Semilinear stochastic equations for symmetric diffusions

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Summary. In this paper we provide new sufficient conditions for the existence of weak solutions to a class of semilinear stochastic equations, which at the same time guarantee that the solution is a symmetric ergodic right process. The invariant measure (which is also the symmetrizing measure) is given explicitly. The emphasis is on the fact that due to the symmetry in our situation, the assumptions on the nonlinear term can be relaxed considerably.

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1 Introduction

Let $\mathbb{M} = \Big(\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta}, \zeta\Big)$ be a stochastic process with lifetime ζ, semigroup $(p_t)_{t\geq 0}$ and state space E. Then $\mathbb{M}$ is called symmetric, if there is a measure μ on E, such that $\int p_t f \cdot g \, d\mu = \int f \cdot p_t g \, d\mu$ for all bounded measurable f, g. If, moreover, $p_t 1 = 1$ μ-a.s., then $\mathbb{M}$ is stationary and reversible w.r.t. the measure $P_\mu := \int P_x \, \mu(dx)$. The aim of this paper is to give sufficient conditions for the existence of a symmetric conservative diffusion $\mathbb{M}$ which is a weak solution (weak in the probabilistic sense) to the stochastic differential equation

$$\begin{aligned} dX_t &= -(\tfrac{1}{2} A X_t + F(X_t))\, dt + dW_t, \\ X_0 &= x \in E, \end{aligned} \tag{1}$$

where E is a separable Hilbert space, F a non-linear map, $(A, D(A))$ a linear operator on E and $(W_t)_{t\geq 0}$ a Wiener process with covariance $Q \geq 0$. For the construction of the solution we use Dirichlet form techniques (the fundamental results were obtained in [AR 91]), which

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are combined with semigroup arguments in order to show that the solution $(X_t)_{t\geq 0}$ is not only a weak solution in the functional analytic sense, but that also $\int_0^t X_s\,ds \in D(A)$ under rather mild assumptions. So far, the symmetric case in infinte dimensions has been dealt with explicitly by J.Zabczyk [Z 89], who constructs mild solutions in the case $Q = Id$ and by I.Shigekawa [Shi 87, Section 4], who considers Girsanov transforms of the infinite dimensional Ornstein-Uhlenbeck process (i.e. $A = Id$). Symmetric mild solutions for $Q = Id$ have also been constructed by R.Marcus in [Mar 74], [Mar 78] and by Gatarek, Goldys [GaGo 95b]. Related results in the Dirichlet form approach have been obtained by S.Albeverio, M.Röckner [AR 91] and by T.S. Zhang [Zh 92]. Here, the task is split up into two parts. The first step is to characterize those operators A, Q for which the linear equation

$$dX_t = -\tfrac{1}{2}AX_t\,dt + dW_t, \quad X_0 = x \in E, \tag{2}$$

has a symmetric solution on E, cf. Theorem 4 below. The special feature of our approach to the second step is that the admissible drift terms F are characterized in terms of the symmetrizing measure μ, which in turn depends on A and Q. It seems that one thus obtains the weakest possible assumptions on F. Some restrictions on F, however, are independent of A, Q. If $Q = Id$, then F has to be a gradient ∇U. This is the classical characterization of Kolmogorov (see [Ko 37]) for the drift term of a stationary process, which can be extended to infinite dimensions, see [BR 95]. If $Q \neq Id$, then at least any drift term F, which can be generated by a Girsanov transformation, necessarily has to be of type $Q\nabla U$, as can be easily seen in the one-dimensional situation. Since these conditions are necessary, they occur in any approach. Apart from these restrictions, however, further conditions are needed that guarantee the existence of a solution. Here, we basically require that U is lower bounded and bounded by an even polynomial q such that $\int \|p(z)\|_E^2\,\mu(dz) < \infty$ for all polynomials p with degree(p) $\leq$degree(q^2). It is very interesting to note that in the case $Q = Id$ T.Funaki obtained a very similar condition [Fu 83, Assumptions 2 and 3 in Section 4], although his techniques are different. He uses a polygonal approximation to obtain a finite dimensional solution and a tightness argument to construct the infinite dimensional one.

In the "semigroup approach", i.e., the construction of mild solutions, different additional regularity assumptions on F are common, which need not be satisfied here; e.g. (local) Lipschitz continuity [Mar 74], [Z 89] or monotonicity [Mar 78]. In the non-symmetric case, mild solutions have been proved to exist also for less regular non-linearities. Among other conditions, it is often assumed that F satisfies a dissipativity condition [dPZ 92], or that F is polynomially bounded and satisfies a one-sided linear growth condition [I 87], [BaGyPa 94], [GyPa 94], or the semigroup generated by $(-A)$ has to be compact (e.g. [GaGo 94]). Here, we give examples, where a *weak* solution exists on E, although none of the latter two conditions is satisfied, cf. Examples 3(iii), 17.