ON CONNECTED SUMS OF FOUR-DIMENSIONAL MANIFOLDS

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Abstract. The monopole map defines an element in an equivariant stable cohomotopy group refining the Seiberg-Witten invariant. A gluing theorem for this stable cohomotopy invariant gives new results on diffeomorphism types of decomposable manifolds.

1. Introduction

For a closed Riemannian four-manifold (X, g) the choice of both an orientation $or(X)$ and a $spin^c$-structure $\mathfrak{s}$ gives rise to an S^1-equivariant monopole map $\Psi = \Psi_{g,or(X),\mathfrak{s}}$ between certain affine Hilbert spaces. The moduli space of monopoles is obtained as the quotient of the zero-set of this map by the S^1-action. It serves as the basic geometric ingredient in the definition of the integer valued Seiberg-Witten invariants for four-dimensional manifolds [13]. For this definition it actually suffices to consider moduli spaces of monopoles up to certain bordisms. If the relevant Hilbert spaces were finite dimensional, the Pontrijagin-Thom construction would relate the corresponding bordism groups to stable homotopy classes of maps. This analogue suggests to interpret Seiberg-Witten invariants as invariants of some kind of homotopy class of the monopole map Ψ.

Theorem 1.1. *The monopole map Ψ defines an element in an equivariant stable cohomotopy group*

$$\pi^b_{S^1,H}(Pic^0(X); \mathrm{ind}(\mathcal{D})),$$

which is independent of the chosen Riemannian metric. For $b > \dim(Pic^0(X)) + 1$, a homology orientation determines a homomorphism of this stable cohomotopy group to $\mathbf{Z}$, which maps $[\Psi]$ to the integer valued Seiberg-Witten invariant.

Here $Pic^0(X)$ denotes the Picard torus $H^1(X; \mathbf{R})/H^1(X; \mathbf{Z})$. The Dirac operator associated to the chosen $spin^c$-structure defines a virtual complex index bundle $\mathrm{ind}(\mathcal{D})$ over the Picard torus, and $b = b_+(X)$ denotes the rank of the positive part of the intersection form on X. The suffix H stands for a universe for the S^1-action, that is a Hilbert space with an orthogonal S^1-action. Its finite dimensional invariant linear subspaces provide the suspension coordinates in the construction of equivariant cohomotopy groups.

The stable cohomotopy invariant in the theorem does not capture all the features of Seiberg-Witten theory. The reason is that the restriction of the monopole map to the S^1-fixed point set is quite special: It is linear. It seems that most, if not all, of the known features of

Seiberg-Witten theory can be recaptured in the stable cohomotopy setting by looking at Ψ up to equivariant homotopy relative to the fixed point set. In case $b > \dim(Pic^0(X)) + 1$ these relative homotopy classes are one-to-one with the homotopy classes which form the stable cohomotopy groups above. In general the stable relative homotopy classes don't admit a natural group structure, but only form a set. Fixing a generic metric, one can define a map of such a set to the integers. The map changes as one changes the metric. In this way chamber structures appear in Seiberg-Witten theory.

The above definition of a refined version of the Seiberg-Witten invariants grew out of my attempts to understand Furuta's work on the $\frac{11}{8}$-conjecture [9]. Actually, when talking to Furuta about my results, I learnt that he was independently working in a similar direction [10]. The main emphasis of the present article lies in the fact that this refined version actually contains more information than the integer valued Seiberg-Witten invariants. This is because the stable cohomotopy invariant defined by the monopole map behaves nicely when taking connected sums. Recall that a *spin*^{c}-structure on a connected sum uniquely decomposes as the sum of *spin*^{c}-structures on the respective summands.

Theorem 1.2. *For a connected sum $X = X_0 \# X_1$ of 4-manifolds, the stable equivariant cohomotopy invariant is the smash product of the invariants of its summands*

$$[\Psi_X] = [\Psi_{X_0}] \wedge [\Psi_{X_1}].$$

Here are two sample applications:

Corollary 1.3. *Let K denote the K3-surface and suppose there is an oriented diffeomorphism $X_1 \# K \# K \cong X_2 \# K \# K$, where the X_i are simply connected Kähler manifolds with $b^+ = 3 \, mod \, 4$. Then the integer Seiberg-Witten invariants of X_1 and X_2 are the same $mod \, 2$. More precisely, there is an isometry of second cohomology groups with integer coefficients of the X_i which maps the characteristic elements of X_1 with odd Seiberg-Witten number to their counterparts in the cohomology of X_2.*

Corollary 1.4. *Suppose the connected sum $\#_{i=1}^m E_i$ of simply connected minimal elliptic surfaces of odd geometric genus is diffeomorphic to a connected sum $\#_{j=1}^n F_j$ of elliptic surfaces. If $m < 4$, then $n = m$ and the F_j and the E_i are diffeomorphic up to permutation.*

2. Fredholm maps and stable homotopy

A *Fredholm map* $f : H' \to H$ in this paper will be a compact perturbation of a linear Fredholm operator between separable Hilbert spaces. This means that f is of the form $f = l + c$, where l is linear Fredholm and the continuous map c maps bounded sets into compact sets.

A theorem, which goes back to A. S. Schwarz [11], associates to certain such Fredholm maps stable homotopy classes of maps between finite dimensional spheres: Let $D' \subset H'$ be a disc with boundary $\partial D'$. Two continuous maps $f_i : \partial D' \to H \setminus \{0\}$ are called *compactly homotopic relative to* l, if there is a continuous and compact map $c : \partial D' \times [0,1] \to H$ with $f_i = l + c_i$ for $i \in \{0,1\}$ and

$$f_t(h') = l(h') + c_t(h') \neq 0$$

for all $t \in [0,1]$ and $h' \in \partial D'$.

Theorem 2.1. *The compact homotopy classes of continuous Fredholm maps relative to* l *are in one-to-one correspondence with elements of the stable homotopy group* $\pi^{st}_{\mathrm{ind}\, l}(S^0)$ *of the sphere.*

This correspondence can be described as follows: Any compact map c on the bounded set $\partial D'$ can be uniformly approximated by maps c_n mapping to finite dimensional linear subspaces $V_n \subset H$ containing $im(l)^\perp$. The correspondence associates to $l + c$ the maps $f_n/||f_n||$ with f_n being the restriction of $l + c_n$ to $l^{-1}(V_n) \cap \partial D'$. A detailed proof of this theorem can be found in [2], p.257f.

In this paper, this concept will be used with a few modifications: Firstly, we will consider Fredholm maps which extend continuously to maps

$$f^+ : H'^+ \to H^+$$

between the one-point completed Hilbert spheres. Equivalently, we suppose f to satisfy a boundedness condition: The preimages of bounded sets are bounded. Secondly, we will consider equivariant maps, which are furthermore parametrized over some space.

First a short discussion of the boundedness condition: In finite dimensions this condition is equivalent to f being proper, i.e. f is closed and the preimage of any point in the target space is compact. Here is a proof that in the setting of Fredholm maps the boundedness condition at least implies properness:

Lemma 2.2. *Let* $l : H' \to H$ *be a continuous linear Fredholm map between Hilbert spaces and let* $c : H' \to H$ *be a compact map. Then the the restriction of the map* $f = l + c$ *to any closed and bounded subset* $A' \subset H'$ *is proper.*
In particular, if preimages of bounded sets in H *under the map* f *are bounded, then* f *is proper and extends to a proper map* $f^+ : H'^+ \mapsto H^+$ *between the one point completions.*

Proof. Let $\rho : H' \to \ker l$ denote the orthogonal projection. Then $f|_{A'}$ factors through an injective, closed and thus proper map $A' \to H \times \overline{q(A')} \times \overline{\rho(A')}$, $a' \mapsto (l(a'), q(a'), \rho(a'))$, a homeomorphism $(h, s, e) \mapsto (h + s, s, e)$ and the projection $H \times \overline{q(A')} \times \overline{\rho(A')} \to H$ which is proper as the two extra factors are compact.

Now we invoke the boundedness condition: As the preimages of points in H are bounded, they are compact by what was already shown. Let $h \in H$ be in the closure of $f(A')$, with A' closed in H'. By the boundedness condition, h is already in the closure of $f(A'_0)$, where A'_0 is a bounded closed subset of A'. From the first part of the proof it follows that h is contained in $f(A'_0) \subset f(A')$. Thus f is proper. But properness extends to f^+: If, for a closed $A' \subset H'^+$, the closure $\overline{f^+(A')}$ contains the point at infinity, then $f^+(A') \cap H = f(A' \cap H')$ is unbounded. Since f is a compact perturbation of a continuous linear map, $A' \cap H'$ is unbounded and thus contains the point at infinity in its closure. In particular, f^+ is closed and thus proper. $\square$

We are now going to associate to a Fredholm map satisfying the boundedness condition a stable homotopy class of maps between spheres. The next lemma will provide the technical foundations. Let's start with fixing notation:
Let $W \subset H$ be a finite dimensional linear subspace and let $W' = l^{-1}(W)$ be its preimage under the linear Fredholm map l. Let $S(W^\perp)$ denote the unit sphere in the orthogonal complement $W^\perp$ of W. As in finite dimensions, the inclusion $W^+ \to H^+ \setminus S(W^\perp)$ is a deformation retract. The retracting map ρ_W can be described as follows: The one-point completed Hilbert space H^+ identifies with the unit sphere $S(\mathbf{R} \oplus H) = S(\mathbf{R} \oplus W \oplus W^\perp)$ in $\mathbf{R} \oplus H$ via the map $h \mapsto (h^2+1)^{-1}(h^2-1, 2h)$. In this identification, the subspace W^+ maps to the "equatorial" subsphere $S(\mathbf{R} \oplus W \oplus 0) \subset S(\mathbf{R} \oplus W \oplus W^\perp)$ and $S(W^\perp)$ maps to the complementary "polar" subsphere $S(0 \oplus 0 \oplus W^\perp)$. The retracting homotopy shrinks the latitutes in $S(\mathbf{R} \oplus W \oplus W^\perp) \setminus S(0 \oplus 0 \oplus W^\perp)$ to the equator. The retraction ρ_W has the following property: For $h \in H \setminus W^\perp$, the vector $\rho_W(h)$ differs from the orthogonal projection $pr_W(h)$ to W by a positive scalar factor $\rho_W(h) = \lambda(h) pr_W(h)$.

Lemma 2.3. *There are finite dimensional linear subspaces $V \subset H$, such that the following statements hold:*

1. *The subspace V spans, together with the image $Im(l)$ of the linear Fredholm map l, the Hilbert space $H = Im(l) + V$.*
2. *For $W \supset V$ with $W = U \perp V$, the restricted map $f|_{W'^+} : W'^+ \to H^+$ misses the unit sphere $S(W^\perp)$ in the orthogonal complement of W.*
3. *The maps $\rho_W f|_{W'}{}^+$ and $\mathrm{id}_U^+ \wedge \rho_V f|_{V'}^+$ are homotopic as pointed maps*
$$W'^+ \cong U^+ \wedge V'^+ \to U^+ \wedge V^+ = W^+.$$

Indeed, if H is separable, then the subspaces V satisfying these three conditions are cofinal in the direct system of finite dimensional subspaces in H.

Proof. The preimage $f^{-1}(D)$ of the unit disc D in H is bounded in H'. So the closure C of its image under the compact map c is compact in H. Cover C by finitely many balls with radius $\varepsilon \leq \frac{1}{4}$, centered at points v_i. Together with the orthogonal complement to the image of the linear Fredholm map l, these points v_i span a finite dimensional linear subspace V of H. Let's check the second condition: Suppose $w \in S(W^\perp)$ is in the image of $f|_{W'}{}^+$. Then $f^{-1}(w) \cap W'^+ \subset f^{-1}(D_1(H))$ will be mapped by $f|_{W'} = l(W) + c(W)$ to a subspace of

$W+C$. So w will be contained both in $S(W^\perp)$ and $W+C$. However, these two subsets of H are at least $1-\varepsilon \geq \frac{3}{4}$ apart.

We will identify W' with the orthogonal sum $U \perp V'$ via the map

$$w' \mapsto (l \circ (1 - pr_{V'})(w'), pr_{V'}(w')).$$

To prove the last claim, it suffices to show that $\mathrm{id}_{U^+} \wedge \rho_V f|^+_{V'}$ and $f|^+_{W'}$ are homotopic as maps $W'^+ \to H^+ \setminus S(W^\perp)$. Let $D' \subset H'$ be a disk, centred at the origin, which contains the preimage $f^{-1}(D)$ of the unit disk in H. Consider the homotopy $h : D' \times [0,3] \to H^+ \setminus S(W^\perp)$, defined by:

$$h_t = \begin{cases} l + ((1-t)id_H + t \cdot pr_V) \circ c & \text{for } 0 \leq t \leq 1, \\ l + pr_V \circ c \circ ((2-t)id_{V'} + (t-1)pr_{V'} & \text{for } 1 \leq t \leq 2, \\ pr_U \circ l + ((3-t)pr_V + (t-2)\rho_V) \circ (l+c) \circ pr_{V'} & \text{for } 2 \leq t \leq 3. \end{cases}$$

Note that the image during the homotopy stays within an ε-neighbourhood of W. The homotopy is chosen in such a way that the image of the sphere $S' = \partial D' \cap W'$ during the homotopy stays away not only from the unit sphere $S(W^\perp)$ in $W^\perp$, but from the whole of $W^\perp$. Before we check this, let's consider the consequences: Since $H^+ \setminus (D \cap W^\perp)$ is contractible, the homotopy h_t can be extended to the complement of $D' \cap W'$ in W'^+, thus defining a homotopy as claimed.

Let s' be an element in the sphere S'. We will track the path its image will take during the homotopy. At starting time, it is mapped to $f(s')$, which is of norm greater or equal to 1, and furthermore, in an ε-neighbourhood of W. In particular, its distance form $W^\perp$ is at least $1-\varepsilon \geq \frac{3}{4}$. During the first part of the homotopy, the image will move at most a distance of ε, so it will definitely stay away from $W^\perp$.

From time $t=1$ on one has $pr_U \circ h_t(s') = pr_U \circ l(s')$. Since $pr_U(W^\perp) = 0$ by definition, we are reduced to checking the case $pr_{W'} \circ l(s') = 0$, that is for $s' \in S' \cap V'$. But for such an element, the image during the second part of the homotopy stays fixed and during the third part moves on a straight line between $pr_V(f(s'))$ and $\rho_V(f(s'))$, which are nonzero vectors in V, differing by a positive real factor. This concludes the proof of 2.3. $\square$

In particular, the restrictions $f|_{l^{-1}(V)}$ to finite dimensional linear subspaces $V \subset H$ as in 2.3 together define an element in the colimit of pointed homotopy classes

$$[f] = \operatorname*{colim}_{V \subset H} [(f|_{l^{-1}(V)})^+] \in \operatorname*{colim}_{V \subset H} [(l^{-1}(V))^+, H^+ \setminus S(V^\perp)].$$

The homotopy equivalences $V^+ \subset (H^+ \setminus S(V^\perp))$ combine to an isomorphism

$$\pi^{st}_{\mathrm{ind}\, l}(S^0) = \operatorname*{colim}_{V \subset H} \; [(l^{-1}(V))^+, V^+] \xrightarrow{\sim} \operatorname*{colim}_{V \subset H} \; [(l^{-1}(V))^+, H^+ \setminus S(V^\perp)].$$

In this way $[f]$ can be identified as an element in the stable homotopy group $\pi^{st}_{\mathrm{ind}\, l}(S^0)$:

Corollary 2.4. *Let $f = l + c : H' \to H$ be a compact perturbation of the linear Fredholm map l such that the preimages of bounded sets under the map f are bounded. Then f defines an element $[f] \in \pi^{st}_{\mathrm{ind}\, l}(S^0)$.* □

In the construction above the linear map l seems to play an essential rôle. In fact it will turn out that the homotopy class $[f]$ basically is independent of the choice of decomposition of f as a sum $f = l + c$. In order to show this, we will have to consider a parametrized version of the above situation and reach back some way:

Let Y be a finite CW-complex. The group $KO^0(Y)$ can be described as follows (cf. [12]):

A (real) Hilbert bundle over Y is a locally trivial fiber bundle with fiber a separable Hilbert space H, whose structure group is the group of linear isometric bijections, equipped with the norm topology. A *cocycle* $\lambda = (E', l, E)$ over Y consists of two Hilbert bundles over Y and a Fredholm morphism $l : E' \to E$ between them. Here a Fredholm morphism is a continuous map which is fiber preserving and fiberwise linear Fredholm over Y. Two cocycles λ_i over Y for $i \in \{0, 1\}$ are homotopic, if there is a cocycle λ over $Y \times [0, 1]$ such that the restriction $\lambda|_{Y\times\{i\}}$ is isomorphic to λ_i. A cocycle (E', l, E) is trivial, if l is invertible. Two cocycles λ_0 and λ_1 are equivalent, if there is a trivial cocycle τ such that $\lambda_0 \oplus \tau$ and $\lambda_1 \oplus \tau$ are homotopic. The group $KO^0(Y)$ is the set of equivalence classes of cycles with addition given by the Whitney sum of cocycles.

Let $f : E' \to E$ be a continuous map between Hilbert bundles of the form $f = l + c$, where $\lambda = (E', l, E)$ is a cocycle over Y and c is fiber preserving and compact, i.e. maps bounded disk bundles in E' to subspaces in E, which are proper over Y. Let's call such a map f a *Fredholm map over* Y. The boundedness condition in this parametrized situation reads: The preimages of bounded disk bundles are contained in bounded disk bundles. An equivalent condition is: The Fredholm map over Y extends to the fiberwise one-point completions of E' and E.

Every Hilbert bundle over the compact space Y is trivial, i.e. $E \cong Y \times H$ by the theorem of Kuiper [8]. The boundedness condition on f thus translates to the condition that the composed map $pr_H \circ f : E' \to H$ extends to the one-point completions, defining a continuous map

$$(pr_H \circ f)^+ : T(E') \to H^+$$

from the Thom space of the Hilbert bundle E' to the Hilbert sphere H^+.

The stage is now set for the definition of stable cohomotopy groups with coefficients: Let λ be a finite dimensional virtual vector bundle over Y. Suppose we are given a presentation $\lambda = F_0 - F_1$ with vector bundles F_i such that $F_1 \cong Y \times V$ is a trivial vector bundle with V a finite dimensional linear subspace of a Hilbert space H. With TF_0 denoting the Thom space of the bundle F_0, stable cohomotopy groups are defined as the colimits

$$\begin{aligned}\pi^n_H(Y;\lambda) &= \operatorname*{colim}_{U\subset V^\perp}\,[U^+\wedge TF_0\ ,\ U^+\wedge V^+\wedge S^n]\\ &= \operatorname*{colim}_{W\subset H}\,[W^+\wedge T\lambda\ ,\ W^+\wedge S^n]\end{aligned}$$

of pointed homotopy classes of maps, where the colimits are over the finite dimensional linear subspaces $U\subset V^\perp\subset H$ and $W=U+V\subset H$, respectively. Here the connecting morphism for $W\subset W_1$ with $U_1=W^\perp\cap W_1$ is the suspension map $(\mathrm{id}_{U_1^+}\wedge\,.)$. The symbol $T\lambda$ stands not anymore for a space, but for a spectrum.

The reason for keeping the Hilbert space H in the notation lies in the equivariant version: For a compact Lie group G we fix a G-universe H, i.e. a real Hilbert space H equipped with an orthogonal G-action such that H contains the trivial representation and, furthermore, the space of equivariant morphisms $Hom_G(V,H)$ for a real G-module V either is zero or infinite dimensional. Let $\lambda=F_0-F_1$ be a virtual equivariant vector bundle over a finite G-CW complex Y such that $F_1\cong Y\times V$ is a trivial bundle with $V\subset H$ a finite dimensional subrepresentation of G. Stable equivariant cohomotopy groups are the colimits

$$\begin{aligned}\pi^n_{G,H}(Y;\lambda) &= \operatorname*{colim}_{U\subset V^\perp}\,[U^+\wedge TF_0\ ,\ U^+\wedge V^+\wedge S^n]^G\\ &= \operatorname*{colim}_{W\subset H}\,[W^+\wedge T\lambda\ ,\ W^+\wedge S^n]^G\end{aligned}$$

of pointed equivariant homotopy classes of maps, where the colimit now is over the finite dimensional subrepresentations $U\subset V^\perp\subset H$ and $W=U+V\subset H$, respectively. This definition of stable equivariant cohomotopy groups differs a little from the usual one as we allow for coefficients λ in the equivariant KO-group $KO^0_G(Y)$ and our universe H need not contain all irreducible representations.

Let $f:E'\to E$ be a G-equivariant Fredholm map between G-Hilbert space bundles over the finite G-CW complex Y such that $E\cong Y\times H$ is a trivialised bundle. Let $f=l+c$ be a presentation of f as a sum of a linear Fredholm morphism and a compact map. For sufficiently large linear G-subspaces $V\subset H$, the cocycle $\lambda=(E',l,E)$ admits a presentation as virtual index bundle

$$\lambda=F_0(V)-F_1(V)$$

with equivariant vector bundles $F_0(V)=(\mathrm{pr}_H\circ l)^{-1}(V)\subset E'$ and $F_1(V)=Y\times V$. The following lemma parallels 2.3. Its proof is omitted, as it is almost verbatim the same.

Lemma 2.5. *There exist finite dimensional linear G-subspaces $V\subset H$ such that the following hold:*

1. *For every $y\in Y$, the subspace V is mapped onto* coker $(l_y:E'_y\to H)$*. In particular, $F_0(V)$ is a bundle over Y and $\lambda=F_0(V)-F_1(V)$ represents the virtual index bundle* $\mathrm{ind}(l)$*.*

2. *For any G-linear $W = W' + V$ with $W' \subset V^\perp$, the restricted map $f(W)^+ = (pr_H \circ f)|^+_{F_0(W)} : TF_0(W) \to H^+$ misses the unit sphere $S(W^\perp)$.*
3. *The maps $\rho_W f(W)^+$ and $\mathrm{id}_{W'^+} \wedge \rho_V f(V)^+$ are G-homotopic as pointed maps*

$$F_0(W)^+ \cong W'^+ \wedge F_0(V)^+ \to W'^+ \wedge V^+ = W^+. \quad \square$$

Theorem 2.6. *An equivariant Fredholm map $f = l + c : E' \to E$ between G-Hilbert space bundles over Y with $E \cong Y \times H$, which extends continuously to the fiberwise one-point completions, defines a stable cohomotopy Euler class*

$$[f] \in \pi^0_{G,H}(Y; \operatorname{ind} l).$$

This Euler class is independent of the presentation of f as a sum.

Proof. The only statement left to prove is the final one. Note that the restriction maps

$$\pi^n_{G,H}(Y \times [0,1], \lambda) \to \pi^n_{G,H}(Y \times \{i\}, \lambda|_{Y \times \{i\}}))$$

are isomorphisms. Thus a homotopy of cocycles naturally induces an isomorphism of the corresponding cohomotopy groups. (An extension of this statement to equivalences of cocycles needs further discussion of universes; it seems unnecessary in the present context.) If $f = l + c = l' + c'$ are two different presentations as a sum, then the constant homotopy $f = f_t = (1-t)(l+c)+t(l'+c')$ defines an Euler class in the cohomotopy group of $Y \times [0,1]$, which restricts for $i \in \{0,1\}$ to the Euler classes defined via the respective presentations of f. $\square$

2.7. Remarks.

- Indeed any element in $\pi^0_{G,H}(Y; \operatorname{ind} l)$ can be realized by a map between Hilbert space bundles satisfying the boundedness condition: Take a finite dimensional representative $p^+ : TF_0^+ \to V^+$. After possibly stabilizing further, this map is homotopic to one where the preimage of the basepoint consists only of the base point. Now take the smash product with the identity on an infinite dimensional Hilbert sphere and remove the base point.
- The stable cohomotopy Euler class has been defined and investigated by Crabb and Knapp [3]. It is related to the standard Euler class the following way: A section of an oriented vector bundle ξ over Y can be regarded as a map $\sigma : Y \times \mathbf{R}^0 \to \xi$. Choosing an bundle isomorphism $\xi \oplus \eta \cong Y \times \mathbf{R}^n$, this section and the projection to the fibers of a trivialized bundle together define a map $(\sigma + id_\eta)^+ : \eta^+ \to (Y \times \mathbf{R}^n)^+ \to S^n$ and thus an element of $\pi^0(Y; -\xi)$. The choice of a Thom class $u \in H^r(Y;\xi) = H^r(D\xi, S\xi)$ corresponds to choosing an orientation of ξ. The standard Euler class is defined by $e(\xi) = \sigma^*(u) \in H^r(Y)$. A generator $1 \in \tilde{H}^0(S^0)$ gives rise to the Hurewicz map $\pi^0(Y; -\xi) \to H^0(Y; -\xi)$, which associates to a stable pointed map $\sigma : T(-\xi) \to S^0$

the image $\sigma^*(1)$. Using the cup product pairing $H^*(Y;-\xi)\times H^*(Y;\xi)\to H^*(Y)$, the singular cohomology Euler class and the stable cohomotopy one are related by

$$e(\xi)=\sigma^*(1)\cdot u.$$

- The approach of 2.1 and the one outlined above obviously are closely related: If $f=l+k:H'\to H$ admits *a priori* estimates and $D'\setminus\partial D'\subset H'$ contains $f^{-1}(0)$, then its compact homotopy class in $\mathcal{C}^0_l(\partial D',H\setminus\{0\})$ corresponds to $[f]\in\pi^0_H(pt;\operatorname{ind} l)=\pi^{st}_{\operatorname{ind} l}(S^0)$.

3. The monopole map

Let S^+ and S^- denote the Hermitian rank-2 bundles associated to the given $Spin^c$ structure on X and let L denote their determinant line bundle. Clifford multiplication $T^*X\times S^\pm\to S^\mp$ defines a linear map $\rho:\Lambda^2\to\mathrm{End}_{\mathbf{C}}(S^+)$ from the bundle of 2-forms to the endomorphism bundle of the positive spinor bundle. The kernel of this homomorphism is the subbundle Λ^- of anti-selfdual 2-forms. Its image is the subbundle of trace-free Hermitian endomorphisms. For a $spin^c$-connection A, denote by $D_A:\Gamma(S^+)\to\Gamma(S^-)$ its associated Dirac operator. The monopole map $\tilde\Psi$ is defined for triples (A,ϕ,a) of a $spin^c$-connection A, a positive spinor ϕ and a 1-form a on X by

$$\begin{array}{rcl}\widetilde\Psi: \mathcal{C}onn\times\Big(\Gamma(S^+)\oplus\Omega^1(X)\Big) & \to & \mathcal{C}onn\times\Big(\Gamma(S^-)\oplus\Omega^+(X)\oplus H^1(X;\mathbf{R})\oplus\Omega^0(X)/\mathbf{R}\Big)\\ (A,\phi,a) & \mapsto & (A,\ D_{A+a}\phi,\ F^+_{A+a}-\sigma(\phi),\ a_{harm},\ \mathrm{d}^*a).\end{array}$$

Here $\sigma(\phi)$ denotes the trace free endomorphism $\phi\otimes\phi^*-\frac{1}{2}|\phi|^2\cdot\mathrm{id}$ of S^+, considered via the map ρ as a selfdual 2-form on X. As a map over the space $\mathcal{C}onn$ of $spin^c$-connections, the monopole map is equivariant with respect to the action of the gauge group $\mathcal{G}=map(X,S^1)$. This group acts on spinors via multiplication with $u:X\to S^1$, on connections via addition of $iu\mathrm{d}u^{-1}$ and trivially on forms. Fixing a base point $*\in X$, the based gauge group $\mathcal{G}_0$ is obtained as the kernel of the evaluation homorphism $map(X,S^1)\to S^1$ at $*$.

Let A be a fixed connection. The subspace $A+ker(\mathrm{d})\subset\mathcal{C}onn$ is invariant under the free action of the based gauge group with quotient space isomorphic to

$$Pic^0(X)=H^1(X;\mathbf{R})/H^1(X;\mathbf{Z}).$$

Let $\mathcal{A}$ and $\mathcal{C}$ denote the quotients

$$\begin{array}{rcl}\mathcal{A} & = & (A+ker\ \mathrm{d})\times\Big(\Gamma(S^+)\oplus\Omega^1(X)\Big)/\mathcal{G}_0\\ \mathcal{C} & = & (A+ker\ \mathrm{d})\times\Big(\Gamma(S^-)\oplus\Omega^+(X)\oplus H^1(X;\mathbf{R})\oplus\Omega^0(X)/\mathbf{R}\Big)/\mathcal{G}_0\end{array}$$

by the pointed gauge group. Both spaces are bundles over $Pic^0(X)$ and the quotient

$$\Psi=\widetilde\Psi/\mathcal{G}_0:\mathcal{A}\to\mathcal{C}$$

of the monopole map is a fiber preserving, S^1-equivariant map over $Pic^0(X)$.

For a fixed $k > 4$, consider the fiberwise L^2_k Sobolev completion $\mathcal{A}_k$ and the fiberwise L^2_{k-1} Sobolev completion $\mathcal{C}_{k-1}$ of $\mathcal{A}$ and $\mathcal{C}$ The monopole map extends to a continuous map $\Psi = \Psi_k : \mathcal{A}_k \to \mathcal{C}_{k-1}$ over $Pic^0(X)$. It is the sum $\Psi = l + c$ of the linear Fredholm map $l = D_A \oplus d^+ \oplus \mathrm{pr}_{harm} \oplus \mathrm{d}^*$ and a term $c : (\phi, a) \mapsto (F^+_A, 0, 0, 0) + (a \cdot \phi, \sigma(\phi), 0)$. This map c is compact as the sum of the constant map F^+_A and the composition of a multiplication map $\mathcal{A}_k \times \mathcal{A}_k \to \mathcal{C}_k$, which is continuous for $k > 2$, and a compact restriction map $\mathcal{C}_k \to \mathcal{C}_{k-1}$. The following statement and its proof are only slight variations of standard ones in Seiberg-Witten theory, compare e.g. [7]:

Proposition 3.1. *Preimages $\Psi^{-1}(B) \subset \mathcal{A}_k$ of bounded disk bundles $B \subset \mathcal{C}_{k-1}$ are contained in bounded disk bundles.*

Proof. It is sufficient to prove this fiberwise for the Sobolev completions of the restriction of the monopole map to the space $\{A\} \times (\Gamma(S^+) \oplus ker(\mathrm{d}^*))$, which maps to $\{A\} \times (\Gamma(S^-) \oplus \Omega^+(X) \oplus H^1(X; \mathbf{R}))$. Using the elliptic operator $D = D_A + d^+$ and its adjoint, define the L^2_k-norm via the scalar product on the respective function spaces through

$$(\, . \, , \, . \,)_i = (\, . \, , \, . \,)_0 + (D \, . \, , D \, . \,)_{i-1}, \qquad (\, . \, , \, . \,)_0 = (\, . \, , \, . \,) = \int_X < \, . \, , \, . \, > *1$$

The norms for the L^p_k-spaces are defined correspondingly. Let $\Psi(A, \phi, a) = (A, \varphi, b, a_{harm}) \in \mathcal{C}_{k-1}$ be bounded by some constant R. The Weitzenböck formula for the Dirac operator associated to the connection $A + a = A'$ reads

$$D^*_{A'} D_{A'} = \nabla^*_{A'} \nabla_{A'} + \frac{1}{4} s - \frac{1}{2} F^+_{A'},$$

with s denoting the scalar curvature of X. As a consequence, there is a pointwise estimate:

$$\begin{aligned} \Delta|\phi|^2 &= 2 < \nabla^*_{A'} \nabla_{A'} \phi, \phi > -2 < \nabla_{A'} \phi, \nabla_{A'} \phi > \\ &\leq 2 < \nabla^*_{A'} \nabla_{A'} \phi, \phi > \\ &= 2 < D^*_{A'} D_{A'} \phi - \frac{s}{4} \phi + \frac{1}{2} F^+_{A'} \phi, \phi > \\ &= < 2 D^*_{A'} \varphi - \frac{s}{2} \phi + (b + \sigma(\phi)) \phi, \phi > \end{aligned}$$

In particular, there are inequalities

$$\begin{aligned} \Delta|\phi|^2 + \frac{s}{2}|\phi|^2 + \frac{1}{2}|\phi|^4 &\leq < 2D^*_A \varphi, \phi > + < 2a \cdot \varphi, \phi > + < b\phi, \phi > \\ &\leq 2(||D^*_A \varphi||_{\mathrm{L}^\infty} + ||a||_{\mathrm{L}^\infty} ||\varphi||_{\mathrm{L}^\infty}) \cdot |\phi| + ||b||_{\mathrm{L}^\infty} \cdot |\phi|^2 \\ &\leq c_1 \Big((1 + ||a||_{\mathrm{L}^\infty}) ||\varphi||_{\mathrm{L}^2_{k-1}} \cdot |\phi| + ||b||_{\mathrm{L}^2_{k-1}} \cdot |\phi|^2 \Big), \end{aligned}$$

with a constant c_1 by applying the Sobolev embedding theorems. To get a bound for the remaining term, use a Sobolev estimate $||a||_{\mathrm{L}^\infty} \leq c_2 ||a||_{\mathrm{L}^p_1}$ for some $p > 4$ and the elliptic estimate $||a||_{\mathrm{L}^p_1} \leq c_3(||d^+ a||_{\mathrm{L}^p_0} + ||a_{harm}||)$. Combination with the equality $d^+ a =$

$b - F_A^+ + \sigma(\phi)$ then leads to an estimate

$$\begin{aligned} ||a||_{\mathrm{L}^\infty} &\leq c_4(||a_{harm}|| + ||b||_{\mathrm{L}_0^p} + ||F_A^+||_{\mathrm{L}_0^p} + ||\sigma(\phi)||_{\mathrm{L}_0^p}) \\ &\leq c_5(||a_{harm}|| + ||b||_{\mathrm{L}_{k-1}^2} + ||F_A^+||_{\mathrm{L}_0^p} + ||\phi||_{\mathrm{L}^\infty}^2). \end{aligned}$$

At the maximum of $|\phi|^2$, its Laplacian is non-negative. So, putting everything together, one obtains a polynomial estimate of the form

$$||\phi||_{\mathrm{L}^\infty}^4 \leq cR\Big((1+R)||\phi||_{\mathrm{L}^\infty} + ||\phi||_{\mathrm{L}^\infty}^2 + ||\phi||_{\mathrm{L}^\infty}^3\Big) + ||s||_{\mathrm{L}^\infty}||\phi||_{\mathrm{L}^\infty}^2.$$

Combining the last two estimates, one obtain bounds for the L^∞-norm and a fortiori for the L_0^p-norm of (ϕ, a) for every $p \geq 1$.

Now comes bootstrapping: For $i \leq k$, assume inductively L_{i-1}^{2p}-bounds on (ϕ, a) with $p = 2^{k-i}$. To obtain L_i^p-bounds, compute:

$$\begin{aligned} ||(\phi,a)||_{\mathrm{L}_i^p}^p - ||(\phi,a)||_{\mathrm{L}_0^p}^p &= ||(D_A\phi, d^+a)||_{\mathrm{L}_{i-1}^p}^p \\ &= ||(\varphi, b, a_{harm})||_{\mathrm{L}_{i-1}^p}^p + ||(a\phi, -F_A^+ - \sigma(\phi))||_{\mathrm{L}_{i-1}^p}^p. \end{aligned}$$

The latter equality holds as $D_{A'} = D_A + a$. The summands in the last expression are bounded by the assumed L_{i-1}^{2p}-bounds on (ϕ, a). □

The proposition in particular implies that the assumptions of 2.6 are satisfied for the monopole map Ψ. The conclusion is spelled out in the following

Corollary 3.2. *The monopole map defines an element $[\Psi]$ in the stable cohomotopy group*

$$\pi_{S^1,H}^0(Pic^0(X); \lambda) = \pi_{S^1,H}^b(Pic^0(X); \mathrm{ind}(D)),$$

where H is a Sobolev completion of the sum $\Gamma(S^- \oplus \Lambda_+^2(T^*X))$ of the vector spaces of negative spinors and selfdual two-forms on X. The virtual index bundle $\lambda = ind(D) \ominus H_+$ is the difference of the complex vitual index bundle of the Dirac operator over $Pic^0(X)$ and the trivial bundle H_+ with fiber $H_+^2(X; \mathbf{R})$, which for a chosen metric on X may be viewed as the space of selfdual harmonic two-forms. The S^1-action on $ind(D)$ is by multiplication with complex numbers and on H_+ is trivial. □

There is a comparison map from the stable equivariant cohomotopy group above to the integers, which relates the element defined by the monopole map with the integer valued Seiberg-Witten invariant associated to it:

Proposition 3.3. *Let X be a closed 4-manifold with $b = b_+ > b_1 + 1$. The choice of a homology orientation (i.e. an orientation of $H^1(X; \mathbf{R}) \oplus H_+^2(X; \mathbf{R})$) then determines a homomorphism $t : \pi_{S^1,H}^b(Pic^0(X); ind(\mathcal{D})) \to \mathbf{Z}$, which maps the class of the monopole map to the integer valued Seiberg-Witten invariant.*

Proof. Let $S(\lambda)^+$ denote the sphere spectrum (with a disjoint base point at the zero level) of the virtual index bundle of the Dirac operator over $Pic^0(X)$. Consider the pair $(Pic^0(X)^+, S(\lambda)^+)$ as the pair of virtual disc and sphere bundle. one can identify stable equivariant cohomotopy groups:

$$\pi^b_{S^1,H}(T(\lambda), pt) = \pi^b_{S^1,H}(Pic^0(X)^+, S(\lambda)^+).$$

The connecting homomorphism

$$\tilde{\pi}^{b-1}_{S^1,H}(S(\lambda)^+) \to \pi^b_{S^1,H}(Pic^0(X)^+, S(\lambda)^+)$$

in the long exact cohomology sequence for the pair $(Pic^0(X)^+, S(\lambda)^+)$ is an isomorphism for $b > b_1+1$. This is because by dimension reasons the groups $\pi^i_{S^1,H}(Pic^0(X)^+)$ have to vanish for $i > dim(Pic^0(X))$. So in this case the monopole map is represented by an equivariant pointed map $\psi : S(\lambda)^+ \wedge (\mathbf{C}^n \oplus \mathbf{R}^m)^+ \to (S^{b-1}) \wedge (\mathbf{C}^n \oplus \mathbf{R}^m)^+$ for some n and m. By abuse of notation we still call the spaces on either side $S(\lambda)^+$ and S^{b-1}, respectively. Now apply reduced equivariant cohomology to the map ψ. Since the S^1-action on $S(\lambda)$ is free, the equivariant cohomology group $\tilde{H}^*_{S^1}(S(\lambda)^+)$ identifies with the nonequivariant cohomology $H^*(P(\lambda))$ of the quotient, which is a projective bundle over $Pic^0(X)$. An orientation of $H^1(X;\mathbf{R})$ together with the standard orientation of complex projective space defines an orientation class $[P(\lambda)]$ of this manifold. Considering the sphere S^{b-1} as the unit sphere in $H^2_+(X)$, the chosen homology orientation of X and the orientation of $Pic^0(X)$ determine the orientation of S^{b-1} and thus a generator in dimension $b-1$ of the reduced equivariant cohomology of S^{b-1} as a free $H^*_{S^1}(*) \cong \mathbf{Z}[x]$-module of rank one. The homomorphism t associates to ψ the degree zero part of $\psi^*(\sum_0^\infty x^i) \cap [P(\lambda)]$. Using the Kuranishi model for the monopole map as in [9], it is straightforward to check that this integer indeed gives the Seiberg-Witten invariant. □

If the first Betti number of X vanishes, the group $\pi^b_{S^1,\mathcal{C}}(Pic^0(X); ind(\mathcal{D}))$ simplifies: The index of the Dirac operator is a complex vector space of complex dimension

$$d = \frac{c(\mathfrak{s})^2 - signature(X)}{8},$$

where $c(\mathfrak{s})$ is the first Chern class of the spinor bundles $S^\pm$ associated to the $spin^c$-structure $\mathfrak{s}$.

Proposition 3.4. *For $i > 1$, the stable equivariant cohomotopy groups $\pi^i_{S^1,H}(*;\mathbf{C}^d))$ are isomorphic to the nonequivariant stable cohomotopy groups $\pi^{i-1}(\mathbf{CP}^{d-1})$ of complex projective $(d-1)$-space. In particular, if X is a closed 4-manifold with $b_1 = 0$ and $b_+ > 1$, then the monopole map determines an element in $\pi^{b-1}(\mathbf{CP}^{d-1})$.*

Proof. The long exact stable cohomotopy sequence for the $(D(\mathbf{C}^d), S(\mathbf{C}^d))$ consisting of the unit disk and sphere in the complex vector space $\mathbf{C}^d$ allows to identify for $i > 1$ the groups

$\pi^i_{S^1,\mathcal{C}}(*;\mathbf{C}^d))$ with $\tilde\pi^{i-1}_{S^1,\mathcal{C}}(S(\mathbf{C}^d)^+)$. But for the free S^1-space $S(\mathbf{C}^d)$ equivariant cohomotopy is isomorphic to the nonequivariant cohomotopy of its quotient [4]. □

To analyse this cohomotopy of projective spaces a little further, consider the Hurewicz map

$$\begin{array}{rcl} \pi^i(Y) & \to & H^i(Y) \\ [f] & \mapsto & f^*(1), \end{array}$$

with $1 \in H^i(S^i) \cong \tilde H^0(S^0)$ defined by the orientation. Rationally it is an isomorphism, as rationally the sphere spectrum is an Eilenberg MacLane spectrum by Serre's theorem. However, nonrationally, the Hurewicz map has both kernel and cokernel. An estimate for the cokernel in each degree was obtained in [1]. The main result of [10] uses similar methods. In what follows, the focus will be on the kernel of the Hurewicz map. In the case of the monopole invariant of a 4-manifold, its image under the Hurewicz map is detected by the integer valued Seiberg-Witten invariants. So to show that the stable cohomotopy invariants are indeed effective generalizations, one has to detect torsion in the kernel of the Hurewicz map. In the following lemma the results are ordered according to k, which can be interpreted as the "expected dimension of the moduli space", i.e. the dimension of the preimage of a generic point in the sphere.

Lemma 3.5. *The Hurewicz map* $h^{2d-2-k}:\pi^{2d-2-k}(\mathbf{CP}^{d-1})\to H^{2d-2-k}(\mathbf{CP}^{d-1})$

0. *for* $k=0$ *is an isomorphism.*
1. *for* $k=1$ *has kernel isomorphic to* $\mathbf{Z}/gcd(2,d)$.
2. *for* $k=2$ *has kernel isomorphic to* $\mathbf{Z}/gcd(2,d)$ *and cokernel isomorphic to* $\mathbf{Z}/gcd(2,d-1)$.
3. *for* $k=3$ *has kernel isomorphic to* $\mathbf{Z}/l$ *with* $l=gcd(24,d-3)/gcd(2,d-1)$ *for* $d>2$, *and* $l=24$ *for* $d=2$.
4. *for* $k=4$ *has trivial kernel.*

Proof. The proof employs the Atiyah-Hirzebruch spectral sequence with E_2-term

$$H^*(Y;\pi^*(pt))\Rightarrow \pi^*(Y)$$

and uses the following facts:
1. The attaching map of the 4-cell in $\mathbf{CP}^2$ is the Hopf map, which is the generator η of $\pi^{-1}(pt)\cong\mathbf{Z}/2$.
2. The group $\pi^{-2}(pt)\cong\mathbf{Z}/2$ is generated by η^2.
3. The attaching map of the 8-cell in $\mathbf{HP}^2$ is again a Hopf map, which is stably the generator ν of $\pi^{-3}(pt)\cong\mathbf{Z}/24$. Furthermore, $\eta^3=12\nu$.
4. The stable homotopy groups $\pi^{-4}(pt)$ and $\pi^{-5}(pt)$ vanish.
5. For even d, there is a projection of the complex projective to the quaternionic projective space.
6. The differentials in the spectral sequence are differentials for the tensor algebra structure on the respective E_i-terms.

7. The spectral sequence is natural in Y. In particular one may use the map between spectral sequences induced by inclusions of the projective spaces into higher dimensional projective spaces and induced by the projection of complex projective spaces to quaternionic projective spaces. $\square$

One observation to be made in the spectral sequence argument above should be singled out for later reference:

Lemma 3.6. *Let κ_d be a generator of the stable cohomotopy group $\pi^{2d-2}(\mathbf{C}P^{d-1}) \cong \mathbf{Z}\kappa_d$. The composition*

$$S^{2d-1} \to S^{2d-1}/S^1 = \mathbf{C}P^{d-1} \xrightarrow{\kappa} S^{2d-2},$$

of κ_d with the quotient map of the free S^1-action is the nontrivial Hopf element $\eta \in \pi_1^{st}(S^0)$ iff d is even. $\square$

4. A gluing theorem

By a folklore theorem the integer valued Seiberg-Witten invariant for a connected sum of 4-manifolds vanishes if either summand has nonzero b_+. This section deals with the corresponding statement for the stable cohomotopy invariant. To state the result, consider a pair $X = X_0 \coprod X_1$ of closed connected Riemannian 4-manifolds. Suppose each component

$$X_i = X_i^- \cup X_i^+$$

is the union of submanifolds along the common boundary $\partial X_i^\pm = S^3$ and suppose the Riemannian metric on X_i is a product $[-L, L] \times S^3$ of an interval and the round three dimensional sphere in a neighbourhood of the submanifold S^3. The length $2L \gg 2$ of this "long neck" $[-L, L] \times S^3$ is to be determined in the course of the proof.

Let $\overline{X} = \overline{X}_0 \coprod \overline{X}_1$ be the manifold obtained from X by interchanging the negative parts of the X_i, that is

$$\overline{X}_i = X_{i\pm 1}^- \cup X_i^+.$$

Spin^c-structures on both components X_i of X induce by gluing *spin*^{c}-structures on the components $\overline{X}_i$ of $\overline{X}$. Fix a *spin*^{c}-structure and a *spin*^{c}-connection A on X, which induces the flat connection on $det(S^\pm)$ over the long neck. Fix once and for all identifications of the spinor bundles and the chosen *spin*^{c}-connection over the two copies $[-1, 1] \times S^3$ in X.

For the gluing, choose a smooth function

$$\mu_c : X \to [0, 1],$$

which is constant 1 on $X_i^{\leq -1}$ and vanishes identically on $X_i^{\geq 1}$. This function μ_c should be a function of the first variable only in the "short neck" $[-1, 1] \times (S^3 \coprod S^3)$. In analogy to sine and cosine, define

$$\mu_s \stackrel{\text{def}}{=} \sqrt{1 - \mu_c^2}.$$

Forms α_i (and similarly spinors ϕ_i) on the components X_i are patched to forms (and spinors, respectively,) on $\overline{X}_i$ via the map

$$V : (\alpha_0\,, \quad \alpha_1) \mapsto (\alpha_0\,, \quad \alpha_1) \cdot \begin{pmatrix} \mu_c & \mu_s \\ -\mu_s & \mu_c \end{pmatrix}.$$

This gluing defines bundle isomorphisms $\mathcal{A} \to \overline{\mathcal{A}}$ and $\mathcal{C} \to \overline{\mathcal{C}}$ of the Hilbert space bundles over an identification $Pic^0(X) \xrightarrow{\cong} Pic^0(\overline{X})$ to be detailed below. All of these isomorphisms will be denoted by V.

Theorem 4.1. *Gluing via the map V induces an isomorphism*

$$\pi^b_{S^1,H}(Pic^0(X); ind(D)) \to \pi^b_{S^1,\overline{H}}(Pic^0(\overline{X}); ind(\overline{D})),$$

which identifies the classes of the monopole maps of X and $\overline{X}$ for corresponding spin^c-structures.

The proof will be given in the next paragraph. In the special case where X_0^- and X_1^+ are both diffeomorphic to the standard four-disk, the theorem above implies the gluing theorem stated in the introduction. This is a consequence of the following statements.

Proposition 4.2. *The monopole map Ψ for a spin^c-structure on $X = X_0 \coprod X_1$, with a base point fixed on either component, is the product of the monopole maps*

$$\Psi = \Psi_0 \times \Psi_1 : \mathcal{A} = \mathcal{A}_0 \times \mathcal{A}_1 \to \mathcal{C}_0 \times \mathcal{C}_1 = \mathcal{C}.$$

Thus the associated stable equivariant cohomotopy element is the smash product

$$[\Psi] = [\Psi_0] \wedge [\Psi_1] \in \pi^b_{S^1 \times S^1, H_0 \oplus H_1}(Pic^0(X); ind(D))$$

of the cohomotopy elements associated to the respective components. The action of the torus $S^1 \times S^1$ on the sum $H_0 \oplus H_1$ is factorwise. □

In the gluing situation, it is convenient to choose a base point in S^3. The gluing map V is S^1-equivariant with respect to the action of the diagonal subgroup of the torus.

Proposition 4.3. *The stable cohomotopy element associated to any spin^c-structure on a four dimensional manifold X with vanishing Betti numbers $b_1 = b_2 = 0$ is the class of the identity map*

$$[\Psi] = [\mathrm{id}] \in \pi^0_{S^1,H}(*) \cong \mathbf{Z}.$$

Proof. In this case, the equivariant index $\lambda \in RO(S^1)$ is zero. Hence, the ring $\pi^0_{S^1,H}(*)$ coincides with the Burnside ring $A(S^1) \cong \mathbf{Z}$. This isomorphism can be described as the map

$$\pi^0_{S^1,H}(*) \to \pi^0_{st}(*) \cong \mathbf{Z}$$

induced by restriction to fixed point sets (cf. [4], 133ff). However, on the S^1-fixed point set, the monopole map is just the linear isomorphism

$$\mathrm{d} + \mathrm{d}^* : \Omega^1(X) \to \Omega^2_+(X) \oplus \Omega^0(X)/\mathbf{R}.$$

□

5. Proof of the gluing theorem

Let $\mathcal{G}_r \subset \mathcal{G}$ denote the subgroup consisting of gauge transformations $u : X \to S^1$ which are trivial over the "short neck" $[-1, 1] \times (S^3 \coprod S^3)$. Let $\ker(d)_r \subset \ker(d)$ be the space of 1-forms on X vanishing identically on the short neck. The group $\mathcal{G}_r$ decomposes into a product of gauge groups, each corresponding to one of the submanifolds $X_i^\pm$ of X. Using the identification of the chosen *spin*^{c}-connections A and $\overline{A}$ over the short neck, the space $A + \ker(d)_r \cong \overline{A} + \ker(d)_r$ can be viewed as a subspace of the space of *spin*^{c}-connections both over X and $\overline{X}$. After suitable Sobolev completion, $A + \ker(d)_r/\mathcal{G}_r$ identifies this way both with $Pic^0(X)$ and $Pic^0(\overline{X})$. In particular, the map

$$(A + \ker(d)_r) \times (\Gamma(S^+) \oplus \Omega^1(X))/\mathcal{G}_r \to \mathcal{A}$$

induced by inclusion, is an isomorphism and similarly for the Hilbert space bundle $\mathcal{C}$ over $Pic^0(X)$. The gluing maps

$$V : \mathcal{A} \to \overline{\mathcal{A}} \qquad \text{and} \qquad V : \mathcal{C} \to \overline{\mathcal{C}}$$

are defined fiberwise over the identification $Pic^0(X) \cong Pic^0(\overline{X})$ by multiplying forms and spinors with the matrix

$$\begin{pmatrix} \mu_c & \mu_s \\ -\mu_s & \mu_c \end{pmatrix}.$$

Let Ψ and $\overline{\Psi}$ denote the monopole maps on the *spin*^{c}-manifolds X and $\overline{X}$, respectively. The diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Psi} & \mathcal{C} \\ {\scriptstyle V}\big\downarrow & & \big\downarrow{\scriptstyle V} \\ \overline{\mathcal{A}} & \xrightarrow{\overline{\Psi}} & \overline{\mathcal{C}} \end{array}$$

is of course not commuting. The theorem claims that it commutes up to suitable homotopy: For a proof of the theorem it suffices to show that Ψ is homotopic to $V^{-1}\overline{\Psi}V$ through a homotopy of Fredholm maps

$$\Psi_t = l_t + c_t$$

such that there is a uniform bound on the solutions during the homotopy, i.e. there is a bound on elements of $\Psi_t^{-1}(0)$, uniform for all t.

In the proof the help of several homotopies in the above sense will be invoked: The first two homotopies from Ψ to an auxiliary map $P : \mathcal{A} \to \mathcal{C}$ will tame the quadratic terms in the monopole map: As an operator on sections over X, the map P differs from the monopole map Ψ only over the long neck. Over the short neck, P is the linearisation of Ψ. The homotopies will be designed in such a way that the solutions stay bounded during the homotopy. This boundedness is achieved by the use of Weitzenböck formulas for both the Dirac operator and the covariant derivative. Positivity of scalar and Ricci curvature, respectively, on the long neck provide sufficient control on the spinor and form components of solutions. In order to tune the estimates for spinors and forms, it may be necessary to stretch the long neck even longer.

Another homotopy then will start from P and end in $V^{-1}PV$, where $\overline{P} : \overline{\mathcal{A}} \to \overline{\mathcal{C}}$ is the corresponding map for the manifold $\overline{X}$.

Let $\rho_R : \mathbf{R} \to [0,1]$ denote a smooth function, which is constant 1 outside the interval $[-R, R]$ and constant 0 in the interval $[-R+1, R-1]$. The functions ρ_R for $R \leq L$ define functions on X and $\overline{X}$, which are constant 1 outside the long neck and are functions in the first variable only on the long neck. The homotopies

$$\rho_{R,t} \stackrel{def}{=} (1-t) + t\rho_R,$$

for the time parameter t in the unit interval, describe a homotopy from the constant map 1 to the function ρ_R on X and on $\overline{X}$.

Consider the homotopy $\Psi_t : \mathcal{A} \to \mathcal{C}$ defined by

$$\Psi_t(A, \phi, a) = (A, D_{A+a}\phi, F^+_{A+a} - \rho_{L,t}\sigma(\phi), a_{harm}, \mathrm{d}^*a).$$

Lemma 5.1. *The preimage $\Psi_t^{-1}(0)$ is uniformly bounded for all times $t \in [0,1]$.*

Proof. The proof of 3.1 applies with minor modifications. Here only the preimage of zero is to be considered. As a consequence, most terms in the estimates in 3.1 vanish. The Weitzenböck formula for the Dirac operator implies a pointwise estimate

$$\Delta|\phi|^2 + \frac{s}{2}|\phi|^2 + \frac{1}{2}\rho_t^L|\phi|^4 \leq 0.$$

for the spinor component of a solution. The scalar curvature over the long neck is positive. So the maximum of $|\phi|^2$ is attained outside the long neck and is bounded by the norm of the scalar curvature. The norm of the term $\rho_{L,t}\sigma(\phi)$ in the rest of the argument is bounded by a multiple of the norm of $\sigma(\phi)$. These bounds are independent of the parameter t. □

The next homotopy moderates the second quadratic term over the short neck:

$$\Psi_{t+1}(A, \phi, a) = (A, D_{(A+\rho_{2,t}a)}\phi, F^+_{A+a} - \rho_L\sigma(\phi), a_{harm}, d^*a)$$

This homotopy starts at Ψ_1 and ends at $\Psi_2 = P$. Note that the latter differential operator is linear on the short neck. This second homotopy is more delicate than the first: In order to get the necessary bounds on the solutions during the homotopy, it may become necessary to stretch the long neck even longer, like playing the trombone.

Lemma 5.2. *If the long neck of X is longer than a threshold length, depending on the geometry of the complement of the long neck, then the preimage $\Psi_t^{-1}(0)$ is uniformly bounded for all times $t \in [1,2]$.*

Proof. Again, the proof of 3.1 applies after suitable modifications:

First, consider the monopole map for the manifold with boundary $Y = \overline{Y}$, which one gets from both X and $\overline{X}$ by removing the open neck of length $2L-2$. It may be suitably defined for one-forms annihilating normal vectors at the boundary. The argument of 3.1 (compare [7]) then provides C^0 bounds on both the spinor and the form component of a solution to the monopole equation on Y. It also provides uniform bounds on solutions in $\Psi_t^{-1}(0)$ during the first homotopy $t \leq 1$. These C^0-bounds on solutions on Y are sufficient for the proof as long as one can assure that the spinor and the one-form component of a solution on X both attain their maximum in Y during the second homotopy.

The one-form component of a solution on X is harmonic in the complement of Y. Because of nonnegative Ricci curvature along the neck, the maximum principle holds for the norm of such a one-form in the complement of Y. Moreover, because of the product structure of the neck, such an harmonic one-form on the neck splits into a sum $a = a_i + a_s$ of harmonic one-forms, according to the direct sum decomposition of the cotangent bundle. The harmonic summand a_s pointing in the sphere direction satisfies an inequality

$$\Delta|a_s|^2 \leq -2 < Ric(a_s), a_s > .$$

Since the Ricci tensor in direction of the sphere is positive definite, $|a_s|^2$ is bounded by a linear combination of $cosh(\delta r)$ and $sinh(\delta r)$ for some $\delta \neq 0$ and $r \in [-L+1, L-1]$. In particular, there is exponential decay of the norm of a_s towards the middle of the neck.

If the spinor component of a solution during the homotopy attains its maximum in the complement of Y, then at that maximum, it satisfies an inequality

$$0 \leq \Delta|\phi|^2 \leq -\frac{s}{2}|\phi|^2 + < (d\rho_{2,t} \wedge a)^+\phi, \phi > .$$

Because of $d\rho_{2,t} \wedge a = d\rho_{2,t} \wedge a_s$, the norm of the latter summand decays exponentially with the length of the long neck. If it is long enough, the scalar curvature summand will dominate and the spinor component cannot attain its maximum in the complement of Y. □

To finish the proof of the gluing theorem, one has to construct a homotopy between P and $V^{-1}PV$. Note that both operators differ only over the short neck in X. Because both differential operators are linear over the short neck, their difference is a multiplication operator:

$$V^{-1}PV = P + dV$$

Let $\mu_{c,t}$ denote the convex combination $\mu_{c,t} = (t-1)+t\mu_c$ and accordingly $\mu_{s,t} = \sqrt{1-\mu_{c,t}^2}$. Consider the matrix

$$\begin{pmatrix} \mu_{c,t} & \mu_{s,t} \\ -\mu_{s,t} & \mu_{c,t} \end{pmatrix}.$$

Multiplication of spinors or forms with this matrix defines a map V_t over the long neck: Pairs of forms or spinors over the long neck are mapped to pairs of forms or spinors over the long neck. Multiplication of pairs of spinors or forms with this matrix will not make sense outside the long neck. However, since multiplication with V_t is the identity outside the short neck, the conjugate $V_t^{-1}PV_t : \mathcal{A} \to \mathcal{C}$ is a well defined operator for $0 \leq t \leq 1$. These operators

$$V_t^{-1}PV_t = P + dV_t$$

provide the final homotopy in the argument. The following lemma finishes the proof of the gluing theorem.

Lemma 5.3. *The preimage $(P+dV_t)^{-1}(0)$ is uniformly bounded for all times $t \in [0,1]$.*

Proof. As for the second homotopy, one only has to show that the spinor and the one-form components of a solution attain their maxima outside the long neck. A solution to $P+dV_t$ admits the same C^0 bounds over the manifold Y as solutions to P, since both operators coincide over Y. It remains to consider the restrictions of solutions to the long neck. However, if (ϕ,α) is a solution to $P+dV_t = V_t^{-1}PV_t$ over the long neck, then $V_t^{-1}(\phi,\alpha)$ is a solution to P over the long neck. In particular, they attain their maxima in Y. □

6. Miscellanea

A *spin*^{c}-structure on a four dimensional manifold is the same as a stably almost complex structure. This follows from the fact that the natural map between the respective classifying spaces $BU \to BSpin^c$, has the appropriate connectivity. Currently, in all known (at least to the author) examples of four dimensional *spin*^{c}-manifolds with nonvanishing integer valued Seiberg-Witten invariant, the *spin*^{c}-structure is indeed associated to an (unstably) almost complex structure. As a consequence, the moduli spaces for all these examples have zero-dimensional expected dimension.

For simplicity, consider from now on only closed, oriented, four-dimensional *spin*^{c}-manifolds with vanishing first Betti number. If the *spin*^{c}-structure of X is associated to an almost complex structure, then the stable map Ψ nonequivariantly is an element of the stable homotopy group $\pi_1^{st}(S_0)$. This group has two elements, the trivial map and the Hopf map η. The lemma 3.6 gives a criterion to distinguish the two elements:

Proposition 6.1. *The cohomotopy invariant of an almost complex manifold with vanishing first Betti number nonequivariantly is the Hopf map if and only if both b_+ is congruent $3 mod\, 4$ and the integer valued Seiberg-Witten invariant is odd.*

The computations of the stable equivariant cohomotopy groups in 3.5 lead to the following statement:

Proposition 6.2. *Let the spin^c-manifold X be a connected sum of at least two, but finitely many, almost complex manifolds X_i with vanishing first Betti numbers. The stable equivariant cohomotopy element of X is nonvanishing if and only if the following three conditions are satisfied: There are at most three summands X_i. Each summand has b_+ congruent $3 \, mod \, 4$. The integer valued Seiberg-Witten invariants are odd for each summand.*

Proof. The "if" part of the statement follows from the fact that the square and the cube of the Hopf map are nontrivial stable homotopy elements.
The second stable homotopy group of the sphere spectrum has as its only nontrivial element the square of the Hopf map. The equivariant cohomotopy element which corresponds to the nontrivial element in the kernel of the Hurewicz map h^{2d-3} of 3.5 nonequivariantly has to be the square of the Hopf map. For $d = 2$ this is by definition of the ring structure on the stable homotopy groups of the sphere spectrum. Otherwise this follows by induction from 3.6 and the fact that every even integer $d > 2$ is the sum of positive even integers. The same argument applies to the kernel of the Hurewicz map h^{2d-4}. Since the kernel of h^{2d-5} is odd if d is even, higher powers of the equivariant Hopf map will be zero. □

Proof. (of 1.3) For Kähler manifolds it is known that solutions to the monopole equations correspond to holomorphic sections of certain line bundles. The Seiberg-Witten invariant is nonzero if such holomorphic sections exist. This in particular implies that the preimage of the zero under the monopole map is empty unless the integer valued Seiberg-Witten invariant is nonzero. As a consequence, the equivariant cohomotopy elements corresponding to *spin*^{c}-structures are detected by the integer valued Seiberg-Witten invariants.

For the K3-surface the SW-invariants are completely known: They vanish except for the one *spin*^{c}-structure which lifts to a *Spin*-structure. For this the value is 1, up to sign convention. From the statements above it follows that for a connected sum $K\#K\#X$ or $K\#X$ for a simply connected Kähler surface X, the *spin*^{c}-structures supporting nontrivial stable cohomotopy invariants of the connected sum correspond to exactly those *spin*^{c}-structures on X having odd SW-invariants (if b_+ is of correct modulus). □

For simply connected manifolds, the *spin*^{c}-structures are detected by the first Chern classes of the associated spinor bundles. The argument above can be rephased the following way: For a connected sum of Kähler surfaces, the nontrivial stable cohomotopy invariants detect the pairs (or triples) consisting of cohomology classes of the summands which support odd Seiberg-Witten invariants. This can be applied in special situations to recognize the summands:

Proof. (of 1.4) The proof is based on the known classification of elliptic surfaces (see e.g. [6]) via SW-invariants. To a simply connected minimal elliptic surface with given b_+ one can associate a pair $1 \leq m \leq n$ of coprime integers which, together with the geometric genus $p_g = \frac{b_+ - 1}{2}$, classify the diffeomorphism type. Note that b_+ is congruent to 3 mod 4 iff the geometric genus is odd. The point in the proof is that one can recognize m, n and p_g for odd geometric genus from the pattern of the cohomology classes corresponding to odd SW-invariants, the "recognizable" classes. Here comes a description, how this can be accomplished combinatorially.

The cohomology classes corresponding to nontrivial SW-invariants are multiples of an indivisible element f in the second cohomology with integer values of the elliptic surface. The multiplicities are of the form $(p_g - 1 - 2a)mn + (m - 2b - 1)n + (n - 2c - 1)m$ for nonnegative integers $a < p_g$, $b < m$ and $c < n$. The value of the SW-invariant for such a multiple of f is $\binom{p_g - 1}{a}$.

Note that the distribution of basic classes is symmetric around the origin and the SW-invariant for the largest such multiple k of f, where $a = b = c = 0$, is odd. So there are at least two recognizable classes except in the case of a K3-surface $p_g = m = n = 1$, where there is exactly one recognizable class.

If there are no more than three recognizable classes, then either $m = n = 1$ or $n = 2m = 2p_g = 2$. In the latter case, the largest multiple is 1, in the former, it is $p_g - 1$, which is even.

In the case of at least four recognizable classes, consider the second but largest multiple. In case $n > 1$, the integer m, which is half the difference, is coprime to the largest multiple. If $m = n = 1$, then there is an $0 < 2a \leq p_g - 1$ with $\binom{p_g - 1}{a}$ odd. This integer a has to be even, because otherwise $\binom{p_g - 1}{a + 1}$, which is obtained from it by multiplication with a rational number having $a + 1$ in its denominator, could not be an integer. In this case, half the difference cannot be coprime to the largest multiple; both are even. This makes it possible to distinguish the $m = n = 1$-cases.

Finally consider the largest multiple k and the second largest multiple. It can be assumed that half the difference is coprime to k and thus equals m. Consider the multiples $(k - 2\lambda m)f$ for $\lambda \geq 1$. The SW-invariants associated to these classes will be 1 for $\lambda < n$ and zero or $p_g - 1$, anyway even, for $\lambda = n$. This characterizes the second integer n. Knowing both integers this way, the geometric genus follows from the formula for k.

In the situation of a connected sum of no more than three elliptic surfaces, as in 1.4, the number of summands can be read off the dimension of the moduli spaces having nontrivial invariants. The cohomology classes associated to nontrivial invariants are situated in a bounded region in a sublattice of the second cohomology of rank at most 3. They form a box and the pattern characterizing the individual summands can be found on the respective edges of the box. ☐

6.3. *The monopole map Ψ may be pertubed quite a bit without changing the resulting stable cohomotopy invariant. For example the term $\sigma(\phi)$ may be replaced by some function $f(|\phi|)\sigma(\phi)$ as long as f does not decay too fast at infinity. Any polynomial with positive leading coefficient will give different moduli spaces, but the same stable cohomotopy invariant.*

A well known theorem of C.T.C. Wall states that any two homeomorphic simply connected four dimensional differentiable manifolds become diffeomorphic after taking connected sum with finitely many copies of $S^2 \times S^2$. Moreover, in many cases of algebraic surfaces it is known that it suffices to take connected sum with only one such copy of $S^2 \times S^2$.

Question 6.4. *Suppose X and Y are homeomorphic, simply connected differentiable four-manifolds. Do they become diffeomorphic after taking connected sum with sufficiently many K3-surfaces?*

Question 6.5. *Are there manifolds realizing other stable cohomotopy elements in $\pi_i^{st}(\mathbf{CP}^{d-1})$, for example the element associated to the stable Hopf map $\nu : S^7 \to S^4$?*

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