A BERRY–ESSEEN BOUND FOR M-ESTIMATORS

V. Bentkus[1] M.Bloznelis[1,2] F.Götze[1]

University of Bielefeld,
Vilnius University and Institute of Mathematics and Informatics,
University of Bielefeld

May 1995

Abstract. We prove a Berry–Esseen bound for general M-estimators under optimal regularity conditions on the score function and the underlying distribution. As an application we obtain Berry–Esseen bounds for the sample median, the L_p-median, $p > 1$ and Huber's estimator of location.

1. Introduction and Main Results

Let X denote a random variable taking values in a measurable space $(\mathcal{X}, \mathcal{F})$ with distribution F. For an open subset T of R, let $f : \mathcal{X} \times T \to R$ be a jointly measurable function. Let $X_1, \dots, X_N$ be independent observations drawn from the distribution F. A random variable $t_N = t_N(X_1, \dots, X_N)$ which minimizes the function

$$Q_N(t) = N^{-1} \sum_{i=1}^{N} f(X_i, t)$$

is called M-estimator. If $\mathbf{E}\,\big|f(X,t)\big| < \infty$, for $t \in T$, and the function $t \to \mathbf{E}\, f(X,t)$ has a unique minimum point, say t_0, then t_N may be used as an estimator of t_0.

[1]Research supported by the SFB 343 at Bielefeld.

[2]Research supported by the ISF grant.

1991 *Mathematics Subject Classification.* Primary 60F05; secondary 62F12.

Key words and phrases. Berry–Esseen bound, Maximum likelihood estimator, M-estimator, sample median, sample quantile, Huber's estimator, convergence rate, Central Limit Theorem.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

The usual requirement for t_N is that it estimates $t_0 \in T$ consistently. Furthermore, under certain conditions $N^{1/2}(t_N - t_0)$ is asymptotically normal, see Huber (1964), Serfling (1980). For recent advances concerning the asymptotic normality we refer to Pollard (1984, 1985) and Hoffmann-Jorgensen (1994).

Our aim is to establish a general Berry–Esseen bound for the statistics $N^{1/2}(t_N - t_0)$ under minimal regularity conditions on the the score function $f(x,t)$. Hence we shall estimate

$$\Delta_N := \sup_{u \in R} \Big| \mathbf{P}\big\{ N^{1/2}(t_N - t_0)/b < u \big\} - \Phi(u) \Big|,$$

where Φ denotes the standard normal distribution function and b is a scale parameter defined in (1.3) below.

Any M-estimator $t_N = t_N(X_1, \dots, X_N)$ is a symmetric function of its arguments $X_1, \dots, X_N$, i.e., it is a symmetric statistic. Berry–Esseen bounds for general symmetric statistics were obtained by van Zwet (1984), Friedrich (1988) and Bolthausen Götze and (1993). These results are applicable (see van Zwet (1984) and Bentkus Götze and van Zwet (1994)) to a wide class of statistics. They apply to M-estimators if the score function f is sufficiently smooth since in this case an explicit approximate solution of the minimization problem is available. However, in a number of important examples the score function is not sufficiently smooth and direct proofs are needed.

Berry–Esseen bounds for M- and related (maximum likelihood, minimum contrast) estimators were considered by Linnik and Mitrofanova (1963), Michel and Pfanzagl (1971), Pfanzagl (1971), Chibisov (1972), Bickel (1974), Matsuda (1983), Radavičius (1990), Paulauskas (1995), etc. Pfanzagl (1971) obtained the bound $\Delta_N = O(N^{-1/2})$ for a minimum contrast estimator in the case where the score function is twice differentiable in t and the second derivative $f''(x,t) = \dfrac{\partial^2}{\partial t^2} f(x,t)$ satisfies Lipschitz condition. In particular cases the smoothness condition on the score function were relaxed in Matsuda (1983) and Radavičius (1990). Matsuda (1983) considered the maximum likelihood (ML) estimator of the shift parameter of a density function and obtained an estimate of Δ_N assuming that the score function is two times differentiable. Radavičius (1990) obtained the bound $\Delta_N = O(N^{-1/2})$ for a ML estimator in the case where the score function $f(x,t)$ is differentiable and convex in t and the derivative $f'(x,t) = \dfrac{\partial}{\partial t} f(x,t)$ satisfies a Lipschitz condition.

We prove a Berry–Esseen bound for general M-estimators. To formulate the result we need notations related to *smoothness,* (*asymptotic*) *convexity* and *consistency.*

Smoothness. Let there exist a neighborhood, say

$$V = (t_0 - \delta, t_0 + \delta) \subset T,$$

of t_0 such that, for any $t \in V$,

$$f'(x,t) = \lim_{\tau \to 0} \big(f(x, t+\tau) - f(x,t) \big) / \tau \quad \text{exists,} \tag{1.1}$$