω are the root and weight lattices, respectively. The lattice $\hat{T}_0$ coincides with Λ_r (respectively Λ_ω) if and only if G_0 is adjoint (respectively simply connected).

0.1. Throughout the paper, G denotes a twisted form of G_0, and $T \subset G$ is the corresponding maximal torus. We always assume that G is an inner twisted form of G_0, and even a little bit more, that is $G = {}_\xi G_0$ for some cocycle $\xi \in Z^1(k, G_0)$. Note that this hypothesis need not be satisfied by all groups isogeneous to G; for instance, it is valid for the simply connected cover G^{sc} of G if and only if G is a strongly inner twisted form of G_0.

0.2. We let $\mathfrak{X}_0$ be the variety of Borel subgroups of G_0, or equivalently of its simply connected cover G_0^{sc}, and $\mathfrak{X} = {}_\xi\mathfrak{X}_0$ the corresponding twisted variety. Both varieties are defined over k, and they are isomorphic over a separable closure k_s of k. The Picard group $\mathrm{Pic}(\mathfrak{X}_0)$ can be computed as follows. Since any character $\lambda \in \hat{T}_0$ extends uniquely to B_0, it defines a line bundle $\mathcal{L}(\lambda)$ over $\mathfrak{X}_0$. Hence, there is a natural map $\hat{T}_0 \to \mathrm{Pic}(\mathfrak{X}_0)$, which is an isomorphism if G_0 is simply connected by [28, Prop. 2.2]. So, we may identify the Picard group $\mathrm{Pic}(\mathfrak{X}_0)$ with the weight lattice Λ_ω.

Algebras with involution. We refer to [23] for definitions and classical facts on algebras with involution. Throughout the paper, (A, σ) always stands for a central simple algebra of even degree $2n$, endowed with an involution of orthogonal type with trivial discriminant. In particular, this implies that the Brauer class $[A]$ of the algebra A is an element of order 2 of the Brauer group $\mathrm{Br}(k)$. Because of the discriminant hypothesis, the Clifford algebra of (A, σ), endowed with its canonical involution, is a direct product $(\mathcal{C}(A, \sigma), \underline{\sigma}) = (\mathcal{C}_+, \sigma_+) \times (\mathcal{C}_-, \sigma_-)$ of two central simple algebras. If moreover n is even, the involutions σ_+ and σ_- also are of orthogonal type.

0.3. We refer to [23, § 6] for the definition of isotropic and hyperbolic involutions. In particular, recall that A has a hyperbolic involution if and only if it decomposes as $A = M_2(B)$ for some central simple algebra B over k. When this occurs, A has a unique hyperbolic involution σ_0 up to isomorphism. Moreover, σ_0 has trivial discriminant, and if additionally the degree of A is divisible by 4, then its Clifford algebra has a split component by [23, (8.31)].

0.4. The connected component of the automorphism group of (A, σ) is denoted by $\mathrm{PGO}^+(A, \sigma)$. Since the involution has trivial discriminant, it is an inner twisted form of PGO^+_{2n}, and hence satisfies (0.1). Both groups are adjoint of type D_n. We recall from Bourbaki [1] the description of their cocenter Λ_ω/Λ_r in terms of the fundamental weights, for $n \geq 3$:

If $n = 2m$ is even, then $\Lambda_\omega/\Lambda_r \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the three non-trivial elements are the classes of ω_1, ω_{2m-1} and ω_{2m}.

If $n = 2m + 1$ is odd, then $\Lambda_\omega/\Lambda_r \simeq \mathbb{Z}/4\mathbb{Z}$, and the generators are the classes of ω_{2m} and ω_{2m+1}. Moreover, the element of order 2 is the class of ω_1.

Tits algebras. Consider the simply connected cover G_0^{sc} of G_0 and the corresponding twisted group G^{sc}. We denote by $\Lambda_\omega^+ \subset \Lambda_\omega$ the cone of dominant weights. Since G is an inner twisted form of G_0, for any $\omega \in \Lambda_\omega^+$ the corresponding irreducible representation $G_0^{\mathrm{sc}} \to \mathrm{GL}(V)$, viewed as a representation of $G^{\mathrm{sc}} \times k_s$, descends to an algebra representation $G^{\mathrm{sc}} \to \mathrm{GL}_1(A_\omega)$, where A_ω is a central simple algebra over k, called a Tits algebra of G (cf. [40, §3,4] or [23, §27]). In particular, to any fundamental weight ω_ℓ corresponds a Tits algebra A_{ω_ℓ}

0.5. Taking Brauer classes, the assignment $\omega \in \Lambda_\omega^+ \mapsto A_\omega$ induces a homomorphism to the Brauer group of k (loc. cit.)

$$\beta : \Lambda_\omega/\hat{T}_0 \to \mathrm{B}r(k).$$

0.6. **Example.** If G is adjoint of type D_n, that is $G = \mathrm{PGO}^+(A,\sigma)$, the Tits algebras A_{ω_1}, $A_{\omega_{2m-1}}$ and $A_{\omega_{2m}}$ are respectively the algebra A and the two components $\mathcal{C}_+$ and $\mathcal{C}_-$ of the Clifford algebra of (A,σ) (see [23, §27.B]). Applying Tits homomorphism, and taking into account the description of $\Lambda_\omega/\hat{T}_0 = \Lambda_\omega/\Lambda_r$, we get the so-called fundamental relations [23, (9.12)] relating their Brauer classes, namely:

If $n = 2m$ is even, that is $\deg(A) \equiv 0 \mod 4$, then $[\mathcal{C}_+]$ and $[\mathcal{C}_-]$ are of order at most 2, and $[A] + [\mathcal{C}_+] + [\mathcal{C}_-] = 0 \in \mathrm{Br}(k)$. In other words, any of those three algebras is Brauer equivalent to the tensor product of the other two.

If $n = 2m+1$ is odd, that is $\deg(A) \equiv 2 \mod 4$, then $[\mathcal{C}_+]$ and $[\mathcal{C}_-]$ are of order dividing 4, and $[A] = 2[\mathcal{C}_+] = 2[\mathcal{C}_-] \in \mathrm{Br}(k)$.

For any $\omega \in \Lambda_\omega$, we denote by $i(\omega)$ the index of the Brauer class $\beta(\bar{\omega})$, that is the degree of the underlying division algebra. For fundamental weights, $i(\omega_\ell)$ is the index of the Tits algebra A_{ω_ℓ}.

1. Characteristic maps and restriction maps

Characteristic map for Chow groups. Let $\mathrm{CH}^*(-)$ be the graded Chow ring of algebraic cycles modulo rational equivalence. Since $\mathfrak{X}_0$ is smooth projective, the first Chern class induces an isomorphism between the Picard group $\mathrm{Pic}(\mathfrak{X}_0)$ and $\mathrm{CH}^1(\mathfrak{X}_0)$ [14, Cor. II.6.16]. Combining with the isomorphism $\Lambda_\omega \simeq \mathrm{Pic}(\mathfrak{X}_0)$ of 0.2, we get an isomorphism, which is the simply connected degree 1 characteristic map:

$$\mathfrak{c}_{\mathrm{sc}}^{(1)} : \Lambda_\omega \xrightarrow{\sim} \mathrm{CH}^1(\mathfrak{X}_0).$$

Hence, the cycles

$$h_i := c_1(\mathcal{L}(\omega_i)), \quad i = 1 \dots n,$$

form a $\mathbb{Z}$-basis of the group $\mathrm{CH}^1(\mathfrak{X}_0)$.

1.1. In general, the degree 1 characteristic map is the restriction of this isomorphism to the character group of T_0,

$$\mathfrak{c}^{(1)} \colon \hat{T}_0 \subset \Lambda_\omega \to \mathrm{CH}^1(\mathfrak{X}_0).$$

Hence, it maps $\lambda = \sum_{i=1}^n a_i\omega_i \in \hat{T}_0$, where $a_i \in \mathbb{Z}$, to $c_1(\mathcal{L}(\lambda)) = \sum_{i=1}^n a_i h_i$. For instance, in the adjoint case, the image of $\mathfrak{c}^{(1)}$ is generated by linear combinations $\sum_j c_{ij}h_j$, where $c_{ij} = \alpha_i^\vee(\alpha_j)$ are the coefficients of the Cartan matrix.

1.2. **Example.** We let $p = 2$ and consider the Chow group with coefficients in $\mathbb{F}_2$ $\mathrm{Ch}^1(\mathfrak{X}_0) = \mathrm{CH}^1(\mathfrak{X}_0) \otimes_{\mathbb{Z}} \mathbb{F}_2$. Assume G_0 is of type D_4. Using the simply connected characteristic map, we may identify the degree 1 Chow group modulo 2 with the $\mathbb{F}_2$-lattice

$$\mathrm{Ch}^1(\mathfrak{X}_0) = \mathbb{F}_2 h_1 \oplus \mathbb{F}_2 h_2 \oplus \mathbb{F}_2 h_3 \oplus \mathbb{F}_2 h_4$$

Moreover, numbering roots as in [1], the Cartan matrix modulo 2 is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Therefore, in the adjoint case the image of the characteristic map $\mathfrak{c}_{\mathrm{ad}}^{(1)}$ with $\mathbb{F}_2$-coefficients is the subgroup

$$\mathrm{im}(\mathfrak{c}_{\mathrm{ad}}^{(1)}) = \mathbb{F}_2 h_2 \oplus \mathbb{F}_2(h_1 + h_3 + h_4) \subset \mathrm{Ch}^1(\mathfrak{X}_0)$$

In the half-spin case, that is when one of the two weights ω_3, ω_4 is in $\hat{T}_0$, say $\omega_3 \in \hat{T}_0$, we get

$$\mathrm{im}(\mathfrak{c}^{(1)}_{\mathrm{hs}}) = \mathbb{F}_2 h_2 \oplus \mathbb{F}_2 h_3 \oplus \mathbb{F}_2(h_1 + h_4) \subset \mathrm{Ch}^1(\mathfrak{X}_0)$$

1.3. The degree 1 characteristic map extends to a characteristic map

$$\mathfrak{c}\colon\ S^*(\hat{T}_0) \to \mathrm{CH}^*(\mathfrak{X}_0),$$

where $S^*(\hat{T}_0)$ is the symmetric algebra of $\hat{T}_0$ (see [13, §4], [3, §1.5]). Its image $\mathrm{im}(\mathfrak{c})$ is generated by the elements of codimension one, that is by the image of $\mathfrak{c}^{(1)}$.

Restriction map for Chow groups.

1.4. Let G and $\xi \in Z^1(k, G_0)$ be as in 0.1, so that $G = {}_\xi G_0$. The cocycle ξ induces an identification $\mathfrak{X} \times_k k_s \simeq \mathfrak{X}_0 \times_k k_s$. Moreover, since $\mathfrak{X}_0$ is split, $\mathrm{CH}^*(\mathfrak{X}_0 \times_k k_s) = \mathrm{CH}^*(\mathfrak{X}_0)$. Hence the restriction map can be viewed as a map

$$\mathrm{res}_{\mathrm{CH}}\colon\ \mathrm{CH}^*(\mathfrak{X}) \to \mathrm{CH}^*(\mathfrak{X} \times_k k_s) \simeq \mathrm{CH}^*(\mathfrak{X}_0).$$

A cycle of $\mathrm{CH}^*(\mathfrak{X}_0)$ is called rational if it belongs to the image of the restriction.

In [22, Thm.6.4(1)], it is proven that, under the hypothesis (0.1), any cycle in the image of the characteristic map $\mathfrak{c}$ is rational, i.e.

$$\mathrm{im}(\mathfrak{c}) \subset \mathrm{im}(\mathrm{res}_{\mathrm{CH}}).$$

(See [22, §7] to compare their $\bar{\varphi}_G$ with our characteristic map.)

1.5. **Remark.** Note that the image of the restriction map does not depend on the choice of G in its isogeny class, while the image of the characteristic map does. But the inclusion holds only if G can be obtained from G_0 by twisting by a cocycle with values in G_0. Such a cocycle cannot always be lifted to a cocycle with values in a larger group isogeneous to G_0. For instance, if G is not a strongly inner form of G_0, then there might be some non rational cycles in the image of the simply connected characteristic map.

For a split group G_0, the restriction map is an isomorphism, and this inclusion is strict, except if $H^1(k, G_0)$ is trivial. On the other hand, generic torsors are defined as the torsors for which it is an equality:

1.6. **Definition.** A cocycle $\xi \in Z^1(k, G_0)$ defining the twisted group $G = {}_\xi G_0$ is said to be *generic* if any rational cycle is in $\mathrm{im}(\mathfrak{c})$, so that

$$\mathrm{im}(\mathfrak{c}) = \mathrm{im}(\mathrm{res}_{\mathrm{CH}}).$$

Observe that a generic cocycle always exists over some field extension of k by [22, Thm.6.4(2)].

Characteristic map for K_0 and the Steinberg basis.

1.7. Using the identification between Λ_ω and $\mathrm{Pic}(\mathfrak{X}_0)$ of 0.2, one also gets a characteristic map for K_0 (see [3, §2.8]),

$$\mathfrak{c}_K\colon \mathbb{Z}[\hat{T}_0] \to K_0(\mathfrak{X}_0),$$

where $\mathbb{Z}[\hat{T}_0] \subset \mathbb{Z}[\Lambda_\omega]$ denotes the integral group ring of the character group $\hat{T}_0$. Any generator e^λ, $\lambda \in \hat{T}_0$, maps to the class of the associated line bundle $[\mathcal{L}(\lambda)] \in K_0(\mathfrak{X}_0)$.

Combining a theorem of Pittie [31] (see also [29, §0]), and Chevalley's description of the representation rings of the simply connected cover G_0^{sc} of G_0 and its Borel subgroup B_0^{sc}, one may check that $K_0(\mathfrak{X}_0)$ is isomorphic to the tensor product $\mathbb{Z}[\Lambda_\omega]\otimes_{\mathbb{Z}[\Lambda_\omega]^W}\mathbb{Z}$. That is, the simply-connected characteristic map $\mathfrak{c}_{K,\mathrm{sc}}:\mathbb{Z}[\Lambda_\omega]\to K_0(\mathfrak{X}_0)$ is surjective, and its kernel is generated by the elements of the augmentation ideal that are invariant under the action of the Weyl group W.

1.8. Moreover, Steinberg described in [39, §2] (see also [29, §12.5]) an explicit basis of $\mathbb{Z}[\Lambda_\omega]$ as a free module over $\mathbb{Z}[\Lambda_\omega]^W$. It consists of the weights ρ_w defined for any w in the Weyl group W by

$$\rho_w=\sum_{\{\alpha_k\in\Pi,\ w^{-1}(\alpha_k)\in\Phi^-\}} w^{-1}(\omega_k),$$

where Φ^- denotes the set of negative roots with respect to Π. Hence, we get that the elements

$$g_w:=\mathfrak{c}_{K,\mathrm{sc}}(e^{\rho_w})=[\mathcal{L}(\rho_w)],\quad w\in W,$$

form a $\mathbb{Z}$-basis of $K_0(\mathfrak{X}_0)$, called the *Steinberg basis*. Note that if w is the reflection $w=s_i$, $1\le i\le n$, associated to the root α_i, we get

$$\rho_{s_i}=\sum_{\{\alpha_k\in\Pi,\ s_i(\alpha_k)\in\Phi^-\}} s_i(\omega_k)=s_i(\omega_i)=\omega_i-\alpha_i.$$

1.9. **Definition.** The elements of the Steinberg basis

$$g_i=[\mathcal{L}(\rho_{s_i})],\quad i=1\dots n$$

are called *special*.

Restriction map for K_0 and the Tits algebras.

1.10. As we did for Chow groups, we use the identification $\mathfrak{X}\times_k k_s\simeq\mathfrak{X}_0\times_k k_s$ to view the restriction map for K_0 as a morphism

$$\mathrm{res}_{\mathrm{K}_0}:K_0(\mathfrak{X})\to K_0(\mathfrak{X}_0)=\bigoplus_{w\in W}\mathbb{Z}\cdot g_w.$$

By Panin's theorem [29, Thm. 4.1], the image of the restriction map, whose elements are called rational bundles, is the sublattice with basis

$$\{i(\rho_w)\cdot g_w,\ w\in W\},$$

where $i(\rho_w)$ is the index of the Brauer class $\beta(\bar{\rho_w})$, that is the index of any corresponding Tits algebra (see 0.5).

1.11. Note that since the Weyl group acts trivially on $\Lambda_\omega/\hat{T}_0$, we have

$$\bar\rho_w=\sum_{\{\alpha_k\in\Pi\mid w^{-1}(\alpha_k)\in\Phi^-\}}\bar\omega_k.$$

Therefore, the corresponding Brauer class is given by

$$\beta(\bar\rho_w)=\Sigma_{\{\alpha_k\in\Pi\mid w^{-1}(\alpha_k)\in\Phi^-\}}\beta(\bar\omega_k).$$

For special elements, we get $\beta(\bar\rho_{s_i})=\beta(\bar\omega_i)$, so that $i(\rho_{s_i})$ is the index of the Tits algebra A_{ω_i}.

Rational cycles versus rational bundles. Since the total Chern class of a rational bundle is a rational cycle, the graded-subring $\mathfrak{B}^*$ of $\mathrm{CH}^*(\mathfrak{X}_0)$ generated by Chern classes of rational bundles consists of rational cycles. We use Panin's description of rational bundles to compute $\mathfrak{B}^*$. The total Chern class of $i(\rho_w)\cdot g_w$ is given by

$$c(i(\rho_w)\cdot g_w) = \big(1 + c_1(\mathcal{L}(\rho_w))\big)^{i(\rho_w)} = \sum_{k=1}^{i(\rho_w)} \binom{i(\rho_w)}{k} c_1(\mathcal{L}(\rho_w))^k$$

Therefore, $\mathfrak{B}^*$ is generated as a subring by the homogeneous elements

$$\binom{i(\rho_w)}{k} c_1(\mathcal{L}(\rho_w))^k, \text{ for } w \in W,\ 1 \le k \le i(\rho_w).$$

Let p be a prime number, and denote by $\mathfrak{i}_w$ the p-adic valuation of $i(\rho_w)$, so that $i(\rho_w) = p^{\mathfrak{i}_w} q$ for some prime-to-p integer q. By Luca's theorem [4, p. 271] the binomial coefficient $\binom{i(\rho_w)}{p^{\mathfrak{i}_w}}$ is congruent to q modulo p. Hence its image in $\mathbb{F}_p$ is invertible. Considering the image in the Chow group modulo p of the rational cycle $\binom{i(\rho_w)}{p^{\mathfrak{i}_w}} c_1(\mathcal{L}(\rho_w))^{p^{\mathfrak{i}_w}}$, we get:

1.12. **Lemma.** *Let p be a prime number. For any w in the Weyl group, the cycle*

$$c_1(\mathcal{L}(\rho_w))^{p^{\mathfrak{i}_w}} \in \mathrm{Ch}(\mathfrak{X}_0) = \mathrm{CH}(\mathfrak{X}_0) \otimes_{\mathbb{Z}} \mathbb{F}_p$$

is rational.

2. The J-invariant

In this section, we recall briefly the definition and the key properties of the J-invariant of an algebraic group, following [34]. The definition involves the choice of a cocycle $\xi \in Z^1(k, G_0)$, such that G is the inner twisted form ${}_\xi G_0$ of the split group G_0. For adjoint groups of type D_4, as opposed to all other types, we show that the J-invariant of G does depend on this cocycle, and should actually be considered as an invariant of the cohomology class of ξ, or of the underlying algebra with involution.

2.1. Let us denote by $\pi : \mathrm{CH}^*(\mathfrak{X}_0) \to \mathrm{CH}^*(G_0)$ the pull-back induced by the natural projection $G_0 \to \mathfrak{X}_0$, where $\mathfrak{X}_0$ is the variety of Borel subgroups of G_0. By [13, §4, Rem. 2], π is surjective and its kernel is the ideal $I(\mathfrak{c}) \subset \mathrm{CH}^*(\mathfrak{X}_0)$ generated by non constant elements in the image of the characteristic map (see 1.3). Therefore, there is an isomorphism of graded rings

$$\mathrm{CH}^*(\mathfrak{X}_0)/I(\mathfrak{c}) \simeq \mathrm{CH}^*(G_0)$$

In particular, in degree 1, we get

$$\mathrm{CH}^1(G_0) \simeq \mathrm{CH}^1(\mathfrak{X}_0)/(\mathrm{im}\,\mathfrak{c}^{(1)}) \simeq \Lambda_\omega/\hat{T}_0. \tag{1}$$

Using this fact, V. Kac computed in [17, Thm. 3] the Chow group of G_0 with $\mathbb{F}_p$-coefficients. Namely, he proved it is isomorphic as an $\mathbb{F}_p$-algebra (and even as a Hopf algebra) to

$$\mathrm{Ch}^*(G_0) \simeq \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}})$$

for some integers r and k_i, for $1 \le i \le r$, which depend on the group G_0. For each i, we let d_i be the degree of the generator x_i. Note in particular that the number of generators of degree 1 is the dimension over $\mathbb{F}_p$ of the vector space $\Lambda_\omega/\hat{T}_0 \otimes_{\mathbb{Z}} \mathbb{F}_p$.

The J-invariant of G is related to the Chow-motif $M(\mathfrak{X})$ of the variety of Borel subgroups of G. Namely, the main result (Thm. 5.13) in [34] asserts that the motif $M(\mathfrak{X})$ splits as a direct sum of twisted copies of some indecomposable motif $R_p(G)$. Moreover, the Poincaré polynomial of $R_p(G)$ over a separable closure of k (see [34, §1.3]) is given by

$$P(R_p(G)\times_k k_s, t) = \prod_{i=1}^{r} \frac{1-t^{d_i p^{j_i}}}{1-t^{d_i}}, \text{ where } 0\le j_i\le k_i. \tag{2}$$

As opposed to r, d_i and k_i for $i \in 1\dots r$, which only depend on G_0, the parameters j_i depend on G, and reflect its splitting properties.

From the values given in the table [17, Table II] (see also [34, §4]), one may check that, except if $p=2$ and G is adjoint of type D_n with n even, the degrees d_i are pairwise distinct. If so, we may assume $d_1 < d_2 < \cdots < d_r$ and we get a well defined r-tuple $J_p(G) = (j_1, j_2, \dots, j_r)$. By (2), this tuple is an invariant of G, and does not depend on the choice of the cocycle ξ.

2.2. We now recall from [34, Def. 4.6] the computation of the indices j_i corresponding to the degree 1 generator(s). Let us fix a cocycle $\xi \in Z^1(k, G_0)$ such that $G = {}_\xi G_0$. Let R_ξ be the pull-back in $\mathrm{Ch}^*(G_0)$ of the rational cycles of $\mathrm{Ch}^*(\mathfrak{X}_0)$, that is the image of the composition

$$R_\xi = \mathrm{Im}\big(\mathrm{Ch}^*(\mathfrak{X}) \xrightarrow{\mathrm{res}_{\mathrm{Ch}}} \mathrm{Ch}^*(\mathfrak{X}_0) \xrightarrow{\pi} \mathrm{Ch}^*(G_0)\big),$$

where the restriction map is as defined in 1.4, and π denotes as above the pull-back with respect to the natural projection $G_0 \mapsto \mathfrak{X}_0$.

Let us denote by s the dimension of $\mathrm{Ch}^1(G_0)$ over $\mathbb{F}_p$, and assume first that $s=1$. We pick a generator $x_1 \in \mathrm{Ch}^1(G_0) \simeq \mathbb{F}_p$. Then the parameter j_1 is the smallest non negative integer a such that $x_1^{p^a}$ belongs to R_ξ (see [34, §4.4]). Observe in particular that $j_1 = 0$ if and only if $x_1 \in R_\xi$.

2.3. We now assume that $p=2$ and G_0 is adjoint of type D_{2m}, with $m\ge 2$, that is $G_0 = \mathrm{PGO}^+_{4m}$. As mentioned earlier, this is the only case when $s>1$. Moreover, in view of (1), choosing two degree 1 generators for $\mathrm{Ch}^*(G_0)$ amounts to the choice of two generators of the cocenter $(\Lambda_\omega/\Lambda_r) = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ of the group. We pick the classes of the fundamental weights ω_1 and ω_{2m} (see 0.4), and denote by x_1 and x_2 the corresponding elements in $\mathrm{Ch}^1(G_0)$,

$$x_1 = \pi(h_1), \quad x_2 = \pi(h_{2m}). \tag{3}$$

By definition [34, §4.4] , j_1 is the smallest non negative integer a such that $x_1^{2^a}$ belongs to R_ξ, and j_2 is the smallest integer b such that

$$x_2^{2^b} + \sum_{0<i\le 2^b} a_i x_1^i x_2^{2^b-i} \in R_\xi \text{ for some } a_i \in \mathbb{F}_2.$$

Note that, since $\omega_1 + \omega_{n-1} + \omega_n \in \Lambda_r = \hat{T}_0$ (cf. 0.4), replacing ω_{2m} by ω_{2m-1} amounts to computing $(x_1+x_2)^{2^b}$ instead of $x_2^{2^b}$, and does not affect the value of j_2.

2.4. It follows from the definition that the values of j_1 and j_2 might decrease under field extension. This also applies to the indices corresponding to generators of higher degrees, as explained in [34, 4.7]. Hence, we have

$$J_p(G_E) \le J_p(G), \text{ for any extension } E \text{ of the base field F}.$$

2.5. We now describe how j_1 and j_2 depend on the choice of a cocycle $\xi \in Z^1(k, G_0)$ such that $G = {}_\xi G_0$, with $G_0 = \mathrm{PGO}^+_{4m}$. The exact sequence

$$1 \mapsto G_0 \to \mathrm{Aut}(G_0) \to \mathrm{Aut}(\Delta) \mapsto 1,$$

where Δ denotes the Dynkin diagram of G_0, induces a map

$$H^1(k, G_0) \mapsto H^1(k, \mathrm{Aut}(G_0)).$$

By [23, §29.F], a cohomology class in $H^1(k, G_0)$ corresponds to the k-isomorphism class of a triple (A, σ, ε) where A is a degree $4m$ central simple algebra over k, σ is an orthogonal involution on A with trivial discriminant, and ε is an isomorphism between the center of the Clifford algebra of (A, σ) and $k \times k$. Moreover, the image in $H^1(k, \mathrm{Aut}(G_0))$ of (A, σ, ε) is the connected component $\mathrm{PGO}^+(A, \sigma)$ of its automorphism group.

Assume first that $n \neq 4$, so that $\mathrm{Aut}(\Delta) = \mathbb{Z}/2$. This group acts on $H^1(k, G_0)$, by sending (A, σ, ε) to $(A, \sigma, \varepsilon')$, where ε' is the composite of ε with the non trivial automorphism of $k \times k$. By [37, §5, Prop.39(ii)], we may replace ξ by a cocycle ξ' representing $(A, \sigma, \varepsilon')$. To prove that this does not affect j_1 and j_2, consider the following commutative diagram:

$$\begin{tikzcd} & \mathrm{Ch}^*(\mathfrak{X}_0) \arrow[r] & \mathrm{Ch}^*(G_0) \\ \mathrm{Ch}^*(\mathfrak{X}) \arrow[r] & \mathrm{Ch}^*(\mathfrak{X} \times_k k_s) \arrow[u, "\simeq"'] \arrow[r] & \mathrm{Ch}^*(G \times_k k_s). \arrow[u, "\simeq"'] \end{tikzcd}$$

The vertical arrows, and only those, depend on the choice of a cocycle ξ. We claim that the antecedent in $\mathrm{Ch}^1(G \times_k k_s)$ of the generator $x_1 \in \mathrm{Ch}^1(G_0)$ does not depend on ξ. Hence the parameter j_1 is a well-defined invariant of the group G.

2.6. Indeed, since we numbered roots as in [1], the generator x_1, which corresponds to the class of ω_1 in $(\Lambda_\omega/\Lambda_r) \otimes \mathbb{F}_2 \simeq \mathrm{Ch}^1(G_0)$ can be characterized as follows: it is the unique element of $\mathrm{Ch}^1(G_0)$ killed by the pull-back map $\mathrm{Ch}^1(G_0) \to \mathrm{Ch}^1(\mathrm{O}^+_{4m})$ induced by taking the quotient map modulo the center. Hence, its image in $\mathrm{Ch}^1(G \times_k k_s)$ is the unique class killed by the pull-back map $\mathrm{Ch}^1(G \times_k k_s) \to \mathrm{Ch}^1(\mathrm{O}^+(A, \sigma) \times_k k_s)$. For a given group $G = \mathrm{PGO}^+(A, \sigma)$, this defines the antecedent of x_1 in $\mathrm{Ch}^1(G \times_k k_s)$ uniquely, independently of the choice of ξ.

As opposed to this, the antecedent of the second generator x_2 does depend on ξ. Hence changing the cocycle amounts to changing the second generator x_2, and we already observed that this does not affect the value of j_2.

2.7. The situation is quite different in the trialitarian case, since the automorphism group of Δ then is the symmetric group S_3. Let us assume now that G is adjoint of type D_4, that is $G_0 = \mathrm{PGO}^+_8$. The J-invariant then is a triple (j_1, j_2, j_3), where the first two parameters correspond to generators of degree 1, and the last one to a generator of degree 3. In this case, there are three possibly non isomorphic algebras with involution (A, σ_A), (B, σ_B) and $(\mathcal{C}, \sigma_C)$ such that

$$G = \mathrm{PGO}^+(A, \sigma_A) = \mathrm{PGO}^+(B, \sigma_B) = \mathrm{PGO}^+(\mathcal{C}, \sigma_C),$$

which leads to 6 cohomology classes in $H^1(k, \mathrm{PGO}^+_8)$ by [23, §42.A]. The very same argument as before now shows that the value of j_1 does depend on the choice of a cocycle, or more precisely on the choice of one of the three possible algebras with involution. Indeed, if the cocycle $\xi \in Z^1(k, \mathrm{PGO}^+_{4m})$ corresponds to one of the triples (A, σ, ε) and $(A, \sigma, \varepsilon')$, then the antecedent of x_1 in $\mathrm{Ch}^1(G \times_k k_s)$ is the unique element killed by the pull back to $\mathrm{Ch}^1(\mathrm{O}^+(A, \sigma))$. We denote by $J(A, \sigma)$ the triple (j_1, j_2, j_3), where j_1 and j_2 are computed as in 2.3, and for such a choice of ξ. It is a well-defined invariant of (A, σ).

2.8. By (2), if $J(A, \sigma_A) = (j_1, j_2, j_3)$, and $G = \mathrm{PGO}^+(A, \sigma_A) = \mathrm{PGO}^+(B, \sigma_B)$, then $J(B, \sigma_B) \in \{(j_1, j_2, j_3), (j_2, j_1, j_3)\}$. In 6.3 and 6.8 below, we give a more precise statement, and provide explicit examples of algebras with involution having isomorphic automorphism groups and different J-invariant.

3. The parameters of degree one and indices of Tits algebras.

In this section, we prove the main results of the paper, which give connections between the indices of the J-invariant corresponding to generators of degree 1 and indices of Tits algebras of the group G (cf. [12, §3]).

3.1. From now on, we let s be the dimension over $\mathbb{F}_p$ of $\Lambda_\omega/\hat{T}_0 \otimes \mathbb{F}_p \simeq \mathrm{Ch}^1(G_0)$, and we fix G_0 and p so that $s \geq 1$. If $s = 1$, we fix an index $i_1 \in \{1, \ldots, n\}$ such that the class of ω_{i_1} generates $\Lambda_\omega/\hat{T}_0 \otimes \mathbb{F}_p$, and we let $x_1 \in \mathrm{Ch}^1(G_0)$ be the corresponding generator. If $p = 2$ and G_0 is adjoint of type D_{2m}, so that $s = 2$, we let $i_1 = 1$, $i_2 = 2m$, and define x_1 and x_2 as in (3). If moreover $m = 2$, that is G_0 is adjoint of type D_4, the J-invariant in this section always refers to the J-invariant of (A, σ), where A is the Tits algebra associated to the weight ω_1.

3.2. Consider the special elements g_i, $i = 1 \ldots n$ of the Steinberg basis of $K_0(\mathfrak{X}_0)$ (see Definition 1.9). Since $c_1(g_i) = h_i - c_1(\mathcal{L}(\alpha_i)) \in \mathrm{Ch}^1(\mathfrak{X}_0)$, we have

$$\pi(c_1(g_i)) = \pi(h_i) - \pi(c_1(\mathcal{L}(\alpha_i))) = \pi(h_i) \in \mathrm{Ch}^1(G_0).$$

Hence the generators x_ℓ, $1 \leq \ell \leq s$ may also be defined by $x_\ell = \pi(c_1(g_{i_\ell}))$. In view of the isomorphism (1), it follows that for any $g \in \mathrm{Pic}(\mathfrak{X}_0)$ its Chern class modulo p can be written as

$$c_1(g) = \sum_{\ell=1}^{s} a_\ell c_1(g_{i_\ell}) \mod \mathrm{im}\, \mathfrak{c}^{(1)} \in \mathrm{Ch}^1(\mathfrak{X}_0) \tag{4}$$

As an immediate consequence of rationality of cycles introduced in Lemma 1.12 we obtain another proof of the first part of [32, Prop. 4.2]:

3.3. **Corollary.** *The first entry j_1 of the J-invariant is bounded*

$$j_1 \leq \mathfrak{i}_{i_1},$$

by the p-adic valuation $\mathfrak{i}_{i_1}$ of the index of the Tits algebra $A_{\omega_{i_1}}$ associated to ω_{i_1}.

Proof. We apply lemma 1.12 to the weight $\rho_{s_{i_1}} = \omega_{i_1} - \alpha_{i_1}$. As noticed in 1.11, the index $i(\rho_{s_{i_1}})$ is equal to the index $i(\omega_{i_1})$ of the Tits agebra $A_{\omega_{i_1}}$. Hence, the cycle $c_1(g_{i_1})^{p^{\mathfrak{i}_{i_1}}}$ is rational, and its image ${x_1}^{p^{\mathfrak{i}_{i_1}}} \in \mathrm{Ch}^*(G_0)$ belongs to R_ξ. The inequality then follows from the definition of j_1 (see 2.2). □

If $s = 2$, the very same argument also applies to ρ_{s_ℓ} for $\ell = 2m - 1$ and $\ell = 2m$. Combining with the definition of j_2 given in 2.3, we get

3.4. **Corollary.** *If $p = 2$ and G is adjoint of type D_{2m}, so that $s = 2$, then the second entry j_2 of the J-invariant is bounded*

$$j_2 \leq \min\{\mathsf{i}_{2m-1}, \mathsf{i}_{2m}\},$$

where i_ℓ is the 2-adic valuation of the index of the Tits algebra A_{ω_ℓ}.

The next result, which gives an inequality in the other direction, uses the notion of common index, which we introduce now.

3.5. **Definition.** Consider the Tits algebras $A_{\omega_{i_\ell}}$ associated to the fundamental weights ω_{i_ℓ}, for $1 \leq \ell \leq s$, where i_ℓ are as in 3.1. We define their *common index* i_J to be the p-adic valuation of the greatest common divisor of all the indices $\mathrm{ind}(A_{\omega_{i_1}}^{\otimes a_1} \otimes \ldots \otimes A_{\omega_{i_s}}^{\otimes a_s})$, where at least one of the a_i is coprime to p.

3.6. **Example.** If $s = 1$, then i_J is the p-adic valuation i_{i_1} of the index of the Tits algebra $A_{\omega_{i_1}}$. Assume for instance that G is adjoint of type D_{2m+1}. As recalled in 0.4, we may take $i_1 = 2m$ or $i_1 = 2m+1$, so that i_J is the 2-adic valuation of any component $\mathcal{C}_+$ or $\mathcal{C}_-$ of the Clifford algebra of (A, σ). From the fundamental relations 0.6, we know that the two components have the same index.

3.7. **Example.** If $s = 2$, we have $p = 2$ and G_0 is adjoint of type D_{2m}. Using 0.6, one may check that i_J is the p-adic valuation of the greatest common divisor of the indices of A_{ω_1}, $A_{\omega_{2m-1}}$ and $A_{\omega_{2m}}$, that is

$$\mathsf{i}_J = \min\{\mathsf{i}_1, \mathsf{i}_{2m-1}, \mathsf{i}_{2m}\}.$$

We will prove:

3.8. **Theorem.** *Let i_J be the common index of the Tits-algebras $A_{\omega_{i_\ell}}$, for $1 \leq \ell \leq s$.*
If $\mathsf{i}_J > 0$, then $j_\ell > 0$ for any ℓ, $1 \leq \ell \leq s$.
If $\mathsf{i}_J > 1$ and $p = 2$, then for any ℓ such that $k_\ell > 1$, the corresponding index also satisfies $j_\ell > 1$.

Consider the ideal $I(\mathrm{res}_{\mathrm{Ch}})$ of $\mathrm{Ch}^*(\mathfrak{X}_0)$ generated by the non constant rational elements. For any integer i, we let $I(\mathrm{res}_{\mathrm{Ch}})^{(i)} \subset \mathrm{Ch}^i(\mathfrak{X}_0)$ be the homogeneous part of degree i. Since the image of the characteristic map consists of rational elements, we have $I(\mathfrak{c}) \subset I(\mathrm{res}_{\mathrm{Ch}})$. The theorem follows easily from the following lemma:

3.9. **Lemma.** *If $\mathsf{i}_J > 0$, then $I(\mathrm{res}_{\mathrm{Ch}})^{(1)} = I(\mathfrak{c})^{(1)} \subset \mathrm{Ch}^1(\mathfrak{X}_0)$.*
If $\mathsf{i}_J > 1$ and $p = 2$, then $I(\mathrm{res}_{\mathrm{Ch}})^{(2)} = I(\mathfrak{c})^{(2)} \subset \mathrm{Ch}^2(\mathfrak{X}_0)$

Indeed, let us assume first that $\mathsf{i}_J > 0$. By the lemma, any element in $\mathrm{im}(\mathrm{res}_{\mathrm{Ch}}^{(1)}) = I(\mathrm{res}_{\mathrm{Ch}})^{(1)}$ belongs to $I(\mathfrak{c})^{(1)}$, which is in the kernel of π. Therefore, the image of the composition

$$R_\xi^{(1)} = \mathrm{im}\big(\mathrm{Ch}^1(\mathfrak{X}) \xrightarrow{\mathrm{res}_{\mathrm{Ch}}^{(1)}} \mathrm{Ch}^1(\mathfrak{X}_0) \xrightarrow{\pi} \mathrm{Ch}^1(G_0)\big)$$

is trivial, $R_\xi^{(1)} = \{0\}$. From the definition 2.3 of j_1 and j_2, this implies that they are both strictly postive.

The proof of the second part follows the same lines. We write it in details for $s = 2$ and $k_1, k_2 > 1$. Assume that $\mathsf{i}_J > 1$. Since the image $\mathrm{im}(\mathrm{res}_{\mathrm{Ch}})^{(2)}$ is contained in $I(\mathrm{res}_{\mathrm{Ch}})^{(2)}$, the lemma again implies that $R_\xi^{(2)} = \{0\}$. On the other hand, the hypothesis on k_1 and k_2 guarantees that in the truncated polynomial algebra $\mathbb{F}_2[x_1, x_2]/(x_1^{2^{k_1}}, x_2^{2^{k_2}}) \subset \mathrm{Ch}^*(G_0)$, the elements x_1^2 and $x_2^2 + a_1x_1x_2 + a_2x_1^2$ are all non trivial. Hence they do not belong to R_ξ, and we get $j_1, j_2 > 1$.

The rest of the section is devoted to the proof of Lemma 3.9. The main tool is the Riemann-Roch theorem, which we now recall.

Filtrations of K_0 and the Riemann-Roch Theorem.

3.10**.** Let X be a smooth projective variety over k. Consider the topological filtration on $K_0(X)$ given by

$$\tau^i K_0(X) = \langle [\mathcal{O}_V],\ \mathrm{codim} V \geq i\rangle,$$

where $\mathcal{O}_V$ is the structure sheaf of the closed subvariety V in X. There is an obvious surjection

$$p:\ \mathrm{CH}^i(X) \to \tau^{i/i+1}K_0(X) = \tau^i K_0(X)/\tau^{i+1}K_0(X),$$

given by $V \mapsto [\mathcal{O}_V]$. By the Riemann-Roch theorem without denominators [6, §15], the i-th Chern class induces a map in the opposite direction

$$c_i:\ \tau^{i/i+1}K_0(X) \to \mathrm{CH}^i(X)$$

and the composite $c_i \circ p$ is the multiplication by $(-1)^{i-1}(i-1)!$. In particular, it is an isomorphism for $i \leq 2$ (see [6, Ex. 15.3.6]).

The topological filtration can be approximated by the so-called γ-filtration. Let $c_i^{K_0}$ be the i-th Chern class with values in K_0 (see [6, Ex. 3.2.7(b)], or [19, §2]). We use the convention $c_1^{K_0}([\mathcal{L}]) = 1 - [\mathcal{L}^v]$ for any line bundle $\mathcal{L}$, where $\mathcal{L}^v$ is the dual of $\mathcal{L}$, so that in the Chow group,

$$c_1(c_1^{K_0}([\mathcal{L}])) = c_1(\mathcal{L}).$$

Similarly, one may compute the second Chern class

$$c_2\big(c_1^{K_0}([\mathcal{L}_1])c_1^{K_0}([\mathcal{L}_2])\big) = -c_1(\mathcal{L}_1)c_1(\mathcal{L}_2). \tag{5}$$

The γ-filtration on $K_0(X)$ is given by the subgroups (cf. [11, §1])

$$\gamma^i K_0(X) = \langle c_{n_1}^{K_0}(b_1)\cdot\ldots\cdot c_{n_m}^{K_0}(b_m) \mid n_1 + \ldots + n_m \geq i,\ b_l \in K_0(X)\rangle,$$

(see [6, Ex.15.3.6], [7, Ch.3,5]). We let $\gamma^{i/i+1}(K_0(X)) = \gamma^i K_0(X)/\gamma^{i+1}K_0(X)$ be the respective quotients, and $\gamma^*(K_0(X)) = \oplus_{i\geq 0}\gamma^{i/i+1}(K_0(X))$ the associated graded ring.

By [19, prop 2.14], $\gamma^i(K_0(X))$ is contained in $\tau^i(K_0(X))$, and they coincide for $i \leq 2$. Hence, by Riemann-Roch theorem, the Chern class c_i with values in $\mathrm{CH}^i(X)$ vanishes on $\gamma^{(i+1)}K_0(X)$, and induces a map

$$c_i:\ \gamma^{i/i+1}(K_0(X)) \to \mathrm{CH}^i(X).$$

In codimension 1 we get an isomorphism

$$c_1\colon \gamma^{1/2}(K_0(X)) \xrightarrow{\simeq} \mathrm{CH}^1(X)$$

which sends for a line bundle $\mathcal{L}$ the class $c_1^{K_0}(\mathcal{L})$ to $c_1(\mathcal{L})$. In codimension 2 the map

$$c_2\colon \gamma^{2/3}(K_0(X)) \twoheadrightarrow \mathrm{CH}^2(X),$$

is surjective and has torsion kernel [19, Cor. 2.15].

Let us now apply this to the varieties $\mathfrak{X}_0$ and $\mathfrak{X}$ of Borel subgroups of G_0 and G respectively. Since $K_0(\mathfrak{X}_0)$ is generated by the line bundles $g_w = [\mathcal{L}(\rho_w)]$ for $w \in W$, one may check that $\gamma^{i/i+1}(\mathfrak{X}_0)$ is generated by the products

$$\{c_1^{K_0}(g_{w_1})\dots c_1^{K_0}(g_{w_i}),\ w_1,\dots,w_i \in W\}.$$

Moreover, the restriction map commutes with Chern classes, so it induces

$$\mathrm{res}_\gamma : \gamma^*(\mathfrak{X}) \to \gamma^*(\mathfrak{X}_0).$$

Using Panin's description of the image of the restriction map $\mathrm{res}_{\mathrm{K}_0}$ recalled in 1.10, we get that the image of $\mathrm{res}_\gamma^{(1)} : \gamma^{1/2}(\mathfrak{X}) \to \gamma^{1/2}(\mathfrak{X}_0)$ is generated by the elements $c_1^{K_0}(i(\rho_w)g_w) = i(\rho_w)c_1^{K_0}(g_w)$, for any $w \in W$, while the image of $\mathrm{res}_\gamma^{(2)}$ is generated by

$$i(\rho_{w_1})i(\rho_{w_2})c_1^{K_0}(g_{w_1})c_1^{K_0}(g_{w_2}) \text{ and } c_2^{K_0}(i(\rho_w)g_w) \text{ for } w_1, w_2, w \in W.$$

If the index $i(\rho_w)$ is 1, then $c_2^{K_0}(i(\rho_w)g_w) = 0$. Otherwise, the Whitney sum formula gives

$$c_2^{K_0}(i(\rho_w)g_w) = \binom{i(\rho_w)}{2} c_1^{K_0}(g_w)^2.$$

Applying the morphisms c_1 and c_2, and using (5), we now get

3.11. **Lemma.** *The subgroup* $c_1\big(\mathrm{im}(\mathrm{res}_\gamma^{(1)})\big) \in \mathrm{CH}^1(\mathfrak{X}_0)$ *is generated by* $i(\rho_w)c_1(g_w)$, *for all* $w \in W$. *The subgroup* $c_2\big(\mathrm{im}(\mathrm{res}_\gamma^{(2)})\big) \in \mathrm{CH}^2(\mathfrak{X}_0)$ *is generated by the elements* $i(\rho_{w_1})i(\rho_{w_2})c_1(g_{w_1})c_1(g_{w_2})$ *and* $\binom{i(\rho_w)}{2}c_1(g_w)^2$ *for all* $w_1, w_2, w \in W$.

Proof of Lemma 3.9. Since the image of the characteristic map consists of rational elements (see 1.4), we already know that $I(\mathfrak{c}) \subset I(\mathrm{res}_{\mathrm{Ch}})$. We now prove the reverse inclusions for the homogeneous parts of degree 1 and 2 under the relevant hypothesis on the common index $\mathfrak{i}_J$. Note that since c_1 and c_2 are both surjective, and commute with restriction maps, one has

$$\mathrm{im}\big(\mathrm{res}_{\mathrm{Ch}}^{(k)}\big) = c_k\big(\mathrm{im}(\mathrm{res}_\gamma^{(k)})\big), \text{ for } k = 1, 2.$$

In degree 1, we have $I(\mathrm{res}_{\mathrm{Ch}})^{(1)} = \mathrm{im}(\mathrm{res}_{\mathrm{Ch}}^{(1)})$, so to prove the first part of the lemma, we have to prove that if $\mathfrak{i}_J > 0$, then for any $w \in W$, the element $i(\rho_w)c_1(g_w)$ belongs, after tensoring with $\mathbb{F}_p$, to $I(\mathfrak{c})^{(1)} = \mathrm{im}\,\mathfrak{c}^{(1)}$. Let us write

$$c_1(g_w) = \sum_{\ell=1}^{s} a_\ell c_1(g_{i_\ell}) \mod \mathrm{im}\,\mathfrak{c}^{(1)},$$

as in (4). If all the $a_\ell \in \mathbb{F}_p$ are trivial, we are done, so we may assume at least one of them is invertible in $\mathbb{F}_p$. The weights ρ_w and ρ_{i_ℓ} satisfy the same relation

$$\rho_w = \sum_{\ell=1}^{s} a_\ell \rho_{i_\ell} \mod \hat{T}_0 \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

Applying the morphism β, we get that the p-primary part of the Brauer class $\beta(\bar{\rho}_w)$ coincides with the p-primary part of the Brauer class of $\otimes_{\ell=1}^{s} A_{\omega_{i_\ell}}^{a_\ell}$ (see 1.11). The hypothesis on $\mathfrak{i}_J$ guarantees that this index of this algebra is divisible by p. Hence $i(\rho_w)$, which is the index of $\beta(\bar{\rho}_w)$ also is divisible by p, so that $i(\rho_w)c_1(g_w) = 0$ in the Chow group $\mathrm{Ch}^1(\mathfrak{X}_0)$ modulo p, and we are done.

Let us now assume that $p = 2$ and $\mathfrak{i}_J > 1$. The homogeneous part $I(\mathrm{res}_{\mathrm{Ch}})^{(2)}$ decomposes as

$$I(\mathrm{res}_{\mathrm{Ch}})^{(2)} = \mathrm{im}\big(\mathrm{res}^{(1)}_{\mathrm{Ch}}\big)\,\mathrm{Ch}^1(\mathfrak{X}_0) + \mathrm{im}\big(\mathrm{res}^{(2)}_{\mathrm{Ch}}\big).$$

By the first part of the Lemma, we already know that

$$\mathrm{im}\big(\mathrm{res}^{(1)}_{\mathrm{Ch}}\big)\,\mathrm{Ch}^1(\mathfrak{X}_0) \subset I(\mathfrak{c}).$$

Hence it remains to prove that $\mathrm{im}(\mathrm{res}^{(2)}_{\mathrm{Ch}}) = c_2(\mathrm{im}\,\mathrm{res}^{(2)}_{\gamma}) \subset I(\mathfrak{c})^{(2)}$. The proof for the degree 1 part already shows that $i(\rho_{w_1})i(\rho_{w_2})c_1(g_{w_1})c_1(g_{w_2})$ belongs to $I(\mathfrak{c})^{(2)}$. The same argument extends to $\binom{i(\rho_w)}{2}c_1(g_w)^2$. Indeed, if the coefficients a_ℓ are not all trivial modulo 2, the condition on the common index now implies that 4 divides $i(\rho_w)$, so that $\binom{i(\rho_w)}{2}$ is zero modulo 2.

4. Applications to quadratic forms

The purpose of this section is to apply our main theorem 3.8 to quadratic forms. The results presented here are not new, but some of them can be recovered very easily from our inequalities. Let φ be a quadratic form of even dimension $2n$. We always assume that φ has trivial discriminant, so that its special orthogonal group $\mathrm{O}^+(\varphi)$ satisfies condition 0.1. We define the J-invariant of φ as follows:

4.1. **Definition.** Let φ be a $2n$ dimensional quadratic form over F, with trivial discriminant. Its J-invariant is

$$J(\varphi) = J_2\big(\mathrm{O}^+(\varphi)\big).$$

4.2. **Remark.** (i) The J-invariant of a quadratic form was initially defined by Vishik in [41, Def 5.11]. The invariant considered here is closely related to Vishik's J-invariant, and also to the dual version given in [5, §88]. We refer the reader to[34, §4.8] for a more precise statement.

(ii) Let φ_0 be any non degenerate subform of φ of codimension 1. The maximal orthogonal grassmanian of φ_0 is isomorphic to any connected component of the maximal orthogonal grassmanian of φ (see [5, 85.2]). So by (2), the forms φ_0 and φ have the same J-invariant (see also [5, §88]). Since any odd-dimensional form can be embedded in an even dimensional form with trivial discriminant, we only consider the even-dimensional case.

The J-invariant is an invariant of the Witt-class of a quadratic form, in the following sense:

4.3. **Proposition.** *Let φ and φ' be two Witt-equivalent even-dimensional quadratic forms with trivial discriminant. All the non trivial indices in their J-invariant are equal.*

Moreover, $J(\varphi) = (0, \dots, 0)$ if and only if φ is hyperbolic.

Proof. In general, the forms φ and φ' have different dimensions, so strictly speaking, their J-invariants are different. Nevertheless, we claim that deleting all the zero indices on both sides, we get the same tuple of integers. In view of (2), this is a direct consequence of [34, 5.18(iii)], since the maximal orthogonal grassmanians of φ and φ' clearly have the same splitting fields.

From [34, 6.7], a quadratic form with trivial J-invariant is hyperbolic over some odd-degree extension of the base field. Hence the second statement follows from Springer's theorem [5, 18.5]. □

In small dimension, the J-invariant of a quadratic form φ can be explicitly computed in terms of the index of its full Clifford algebra $\mathcal{C}(\varphi)$. More precisely, we let $\mathfrak{i}_S$ be the 2-adic valuation of the index of $\mathcal{C}(\varphi)$. Since φ has trivial discriminant, its even Clifford algebra has center $F \times F$ and splits as $\mathcal{C}_0(\varphi) = C \times C$ for some central simple algebra C which is Brauer-equivalent to $\mathcal{C}(\varphi)$, that is $\mathcal{C}(\varphi) \simeq M_2(C)$. In particular, it follows that $\mathfrak{i}_S \leq n-1$. In this setting, we have $s = 1$, and C is the Tits algebra associated to a generator of $\Lambda_\omega/\hat{T}_0 \otimes \mathbb{F}_2$. So the common index $\mathfrak{i}_J$ is given by $\mathfrak{i}_J = \mathfrak{i}_S$. With this notation, the inequalities 3.3 and 3.8 can be translated as follows:

4.4. **Corollary.** *Let φ be a $2n$ dimensional quadratic form with trivial discriminant. The 2-adic valuation $\mathfrak{i}_S$ of its Clifford algebra and the first index j_1 of its J-invariant are related as follows:*

(1) $j_1 \leq \mathfrak{i}_S$
(2) *If $n \geq 2$, and $\mathfrak{i}_S > 0$, then $j_1 > 0$.*
(3) *If $n \geq 3$ and $\mathfrak{i}_S > 1$, then $j_1 > 1$.*

Assume now that φ has dimension 4 or 6, that is $n = 2$ or 3. By table [34, 4.13], the J invariant of φ consists of a single integer, $J(\varphi) = (j_1)$, which is bounded by 1 (respectively 2) if φ has dimension 4 (respectively 6). Therefore, by Corollary 4.4, we have:

4.5. **Proposition.** *Let φ be a quadratic form of dimension 4 or 6 with trivial discriminant. Its J-invariant is $J(\varphi) = (\mathfrak{i}_S)$.*

We can give a more precise description of the quadratic form φ in each case. In dimension 4, the Clifford algebra of φ is Brauer-equivalent to a quaternion algebra Q over F, and φ is similar to the norm form n_Q of Q, which is a 2-fold Pfister form. It is hyperbolic if Q is split and anisotropic otherwise.

If φ has dimension 6, its Clifford algebra is a biquaternion algebra B over F, and φ is an Albert form of B. If $J(\varphi) = (2)$, or equivalently B is division, then φ is anisotropic. If $J(\varphi) = (1)$, or equivalently B is Brauer equivalent to a non split quaternion algebra Q, then φ is similar to $n_Q \oplus \mathbb{H}$. If $J(\varphi) = (0)$, or equivalently B is split, then φ is hyperbolic. Hence we get:

4.6. **Corollary.** *A quadratic form φ of dimension 6 and trivial discriminant is anisotropic if and only if its J-invariant is $J(\varphi) = (2)$; it is isotropic and non-hyperbolic if and only if $J(\varphi) = (1)$.*

Let us now consider quadratic forms of dimension 8 with trivial discriminant. Their special orthogonal group have type D_4, and table [34, 4.13] now says $J(\varphi) = (j_1, j_2)$ with $0 \leq j_1 \leq 2$ and $0 \leq j_2 \leq 1$. We have:

4.7. **Proposition.** *Let φ be a quadratic form of dimension 8 with trivial discriminant, and consider its J-invariant $J(\varphi) = (j_1, j_2)$. The first index j_1 is given by $j_1 = \min\{\mathfrak{i}_S, 2\}$. Moreover, j_2 is 0 if φ is isotropic and 1 if φ is anisotropic.*

Proof. Again, the first assertion follows instantly from Corollary 4.4. To prove the second, let us first assume that φ is isotropic. If it is hyperbolic, we already know that $J(\varphi) = (0, 0)$, so in particular $j_2 = 0$. Otherwise, φ splits as $\varphi = \varphi_0 \oplus \mathbb{H}$ for some 6-dimensional non hyperbolic form φ_0 with trivial discriminant. By 4.3, this implies that one of the two indices j_1 and j_2 of $J(\varphi)$ is zero. Since the Clifford algebra of φ is Brauer-equivalent to the Clifford algebra of φ_0, which is non split by 4.6, $j_1 = \mathfrak{i}_S \neq 0$. Therefore, we get $j_2 = 0$ as expected.

To prove the converse, we distinguish two cases, depending on the value of $\mathfrak{i}_S$. Let us first assume that φ is anisotropic and $\mathfrak{i}_S \leq 2$. Consider a generic splitting F_C of the Clifford algebra of φ. By a theorem of Laghribi [25, Thm. 4], the form φ remains anisotropic after scalar extension to F_C. Hence the J-invariant of φ_{F_C} is non-trivial. On the other hand, since its Clifford algebra is split, the first index is zero. Therefore the second index j_2 is 1, and this is a fortiori the case over the base field.

Finally, let us assume $\mathfrak{i}_S = 3$, which implies in particular that φ is anisotropic. Any field extension over which the quadratic form φ becomes hyperbolic splits its Clifford algebra. Therefore the index of such a field extension has to be at least $8 = 2^3$. Hence, by [34, 6.6], we have $3 \leq j_1 + j_2$, so that $J(\varphi) = (2, 1)$. □

In his paper [15], Detlev Hoffmann classified quadratic forms of small dimension in terms of their splitting pattern. Using his classification, one can give a precise description of quadratic forms of dimension 8 with trivial discriminant, depending on the value of their J-invariant. The results are summarized in the table below. The notation $J_v(\varphi)$ stands for Vishik's J-invariant, as defined in [5, §88]. The index $\mathfrak{i}$ is the 2-adic valuation of the greatest common divisor of the degrees of the splitting fields of φ. In the explicit description, Pf_k stands for a k-fold Pfister form, $s_{l/k}(Pf_2)$ for the Scharlau transfer of a 2-fold Pfister form with respect to a quadratic field extension, and Al_6 for an Albert form.

$J(\varphi)$	$J_v(\varphi)$	$\mathfrak{i}_S$, $\mathfrak{i}$	Splitting Pattern	Description
(0)	$\emptyset$	$\mathfrak{i}_S = \mathfrak{i} = 0$	(4)	hyperbolic
(1,0)	$\{1\}$	$\mathfrak{i}_S = \mathfrak{i} = 1$	(2,4)	$Pf_2 \perp 2\mathbb{H}$
(2,0)	$\{1, 2\}$	$\mathfrak{i}_S = \mathfrak{i} = 2$	(1,2,4)	$Al_6 \perp \mathbb{H}$
(0,1)	$\{3\}$	$\mathfrak{i}_S = 0$; $\mathfrak{i} = 1$	(0,4)	Pf_3
(1,1)	$\{1, 3\}$	$\mathfrak{i}_S = \mathfrak{i} = 1$	(0,2,4)	$q = \langle 1, -a\rangle \otimes q'$
(2,1)	$\{1, 2, 3\}$	$\mathfrak{i}_S = \mathfrak{i} = 2$	(0,1,2,4)	$Pf_2 \perp Pf_2$ or $s_{l/k}(Pf_2)$
		$\mathfrak{i}_S = \mathfrak{i} = 3$	"	generic

In particular, all possible values for the J-invariant do occur in this setting. Moreover, it follows from the table that the J-invariant and the splitting pattern uniquely determine each other. This is not true anymore in higher degree. Indeed, consider a 10-dimensional quadratic form φ over F with splitting pattern $\{0, 2, 3, 5\}$. By Hoffmann's classification theorem [15, Thm 5.1], φ has trivial discriminant, its Clifford algebra has index 4, so that $\mathfrak{i}_S = 2$ and it is a Pfister neighbor. We claim that its J-invariant is $J(\varphi) = (2, 0)$. To prove this, one may use the relation between the splitting pattern and Vishik's J invariant of a quadratic form as described in [5, 88.8]. Once translated in terms of $J(\varphi)$ following [34, 4.8], we get that $J(\varphi)$ is $(j_1, 0)$ for some integer j_1, $1 \leq j_1 \leq 3$. Since $\mathfrak{i}_S = 2$, Corollary 4.4 give $j_1 = 2$. From the classification in degree 8, the value $(2, 0)$ also is the J invariant of $\pi + 2\mathbb{H}$ for any Albert form π. On the other hand, such a form has splitting pattern $\{2, 3, 5\}$.

5. J-invariant of an algebra with involution

Recall that (A, σ) is a degree $2n$ central simple algebra over k, endowed with an involution of orthogonal type and trivial discriminant. In particular, this implies that A has exponent 2, so that it has index $2^{\mathfrak{i}_A}$ for some integer $\mathfrak{i}_A$. The connected component $\mathrm{PGO}^+(A, \sigma)$ of the automorphism group of (A, σ) is an adjoint group of type D_n. Because of the discriminant hypothesis, it is an inner twisted form of PGO^+_{2n}. If the degree of A is different from 8, we define

$$J(A,\sigma) = J_2\big(\mathrm{PGO}^+(A,\sigma)\big).$$

In degree 8, $J(A,\sigma)$ was defined in 2.7 as the J-invariant of $\mathrm{PGO}^+(A,\sigma)$, computed with a suitable cocycle. Therefore, from the table [34, 4.13], one may check that $J(A,\sigma)$ is an r-tuple $J(A,\sigma) = (j_1, j_2, \dots, j_r)$, with $r = m+1$ if $n = 2m$ and $r = m$ if $n = 2m+1$. Note that our notation slightly differs from the notation in the table, where in the n-odd case, they have an additional index, but which is bounded by $k_1 = 0$. So, for n odd, our $(j_1,\dots,j_r)$ coincides with $(j_2,\dots,j_{r+1})$ in [34]. In particular, the indices corresponding to generators of degree 1 are j_1 if n is odd and j_1 and j_2 if n is even.

Since σ has trivial discriminant, its Clifford algebra splits as a direct product $\mathcal{C}(A,\sigma) = \mathcal{C}_+ \times \mathcal{C}_-$ of two central simple algebras over k. We let i_A (respectively i_+, i_-) be the 2-adic valuation of the index of A (respectively $\mathcal{C}_+$, $\mathcal{C}_-$). From Examples 3.6 and 3.7, the common index i_J is

$$\mathsf{i}_J = \begin{cases} \mathsf{i}_+ = \mathsf{i}_- & \text{if } n \text{ is odd,}\\ \min\{\mathsf{i}_A, \mathsf{i}_+, \mathsf{i}_-\} & \text{if } n \text{ is even.}\end{cases}$$

Hence, Corollaries 3.3 and 3.4 and Theorem 3.8 translate as follows:

5.1. **Corollary.** *Assume that n is odd, so that $\deg(A) \equiv 2[4]$, and let $\mathsf{i}_S = \mathsf{i}_+ = \mathsf{i}_-$. We have:*

(1) $j_1 \le \mathsf{i}_S$*;*
(2) *If* $\mathsf{i}_S > 0$*, then* $j_1 > 0$*;*
(3) *If* $\deg(A) \ge 6$ *and* $\mathsf{i}_S > 1$*, then* $j_1 > 1$*.*

5.2. **Corollary.** *Assume now that n is even, that is $\deg(A) \equiv 0[4]$, and let $\mathsf{i}_J = \min\{\mathsf{i}_A, \mathsf{i}_+, \mathsf{i}_-\}$. We have:*

(1) $j_1 \le \mathsf{i}_A$*;*
(2) $j_2 \le \min\{\mathsf{i}_+, \mathsf{i}_-\}$*;*
(3) *If* $\mathsf{i}_J > 0$*, then* $j_1 > 0$ *and* $j_2 > 0$*.*
(4) *If* $\deg(A) \equiv 0[8]$ *and* $\mathsf{i}_J > 1$*, then* $j_1 > 1$*.*
(5) *If* $\deg(A) \ge 8$ *and* $\mathsf{i}_J > 1$*, then* $j_2 > 1$*.*

The additional conditions on the degrees are obtained from the table [34, 4.13], and guarantee that $k_1 > 1$ or $k_2 > 1$.

Split case. If A is split, the involution σ is adjoint to a quadratic form φ over k. We then have:

5.3. **Proposition.** *If A is split and σ is adjoint to the quadratic form φ, the J-invariants of (A,σ) and φ are related as follows:*

$$J(A,\sigma) = \begin{cases} J(\varphi) & \textit{if } \deg(A) \equiv 2[4]\\ \big(0, J(\varphi)\big) & \textit{if } \deg(A) \equiv 0[4]\end{cases}$$

Proof. Assume A is split, and σ is adjoint to the quadratic form φ. Since $d(\varphi) = d(\sigma) = 1$, the J invariant of φ is defined,

$$J(\varphi) = J_2\big(\mathrm{O}^+(\varphi)\big).$$

Moreover, the groups $\mathrm{O}^+(\varphi) = \mathrm{O}^+(A,\sigma)$ and $\mathrm{PGO}^+(A,\sigma) = \mathrm{PGO}^+(\varphi)$ are isogeneous. Therefore, the corresponding varieties of Borel subgroups are the same, and by (2), the non trivial indices in the J-invariants of φ and (A,σ) are the same. If n is odd, this is enough to conclude that the J-invariants are equal. If n is even, the only difference comes from the presentations of $\mathrm{Ch}^*(\mathrm{O}^+_{2n})$ and $\mathrm{Ch}^*(\mathrm{PGO}^+_{2n})$: the second group has two generators of degree 1, while the first has only one. Since we precisely defined x_1 to be the generator of $\mathrm{Ch}^1(\mathrm{PGO}^+_{2n})$ killed by pull-back to $\mathrm{Ch}^1(\mathrm{O}^+_{2n})$ (see § 2.6), we get that the index j_1 of $J(A,\sigma)$ is trivial, and this proves the proposition. $\square$

Half-spin case. We now assume that the Clifford algebra $\mathcal{C}(A,\sigma)=\mathcal{C}_+\times\mathcal{C}_-$ has a split component. If $\deg(A)\equiv 2[4]$, by the fundamental relations 0.6, the algebra A is split, so that $J(A,\sigma)=J(\varphi)$ for a suitable quadratic form φ. So we may assume $\deg(A)\equiv 0[4]$. Let us pick one of the two (isomorphic) half-spin groups $\mathrm{Spin}^+_{2n}\subset\mathrm{Spin}_{2n}$. As explained in [10, Lem. 4.1], $\mathcal{C}(A,\sigma)$ has a split component if and only if at least one of the cocycles ξ and $\xi'\in Z^1(k,\mathrm{PGO}^+_{2n})$ corresponding to the classes of (A,σ,ε) and (A,σ,ε') lifts to a cocycle $\eta\in Z^1(k,\mathrm{Spin}^+_{2n})$. Let $\mathrm{Spin}^+(A,\sigma)$ be the corresponding half-spin group, $\mathrm{Spin}^+(A,\sigma)={}_\eta\mathrm{Spin}^+_{2n}$. It satisfies condition 0.1, and its J-invariant is well defined. For the same reason as before, the non trivial indices of this J-invariant are equal to the non trivial indices of $J(A,\sigma)$, and again the only difference comes from the presentations of $\mathrm{Ch}^*(\mathrm{Spin}^+_{2n})$ and $\mathrm{Ch}^*(\mathrm{PGO}^+_{2n})$: one of the two generators x_2 and x_1+x_2 is killed by the pull-back map $\mathrm{Ch}^1(\mathrm{PGO}^+_{2n})\mapsto\mathrm{Ch}^1(\mathrm{Spin}^+_{2n})$ by 1.2. Hence, we have proven:

5.4. **Proposition.** *Assume that* $\deg(A)\equiv 0[4]$. *The Clifford algebra* $\mathcal{C}(A,\sigma)=\mathcal{C}_+\times\mathcal{C}_-$ *has a split component if and only if one of the two half-spin groups, say* $\mathrm{Spin}^+(A,\sigma)$*, satisfies condition 0.1. If so, the algebra with involution* (A,σ) *is said to be half-spin, and its* J*-invariant satisfies*

$$J(A,\sigma)=(j_1,0,j_3,\dots,j_r),\ \textit{where}\ (j_1,j_3,\dots,j_r)=J_2\big(\mathrm{Spin}^+(A,\sigma)\big).$$

If (A,σ) is half-spin, we can refine the inequalities given in 5.2 by applying Theorem 3.8 to the half-spin group $\mathrm{Spin}^+(A,\sigma)$. We get the following:

5.5. **Corollary.** *Assume* $\deg(A)\equiv 0[4]$ *and* (A,σ) *is half-spin, that is its Clifford algebra has a split component. The following hold:*

(1) *If* $\mathfrak{i}_A>0$*, then* $j_1>0$.
(2) *If* $\mathfrak{i}_A>1$*, then* $j_1>1$.

Indeed, in degree 1, the Chow group modulo 2 of a half-spin group is

$$\mathrm{Ch}^1(\mathrm{Spin}^+_{2n})=\mathrm{Ch}^1(\mathfrak{X}_0)/\operatorname{im}(\mathfrak{c}^{(1)}_{hs})=\Lambda_\omega/\hat{T}_0\otimes_{\mathbb{Z}}\mathbb{F}_2.$$

It is generated by $x_1=\pi(h_1)=\pi\big(c_1(\mathcal{L}(\omega_1))\big)$. Hence the common index in this case is $\mathfrak{i}_J=\mathfrak{i}_A$. Moreover, if $\mathfrak{i}_A>1$, then $4|\deg(A)$ and $k_1>1$.

Witt-equivalent algebras with involution. The arguments of 4.3 also apply to algebras with involution. Consider two Brauer-equivalent algebras A and B, respectively endowed with the orthogonal involutions σ and τ. They can be represented as $(\mathrm{End}_D(M),\mathrm{ad}_h)$ and $(\mathrm{End}_D(M'),\mathrm{ad}_{h'})$, for some hermitian modules (M,h) and (M',h') over a division algebra with orthogonal involution $(D,\bar{\ })$, Brauer-equivalent to A. The algebras (A,σ) and (B,τ) are said to be Witt-equivalent if the hermitian modules (M,h) and (M',h') are Witt-equivalent. If so, (A,σ) and (B,τ) are split hyperbolic over the same fields. Hence, the corresponding twisted

Borel varieties $\mathfrak{X}_A$ and $\mathfrak{X}_B$ are split over the function field of each other. So [34, 5.18(iii)] applies and we get:

5.6. **Proposition.** *Let (A, σ) and (B, τ) be two Witt-equivalent algebras with involution. All the non trivial indices in their J-invariant are equal.*

This proposition will prove useful to complete the classification in degree 8. Before, we study the degree 6 and degree 4 cases.

Classification results in degree 6. Assume that $\deg(A) \equiv 2[4]$. From the table [34, 4.13], 6 is the smallest value for which the J-invariant may be non trivial. This is not surprising, since in degree 2, any quaternion algebra endowed with an orthogonal involution with trivial discriminant is split hyperbolic [23, (7.4)]. In degree 6, the J-invariant is given by $J(A, \sigma) = (j_1)$, with $0 \leq j_1 \leq 2$. It can be computed as follows:

5.7. **Theorem.** *Let A be a degree 6 algebra endowed with an orthogonal involution σ with trivial discriminant. Its J-invariant is given by*

$$J(A, \sigma) = (\mathsf{i}_S),$$

where, as before, $\mathsf{i}_S = \mathsf{i}_+ = \mathsf{i}_-$ is the 2-adic valuation of any component of the Clifford algebra $\mathcal{C}(A, \sigma)$. So, we have:

1. *$J = (0) \iff (A, \sigma)$ is split hyperbolic.*
2. *$J = (1) \iff (A, \sigma)$ is split isotropic and non hyperbolic.*
3. *$J = (2) \iff (A, \sigma)$ is anisotropic.*

Note that the algebra can be split or non split in the last case.

Proof. For degree reasons, the index i_s is bounded by 2. Hence the equality $J(A, \sigma) = (\mathsf{i}_S)$ is a direct consequence of corollary 5.1. Moreover, the fundamental relations 0.6 show that if $\mathsf{i}_S \leq 1$, the algebra A is split. If so, σ is adjoint to a quadratic form φ, and the J invariant of (A, σ) is $J(A, \sigma) = J(\varphi)$ (see 5.3). By 4.6, this proves (1) and (2), and also (3) in the split case. To finish the proof, it is enough to check that if A is non split, then σ is anisotropic. For the sake of contradiction, assume A is non split, that is $A = M_3(Q)$ for some division quaternion algebra Q over k, and σ is isotropic. Since A is the endomorphism ring of a 3-dimensional Q-module, (A, σ) is a hyperbolic extension, in the sense of [9, 3.1], of $(Q, \sigma_{\mathrm{an}})$, for some orthogonal involution σ_{an} of Q. Moreover, by [23, (7.5)], we have $d(\sigma_{\mathrm{an}}) = d(\sigma) = 1$. This is impossible if Q is non split (see [23, (7.4)]). □

Classification in degree 4. Assume the degree of A satisfies $\deg(A) \equiv 0[4]$. The first two parameters of $J(A, \sigma) = (j_1, j_2, \ldots, j_r)$ correspond to generators of degree 1, and we now have:

5.8. **Lemma.** *Assume the algebra A has degree $\deg(A) \equiv 0[4]$, and consider the first indices j_1 and j_2 of its J-invariant. We have:*

1. *$j_1 = 0 \iff A$ is split;*
2. *$j_2 = 0 \iff (A, \sigma)$ is half-spin, i.e. its Clifford algebra $\mathcal{C}(A, \sigma)$ has a split component.*

Proof. We already know from 5.2 that $j_1 = 0$ if A is split and $j_2 = 0$ if (A, σ) is half-spin. To prove the converse, assume first that A is non split, that is $\mathsf{i}_A > 0$. If $\min\{\mathsf{i}_+, \mathsf{i}_-\} > 0$, Corollary 5.2(3) shows that $j_1 > 0$. Otherwise, we are in the half spin case, so we can apply Corollary 5.5, which also gives $j_1 > 0$. Similarly, assume that $\mathfrak{i}_+ > 0$, and $\mathfrak{i}_- > 0$. If $\mathfrak{i}_A > 0$, Corollary 5.2(3) gives $j_2 > 0$. If A is split, Corollary 4.4 gives the conclusion since, by Proposition 5.3, j_2 is the first index of the J-invariant of the underlying quadratic form.

□

For algebras of degree 4, the J-invariant is given by $J(A, \sigma) = (j_1, j_2)$ with $0 \leq j_1, j_2 \leq 1$. Hence the previous lemma suffices to determine $J(A, \sigma)$. Moreover, it is well-known that (A, σ) is hyperbolic if and only if one component of $\mathcal{C}(A, \sigma)$ is split (see [23, (15.14)]). So we can rephrase the result as follows:

5.9. **Lemma.** *Let A be a degree 4 algebra endowed with an orthogonal involution σ with trivial discriminant. Its J-invariant $J(A, \sigma) = (j_1, j_2)$ can be computed as follows:*

(1) *j_1 is 0 if A is split and 1 otherwise;*
(2) *j_2 is 0 if σ is hyperbolic, and 1 otherwise.*

We can give a precise description of (A, σ) is each case. As explained in [23, (15.14)], since A has degree 4 and σ has trivial discriminant, (A, σ) decomposes as $(A, \sigma) = (Q_1, \bar{\ }) \otimes_k (Q_2, \bar{\ })$, where the quaternion algebras Q_1 and Q_2 are the two components of the Clifford algebra $\mathcal{C}(A, \sigma)$, each endowed with its canonical involution. The algebra A is split if and only if Q_1 and Q_2 are isomorphic, in which case σ is adjoint to the norm form of $Q_1 = Q_2$, which is a 2-fold Pfister form. So $J = (0, 0)$ if and only if $Q_1 = Q_2$ is split, and σ is hyperbolic. Otherwise, $Q_1 = Q_2$ is division, its norm form is anisotropic, and $J = (0, 1)$. If A is non split, then Q_1 and Q_2 are not isomorphic. If one of them, say Q_1 is split, then A has index 2, $A = M_2(Q_2)$, σ is hyperbolic, and $J = (1, 0)$. Otherwise, $J = (1, 1)$, the involution is anisotropic, and A has index 2 or 4. Using this description, one may easily construct explicit examples for each possible value of the J-invariant.

6. The trialitarian case

From now on, we assume that (A, σ) has degree 8. The J-invariant of (A, σ) is a triple $J(A, \sigma) = (j_1, j_2, j_3)$ with $0 \leq j_1, j_2 \leq 2$ and $0 \leq j_3 \leq 1$. In this section, we will explain how to compute $J(A, \sigma)$. As a consequence of our results, we will prove:

6.1. **Corollary.** *(i) There is no algebra of degree 8 with orthogonal involution with trivial discriminant having J-invariant equal to $(1, 2, 0)$, $(2, 1, 0)$ or $(2, 2, 0)$.*

(ii) All other possible values do occur.

In particular, this shows that the restrictions described in the table [34, 4.13] (see also § 8), which were obtained by applying the Steenrod operations on $\mathrm{Ch}^*(G_0)$ (*loc. cit.* 4.12) are not the only ones.

Recall that the group $\mathrm{PGO}^+(A, \sigma)$ is of type D_4. To complete the classification in this case, we need to understand the action of the symmetric group S_3 on the J-invariant (see 2.8). Let (B, τ) and (C, γ) be the two components of the Clifford algebra $\mathcal{C}(A, \sigma)$, each endowed with its canonical involution. It follows from the structure theorems [23, (8.10) and(8.12)] that both are degree 8 algebras with orthogonal involutions. The triple $\big((A, \sigma), (B, \tau), (C, \gamma)\big)$ is a trialitarian triple in the sense of *loc.cit.* § 42.A, and in particular, the Clifford algebra of any of those three algebras with involution is the direct product of the other 2. Hence, if one of them, say (A, σ) is split, then the other two are half-spin.

6.2. **Definition.** The trialitarian triple $\big((A, \sigma), (B, \tau), (C, \gamma)\big)$ is said to be ordered by indices if the indices of the algebras A, B and C satisfy

$$\operatorname{ind}(A) \leq \operatorname{ind}(B) \leq \operatorname{ind}(C).$$

The J invariant of such a triple can be computed as follows:

6.3. **Theorem.** *Let $\big((A, \sigma), (B, \tau), (C, \gamma)\big)$ be a trialitarian triple ordered by indices, so that $\mathsf{i}_A \leq \mathsf{i}_B \leq \mathsf{i}_C$. The J-invariants are given by*

$$J(A, \sigma) = (j, j', j_3) \quad \textit{and} \quad J(B, \tau) = J(C, \gamma) = (j', j, j_3),$$

where $j = \min\{\mathsf{i}_A, 2\}$ and $j' = \min\{\mathsf{i}_B, \mathsf{i}_C, 2\} = \min\{\mathsf{i}_B, 2\}$.

Moreover, the third index j_3 is 0 if the involution is isotropic and 1 otherwise.

6.4. **Remark.** (i) The first index of the J-invariant of an algebra with involution (D, ρ) is $\min\{\mathsf{i}_D, 2\}$ if D is not of maximal index in its triple. But it might be strictly smaller in general. In 6.9 below, we will give an explicit example where $j_1 < \mathsf{i}_D = 2$.

(ii) By 2.8, we already know that j_3 does not depend on the choice of an element of the triple. On the other hand, as explained in [8], the involutions σ, τ and γ are either all isotropic or all anisotropic. The triple is said to be isotropic or anisotropic accordingly.

Proof. To start with, let us compute the first two indices j_1 and j_2 of the J-invariant of (A, σ). Since we are in degree 8, they are both bounded by 2. Moreover, the triple being ordered by indices, the common index is given by $\mathsf{i}_J = \mathsf{i}_A$. So the equality $j_1 = j$ follows directly from the inequalities of Corollary 5.2. If additionally $j' = j$, the very same argument gives $j_2 = j'$. Assume now that j and j' are different, that is $j < j'$. If so, $j = 0$ or $j = 1$. In the first case, we have $\mathsf{i}_A = 0$ so that the algebra A is split, and the result follows from 4.7 and 5.3. The only remaining case is $j = \mathsf{i}_A = 1$ and $\mathsf{i}_B \geq 2$, so that $j' = 2$. Consider the function field F_A of the Severi-Brauer variety of A, which is a generic splitting field of A. By the fundamental relations 0.6, the algebra C is Brauer equivalent to $A \otimes B$. Hence Merkurjev's index reduction formula [26] says

$$\operatorname{ind}(B_{F_A}) = \min\{\operatorname{ind}(B), \operatorname{ind}(B \otimes A)\} = \operatorname{ind}(B).$$

So the values of i_B and j' are the same over F and F_A. We know the result hold over F_A by reduction to the split case. Since the index j_2 can only decrease under scalar extension, we get $j_2 \geq j' = 2$, which concludes the proof in this case.

So the J-invariant of (A, σ) is given by $J(A, \sigma) = (j, j', j_3)$ for some integer j_3. It remains to compute the J-invariant of (B, τ) and (C, γ). Recall from 2.8 that (j, j', j_3) and (j', j, j_3) are the only possible values. So, if $j = j'$, there is no choice and we are done. Again, there are two remaining cases. Assume first that $j = \mathsf{i}_A = 0$ and $j' \geq 1$, so that $J(A, \sigma) = (0, j', j_3)$. Since A is split, (B, τ) and (C, γ) are half-spin, so they have trivial j_2 and this gives the result. Assume now that $j = 1$ and $j' = 2$, so that $J(A, \sigma) = (1, 2, j_3)$. By the previous case, over the field F_A, both (B, τ) and (C, γ) have J-invariant $(2, 0, j_3)$. So the value over F has to be $(2, 1, j_3)$.

To conclude the proof, it only remains to compute j_3. If A is split, this was done in 4.7. In the anisotropic case, we can reduce to the split case by generic splitting.

Indeed, by [18] in the division case, [38, Prop. 3] in index 4, and [30, Cor. 3.4] in index 2 (see also [20]) the triple remains anisotropic after scalar extension to a generic splitting field F_A of the algebra A. Hence j_3 is equal to 1 over F_A, and this implies $j_3 = 1$. In the isotropic case, if $\mathfrak{i}_C \geq 2$, then we actually are in the split case. Indeed, if $\mathrm{ind}(C) \geq 4$ and γ is isotropic, then $C = M_2(D)$ for some degree 4 division algebra D, and γ has to be hyperbolic. So by 0.3, (C, γ) is half-spin, that is A is split. The only remaining case is $\mathfrak{i}_A = \mathfrak{i}_B = \mathfrak{i}_C = 1$ and all three involutions are isotropic. In this case, (A, σ) is Witt-equivalent to a non-split algebra of degree 4 with anisotropic involution, which has J-invariant $(1, 1)$ by 5.9. Hence, in view of 5.6, the J-invariant of (A, σ), which already has $j_1 = j_2 = 1$ must have $j_3 = 0$. □

The first part of Corollary 6.1 follows easily from Theorem 6.3. Indeed, if one of j_1, j_2 is 2 and the other one is ≥ 1, then the algebras A, B and C are all three non split, and B and C have index ≥ 4. By 0.3, since A and B are non split, the involution γ on C is not hyperbolic, so it is anisotropic, and the theorem gives $j_3 = 1$.

Explicit examples. We now prove the second part of Corollary 6.1. Recall from 5.3 that if A is split, and σ is adjoint to a quadratic form φ, then $J(A, \sigma) = (0, J(\varphi))$. Hence any triple with $j_1 = 0$ is obtained for a suitable choice of φ by 4.7. Considering the components of the even Clifford algebra of those quadratic forms, we also obtain all triples with $j_2 = 0$ by Theorem 6.3. The maximal value $(2, 2, 1)$ is obtained from a generic cocycle; such a cocycle exists by [22, Thm. 6.4(ii)]. Hence, it only remains to prove that the values $(1, 1, 0)$, $(1, 1, 1)$, $(1, 2, 1)$ and $(2, 1, 1)$ occur. For any of those, we will produce an explicit example, inspired by the trialitarian triple constructed in [35, Lemma 6.2]

Our construction uses the notion of direct sum for algebras with involution, which was introduced by Dejaiffe [2]. Consider two algebras with involution (E_1, θ_1) and (E_2, θ_2) which are Morita-equivalent, that is E_1 and E_2 are Brauer equivalent and the involutions θ_1 and θ_2 are of the same type. Dejaiffe defined a notion of Morita equivalence data, and explains how to associate to any such data an algebra with involution (A, σ), which is called a direct sum of (E_1, θ_1) and (E_2, θ_2). In the split orthogonal case, if θ_1 and θ_2 are respectively adjoint to the quadratic forms φ_1 and φ_2, any direct sum of (E_1, θ_1) and (E_2, θ_2) is adjoint to $\varphi_1 \oplus \langle\lambda\rangle\varphi_2$ for some $\lambda \in F^\times$, and the choice of a Morita-equivalence data precisely amounts to the choice of a scalar λ. In general, there exist non isomorphic direct sums of two given algebras with involution. We will use the following characterization of direct sums [35, Lemma 6.3] :

6.5. **Lemma.** *The algebra with involution (A, σ) is a direct sum of (E_1, θ_1) and (E_2, θ_2) if and only if there is an embedding of the direct product $(E_1, \theta_1) \times (E_2, \theta_2)$ in (A, σ) and* $\deg(A) = \deg(E_1) + \deg(E_2)$.

Slightly extending Garibaldi's 'orthogonal sum lemma' [9, Lemma 3.2], we get:

6.6. **Proposition.** *Let Q_1, Q_2, Q_3 and Q_4 be quaternion algebras such that $Q_1 \otimes Q_2$ and $Q_3 \otimes Q_4$ are Brauer equivalent. If (A, σ) is a direct sum of $(Q_1, \bar{\ }) \otimes (Q_2, \bar{\ })$ and $(Q_3, \bar{\ }) \otimes (Q_4, \bar{\ })$ then one of the two components of the Clifford algebra of (A, σ) is a direct sum of $(Q_1, \bar{\ }) \otimes (Q_3, \bar{\ })$ and $(Q_2, \bar{\ }) \otimes (Q_4, \bar{\ })$, while the other is a direct sum of $(Q_1, \bar{\ }) \otimes (Q_4, \bar{\ })$ and $(Q_2, \bar{\ }) \otimes (Q_3, \bar{\ })$.*

6.7. **Remark.** If one of the four quaternion algebras is split, as we assumed in [35], then all three direct sums have a hyperbolic component. Hence they are uniquely defined. This is not the case anymore in the more general setting considered here. The algebra with involution (A, σ) does depend on the choice of an equivalence data. Nevertheless, once such a choice is made, its Clifford algebra is well defined. So the equivalence data defining the other two direct sums are determined by the one we have chosen.

Proof. Denote $(E_1, \theta_1) = (Q_1, ^-) \otimes (Q_2, ^-)$ and $(E_2, \theta_2) = (Q_3, ^-) \otimes (Q_4, ^-)$. By [23, (15.12)], their Clifford algebras with canonical involution are $(Q_1, ^-) \times (Q_2, ^-)$, and $(Q_3, ^-) \times (Q_4, ^-)$ respectively. The embedding of the direct product $(E_1, \theta_1) \times (E_2, \theta_2)$ in (A, σ) induces an embedding of the tensor product of their Clifford algebras in the Clifford algebra of (A, σ):

$$\big((Q_1, ^-) \times (Q_2, ^-)\big) \otimes \big((Q_3, ^-) \times (Q_4, ^-)\big) \hookrightarrow (\mathcal{C}(A, \sigma), \underline{\sigma}).$$

This tensor product splits as a direct product of four tensor products of quaternion algebras with canonical involution; for degree reasons, two of them embed in each component of $\mathcal{C}(A, \sigma)$. To identify them, it is enough to look at their Brauer classes. From the hypothesis, we have Brauer equivalences $Q_1 \otimes Q_3 \sim Q_2 \otimes Q_4$ and $Q_1 \otimes Q_4 \sim Q_2 \otimes Q_3$. If $Q_1 \otimes Q_3$ and $Q_1 \otimes Q_4$ are not Brauer equivalent, that is if A is non split, this concludes the proof. Otherwise, all four tensor products are isomorphic, and the result is still valid. $\square$

With this in hand, we now give explicit examples of algebras with involution having J-invariant $(1, 2, 1)$, $(2, 1, 1)$, $(1, 1, 1)$ and $(1, 1, 0)$.

6.8. **Example.** Let $F = k(x, y, z, t)$ be a function field in 4 variables over k, and consider the following quaternion algebras over F:

$$Q_1 = (x, zt),\ Q_2 = (y, zt),\ Q_3 = (xy, z) \text{ and } Q_4 = (xy, t).$$

We let (A, σ) be a direct sum of $(Q_1, ^-) \otimes (Q_2, ^-)$ and $(Q_3, ^-) \otimes (Q_4, ^-)$ as in 6.6, and denote by (B, τ), and respectively (C, γ), the component of $\mathcal{C}(A, \sigma)$ Brauer equivalent to $Q_1 \otimes Q_3 \sim (x, t) \otimes (y, z)$ and $Q_1 \otimes Q_4 \sim (x, z) \otimes (y, t)$. The algebras A, B and C have index 2, 4 and 4, so that $\big((A, \sigma), (B, \tau), (C, \gamma)\big)$ is a trialitarian triple ordered by indices. By Theorem 6.3, we get $J(A, \sigma) = (1, 2, j_3)$ and $J(B, \tau) = J(C, \gamma) = (2, 1, j_3)$ for some j_3. Finally, assertion (i) of Corollary 6.1 implies $j_3 = 1$; in other words, this triple is anisotropic.

6.9. **Example.** This example is obtained from the previous one by scalar extension. Consider the Albert form $\varphi = \langle x, t, -xt, -y, -z, yz \rangle$ associated to the biquaternion algebra $Q_1 \otimes Q_3$. We let F' be its function field, $F' = F(\varphi)$, and denote by (A', σ'), (B', τ') and (C', γ') the extensions of (A, σ), (B, τ) and (C, γ) to F'. Since B is Brauer equivalent to $Q_1 \otimes Q_3$, the algebra B' has index 2. On the other hand, it follows from Merkurjev's index reduction formula [26, Thm. 3] that the indices of A and C are preserved by scalar extension to F', so that A' and C' have indices 2 and 4 respectively. Hence $\big((A', \sigma'), (B', \tau'), (C', \gamma')\big)$ again is a trialitarian triple ordered by indices and Theorem 6.3 now gives $J(A', \sigma') = J(B', \tau') = J(C', \gamma') = (1, 1, j_3)$ for some j_3. The same argument as in the proof of the first assertion of Corollary 6.1 applies here: since A' and B' are non split and C' has index 4, the involutions are anisotropic and Theorem 6.3 gives $j_3 = 1$. Note that, in particular, we have $J(C', \gamma') = (1, 1, 1)$, even though C' has index $4 = 2^2$.

6.10. **Example.** We now produce another example of an anisotropic trialitarian triple having J-invariant $(1,1,1)$ in which all three algebras have index 2. Namely, consider the F-quaternion algebras

$$Q_1 = (x,y),\ Q_2 = (x,z),\ Q_3 = (x,t) \text{ and } Q_4 = (x,yzt).$$

Pick an arbitrary orthogonal involution ρ on $H = (x,yz)$ over F. Since $Q_1 \otimes Q_2$ is isomorphic to 2 by 2 matrices over H, the tensor product of the canonical involutions of Q_1 and Q_2 is adjoint to a 2-dimensional hermitian form h_{12} over (H,ρ). Similarly, $(Q_3, \overline{}) \otimes (Q_4, \overline{})$ is isomorphic to $M_2(H)$ endowed with the adjoint involution with respect to some hermitian form h_{34}. Since h_{12} and h_{34} are both anisotropic, the hermitian form $h = h_{12} \oplus \langle u \rangle h_{34}$ over $H'' = H \otimes F(u)$, for some indeterminate u, also is anisotropic. We define

$$(A,\sigma) = (M_4(H''), \mathrm{ad}_h).$$

It is clear from the definition that (A,σ) is a direct sum of $(Q_1, \overline{}) \otimes (Q_2, \overline{})$ and $(Q_3, \overline{}) \otimes (Q_4, \overline{})$. Hence, by 6.6, the two components (B,τ) and (C,γ) of its Clifford algebra are Brauer equivalent to (x,yt) and (x,zt). This shows that all three algebras have index 2. Since the involutions are anisotropic, by Theorem 6.3, their J-invariant is $(1,1,1)$.

6.11. **Remark.** Note that there are many other examples, and not all of them can be described as in 6.6. In particular, any triple which includes a division algebra cannot be obtained from this proposition. Consider for instance the algebra with involution (A,σ) described in [36, Exple 3.6], and let (B,τ) and (C,γ) be the two components of its Clifford algebra. As explained there, A is a division indecomposable algebra, and one component of its Clifford algebra, say B, has index 2. Since A is Brauer equivalent to $B \otimes C$, its indecomposability guarantees that C is division, and we get $J(A,\sigma) = J(C,\gamma) = (2,1,1)$ and $J(B,\tau) = (1,2,1)$.

To produce examples of algebras with involution having J-invariant $(1,1,1)$, we now construct examples of isotropic non split and non half-spin triples. As opposed to the previous examples, they can always be described using Proposition 6.6, as we now prove:

6.12. **Proposition.** *If $\big((A,\sigma),(B,\tau),(C,\gamma)\big)$ is an isotropic trialitarian triple with A, B and C non split, then there exists division quaternion algebras Q_1, Q_2 and Q_3 such that $Q_1 \otimes Q_2 \otimes Q_3$ is split and the triple is described as in 6.6 with $Q_4 = M_2(k)$.*

Proof. Since B and C are non split, the involution σ is not hyperbolic by 0.3. Hence A has index 2, $A = M_4(Q_1)$ for some quaternion algebra Q_1 over k. Fix an orthogonal involution ρ_1 on Q_1; the involution σ is adjoint to a hermitian form $h = h_0 \oplus h_1$ over (Q_1,ρ_1), with h_0 hyperbolic, h_1 anisotropic and both of dimension 2 and trivial discriminant. Therefore, (A,σ) is a direct sum of $(M_2(Q_1), \mathrm{ad}_{h_0})$ and $(M_2(Q_1), \mathrm{ad}_{h_1})$. Since the first summand is hyperbolic, it is isomorphic to $(M_2(k), \overline{}) \otimes (Q_1, \overline{})$. The second is $(Q_2, \overline{}) \otimes (Q_3, \overline{})$, where Q_2 and Q_3 are the two components of the Clifford algebra ad_{h_1}, and this concludes the proof. □

We refer the reader to [35, §6] for a more precise description of those triples. They are the only ones for which the J-invariant is $(1,1,1)$.

7. Generic properties

In the present section we investigate the relationship between the values of the J-invariant of an algebra with involution (A,σ) and the J-invariant of the respective adjoint quadratic form φ_σ over the function field F_A of the Severi-Brauer variety of A, which is a generic splitting field of A.

7.1. **Definition.** We say (A,σ) is generically Pfister iff φ_σ is a Pfister form. Observe that in this case $\deg A$ is always a power of 2 and the J-invariant over F_A has the form:

$$J\big((A,\sigma)_{F_A}\big) = (0,\dots,0,*)$$

(all zeros except possibly the last entry which is 0 or 1).

We say (A,σ) is in I^s, $s>2$, if and only if φ_σ belongs to the s-th power $I^s(F_A)$ of the fundamental ideal $I(F_A)\subset W(F_A)$ of the Witt ring of F_A.

7.2. **Theorem.** *Let (A,σ) be an algebra with orthogonal involution with trivial discriminant.*

(a) *If (A,σ) is in I^s, $s>2$, then*

$$J(A,\sigma) = (j_1,\ \underbrace{0,\dots,0}_{2^{s-2}-1\ times}\ ,*,\dots,*)$$

(b) *In particular, if (A,σ) is generically Pfister, then $J(A,\sigma)=(*,0,\dots,0,*)$.*

Proof. (a) Let $X=D_n/P_i$ be the variety of maximal parabolic subgroups of type $i:=2\cdot[\frac{n+1}{2}]-2^{s-1}+1$. Since i is odd, $A_{k(X)}$ splits, and therefore the quadratic form φ_σ is defined over $k(X)$. By assumption $\varphi_\sigma\in I^s\big(k(X)\big)$. The Witt index of φ_σ is at least i. Therefore the anisotropic part of φ_σ has dimension at most $2(n-i)<2^s$. Thus, by the Arason-Pfister theorem φ_σ is hyperbolic. In particular, the variety X is generically split. Therefore by [33, Theorem 2.3] we obtain the desired expression for the J-invariant.

(b) Finally, if (A,σ) is generically Pfister, then $\varphi_\sigma\in I^s\big(k(X)\big)$, where $2^s=2n$ and (b) follows from (a). □

7.3. **Remark.** Let $(j_2,\dots,j_r)$ be the J-invariant of φ_σ over F_A, $r=[\frac{n+2}{2}]$. In view of the theorem one may conjecture that the J-invariant of (A,σ) is obtained from $J(\varphi_\sigma)$ just by adding an arbitrary left term, i.e.

$$J(A,\sigma)=(*,j_2,\dots,j_r).$$

For example, if φ_σ is excellent, then the J-invariant has to be equal

$$J(A,\sigma)=(*,0,\dots,0,*,0,\dots,0),$$

where the second $*$ has degree 2^s-1 for some s and equals either 0 or 1.

Observe that this holds for algebras of degree 8 (see §6).

8. Appendix

The following table provides the values of the parameters of the J-invariant for all orthogonal groups (here $p=2$).

G_0	r	d_i	k_i	j_i
O_n^+	$[\frac{n+1}{4}]$	$2i-1$	$[\log_2 \frac{n-1}{d_i}]$	if $d_v+l=2^s d_u$ and $2\nmid\binom{d_v}{l}$, then $j_u\le j_v+s$
PGO_{4m}^+	$m+1$	1, $i=1$ $2i-3$, $i\ge 2$	$2^{k_1-1}\parallel m$ $[\log_2 \frac{4m-1}{d_i}]$	if $d_v+l=2^s d_u$ and $2\nmid\binom{d_v}{l}$, for $u,v>1$, then $j_u\le j_v+s$

The $Spin_n$-case is obtained from the O_n^+-case by removing the first entry, i.e. replacing r by $r-1$ and i by $i+1$. The PGO_{4m+2}^+-case is the same as the O_{4m+2}^+-case. The $Spin_{4m}^{\pm}$-case is the same as the O_{4m}^+-case.

Note that this table coincides with [34, Table 4.13] except of the last column which in our case contains more restrictive conditions. For the groups O_n^+ the conditions of the last column coincide with the ones of [41, Prop. 5.12].

All values of the J-invariant which satisfy the restrictions given in the table will be called admissible.

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