Connecting paracontractivity and convergence of products

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October 8, 1996

Abstract

In [2] the LCP-property of a finite set Σ of square complex matrices was introduced and studied. Σ is an LCP-set if all left infinite products formed from matrices in Σ are convergent. It had been shown earlier in [3] that a set Σ paracontracting with respect to a fixed norm is an LCP-set. Here we prove a converse statement: If Σ is an LCP-set with a continuous limit function then there exists a norm such that all matrices in Σ are paracontracting with respect to this norm.

In addition we introduce the stronger property of l-paracontractivity. It is shown that common l-paracontractivity of a set of matrices has a simple characterization. It turns out that in the above mentioned converse statement the norm can be chosen such that all matrices are l-paracontracting.

It is shown that for Σ consisting of two projectors the LCP-property is equivalent to l-paracontractivity, even without requiring continuity.

1 Introduction

In the investigation of chaotic iteration procedures for linear consistent systems matrices which are paracontracting with respect to some vector norm play an important role. It was shown in [3], that if $A_1, \ldots, A_m$ are finitely many $k \times k$ complex matrices which are paracontracting with respect to the same norm, then for any sequence $d_i,\ 1 \le d_i \le m,\ i = 1, 2, \ldots$ and any x_0 the sequence

$$x_i = A_{d_i} x_{i-1} \quad i = 1, 2, \ldots \tag{1}$$

is convergent. In particular $A^{(d)} = \lim_{i\to\infty} A_{d_i} \ldots A_{d_1}$ exists for all sequences $\{d_i\}_{i=1}^{\infty} = d$. Hence those sets are examples of sets of matrices all infinite products of which converge. Such sets have been studied in [2]. Following [2], we call them LCP–sets.

In this note we investigate the question of necessity. As our main result we show that under the additional assumption that the mapping

$$d = \{d_i\}_{i=1}^{\infty} \to A^{(d)} = \lim_{i\to\infty} A_{d_i} A_{d_{i-1}} \ldots A_{d_1} \tag{2}$$

is continuous (which is equivalent to the set of fixed points of A_i being the same for all $1 \le i \le m$), an LCP–set is necessarily paracontracting with respect to some norm. In this sense paracontractivity is equivalent to the LCP–property. We will show in addition that continuity implies even the stronger property of l-paracontractiveness.

In the last section we consider the case $m = 2$. It is shown that for two projectors the equivalence of LCP-property and l-paracontractivity holds even without continuity.

Some parts of this paper are contained in [7].

2 Notations and known results

Let $|| \; ||$ denote a vector norm in C^k. A $k \times k$ matrix P is **paracontracting** with respect to $|| \; ||$, if for all x

$$Px \ne x \Leftrightarrow ||Px|| < ||x||. \tag{3}$$

We denote by $\mathcal{N}(|| \; ||)$ the set of all $k \times k$ matrices paracontracting w.r.t. $|| \; ||$. We call P **l-paracontracting** w.r.t. $|| \; ||$, if there exists $\gamma > 0$ such that

$$||Px|| \le ||x|| - \gamma ||Px - x|| \tag{4}$$

holds for all $x \in C^k$ and denote this set of matrices by $\mathcal{N}_\gamma(|| \; ||)$. Obviously

$$\mathcal{N}_\gamma(|| \; ||) \subset \mathcal{N}(|| \; ||). \tag{5}$$