Computation of the Metaplectic Kernel

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Dedicated to G. D. Mostow

Introduction

Let G be an absolutely simple, simply connected algebraic group defined over a global field K. Given a finite (possibly, empty) set S of places of K, we let $G(A(S))$ denote the group of S-adeles (i.e. adeles without the components corresponding to the places in S) of G. Our main concern in this paper is a description of topological central extensions of the form:

$$1 \to I \longrightarrow E \xrightarrow{\pi} G(A(S)) \to 1,$$

where $I = \mathbf{R}/\mathbf{Z}$ is the 1-dimensional compact torus, which split over the group $G(K)$ of K-rational points of G. Study of such central extensions is required for the theory of automorphic forms of fractional weights (the case $S = \emptyset$) and for a solution of the congruence subgroup problem (for applications to the congruence subgroup problem it is enough to consider the case where S contains V_∞^K, the set of archimedean places of K). Since the above extension admits a measurable cross-section, one can show (cf. Mackey [Mc]) that the set of equivalence classes of such extensions is in one-to-one correspondence with the elements of the kernel $M(S,G)$ of the following restriction map: $H^2(G(A(S))) \to H^2(G(K))$, where $H^2(G(A(S)))$ (resp., $H^2(G(K))$) is the second cohomology group of the group $G(A(S))$ (resp., of $G(K)$) defined in terms of measurable (resp., abstract) cochains, with values in I. As the unique nontrivial two-sheeted cover of $G(A)$, for $G = \mathbf{Sp}_{2n}$, $A = A(\emptyset)$, which splits over $G(K)$, was named the metaplectic group by André Weil, $M(S,G)$ is called the *metaplectic kernel.* It is always finite (Theorem 2.4) and its precise computation is the main objective of the present paper.

We shall now briefly recall the relation between the congruence subgroup problem and the metaplectic kernel. Assume that $S \supset V_\infty^K$. The congruence kernel $C(S,G)$, as defined by J-P. Serre, is the kernel of the natural surjective homomorphism $\hat{G} \to \bar{G}$ from the completion $\hat{G}$ of $G(K)$ with respect to the S-arithmetic topology to its completion $\bar{G}$ with respect to the S-congruence subgroup topology. It is well known that if $C(S,G)$ is central in $\hat{G}$ and $G(K)$ is perfect, then $C(S,G)$ is isomorphic to the dual of the metaplectic kernel. We note here that either if G/K is isotropic and is not an outer form of type E_6 of K-rank one, or G/K is anisotropic but not of type A_r, E_6 or ${}^{3,6}D_4$, then $G(K)$ is projectively simple, i.e. it is simple modulo its center (cf. [PR2], Ch. 9). Also, for most of these groups $C(S,G)$ is known to be central (in $\hat{G}$) provided that $\sum_{v\in S} K_v\text{-rank}G \geq 2$ (cf. [Ra3], [R3]).

According to an interesting result of Pierre Deligne, if $C(S,G)$ is central, then no S-arithmetic subgroup in a covering of $\prod_{v\in S} G(K_v)$, of degree larger than the order of $M(\emptyset,G)$, is residually finite. Thus our results on $M(\emptyset,G)$ imply that S-arithmetic subgroups in nonlinear semi-simple groups often fail to be residually finite. A known result in this direction, proved by Raghunathan ([Ra1]), has been used by Toledo ([T]) to construct an example of a smooth complex projective variety whose fundamental group is not residually finite.

In a fundamental work [Mo], Moore, making use of the description given by Robert Steinberg of the (abstract) universal central extension of the group of rational points of a simply connected Chevalley (i.e. split) group over an arbitrary field, found a relation between the norm residue symbols (of local class field theory) and the topological central extensions of $\mathbf{SL}_2(k)$, where k is a local field. Using the uniqueness of the reciprocity law (see Appendix B), which he proved in the same paper, he was able to compute the metaplectic kernel $M(S,G)$ for $G = \mathbf{SL}_2$ and also show that for any Chevalley group the metaplectic kernel is trivial if S contains a noncomplex place and it is a subgroup of $\hat{\mu}(K)$, the dual group of the group $\mu(K)$ of roots of unity in K, if all the places in S are complex. Soon afterwards Matsumoto [Ma], by explicitly constructing certain topological central extensions of $G(A(S))$, was able to prove that in the latter case $M(S,G)$ is in fact equal to $\hat{\mu}(K)$ for any Chevalley group G. Deodhar [D] extended the results of Steinberg and Moore for split groups to quasi-split groups and in particular determined the metaplectic kernel for this class of groups.

The metaplectic kernel for arbitrary K-isotropic groups was computed by Prasad and Raghunathan [PrRa1] who used the general injectivity results of [PrRa2] (cf. Theorem 1.2 below) to reduce the problem first to the groups of K-rank one, and then further to the groups of the form $R_{L/K}(\mathbf{SL}_2)$, where $R_{L/K}$ is the restriction of scalars functor, for which one can use the results of [Mo]. (Note that the papers [PrRa2], [PrRa4] and [Pr2] determine the topological central extensions of the group of rational points of an arbitrary simple, simply connected group over a nonarchimedean local field.) The metaplectic kernel for the group $\mathbf{SL}_{n,D}, n \geq 2$ was computed independently by Bak and Rehmann [BR] using (algebraic) K-theoretic methods. Bak [B] has announced its computation for the classical groups of K-rank≥ 2.

The computation of the metaplectic kernel for anisotropic groups in the present paper has required some new arithmetic, geometric and group-theoretic ideas. The main problem is that though one can still use the local results of [Mo], [D], [PrRa 2, 4] and [Pr2] to describe the topological central extensions of the S-adele group $G(A(S))$, it is very difficult to determine the precise conditions under which such an extension