EXACT CATEGORIES AND VECTOR SPACE CATEGORIES

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with an appendix by B. KELLER

Abstract. In a series of papers starting with [ASo] additive subbifunctors F of the bifunctor $\mathrm{Ext}_\Lambda(\ ,\ )$ are studied in order to establish a relative homology theory for an artin algebra Λ. On the other hand, one may consider the elements of $F(X,Y)$ as short exact sequences. We observe that these exact sequences make $\mathrm{mod}\,\Lambda$ into an exact category if and only if F is closed in the sense of [BH].

Concerning the axioms for an exact category we refer to [GR]. In fact, for our general results we work with subbifunctors of the extension functor for arbitrary exact categories.

In order to study projective and injective objects for exact categories it turns out to be convenient to consider categories with almost split exact pairs because many of the results from [ASo] can easily be adapted to this situation.

Exact categories arise in representation theory for example if one studies categories of representations of bimodules. Representations of bimodules gained their importance in studying questions about representation types. They appear as domains of certain reduction functors defined on modules categories. These reduction functors are frequently closely related to the functor $\mathrm{Ext}_\Lambda(\ ,\ )$ and in general do not preserve at all the usual exact structure of $\mathrm{mod}\,\Lambda$.

By showing the closedness of suitable subbifunctors of $\mathrm{Ext}_\Lambda(\ ,\ )$, we can equip $\mathrm{mod}\,\Lambda$ with an exact structure such that some reduction functors actually become 'exact'. This allows us to derive informations about the projective and injective objects in the respective categories of representations of bimodules appearing as domains and even show existence of almost split sequences for them.

Examples of such domains appearing in practice are the subspace categories of a vector space category with bonds used in [G-V]. We provide an example that existence of almost split sequences for them is not a general fact but may even fail if the vector space category is finite.

1. Exact categories

This section is devoted to transferring some definitions and basic results on relative theory developed for abelian categories in [BH] and for finitely generated modules over artin algebras in [ASo] to the context of exact categories. Furthermore, we show how one can construct a new exact structure on a category from closed subbifunctors of the extension bifunctor induced by a given exact structure. Moreover, we study when an exact structure on a category $\mathcal{A}$ induces an exact structure on a factor category of $\mathcal{A}$.

1.1. **Axioms for exact categories.** Let $\mathcal{A}$ be an additive category with split idempotents. From [GR] we recall the following notions. A pair (i,d) of composable morphisms $X \xrightarrow{i} Y \xrightarrow{d} Z$ in $\mathcal{A}$ is called *exact* if i is a kernel of d and d is a cokernel of i.

Let $\mathcal{E}$ be a class of exact pairs $X \xrightarrow{i} Y \xrightarrow{d} Z$ which is closed under isomorphisms. The morphisms i and d appearing in a pair (i,d) in $\mathcal{E}$ are called an *inflation* and a *deflation* of $\mathcal{E}$, respectively. The class $\mathcal{E}$ is said to be an *exact structure* on $\mathcal{A}$ and $(\mathcal{A},\mathcal{E})$ an *exact category* if the following axioms are satisfied:

E1 The composition of two deflations is a deflation.
E2 For each f in $\mathcal{A}(Z',Z)$ and each deflation d in $\mathcal{A}(Y,Z)$, there is some Y' in $\mathcal{A}$, an f' in $\mathcal{A}(Y',Y)$ and a deflation $d'\colon Y'\to Z'$ such that $df'=fd'$.
E3 Identities are deflations. If de is a deflation, then so is d.
E3$^{\mathrm{op}}$ Identities are inflations. If ji is a inflation, then so is i.

This set of axioms is proved to be equivalent to the following one (see [Ke2]):

Ex0 The identity morphism of the zero object id_0 is a deflation.
Ex1 The composition of two deflations is a deflation.
Ex2 For each f in $\mathcal{A}(Z',Z)$ and each deflation d in $\mathcal{A}(Y,Z)$, there is a pullback diagram

$$\begin{array}{ccc} Y' & \xrightarrow{d'} & Z' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{d} & Z \end{array}$$

where d' is a deflation.
Ex2$^{\mathrm{op}}$ For each f in $\mathcal{A}(X,X')$ and each inflation i in $\mathcal{A}(X,Y)$, there is a pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X' & \xrightarrow{i'} & Y' \end{array}$$

where i' is an inflation.

Since the morphisms d and d' in Ex2 are deflations, there are corresponding inflations $i\colon X\to Y$, $i'\colon X'\to Y'$ such that (i,d) and (i',d') belong to $\mathcal{E}$. Using the pullback property one sees that $f'i'$ is a kernel of d. Hence by possibly modifying i' by a unique isomorphism we may suppose $f'i'=i$. We call the pair (i',d') a *pullback pair* of (i,d) along f. Dually, we use the notion of a *pushout pair*.

It is shown in [Ke1, Appendix A] that each skeletally small exact category $(\mathcal{A},\mathcal{E})$ admits an equivalence, $F\colon\ \mathcal{A}\to\mathcal{B}$ with a full subcategory $\mathcal{B}$ of an abelian category $\mathcal{C}$ such that $\mathcal{B}$ is extension closed and an exact pair (i,d) is in $\mathcal{E}$ if and only if

$$0\to F(X)\xrightarrow{F(i)}F(Y)\xrightarrow{F(d)}F(Z)\to 0$$

is a short exact sequence in $\mathcal{C}$. Moreover, it is established that an exact category also satisfies Quillen's axioms from [Qu]. As a particular consequence also the dual axiom of E1 is satisfied. Thus the duals of all axioms are satisfied. We sum this up in the following proposition which is used later.

Proposition 1.1. *If $(\mathcal{A},\mathcal{E})$ is an exact category, then also $(\mathcal{A}^{\mathrm{op}},\mathcal{E}^{\mathrm{op}})$ is an exact category where $\mathcal{A}^{\mathrm{op}}$ is the opposite category of $\mathcal{A}$ and $\mathcal{E}^{\mathrm{op}}$ consists of the pairs (d,i) such that (i,d) is in $\mathcal{E}$.*

There is always a unique minimal exact structure $\mathcal{E}_0$ on a given additive category $\mathcal{A}$ with splitting idempotents. Namely, $\mathcal{E}_0$ consists of all exact pairs (i,d) such that d is a retraction and i is a kernel of d (or alternatively i is a section and d is a cokernel of i). The exact structure $\mathcal{E}_0$ is called the *split structure*. On the other hand, if $\mathcal{A}$ happens to be abelian, then the collection $\mathcal{E}_{\mathcal{A}}$ of all exact sequences in $\mathcal{A}$ is an exact structure which obviously is the unique maximal exact structure on $\mathcal{A}$.