THE HOMOLOGICAL QUADRATIC FORM OF A BIEXTENSION ALGEBRA

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Let A be a finite dimensional algebra over an algebraically closed field k. One of the fundamental problems in Representation Theory of Algebras is to determine the representation type of A and in case A is tame, to classify all the indecomposable A-modules.

We denote by mod_A the category of finite dimensional left A-modules. Roughly speaking, A is *tame* if for each dimension, the indecomposable A-modules occur (up to isomorphism) in a finite number of discrete and one-parameter families. If A is not tame, then the representation theory of A is as complicated as the study of vector spaces equipped with two non-commuting endomorphism, for which the classification of indecomposables is a famously unsolved problem.

Important classes of algebras are known to be tame: path algebras $k\Delta$ where Δ is a quiver with underlying graph of Dynkin or extended-Dynkin type; tubular algebras (see for example [7, 16]). For tame algebras of polynomial growth, the structure of the module category is well understood [15, 18], while the representation theory of tame non-polynomial algebras only in few cases is known [4, 6, 10]. Among the tools for the study of these algebras and their module categories we find integral quadratic forms, vector space categories, derived categories and others.

For the study of non-polynomial growth tame algebras, the following construction was introduced in [6]. Let $R_1, \dots, R_t$ and $R'_1, \dots, R'_s$ be two sequences of modules over an algebra B. We consider $R = \overset{t}{\underset{i=1}{\oplus}} R_i$ as a $B - k^t$-bimodule and $R' = \overset{s}{\underset{j=1}{\oplus}} R'_j$ as a $B - k^s$-bimodule. The *biextension* $[R']B[R]$ of B by the two sequences $R_1, \dots, R_t$ and $R'_1, \dots, R'_s$ is by definition the matrix algebra

$$\begin{pmatrix} k^t & 0 & 0 \\ R & B & 0 \\ DR' \otimes_B R & DR' & k^s \end{pmatrix}$$

equipped with the obvious addition and multiplication, where we denote by $D = \mathrm{Hom}_k(-,k)$ the usual duality. For any set of pairs of indices $L \subset \{1, \dots, t\} \times \{1, \dots, s\}$, consider the subspace $V = \underset{(i,j)\in L}{\oplus} DR_j \otimes_B R_i$ of the space $DR' \otimes_B R$ and the associated ideal

$$J(V) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ V & 0 & 0 \end{pmatrix}$$

in $[R']B[R]$. The algebra $A = [R']B[R]/J(V)$ is called a *truncated biextension* of B.

For an algebra A with finite global dimension we consider the non-commutative bilinear form $\langle -,-\rangle_A\colon K_0(A) \times K_0(A) \to \mathbb{Z}^n$ defined on the Grothendieck group $K_0(A) \cong \mathbb{Z}^n$ as

$$\langle v, w\rangle_A = \sum_{s=0}^{\infty} \sum_{i,j=1}^{n} (-1)^s v(i) w(j) \dim_k \operatorname{Ext}_A^s(S_i, S_j),$$

where $S_1, \ldots, S_n$ is a complete set of the isoclasses of simple A-modules. The associated quadratic form $\chi_A(v) = \langle v, v\rangle_A$ is called the *homological form* of A (also called the *Euler form* of A). In several instances, arithmetical properties of χ_A are related with the representation type of A [5, 13, 14, 15]. The main result of this work is the following.

Theorem. *Let B be a strongly simply connected algebra with at least 6 vertices and $R_1, \ldots, R_t$; $R'_1, \ldots, R'_s$ be two sequences of indecomposable B-modules. Let $R = \bigoplus_{i=1}^{t} R_i$ and $R' = \bigoplus_{j=1}^{s} R'_j$ and consider a truncated biextension $A = [R']B[R]/J(V)$ of B.*

If χ_A is non-negative with $\operatorname{corank} \chi_A = 1 + s + t$ and $\operatorname{corank} \chi_B = 1$, then A is tame.

The proof of the Theorem uses heavily the results in [6], we present it in section 2. In section 1, besides some introductory remarks, we present many examples of algebras satisfying the hypothesis of the Theorem. In particular, we show that tree algebras A with global dimension 2 and non-negative Euler form χ_A accepting a positive sincere isotropic vector are in this class (hence they are tame, a result first shown in [5]).

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1. Some examples.

1.1. Let A be a basic, finite dimensional algebra over the fixed algebraically closed field k. It is well-known that A may be written as a quotient kQ/I of a path algebra kQ where Q is a finite quiver. We shall consider also A as a *spectroid*, that is, a k-category whose objects are the vertices $Q_0 = \{1, \ldots, n\}$ of Q and the space of maps from x to y is $A(x, y) = e_y A e_x$, where e_x denotes the primitive idempotent associated to the vertex x. See [7].

We recall that the *one-point extension* $A[M]$ of the algebra A by a module M is the spectroid with objects $Q_0 \cup \{s\}$ and morphism spaces $A[M](s, x) = M(x)$ for $x \neq s$, $A[M](s, s) = k$ and such that A is a full convex (= path closed) subcategory. We denote by $P_s = Ae_s$ the indecomposable projective cover of the simple module S_s. Observe that $M = \operatorname{rad} P_s$. See [16].