Schurian vector space categories of polynomial growth

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Abstract

The purpose of this work is to prove criteria of polynomial growth and domestic representation type of schurian vector space categories over an algebraically closed field. A list of all infinitely sincere schurian vector space categories of polynomial growth is obtained.

Introduction

Throughout the paper k is an algebraically closed field. A vector space category $\mathcal{K} = (\mathcal{K}, |-|)$ is a Krull-Scmidt k-category (see [Ri2]) together with a k-linear faithful functor $|-| : \mathcal{K} \to \operatorname{mod} k$.

The factor space category $\mathcal{V}(\mathcal{K})$ of $\mathcal{K}$ is defined as follows. The objects of $\mathcal{V}(\mathcal{K})$ are triples (X, V, φ), where $X \in \operatorname{Ob} \mathcal{K}$, V is a finite-dimensional k-vector space, and $\varphi : |X| \to V$ is a k-linear map. A morphism $(X, V, \varphi) \to (X', V', \varphi')$ is a pair (f, g), $f \in \mathcal{K}(X, X')$, $g \in \operatorname{Hom}_k(V, V')$ such that $\varphi'|f| = g\varphi$.

The concept of a factor space category was introduced by Nazarova and Roiter in [NR1]. It plays an important role in the representation theory of finite-dimensional associative algebras (see, for example, [NR1],[Ri1],[Ri2],[Si],[B1],[B2]).

We shall assume that $\mathcal{K}$ has a finite number of pairwise nonisomorphic indecomposable objects. We fix a set of representatives of the isomorphism classes of indecomposable objects in $\mathcal{K}$ and denote it by $\operatorname{ind} \mathcal{K}$. Denote by $\operatorname{ind}_m \mathcal{K}$ the set of objects which have the dimension m. Set $S\mathcal{K} = \operatorname{ind}_1 \mathcal{K}$, $L\mathcal{K} = \operatorname{ind}_2 \mathcal{K}$.

The vector space category $\mathcal{K}$ is called *schurian* (resp., *linear*) if $\operatorname{End}_{\mathcal{K}}(X) \cong k$ (resp., $\dim_k |X| = 1$) for any indecomposable object X of $\mathcal{K}$.

Let $\leq$ be the relation on $\operatorname{ind} \mathcal{K}$ satisfying $X \leq Y$ iff $\mathcal{K}(Y, X) \neq 0$. It should be noted that the relation $\leq$ is not transitive. The restriction of $\leq$ to $S\mathcal{K}$ is a partial order, i.e. $S\mathcal{K}$ is a poset (i.e. partially ordered set). We shall write $X < Y$ if $X \leq Y$ and $X \neq Y$.

For $(X, V, \varphi) \in \operatorname{Ob} \mathcal{V}(\mathcal{K})$, we define its dimension vector $\vec{d} = \vec{d}\,((X, V, \varphi))$ by setting $\vec{d} = (d_0; (d_Y)_{Y \in \operatorname{ind} \mathcal{K}})$, where $d_0 := \dim_k V$ and, for all $Y \in \operatorname{ind} \mathcal{K}$, d_Y is the multiplicity of Y as the summand in a decomposition of X into indecomposable direct summands. We set $d = d\,((X, V, \varphi)) = d_0 + \sum_{Y \in \operatorname{ind} \mathcal{K}} d_Y$.

The criterion of tameness for linear schurian vector space categories (= posets) was obtained by Nazarova in [N]. The criteria of polynomial growth and domestic representation type in this case were obtained in [ZN1] and [BZN]. In the case of nonlinear vector space categories, a problem of a characterization of tame, polynomial growth and domestic categories remains an open problem. The important classes of such categories are successfully studied in [Ri2],[NBR].

Let us remark that a tame criterion for schurian vector space categories was announced by Nazarova (see, for example, [NBR], [Ro2]). A finite growth criterion for schurian vector space categories was announced by Bondarenko, Nazarova and Roiter (see, for example, [NBR],[Ro2]).

A factor space (X, V, φ) of dimension vector $\vec{d}$ is called *sincere* if $d_0 \neq 0$ and $d_Y \neq 0$ for all $Y \in \operatorname{ind}\mathcal{K}$. The vector space category $\mathcal{K}$ is called *sincere* if there exists a sincere indecomposable factor space of $\mathcal{K}$. The vector space category $\mathcal{K}$ is called *infinitely sincere* if there exist infinitely many indecomposable pairwise nonisomorphic sincere factor spaces in some dimension $\vec{d}$.

For a given object $X \in L\mathcal{K}$, we shall identify the set of all one-dimensional subspaces of $|X|$ with the points of the one-dimensional projective space and denote it by $\mathbf{P}^1(X)$. For $X \in L\mathcal{K}$ and $V \in \mathbf{P}^1(X)$, we denote by $S_{X,V} = S\mathcal{K}_{X,V}$ (resp., $S^{X,V} = S\mathcal{K}^{X,V}$) the set of all $Z \in S\mathcal{K}$ such that $\mathcal{K}(X, Z) \neq 0$ (resp., $\mathcal{K}(Z, X) \neq 0$) and $V = \operatorname{Ker}|\varphi|$ (resp., $V = \operatorname{Im}|\varphi|$) for any $0 \neq \varphi \in \mathcal{K}(X, Z)$ (resp., $0 \neq \varphi \in \mathcal{K}(Z, X)$). Since $S_{X,V} \subseteq S\mathcal{K}$ (resp., $S^{X,V} \subseteq S\mathcal{K}$), the elements of $S_{X,V}$ (resp., $S^{X,V}$) form a poset.

For an object $X \in L\mathcal{K}$, we denote by $\Omega_X = \Omega\mathcal{K}_X$ (resp., $\Omega^X = \Omega\mathcal{K}^X$) the set of all $V \in \mathbf{P}^1(X)$ such that $S_{X,V} \neq \emptyset$ (resp., $S^{X,V} \neq \emptyset$). We set $S\mathcal{K}^X = \cup_{V\in\Omega^X} S\mathcal{K}^{X,V}$, $S\mathcal{K}_X = \cup_{V\in\Omega_X} S\mathcal{K}_{X,V}$.

The aim of the present paper is to prove the following seven theorems.

Theorem 1. *The schurian vector space category $\mathcal{K}$ is of polynomial growth if and only if the following conditions are satisfied:*

(a) $\dim_k|X| \leq 2$ *for all* $X \in \operatorname{ind}\mathcal{K}$*;*

(b) $\dim_k\mathcal{K}(X, Y) + \dim_k\mathcal{K}(Y, X) > 0$ *for all* $X \in L\mathcal{K}$, $Y \in S\mathcal{K}$*;*

(c) $\dim_k\mathcal{K}(X, Z) + \dim_k\mathcal{K}(Z, X) \geq 3$ *for all* $X, Z \in L\mathcal{K}, X \neq Z$ *;*

(d) $S_{X,V} \cap S^{Y,U} = \emptyset$ *for all* $X, Y \in L\mathcal{K}$, $V \in \mathbf{P}^1(X)$, $U \in \mathbf{P}^1(Y)$ *such that* $|\mathcal{K}(X, Y)|(V) = U$*;*

(e) $S_{X,V}$ *and* $S^{X,V}$ *do not contain the posets* $(1, 1, 1)$ *and* $(1, 2)$ *for any* $X \in L\mathcal{K}$ *and any* $V \in \mathbf{P}^1(X)$*;*

(f) if $1 < \operatorname{ord}\Omega_X < 4$ *(resp.,* $1 < \operatorname{ord}\Omega^X < 4$*), then* $S\mathcal{K}_X$ *(resp.,* $S\mathcal{K}^X$*) does not contain the poset* $(1, 1, 1, 1)$;

(g) S^{X_1,V_1} *or* S_{X_n,V_n} *does not contain the poset* $(1, 1)$ *for any* $X_1, \ldots, X_n \in L\mathcal{K}$, $V_i \in \mathbf{P}^1(X_i)$, $1 \leq i \leq n$, $n \in \mathbf{N}$, *such that* $|\mathcal{K}(X_i, X_{i+1})|(V_i) = V_{i+1}$ *for* $1 \leq i < n$.

(h) $S\mathcal{K}$ *does not contain the following posets:* $(1, 1, 1, 1, 1)$, $(1, 1, 1, 2)$, $(2, 2, 3)$, $(1, 3, 4)$, $(1, 2, 6)$, $(N, 5)$ *and* $\{A < B > C < D > A, S, T\}$.

Let $\mu_{\mathcal{K}}(\vec{d})$ be the number of series of the given dimension $\vec{d}$ (see Subsection 1.2).

Theorem 2. *Let $\mathcal{K}$ be a schurian vector space category of polynomial growth. Then $\mu_{\mathcal{K}}(\vec{d}) \leq 1$ for any $\vec{d}$.*

Theorem 3. *Let $\mathcal{K}$ be a schurian vector space category. The following conditions are equivalent:*

(a) $\mathcal{K}$ is of polynomial growth;

(b) $\mathcal{K}$ is of linear growth;

(c) $\mathcal{K}$ is of finite growth.