Exactly definable categories

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Given a skeletally small abelian category $\mathcal{C}$ we denote by $\mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ the category of exact additive functors from $\mathcal{C}^{\mathrm{op}}$ to the category Ab of abelian groups. We call a category exactly definable provided that it is equivalent to $\mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ for some skeletally small abelian category $\mathcal{C}$.

This definition and our interest in this type of categories is motivated by the following example. Let Λ be a ring and denote by $C(\Lambda) = \mathrm{fp}(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ the category of finitely presented functors from the category of finitely presented Λ^{op}-modules to the category of abelian groups. Then $C(\Lambda)$ is skeletally small abelian. In fact, it is the free abelian category over Λ viewed as a category with a single object [5], and the functor

$$\mathrm{Mod}(\Lambda) \to \mathrm{Ex}(C(\Lambda)^{\mathrm{op}}, \mathrm{Ab}), \quad M \mapsto \mathrm{Hom}(\ , M \otimes_\Lambda -)$$

is an equivalence. Thus we can replace a Λ-module by an exact functor $C(\Lambda)^{\mathrm{op}} \to \mathrm{Ab}$. Of course, since the work of Auslander it has been common practice to identify a Λ-module M either with $\mathrm{Hom}_\Lambda(\ , M)$ or $M \otimes_\Lambda -$ [1]. However, these functors are not exact in general. The aim of this paper is to show that certain properties of Λ-modules become more transparent if one views them as exact functors. In particular, one can use the machinery of localization theory for locally coherent Grothendieck categories because $\mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is a subcategory of the locally coherent category $\mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ of left exact functors. Another reason for introducing the concept of an exactly definable category is the fact that for any ring Λ there are plenty of categories related to $\mathrm{Mod}(\Lambda)$ which are exactly definable but not necessarily module categories.

We give now a brief survey of the basic notions and results of this paper. To this end fix an exactly definable category $\mathcal{A}$ and we may assume that $\mathcal{A} = \mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ for some skeletally small abelian category $\mathcal{C}$. The first sections are devoted to properties of the inclusion $d\colon \mathcal{A} \to D(\mathcal{A}) = \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. This leads to a notion of purity for $\mathcal{A}$ which coincides with the usual definition if $\mathcal{A}$ is a category of modules. A sequence $0 \to L \to M \to N \to 0$ of morphisms in $\mathcal{A}$ is pure exact if the sequence $0 \to L(X) \to M(X) \to N(X) \to 0$ is exact for every $X \in \mathcal{C}$, and it is shown that an object M is pure-injective if and only if for every set I the summation morphism $\coprod_I M \to M$ factors through the canonical morphism $\coprod_I M \to \prod_I M$. Moreover, the functor d identifies pure-exactness and pure-injectivity in $\mathcal{A}$ with exactness and injectivity in $D(\mathcal{A})$.

Call a full subcategory of $\mathcal{A}$ definable if it consists of those functors in $\mathcal{A}$ which vanish on some fixed subcategory of $\mathcal{C}$. This is a natural concept because a definable subcategory is again exactly definable. Also it is closed under forming direct limits, products and pure subobjects, and we shall prove that this property characterizes a definable subcategory. The dual concept is that of a definable quotient category which is discussed in the following section. We describe the full subcategory of pure-injective objects of a definable quotient category and use a result about localizing subcategories which seems to be of independent interest.

Next we exhibit the Ziegler spectrum $\mathrm{Zsp}(\mathcal{A})$ of $\mathcal{A}$ which is, by definition, the set of isomorphism classes of indecomposable pure-injective objects in $\mathcal{A}$. The subsets which are of the form $\mathcal{B} \cap \mathrm{Zsp}(\mathcal{A})$ for some definable subcategory $\mathcal{B}$, are precisely the closed sets of a topology on $\mathrm{Zsp}(\mathcal{A})$.

Exactly definable categories have direct limits and products and it is therefore natural to study functors between exactly definable categories which commute with direct limits and products. We prove that any such functor $f\colon \mathcal{A} \to \mathcal{B} = \mathrm{Ex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ is of the form $X \mapsto X \circ g$ for some exact functor $g\colon \mathcal{D} \to \mathcal{C}$, and we deduce several properties from this fact. For instance, f induces a continuous and closed map $\mathrm{Zsp}(\mathcal{A}) \to \mathrm{Zsp}(\mathcal{B})$ provided that f preserves indecomposability.

Given any object $M \in \mathcal{A}$, the endomorphism ring $\Gamma = \mathrm{End}(M)^{\mathrm{op}}$ acts on $M(X)$ for every $X \in \mathcal{C}$, and we obtain therefore an induced functor $\bar{M}\colon (\mathcal{C}/\operatorname{Ker}(M))^{\mathrm{op}} \to \mathrm{Mod}(\Gamma)$ which is faithful and exact. We call its image the endocategory $\mathcal{E}_M$ of M. This is a useful invariant since several local properties of M depend only on $\mathcal{E}_M$. For example, M is Σ-pure-injective if and only if every object in $\mathcal{E}_M$ is artinian, and M is endofinite if and only if every object in $\mathcal{E}_M$ is of finite length.

The next part of the paper is devoted to various finiteness conditions for exactly definable categories. Some of them are related to filtrations of the form $(\mathcal{A}_\alpha)_\alpha$ where each $\mathcal{A}_\alpha$ is a definable subcategory of $\mathcal{A}$. We give a complete classification of the indecomposable pure-injective objects in $\mathcal{A}$ under the assumption that a certain filtration of $\mathcal{A}$ exists.

Our discussion of exactly definable categories is mainly inspired and motivated by results from module theory. In fact the category of modules over a ring is the prototype of an exactly definable category, even though we have choosen a more abstract level for our exposition. To illustrate this point we devote the final section of this paper to epimorphisms in the category of rings. Using techniques and results from the previous sections it is shown that any epimorphism $\Lambda \to \Gamma$ is induced by a suitable localization of the Grothendieck category $D(\Lambda) = (\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$. We shall also discuss some consequences. For instance, we show that each epimorphism $\Lambda \to \Gamma$ is determined by a closed subset of the Ziegler spectrum of $\mathrm{Mod}(\Lambda)$. Further applications and examples of the material presented in this paper, in