Superlinear Period Lengths in Some Subtraction Games

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Abstract

Subtraction games are "simple" variants of the famous Nim game [Bou 01]. In this note we will show that in some subtraction games the sequences of Win/Loss states have superlinear period lengths. Our most prominent observation is:

> For all s with $1 \leq s \leq 26$ the (s,4s,12s+1,16s+1)-game has the cubic period length $56s^3 + 52s^2 + 9s + 1$.

Possibly, the (s,8s,30s+1,37s+1,38s+1)-games with $s \in \mathbb{N}$ have superpolynomial period lengths.

1 Introduction and Results

In the $(s_1, \ldots, s_k)$-subtraction game there is a heap of size $m > 0$ and natural numbers $s_1, \ldots, s_k$, the legal moves. Without loss of generality we always assume $0 < s_1 < s_2 < \ldots < s_k$ and $k \geq 2$.

Two players move alternately. A move consists in substracting some s_i from m. The player who makes the heap size nonpositive is the winner. Assuming optimal strategies for both players, we get a sequence $(v(m))_{m=1}^{\infty}$ of game theoretic values:

$v(m)$=1, if the starting position with heap size m is a win for the player to move.

$v(m)$=0, if the starting position with heap size m is a loss for the player to move.

The starting values are $v(1) = \ldots = v(s_k) = 1$, as the player wins directly by subtracting s_k . For all $m > s_k$ the values obey the recursive condition

$$v(m) = 1 - \min\{v(m - s_1), \ldots, v(m - s_k)\}. \tag{1}$$

A sequence $(v(m))$ is called periodic if there are some $M \geq 0$ and $P > 0$ such that $v(M + t) = v(M + P + t)$ for all $t \geq 0$. The smallest such values for M and P are called the starting point and the period, respectively.

The sequence $(v(m))_{m=s_{k+1}}^{\infty}$ in the $(s_1, \ldots, s_k)$-game has to become periodic with period $P(s_1, \ldots, s_k) \leq 2^{s_k}$. Let M be the starting point of the period. Then we call $PP(s_1, \ldots, s_k) = M - (s_k + 1)$ the preperiod of the $(s_1, \ldots, s_k)$-game. In the case of $PP = 0$ we say that the game has no preperiod.

We are interested in subtraction games with small numbers of legal moves and *long* periods and/or preperiods. In the rest of this section we list our theoretical and experimental results. Section 2 contains some remarks on the proofs. Section 3 demonstrates the structure of the $(s, 4s, 12s+1, 16s+1)$-games for $1 \leq s \leq 26$. In Section 4 we formulate conjectures and open questions.