The chromatic numbers of distance graphs

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December 5, 1995

Abstract

Suppose D is a subset of all positive integers Z. The distance graph $G(\mathbb{Z}, D)$ is the graph with vertex set Z and two vertices x, y are adjacent if $|x-y| \in D$. This paper studies the chromatic number of $G(\mathbb{Z}, D)$. We shall concentrate on the case that $|D| = 3$ and give a complete classification of those $D = \{a, b, c\}$ for which b is a multiple of a and $\chi(G(\mathbb{Z}, D)) = 3$.

1 Introduction

Let $\mathbb{Z}$ be the set of all integers. For a subset D of $\mathbb{Z}$, the distance graph $G(\mathbb{Z}, D)$ is the graph with vertex set $\mathbb{Z}$ in which two vertices x, y are adjacent if $|x-y| \in D$. Distance graphs were introduced and studied by Eggleton, Erdös and Skilton [4]. They were motivated by the well known plane colouring problem: What is the least number of colours which can be used to colour all points of the Euclidean plane so that vertices of unit distance are coloured with distinct colours. Moser and Moser [8] showed that four colours are necessary, and Hadwiger, Debrunner and Klee [7] showed that seven colours are sufficient. Essentially no further progress has been made on the problem. The plane colouring problem is naturally generalized to higher dimension spaces. The gap between the known upper and lower bounds of the corresponding chromatic number increases as the dimension increases [3]. On the other hand, the corresponding problem for one dimensional space, i.e., for the real line, is trivial. It is easy to see that this problem is equivalent to find the chromatic number of the distance graph $G(\mathbb{Z}, D)$ for which $|D| = 1$. Obviously, $\chi(G(\mathbb{Z}, D)) = 2$ if $|D| = 1$.

Eggleton, Erdös and Skilton [4, 5] studied, among other things, the chromatic number of $G(\mathbb{Z}, D)$ for special classes of sets D. They proved that if D is the set of all primes, then $\chi(G(\mathbb{Z}, D)) = 4$; if D contains $2, 3$ and a pair of twin primes, then $\chi(G(\mathbb{Z}, D)) = 4$; if $|D| = 2$ and it contains two coprime integers, then $\chi(G(\mathbb{Z}, D)) = 3$ if the two integers are of opposite parity and $\chi(G(\mathbb{Z}, D)) = 2$ if the two integers are odd. Using a computer, they also calculated the chromatic number and the smallest period for a periodic colouring of graphs $G(\mathbb{Z}, D)$ for which $D = \{2, 3, p\}$ where p is a prime number less than 1000. There are some other special sets D for which the chromatic number $\chi(G(\mathbb{Z}, D))$ was determined, [4, 5, 6].

The chromatic number of distance graphs $G(\mathbb{Z}, D)$ was also studied by Chen, Chang and Huang [2]. Our present paper is motivated by the results and a conjecture in [2]. Chen, Chang and Huang [2] concentrated on the case that $|D| = 3$. They observed in [2] $\chi(G(\mathbb{Z}, D)) \leq |D| + 1$ for any finite set D. Thus $\chi(G(\mathbb{Z}, D)) \leq 4$ if $|D| = 3$. Also if $D = \{a, b, c\}$ and $\gcd(a, b, c) = 1$ (which is the only case of interest), then $\chi(G(\mathbb{Z}, D)) = 2$ if and only if all three integers a, b, c are odd. Therefore the only difficulty is to decide for which D $\chi(G(\mathbb{Z}, D)) = 3$ and for which D $\chi(G(\mathbb{Z}, D)) = 4$. For the remaining part of this paper, we assume that $D = \{a, b, c\}$, $a < b < c$ and that $\gcd(a, b, c) = 1$. The following results were obtained in [2]:

If $c = a + b$, then $\chi(G(\mathbb{Z}, D)) = 4$ when $a \not\equiv b(\text{mod}3)$ and $\chi(G(\mathbb{Z}, D)) = 3$ when $a \equiv b(\text{mod}3)$; If $a = 1, b = 2$, then $\chi(G(\mathbb{Z}, D)) = 4$ when $c \equiv 0(\text{mod}3)$ and $\chi(G(\mathbb{Z}, D)) = 3$ when $c \not\equiv 0(\text{mod}3)$; If $a \geq 2, b = a + 1$, then $\chi(G(\mathbb{Z}, D)) = 3$ when $c \neq a + b$.

They conjectured that $\chi(G(\mathbb{Z}, D)) = 4$ if and only if either $c = a + b$ and $a \not\equiv b(\text{mod}3)$ or $a = 1, b = 2$ and $c \equiv 0(\text{mod}3)$.

In this paper we also study the chromatic number of graphs $G(\mathbb{Z}, D)$ with $|D| = 3$. We take the above results of [2] as the starting point. Our results are in support of the above conjecture. In particular we shall prove that the conjecture is true if b is a multiple of a. In Section 2, we present a general method of colouring distance graphs $G(\mathbb{Z}, D)$ with $|D| = 3$. The method is simple, and the idea is to color the whole real line $\mathbb{R}$ instead of $\mathbb{Z}$ in a periodic way using certain parameters. It turns out many graphs $G(\mathbb{Z}, D)$