On the Hajós number of graphs

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Abstract

A graph G is said to have property P_m if it contains no subdivision of K_{m+1} and no subdivision of $K_{\lceil m/2\rceil+1,\lfloor m/2\rfloor+1}$. Chartrand, Geller and Hedetniemi [3] (see also Problem 6.3 in [11]) conjectured that the set of vertices (respectively, edges) of any graph with property P_m can be partitioned into $m-n+1$ subsets such that each of these subsets induces a graph with property P_n, provided $m \geq n \geq 1$ (respectively, $m \geq n \geq 2$). We prove that both conjectures fail when $m > cn^2$ for some positive constant c. In fact, we prove that under the condition $m > cn^2$, there exists a graph G with property P_m such that in every colouring of its vertices or edges with m colours there is a monochromatic subgraph H with Hajós number $h(H) > n$, that is, with a subdivision of K_{n+1}. In addition, we prove bounds of Nordhaus-Gaddum type for the Hajós number.

1 Introduction

For a graph G, let *the Hajós number* $h(G)$ denote the maximum k such that G contains a subdivision of K_k.

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For a positive integer m, Chartrand, Geller and Hedetniemi [3] denote by P_m the property of a graph G to have $h(G) \leq m$ and not to contain a subdivision of the complete bipartite graph $K_{\lceil m/2 \rceil+1, \lfloor m/2 \rfloor+1}$ either. Let $f_1(m,n)$ (respectively, $f_2(m,n)$) denote the minimum k such that for any graph G with property P_m, there exists a partition of its vertices (respectively, edges) into k sets such that each set induces a subgraph with property P_n. The conjecture in [3] (see also Problem 6.3 in [11]) that $f_1(m,n) \leq m-n+1$ for each $m \geq n \geq 1$ is sometimes referred to as the (m,n)-conjecture, and the conjecture that $f_2(m,n) \leq m-n+1$ for each $m \geq n \geq 2$ as the $[m,n]$-conjecture. Chartrand, Geller and Hedetniemi proved the (m,n)-conjecture for all n and m satisfying $1 \leq n \leq m \leq 4$ except for the case $(4,1)$ which is the 4CT. They also proved the $[m,n]$-conjecture for all n and m satisfying $2 \leq n \leq m \leq 4$, except the case $[4,3]$. The $[4,3]$-conjecture was proved by Heath [10]. The (m,n)- and $[m,n]$-conjectures were mentioned in the excellent survey [19]. Woodall [19] remarked that perhaps the corresponding conjectures in terms of contractions, rather than subdivisions, are more likely to be true.

Indeed, Jørgensen [12], and Hanson and Toft [9] proved that for each n the (m,n)-conjecture fails for almost all graphs. They used the ideas of Erdős and Fajtlowicz [5] who showed that for almost all graphs on p vertices, the Hajós number is at most $(2+\epsilon)\sqrt{p}$. The first aim of this paper is to show that both the (m,n)-conjecture and the $[m,n]$-conjecture are not true when $m > cn^2$, and that also even the contraction version of the $[m,n]$-conjecture is false for m large enough with respect to n.

Theorem 1 *There exists a constant c such that if $m > cn^2$, then there exists a graph G with property P_m such that for all partitions of its vertices or of its edges into m colour classes, there is a monochromatic subgraph H with Hajós number $h(H) > n$.*

Theorem 2 *For each $n > 1$, there exists an m_0 such that for every $m > m_0$, there exists a graph G having neither K_{m+1} nor $K_{\lceil m/2 \rceil+1, \lfloor m/2 \rfloor+1}$ as a minor such that for all partitions of its edges into m colour classes at least one monochromatic subgraph H has K_{n+1} as a minor. (It follows that the $[m,n]$-conjecture is false, even in terms of contractions).*

The second aim of the paper is to find bounds of Nordhaus-Gaddum type for the Hajós number. Zelinka [20] conjectured that $\eta(G) + \eta(\bar{G}) \leq n+1$ for each graph G on n vertices, where $\eta(G)$ is the Hadwiger number of G. This is not true (see [15]), but for the Hajós number instead of the Hadwiger number the bound is true.

Theorem 3 *For each positive integer n,*

$$\max\{h(G) + h(\bar{G}) \mid |V(G)| = n\} = n+1, \tag{1}$$

$$\max\{h(G) \times h(\bar{G}) \mid |V(G)| = n\} = \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor. \tag{2}$$