A NOTE ON INVARIANT MEASURES OF INFINITE DIMENSIONAL DIFFUSIONS

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Abstract

We construct two examples of a nondegenerate diffusion on a Hilbert space X governed by the SDE

$$d\xi_t = \sigma(\xi_t)dW_t + B(\xi_t)dt,$$

with $\sigma(x) = I + \sigma_0(x)$ and $B(x) = \Lambda x + v(x)$, where $\Lambda : X \to X$ is a bounded linear operator, σ_0 and v are smooth (Fréchet differentiable on X) mappings with values in the space of nuclear operators on X and X, respectively, such that:

(i) in the first example $\Lambda x = -\frac{1}{2}x$ and the diffusion ξ_t has the transition probabilities and an invariant measure (which is unique) without directions of differentiability (or even continuity);

(ii) in the second example, ξ_t has two different invariant probabilities ν_1 and ν_2 such that ν_1 is equivalent to some Gaussian measure and is differentiable, while ν_2 has no directions of differentiability (or even continuity).

In addition, given $\varepsilon > 0$, both σ_0 and v can be chosen vanishing outside the ε-ball around zero and with the norms (respectively, in the space of trace operators on X and X) bounded by ε. Some natural modification of our construction makes the appropriate measures to be exceptionally supported (in the sense of Aronszajn) and ultimately singular.

1. Introduction

Analytic properties of the transition probabilities and invariant measures of infinite dimensional diffusions have been actively studied over the last three decades. Under various restrictions on the coefficients, some positive results have been obtained in [16], [18], [19], [21], [1], [17], [24], [2], [13], [14], [15], [24], [10], [11], [8], [9] (see also references in [7]).

It has been shown in [10], [11] that if μ is an invariant probability for a diffusion ξ on R^n with constant diffusion operator and drift $B \in L^2(\mu, R^n)$, then μ admits

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a density p from the Sobolev class $H^{1,1}(R^n)$ and, moreover, it has the logarithmic gradient $\beta^\mu := \nabla p/p$ belonging to $L^2(\mu, R^n)$. In [9] this result has been extended to diffusions with non-constant uniformly Lipschitzian and uniformly nondegenerate diffusion operators σ. In fact, the corresponding results have been proved even in a more general setting of weak elliptic equations

$$L^*\mu = 0, \tag{$*$}$$

where

$$Lf = trace(\sigma f'') + (B, f'),$$

and the equation $(*)$ is understood in the weak sense

$$\int Lf(x)\,\mu(dx) = 0 \qquad \forall f \in C_0^\infty(R^n).$$

In [9] and [11] also some infinite dimensional analogues of these results have been obtained. More specifically, let (X, H, γ) be an abstract Wiener space (with a centered Gaussian measure γ), $B(x) = -x + v(x)$, where $v : X \to H$ is a bounded Borel mapping, and $\sigma(x) = I + \sigma_0(x)$, where $\sigma_0(\,\cdot\,)$ is a measurable map with values in the space of nuclear operators on H satisfying certain additional technical assumptions. Suppose that a probability measure μ on X satisfies the equation $(*)$ with

$$Lf = trace_H(\sigma f'') + \langle f', B\rangle$$

with $C_0^\infty(R^n)$ replaced by the class $\mathcal{FC}^\infty$ of all functions of the form

$$f(x) = \varphi(l_1(x), \dots, l_n(x)), \quad \varphi \in C_b^\infty(R^n),\ l_i \in X^*.$$

Then μ is absolutely continuous with respect to γ, admits a representation $\mu = \varrho^2\gamma$ with ϱ from the Sobolev class $H^{2,1}(\gamma)$, and μ has the vector logarithmic derivative β^μ (in particular, μ is differentiable along all directions in H). Note that this result gave a positive solution of the well-known Shigekawa problem [24].

Certainly, a question arises whether the finite dimensional results can be extended to infinite dimensions in the same generality. The aim of this note is construct two examples showing that in infinite dimensions certain extra assumptions (of the type considered in [9]) are necessary. We consider Hilbert spaces X.

In the first example we construct, for arbitrary $\varepsilon > 0$, a smooth map σ_0 with values in the space of nuclear operators on X and a smooth map $v : X \to X$ with $\|\sigma_0(\,\cdot\,)\|_{L^{(1)}(X)} \le \varepsilon$ and $\|v(\,\cdot\,)\| \le \varepsilon$ such that there is a unique probability measure μ invariant for the diffusion ξ generated by the stochastic differential equation

$$d\xi_t = \sigma(\xi_t)dW_t + B(\xi_t)dt$$

with $\sigma(x) = I + \sigma_0(x)$, $B(x) = -\frac{1}{2}x + v(x)$, but μ has no nonzero directions of differentiability (and even continuity).