Pseudo-finite generalized triangle groups

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A generalized triangle group (g.t.g.) is a group Γ with a fixed presentation of the form

$$\Gamma = \langle\ x,\ y\ \mid\ x^k = y^l = W(x,y)^m = 1\ \rangle, \tag{1}$$

where $k,\, l,\, m \geq 2$ and

$$W(x,y) = x^{k_1}\, y^{l_1}\, x^{k_2}\, y^{l_2}\ \ldots\ x^{k_s}\, y^{l_s} \tag{2}$$

with $0 < k_i < k, \quad 0 < l_i < l, \quad s \geq 1$. It is also required that the word W should not be a power of a shorter word.

G.t.g. were introduced in [FR] and [BMS]. They have been intensively studied by many authors (see [HMT] and references there). In particular, all finite g.t.g. were found in [HMT] and [LRS].

One of the main tools for studying g.t.g. is constructing their essential homomorphisms to $\mathrm{PSL}_2(\mathbb{C})$. A homomorphism $\varphi : \Gamma \to G$ is called *essential*, if

$$\mathrm{ord}\ \varphi(x) = k, \quad \mathrm{ord}\ \varphi(y) = l, \quad \mathrm{ord}\ \varphi\,(W(x,y)) = m.$$

It was proved in [BMS] and [FHR] that any g.t.g. admits an essential homomorphism to $\mathrm{PSL}_2(\mathbb{C})$.

Most of g.t.g. admit an essential homomorphism to $\mathrm{PSL}_2(\mathbb{C})$ with an infinite image. This is a key step in the classification of finite g.t.g. There are, however, infinite g.t.g. that do not admit such a homomorphism. Let us call a g.t.g. Γ *pseudo-finite* if the image of any essential homomorphisms $\varphi : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ is finite.

In this work we present some partial results on the classification of pseudo-finite g.t.g. This problem was originally motivated by the classification problem for finite groups defined by periodic paired relations (see the definition in [V1]). It seems, however, that it is interesting in its own right.

Our results cover the following cases:
1) $m \geq 3$ (see Propositions 4 and 5);
2) $s \leq 3$ (see Propositions 4 and 7 - 10).

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1 Preliminaries

A pair of matrices $X,\ Y \in \mathrm{SL}_2(\mathbb{C})$ is called *irreducible*, if they do not have a common eigenvector, or, equivalently, if they generate an irreducible linear group.

The following facts can be found , e.g., in [VMH, Appendix] and [V2].

An irreducible pair (X, Y) is defined up to conjugacy by the numbers $\operatorname{tr} X,\ \operatorname{tr} Y,\ \operatorname{tr} XY$. Moreover, for any complex numbers $a,\ b,\ c$ there are matrices $X,\ Y \in \mathrm{SL}_2(\mathbb{C})$ such that $\operatorname{tr} X = a,\ \operatorname{tr} Y = b,\ \operatorname{tr} XY = c$.

A pair (X, Y) is irreducible if and only if the matrix

$$\begin{pmatrix} 2 & \operatorname{tr} X & \operatorname{tr} Y \\ \operatorname{tr} X & 2 & \operatorname{tr} XY \\ \operatorname{tr} Y & \operatorname{tr} XY & 2 \end{pmatrix} \tag{3}$$

is non-degenerate. An irreducible pair is conjugate to a pair of matrices of SU_2 if and only if $\operatorname{tr} X,\ \operatorname{tr} Y,\ \operatorname{tr} XY \in \mathbb{R}$ and the (symmetric) matrix (3) is positive definite. The latter means that

$$\begin{gathered} \operatorname{tr} X = 2\cos\alpha, \quad \operatorname{tr} Y = 2\cos\beta \quad (\alpha, \beta \in (0, \pi)), \\ \operatorname{tr} XY \in (2\cos(\alpha+\beta), \quad 2\cos(\alpha-\beta)). \end{gathered} \tag{4}$$

The boundary cases $\operatorname{tr} XY = 2\cos(\alpha \pm \beta)$ are realized for the pairs of diagonal matrices

$$X = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \quad Y = \begin{pmatrix} e^{\pm i\beta} & 0 \\ 0 & e^{\mp i\beta} \end{pmatrix}, \tag{5}$$

but also for the pairs of non-commuting matrices

$$X = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \quad \widetilde{Y} = \begin{pmatrix} e^{\pm i\beta} & 1 \\ 0 & e^{\mp i\beta} \end{pmatrix}. \tag{6}$$

In particular, if X, Y generate an irreducible finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, the conditions (4) hold.

Since

$$\mathrm{tr}\, X^{-1} = \mathrm{tr} X, \quad \mathrm{tr}\, Y^{-1} = \mathrm{tr}\, Y, \quad \mathrm{tr}\, X^{-1}Y^{-1} = \mathrm{tr}\, YX = \mathrm{tr}\, XY,$$

any irreducible pair (X, Y) is conjugate to the pair (X^{-1}, Y^{-1}).

For any matrix $X \in \mathrm{SL}_2(\mathbb{C})$ we shall denote by $[X]$ the corresponding element $\{\pm X\}$ of $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm E\}$. The element $[X]$ has order $n \geq 2$ if and only if the eigenvalues of X have the form $e^{\pm \frac{\pi i u}{n}}$ with $(u, n) = 1$ or, equivalently, if

$$\mathrm{tr}\, X = 2 \cos \frac{\pi u}{n}.$$

A pair of elements $[X], [Y] \in \mathrm{PSL}_2(\mathbb{C})$ is called *irreducible*, if the pair (X, Y) is irreducible in the above sense. An irreducible pair $([X], [Y])$ is still defined up to conjugacy by the numbers $\mathrm{tr}\, X$, $\mathrm{tr}\, Y$, $\mathrm{tr}\, XY$, but these numbers are defined by the elements $[X]$, $[Y]$ only up to multiplying any two of them by -1.

This can be applied to constructing essential homomorphisms $\varphi : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$, where Γ is the g.t.g. defined by (1). Set

$$\varphi(x) = [X], \quad \varphi(y) = [Y] \quad (X, Y \in \mathrm{SL}_2(\mathbb{C})). \tag{7}$$

We are to choose X and Y satisfying the conditions

$$\mathrm{tr}\, X = 2 \cos \frac{\pi u}{k}, \quad \mathrm{tr}\, Y = 2 \cos \frac{\pi v}{l}, \tag{8}$$

$$\mathrm{tr}\, W\,(X, Y) = 2 \cos \frac{\pi w}{m}, \tag{9}$$

where u is prime to k, v is prime to l, and w is prime to m. Since multiplying X (resp. Y) by -1 leads to replacing u (resp. v) with $k - u$ (resp. $l - v$), we may assume that

$$0 < u \leq \frac{k}{2}, \quad 0 < v \leq \frac{l}{2}. \tag{10}$$

For any matrix $Z \in \mathrm{SL}_2(\mathbb{C})$, the Hamilton - Cayley equation gives

$$Z^2 = (\mathrm{tr}\, Z)\, Z - E.$$

It follows that

$$Z^n = P_n\,(\mathrm{tr}\, Z)\, Z - P_{n-1}\,(\mathrm{tr}\, Z)\, E,$$

where $P_1(z),\ P_2(z),\ \ldots$ are the polynomials defined by

$$P_1(z) = 1, \quad P_2(z) = z, \quad P_{n+1}(z) = z\,P_n(z) - P_{n-1}(z).$$

(These are the Chebyshev polynomials of second kind up to a linear substitution.)

Making use of this formula, one can express $\operatorname{tr} W(X,Y)$ as a polynomial in $\operatorname{tr} X,\ \operatorname{tr} Y,\ \operatorname{tr} XY$ (with integral coefficients). Substituting the values of $\operatorname{tr} X$ and $\operatorname{tr} Y$ from (8) and $\operatorname{tr} XY = t$, we obtain a polynomial f of degree s in t [BMS]. For any (complex) root λ of the algebraic equation

$$f(t) = 2\cos\frac{\pi w}{m} \tag{11}$$

there exists an essential homomorphism $\varphi : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ satisfying (8) and (9) such that $\operatorname{tr} XY = \lambda$. Moreover, if $\lambda \neq 2\cos(\frac{\pi u}{k} \pm \frac{\pi v}{l})$, the pair (X,Y) is irreducible, so this homomorphism is uniquely defined up to conjugacy.

If $\lambda = 2\cos(\frac{\pi u}{k} \pm \frac{\pi v}{l})$, there is an essential homomorphism of Γ to a cyclic group of diagonal matrices. This situation will be investigated in the following section.

Another way to find the polynomial f is as follows.
Set

$$X = \begin{pmatrix} e^{i\alpha} & 1 \\ 0 & e^{-i\alpha} \end{pmatrix}, \qquad Y = \begin{pmatrix} e^{i\beta} & 0 \\ \tau & e^{-i\beta} \end{pmatrix}.$$

with $\alpha = \frac{\pi u}{k}, \quad \beta = \frac{\pi v}{l}$. Then

$$\operatorname{tr} X = 2\cos\alpha, \quad \operatorname{tr} Y = 2\cos\beta, \quad \operatorname{tr} XY = \tau + 2\cos(\alpha+\beta).$$

Making use of the formulas

$$X^p = \begin{pmatrix} e^{ip\alpha} & \frac{\sin p\alpha}{\sin\alpha} \\ 0 & e^{-ip\alpha} \end{pmatrix}, \qquad Y^q = \begin{pmatrix} e^{iq\beta} & 0 \\ \frac{\sin q\beta}{\sin\beta}\tau & e^{-iq\beta} \end{pmatrix},$$

one can express $\operatorname{tr} W(X,Y)$ as a polynomial in τ. Substituting

$$\tau = t - 2\cos(\alpha+\beta)$$

we obtain the polynomial $f(t)$.

Sometimes we shall extend the notation $[X]$ to any $X \in \mathrm{GL}_2(\mathbb{C})$. More precisely, for any $X \in \mathrm{GL}_2(\mathbb{C})$ we shall denote by $[X]$ the set $\{\lambda X : \lambda \in \mathbb{C}^*\}$ as an element of the group $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{PSL}_2(\mathbb{C})$.

2 G.t.g. admitting an essential homomorphism to a cyclic group.

Any finite subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is one of the following groups:

C_n, the cyclic group of order n;
D_n, the dihedral group of order $2n$;
T, the tetrahedral group of order 12;
O, the octahedral group of order 24;
I, the icosahedral group of order 60.

In this section, we consider g.t.g. admitting an essential homomorphism to C_n.

Let Γ be the g.t.g. defined by (1).

Proposition 1 ([BMS]) *If there exists an essential homomorphism $\varphi : \Gamma \to C_n$, then there exists an essential homomorphism $\widetilde{\varphi} : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ with an infinite image.*

Proof. One can interpret φ as a homomorphism of Γ to $\mathrm{PSL}_2(\mathbb{C})$, taking x to $[X]$ and y to $[Y]$, where

$$X = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \qquad Y = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix}$$

with some $\alpha, \beta \in (0, \pi)$. Replacing Y with

$$\widetilde{Y} = \begin{pmatrix} e^{i\beta} & 1 \\ 0 & e^{-i\beta} \end{pmatrix},$$

we obtain the required homomorphism. $\square$

Let us find a criterion for Γ to admit an essential homomorphism to a cyclic group.

For any prime p and non-zero integer n, set

$$\nu_p(n) = \max\{\, \alpha \in \mathbb{Z}_+ : \; p^\alpha \,|\, n \,\}.$$

Let n_1, n_2, n_3 be three divisors of n.

Lemma 1 *The congruence*

$$n_1\, u_1 + n_2\, u_2 + n_3\, u_3 \equiv 0 \pmod n \tag{12}$$

has a solution with u_1, u_2, u_3 *prime to* n *if and only if the following conditions are satisfied:*

(B1′) *for any odd prime* p*, at least two of the numbers* $\nu_p(n_1),\ \nu_p(n_2),\ \nu_p(n_3)$ *are minimal among them;*

(B2′) *exactly two of the numbers* $\nu_2(n_1),\, \nu_2(n_2),\, \nu_2(n_3)$ *are minimal among them, unless they all equal* $\nu_2(n)$*.*

Proof. Decomposing n into primes, we reduce to the case, when n is a power of a prime p. Then, cancelling the congruence (12) by a power of p, we reduce to the case, when

$$\min\{\, \nu_p(n_1),\, \nu_p(n_2),\, \nu_p(n_3)\,\} = 0,$$

i.e. one of the numbers n_1, n_2, n_3 equals 1.

If exactly one of these numbers equals 1 (and two others are non-trivial powers of p), then the congruence (12) has no solutions with u_1, u_2, u_3 prime to p. If at least two of them equal 1, one can easily find a solution of (4) with u_1, u_2, u_3 prime to p , except for the case, when $p = 2,\ n_1 = n_2 = n_3 = 1,\ n > 1$. This proves the lemma. □

Let us call a triple of non-zero integers $\{n_1, n_2, n_3\}$ *balanced* , if it satisfies the following conditions:

(B1) for any odd prime p, at least two of the numbers $\nu_p(n_1),\ \nu_p(n_2),\ \nu_p(n_3)$ are maximal among them;

(B2) exactly two of the numbers $\nu_2(n_1),\, \nu_2(n_2),\, \nu_2(n_3)$ are maximal among them, unless they all equal 0.

Set

$$K = k_1 + \ldots + k_s, \qquad L = l_1 + \ldots + l_s. \tag{13}$$

Proposition 2 *The group* Γ *admits an essential homomorphism to a cyclic group if and only if the triple*

$$\{\, \frac{k}{(k,K)}, \quad \frac{l}{(l,L)}, \quad m\,\}$$

is balanced.

Proof. If $\varphi : \Gamma \to C_n$ is an essential homomorphism, then $\varphi(\Gamma) = \langle \varphi(x),\, \varphi(y) \rangle$ is a cyclic group , whose order is the least common multiple $[k, l]$ of k and l. This shows that we may assume n to be any common multiple of k and l.

Let us try to construct an essential homomorphism $\varphi : \Gamma \to \mathbb{Z}_n \doteqdot \mathbb{Z} \,/\, n\mathbb{Z}$, where n is a common multiple of $k,\, l,\, m$.

We shall denote a coset $r + n\mathbb{Z}$ by $[r]_n$.

If $\varphi(x) = a$ and $\varphi(y) = b$, then $\varphi\,(W(x, y)) = Ka + Lb$. When a runs over all elements of order k of $\mathbb{Z}_n$, Ka runs over all elements of order $\frac{k}{(k,K)}$, i.e. the elements of the form $[\,\frac{n\,(k,K)}{k}\, u\,]_n$, where u is prime to n.

In an analogous way, when b runs over all the elements of order l, Lb runs over all the elements of the form $[\,\frac{n\,(l,L)}{l}\, v\,]_n$, where v is prime to n. The order of an element of $\mathbb{Z}_n$ equals m if and only if it has the form $[\,\frac{n}{m}\, w\,]_n$, where w is prime to n. Thus, the group Γ admits an essential homomorphism to $\mathbb{Z}_n$ if and only if the congruence

$$\frac{n\,(k, K)}{k}\, u + \frac{n\,(l, L)}{l}\, v \equiv \frac{n}{m}\, w \pmod{n}$$

has a solution with $u,\, v,\, w$ prime to n. According to Lemma 1, this takes place if and only if the triple

$$\{\, \frac{n\,(k, K)}{k}, \quad \frac{n\,(l, L)}{l}, \quad \frac{n}{m}\,\}$$

satisfies the conditions (B1$'$) and (B2$'$) or , equivalently, if the triple

$$\{\, \frac{k}{(k, K)}, \quad \frac{l}{(l, L)}, \quad m\,\}$$

satisfies the conditions (B1) and (B2). □

3 Generating pairs of irreducible finite subgroups of $\mathrm{PSL}_2(\mathbb{C})$

Let Γ be the g.t.g. defined by (1) and $\varphi : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ an essential homomorphism, whose image is an irreducible finite group $F \subset \mathrm{PSL}_2(\mathbb{C})$, i.e. one of the groups $D_n,\, T,\, O,\, I$. Then

$$\operatorname{ord} \varphi(x) = k, \quad \operatorname{ord} \varphi(y) = l,$$

and $\varphi(x), \varphi(y)$ generate F.

It is not difficult to enumerate all generating pairs of each group F of the above list up to conjugacy in $\mathrm{PSL}_2(\mathbb{C})$ or, equivalently, in the normalizer $N(F)$ of F in $\mathrm{PSL}_2(\mathbb{C})$. Note that

$$N(D_n) = D_{2n}, \quad N(T) = N(O) = O, \quad N(I) = I.$$

In order to simplify the task, let us for any generating pair $([X],[Y])$ of a subgroup $F \subset \mathrm{PSL}_2(\mathbb{C})$ consider the element $[Z] \in F$, where $Z \in \mathrm{SL}_2(\mathbb{C})$, satisfying the condition

$$XYZ = 1. \tag{14}$$

Then $([X],[Y])$ and $([Z],[X])$ will also be generating pairs of F. So any generating triple $([X],[Y],[Z])$ of F satisfying the condition (14), gives rise to 3 generating pairs. (Of course, some of them may be conjugate.)

Note that the pair $([Y],[X])$ is conjugate to the pair $([Y]^{-1},[X]^{-1})$, which is obtained from the "inverse" triple $(\,[Z]^{-1},\ [Y]^{-1},\ [X]^{-1}\,)$ still satisfying the condition (14).

Thus, the problem reduces to a classification of generating triples of irreducible subgroups $F \subset \mathrm{PSL}_2(\mathbb{C})$, satisfying the condition (14), up to conjugacy, cyclic permutations and inversion. Below is a table of all such triples.

We use the following presentation for D_n:

$$D_n = \langle\, a, b \quad | \quad a^n = b^2 = (ab)^2 = 1 \,\rangle,$$

and we identify the groups $T,\ O,\ I$ with $A_4,\ S_4,\ A_5$, respectively, via well-known isomorphisms.

Note that

$$2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2}, \quad 2\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{2}.$$

In the column "Type" we provide a notation for each type of triples which includes the notation of the corresponding group F. In the column "Orders" the orders of $[X],\ [Y],\ [Z]$ are indicated.

For $F = D_n$, the number u is prime to n, and one may assume that $0 < u \leqslant \frac{n}{2}$.

Table 1.

Type	$[X]$	$[Y]$	$[Z]$	Orders	$\operatorname{tr} X$	$\operatorname{tr} Y$	$\operatorname{tr} Z$
$D_n(u)$	a^u	b	$a^u b$	$n, 2, 2$	$2\cos\frac{\pi u}{n}$	0	0
$T(1)$	(123)	(234)	(12)(34)	3, 3, 2	1	1	0
$T(2)$	(123)	(243)	(142)	3, 3, 3	1	1	1
$O(1)$	(1234)	(132)	(14)	4, 3, 2	$\sqrt{2}$	1	0
$O(2)$	(1234)	(1243)	(123)	4, 4, 3	$\sqrt{2}$	$\sqrt{2}$	0
$I(1)$	(12345)	(142)	(15)(34)	5, 3, 2	$\frac{1+\sqrt{5}}{2}$	1	0
$I(2)$	(12354)	(152)	(14)(35)	5, 3, 2	$\frac{1-\sqrt{5}}{2}$	1	0
$I(3)$	(12345)	(132)	(154)	5, 3, 3	$\frac{1+\sqrt{5}}{2}$	1	1
$I(4)$	(12354)	(132)	(145)	5, 3, 3	$\frac{1-\sqrt{5}}{2}$	1	1
$I(5)$	(12345)	(12354)	(13)(24)	5, 5, 2	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0
$I(6)$	(12345)	(14352)	(135)	5, 5, 3	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	1
$I(7)$	(12354)	(15342)	(134)	5, 5, 3	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	1
$I(8)$	(12345)	(14532)	(145)	5, 5, 3	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1
$I(9)$	(12345)	(12534)	(12453)	5, 5, 5	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
$I(10)$	(12354)	(12435)	(12543)	5, 5, 5	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$

It is clear from the very beginning, and it is seen from the table, that the set of all unordered triples $\{\operatorname{tr} X,\ \operatorname{tr} Y,\ \operatorname{tr} Z\}$ considered up to multiplying by -1 of any two members, is invariant under the Galois group (of a sufficiently large algebraic number field including all the involved numbers).

For any generating pair $([X],[Y])$ of a finite subgroup $F \subset \mathrm{PSL}_2(\mathbb{C})$ and for any $d \mid \operatorname{ord}[X]$, $e \mid \operatorname{ord}[Y]$ it is important to know the subgroup generated by $[X]^d$ and $[Y]^e$, and the type of the corresponding triple $([X]^d,\ [Y]^e,\ [Y]^{-e}[X]^{-d})$. There are only few non-trivial cases, which are presented in the following table.

Table 2.

Type of the original triple	Generators of the subgroup	Their orders	Type of the obtained triple
$D_n(u)$	$[X]^d,\ [Y]$	$\frac{n}{d}$, 2	$D_{n/d}(u)$
$O(1)$	$[X]^2,\ [Y]$	2, 3	$T(1)$
$O(1)$	$[X]^2,\ [Z]$	2, 2	$D_4(1)$
$O(2)$	$[X]^2,\ [Y]$	2, 4	$D_4(1)$
$O(2)$	$[X]^2,\ [Z]$	2, 3	$T(1)$
$O(2)$	$[X]^2,\ [Y]^2$	2, 2	$D_2(1)$

4 Admissible transformations. Imprimitive pseudo-finite g.t.g.

Some obvious transformations of the data defining a g.t.g. Γ lead to isomorphic g.t.g. which are, in particular, pseudo-finite if and only if Γ is pseudo-finite.

First, one can multiply modulo k the exponents $k_1, \dots, k_s$ by a factor prime to k and, in a similar way, multiply modulo l the exponents $l_1, \dots, l_s$ by a factor prime to l. These transformations can be interpreted as changes of the generators of the cyclic groups $\langle x\rangle$ and $\langle y\rangle$.

Second, one can cyclically shift the sequence $(k_1, l_1, \dots, k_s, l_s)$ by an even number. This replaces the relation $W(x,y)^m = 1$ with an equivalent one.

Third, one can interchange k and l and simultaneously shift the sequence $(k_1, l_1, \dots, k_s, l_s)$ by an odd number. This can be interpreted as interchanging x and y.

Fourth, one can replace the sequence $(k_1, l_1, \dots, k_s, l_s)$ with $(k_s, l_{s-1}, k_{s-1}, \dots, l_1, k_1, l_s)$. This replaces the relation $W(x,y)^m = 1$ with $(W(x^{-1}, y^{-1})^{-1})^m = 1$, which can be interpreted as replacing the relation $(W(x,y))^m = 1$ with the equivalent relation $(W(x,y)^{-1})^m = 1$, combined with changing the generators x and y for x^{-1} and y^{-1}.

Transformations of these four types and their combinations are called *admissible*. G.t.g. obtained from each other by admissible transformations are called *equivalent*. It is reasonable to classify pseudo-finite g.t.g. up to equivalence.

A g.t.g. is called *primitive* if

$$(k_1, \dots, k_s,\ k) = 1, \quad (l_1, \dots, l_s,\ l) = 1.$$

In the general case, set

$$(k_1, \dots, k_s,\ k) = d, \quad (l_1, \dots, l_s,\ l) = e,$$

and consider the primitive g.t.g.

$$\bar\Gamma = \langle\ \bar x,\ \bar y \mid \bar x^{\bar k} = \bar y^{\bar l} = \overline{W}(\bar x, \bar y)^m = 1\ \rangle,$$

where $\bar k = k/d$, $\bar l = l/e$ and $\overline{W}(\bar x, \bar y) = \bar x^{\bar k_1}\ \bar y^{\bar l_1}\ \dots\ \bar x^{\bar k_s}\ \bar y^{\bar l_s}$ with $\bar k_i = k_i/d$, $\bar l_i = l_i/e$. There is a natural homomorphism

$$\pi\ :\ \bar\Gamma \to \Gamma,$$

taking $\bar x$ to x^d and $\bar y$ to y^e. If $\varphi\ :\ \Gamma \to G$ is an essential homomorphism, then $\bar\varphi = \varphi\pi\ :\ \bar\Gamma \to G$ is also an essential homomorphism, and $\bar\varphi(\bar\Gamma) = \langle\ \varphi(x)^d,\ \varphi(y)^e\ \rangle \subset \varphi(\Gamma)$.

Conversely, let $\bar\varphi : \bar\Gamma \to \mathrm{PSL}_2(\mathbb{C})$ be an essential homomorphism. Let $[X]$ be any d-th root of $\bar\varphi(\bar x)$ and $[Y]$ any e-th root of $\bar\varphi(\bar y)$. Then there is a homomorphism $\varphi : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$, taking x to $[X]$ and y to $[Y]$. Obviously, φ is essential and $\bar\varphi = \varphi\pi$.

It follows that if Γ is pseudo-finite, then $\bar\Gamma$ is also pseudo-finite.

Let us call a g.t.g. Γ *pseudo-dihedral* (resp. *pseudo-tetrahedral*) if the image of any essential homomorphism $\varphi : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ is a dihedral (resp. tetrahedral) group.

Proposition 3 . *Imprimitive pseudo-finite g.t.g. are (up to equivalence) exactly the groups of the following six types :*

(I1) $\Gamma = \langle\ x,\ y \quad|\quad x^k = y^2 = (x^{k_1}y \dots x^{k_s}y)^2 = 1\ \rangle$,
where $d > 1$ *and the group* $\bar\Gamma$ *is pseudo-dihedral;*

(I2) $\Gamma = \langle\ x,\ y \quad|\quad x^4 = y^4 = (x^{k_1}\, y^2\ \dots\ x^{k_s}\, y^2)^2 = 1\ \rangle$,
where not all of the exponents $k_1, \dots, k_s$ *are even and the group* $\bar\Gamma$ *is pseudo-dihedral;*

(I3) $\Gamma = \langle\ x,\ y \quad|\quad x^4 = y^4 = (\ x^2\, y^2)^2 = 1\ \rangle$*;*

(I4) $\Gamma = \langle\ x,\ y \quad|\quad x^3 = y^4 = (x^{k_1}y^2\ \dots\ x^{k_s}\, y^2)^3 = 1\ \rangle$,
where the group $\bar\Gamma$ *is pseudo-tetrahedral;*

(I5) $\Gamma = \langle\, x,\ y \quad | \quad x^3 = y^4 = (x^{k_1} y^2 \,\ldots\, x^{k_s} y^2)^2 = 1 \,\rangle$,
where the group $\bar\Gamma$ *is pseudo-tetrahedral;*

(I6) $\Gamma = \langle\, x,\ y \quad | \quad x^2 = y^4 = (x\, y^2)^2 = 1 \,\rangle$.

Proof. If an imprimitive g.t.g. Γ is pseudo-finite, then for any essential homomorphism $\bar\varphi : \bar\Gamma \to \mathrm{PSL}_2(\mathbb{C})$ not only the group $\bar\varphi\,(\bar\Gamma) = \langle\, \bar\varphi(\bar x),\ \bar\varphi(\bar y) \,\rangle$ is finite, but the group, generated by a d-th root of $\bar\varphi(\bar x)$ and a e-th root of $\bar\varphi(\bar y)$, is still finite. All such possibilities are enumerated in Table 2. Consider them case-by-case. We shall use the notation (7) and set $\bar X = X^d$, $\bar Y = Y^e$.

If $d > 2$ or $k > 4$, then $l = 2$, i.e.

$\Gamma = \langle\, x,\ y \quad | \quad x^k = y^2 = (x^{k_1} y \,\ldots\, x^{k_s} y)^m = 1 \,\rangle$,
and $\mathrm{tr}\, \bar X \bar Y = 0$ for any $\bar\varphi$. But if $m > 2$, there are at least two possibilities for $\mathrm{tr}\, \overline{W}(\bar X, \bar Y)$ and, thereby, at least 2 possibilities for the equation (11) for $\bar\Gamma$, which differ only by constant term. At least one of these polynomials does not vanish at 0. Hence, $m = 2$, and we come to the case (I1).

The cases, when $e > 2$ or $l > 4$, are obtained by interchanging x and y. In all the other cases $d,\ e \le 2$ and $k,\ l \le 4$. By symmetry, we may (and shall) assume that $e = 2$ and $l = 4$.

Under these conditions, if $k = 4$ and $d = 1$, then again $\mathrm{tr}\, \bar X \bar Y = 0$ for any $\bar\varphi$. Reasoning as above, we can conclude that $m = 2$, which gives the case (I2).

If $k = 4$ and $d = 2$, we obtain the case (I3).

If $k = 3$, then $d = 1$ and $\bar\varphi(\bar\Gamma)$ must be a tetrahedral group. It follows that $m \le 3$, so we obtain the case (I4) or (I5).

Finally, if $k = 2$, we obtain the case (I6). □

Note that any imprimitive g.t.g. is infinite as a non-trivial amalgamated product.

Now we are able to describe all pseudo-finite g.t.g. with $s = 1$.

Proposition 4 . *All the pseudo-finite g.t.g. with* $s = 1$ *are, up to equivalence, the usual triangle groups*

$$\langle\, x, y \quad | \quad x^k = y^l = (x\, y)^m = 1 \,\rangle$$

with $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} > 1$ *and the following imprimitive g.t.g. :*

1) $\langle\, x,\ y \quad | \quad x^k = y^2 = (x^d\, y)^2 = 1 \,\rangle, \quad (\ d \mid k,\ \ d > 1\ );$

2) $\langle\, x,\; y \quad | \quad x^4 = y^4 = (x\,y^2)^2 = 1 \quad \rangle$;

3) $\langle\, x,\; y \quad | \quad x^4 = y^4 = (x^2\,y^2)^2 = 1 \,\rangle$;

4) $\langle\, x,\; y \quad | \quad x^3 = y^4 = (x\,y^2)^3 = 1 \quad \rangle$;

5) $\langle\, x,\; y \quad | \quad x^2 = y^4 = (x\,y^2)^2 = 1 \quad \rangle$.

.

Proof. Changing the generators of the cyclic groups $\langle x\rangle$ and $\langle y\rangle$, we may assume that

$$k_1 = d \mid k, \quad l_1 = e \mid l.$$

If Γ is primitive, i.e. $d = e = 1$, then Γ is a usual triangle group and, as it well-known, it is embedded into $\mathrm{PSL}_2(\mathbb{C})$. Therefore, it is pseudo-finite if and only if it is finite, which take place if and only if $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} > 1$.

If Γ is imprimitive, it belongs to one of the types (I1) - (I6) of Proposition 3. The type (I5) is not realized, because the group $\bar{\Gamma}$ in this case is the dihedral group D_3. The other types constitute the above list. $\square$

5 Case $m \geq 3$

Constructing an essential homomorphism of a g.t.g. Γ to $\mathrm{PSL}_2(\mathbb{C})$, we can vary the parameters u, v, w in (8) and (9). Let us fix u, v and vary w. We shall obtain $\varphi(m)$ different algebraic equations of the form (11) with one and the same polynomial f of degree s in the left hand side. Obviously, they do not have common roots. Let N be the total number of their different roots. Then the total number of their roots with multiplicities, which is surely equal to $s\,\varphi\,(m)$, does not exceed N plus the number of roots (with multiplicities) of f' , whence

$$s\,(\varphi\,(m) - 1) \leq N - 1. \tag{15}$$

On the other hand, assuming the group Γ to be pseudo-finite, one can extract from Table 1 all possible values of $\mathrm{tr}\, XY$ for any fixed values of $\mathrm{tr}\, X$ and $\mathrm{tr}\, Y$. They are presented in the following table. It contains all possible values of $\mathrm{tr}\, X$ and $\mathrm{tr}\, Y$ up to interchanging them, multiplying by -1 , and acting by the Galois group.

Table 3.

k	l	$\operatorname{tr} X$	$\operatorname{tr} Y$	$\operatorname{tr} XY$
≥ 6	2	$2\cos\frac{\pi}{k}$	0	0
5	5	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$1,\ \frac{1+\sqrt{5}}{2}$
5	5	$\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$0,\ 1$
5	3	$\frac{1+\sqrt{5}}{2}$	1	$0,\ 1,\ \frac{1+\sqrt{5}}{2},\ \frac{-1+\sqrt{5}}{2}$
5	2	$\frac{1+\sqrt{5}}{2}$	0	$0,\ \pm 1,\ \pm\frac{1-\sqrt{5}}{2}$
4	4	$\sqrt{2}$	$\sqrt{2}$	1
4	3	$\sqrt{2}$	1	$0,\ \sqrt{2}$
4	2	$\sqrt{2}$	0	$0,\ \pm 1$
3	3	1	1	$0,\ 1,\ \frac{1+\sqrt{5}}{2},\ \frac{1-\sqrt{5}}{2}$
3	2	1	0	$0,\ \pm 1,\ \pm\sqrt{2},\ \pm\frac{1+\sqrt{5}}{2},\ \pm\frac{1-\sqrt{5}}{2}$
2	2	0	0	$2\cos\frac{\pi u}{m},\quad (u,m)=1$

In the last case

$$\Gamma = \langle\, x,\ y \quad | \quad x^2 = y^2 = (x\,y)^m = 1 \,\rangle,$$

so $s = 1$ and Γ is a dihedral group.

In all the other cases $N \leq 9$ and for $m \geq 3$ the inequality (15) gives an upper bound for s. This bound can be slightly improved with help of

Lemma 2 ([HMT], the proof of Theorem 6.4). *If a g.t.g. Γ admits essential homomorphisms onto D_3 and T, it admits an essential homomorphism onto C_6 and, hence, is not pseudo-finite.*

Proof. Let $\varphi : \Gamma \to D_3$ and $\psi : \Gamma \to T$ be essential homomorphisms. Then the homomorphism

$$\Gamma \to (D_3/C_3) \times (T/D_2) \simeq C_6,$$

$$\gamma \mapsto (\, \varphi(\gamma)C_3,\ \psi(\gamma)D_2\,),$$

is also essential. $\square$

Proposition 5 *All the pseudo-finite g.t.g. with $m \geq 3$ and $s \geq 2$ are, up to equivalence, the following two groups:*

1) $\langle\, x,\ y \quad | \quad x^3 = y^2 = (x\,y\,x^2\,y)^3 = 1\,\rangle$;

2) $\langle\, x,\ y \quad | \quad x^3 = y^2 = (x\,y\,x\,y\,x^2\,y)^3 = 1\,\rangle$.

The first of these groups is pseudo-tetrahedral. It gives rise to the following imprimitive pseudo-finite g.t.g. according to the case (I4) of Proposition 3:
$\langle\, x,\ y \quad | \quad x^3 = y^4 = (x\,y^2\,x^2\,y^2)^3 = 1\,\rangle$.

As it follows from the classification of finite g.t.g., the first of these groups is infinite, while the second one is finite.

Proof. Under our restrictions, the inequality (15) can be satisfied only in five cases of Table 3. In all these cases, the group $\varphi(\Gamma)$ is one of the groups D_3, D_4, D_5, T, O, I. The orders of elements of these groups do not exceed 5. It follows that $m \leq 5$. More precisely, all possible cases are presented in the following table.

Table 4.

k	l	The possible groups $\varphi\,(\Gamma)$	m	s
5	3	I	3	2, 3
5	2	I	3	2, 3, 4
4	2	D_4, O	4	2
3	3	T, I	3	2, 3
3	2	D_3, T, O, I	3	2 - 7

The case $k = 4,\ l = 2,\ m = 3$ is impossible, since in this case $\varphi(\Gamma)$ cannot be the group D_4, which leaves only two possibilities for $\operatorname{tr} XY$. The case $k = 3,\ l = 2,\ m = 4$ is impossible, since in this case $\varphi(\Gamma)$ can be only the group O, which again leaves only two possibilities for $\operatorname{tr} XY$.

In all the cases, enumerated in Table 4, and all, up to admissible transformations, exponents $k_1, \ldots, k_s,\ l_1, \ldots, l_s$, we explicitly wrote the equations (11) and found out if all their roots are among the admissible numbers indicated in Table 3. This turned to be true only in the two cases of the proposition. $\square$

6 Case $\quad m = 2, \quad k, l \geq 3$

For $m = 2$ the equation (11) takes the form

$$f(t) = 0.$$

If the group Γ is pseudo-finite, all the roots of the polynomial f (for any u, v) must be real and lie in the interval

$$\left(2\cos(\frac{\pi u}{k} + \frac{\pi v}{l}), \quad 2\cos(\frac{\pi u}{k} - \frac{\pi v}{l}) \right)$$

(see (4)), whence

$$\operatorname{sgn} f(\,2\cos(\frac{\pi u}{k} + \frac{\pi v}{l}))\, f\,(\,2\cos(\frac{\pi u}{k} - \frac{\pi v}{l})) = (-1)^s\,.$$

We deduce from this

Proposition 6 *If Γ is a pseudo-finite g.t.g. with $m = 2$, then*

$$\operatorname{sgn}\left(\cos\frac{2\pi K u}{k} + \cos\frac{2\pi L v}{l} \right) = (-1)^s \tag{16}$$

for any u prime to k and v prime to l.

(For the notation K and L see (13).)

Proof. Set $\alpha = \frac{\pi u}{k}, \quad \beta = \frac{\pi v}{l}$. Then the commuting matrices X and Y from (5) satisfy the conditions

$$\operatorname{tr} X = 2\cos\frac{\pi u}{k}, \quad \operatorname{tr} Y = 2\cos\frac{\pi v}{l}, \quad \operatorname{tr} XY = 2\cos(\frac{\pi u}{k} \pm \frac{\pi v}{l}),$$

and, hence,

$$f\,(2\cos(\frac{\pi u}{k} \pm \frac{\pi v}{l})) = \operatorname{tr} W(X,\, Y) = 2\cos(\frac{\pi K u}{k} \pm \frac{\pi L v}{l}).$$

It follows that

$$f\,(2\cos(\frac{\pi u}{k} + \frac{\pi v}{l}))\, f\,(2\cos(\frac{\pi u}{k} - \frac{\pi v}{l})) =$$

$$4\cos(\frac{\pi K u}{k} + \frac{\pi L v}{l})\cos(\frac{\pi K u}{k} - \frac{\pi L v}{l}) = 2(\cos\frac{2\pi K u}{k} + \cos\frac{2\pi L v}{l}). \qquad \square$$

Corollary. *If* $k = l = 5$, *then* s *is even.*

Proof. Suppose s is odd. Then

$$\cos\frac{2\pi Ku}{5} + \cos\frac{2\pi Lv}{5} < 0 \qquad (17)$$

for any $u, v \in \{1, 2, 3, 4\}$. If K is divisible by 5, then the first summand is equal to 1 and the inequality (17) cannot hold. Hence, K is not divisible by 5. In the same way, L is not divisible by 5. Consequently, one can choose u, v so that

$$Ku \equiv Lv \equiv 1 \pmod 5$$

Then (17) does not hold. □

If one of the numbers k, l equals 2 and Γ does not admit an essential homomorphism to a cyclic group, then the condition (16) holds automatically. However, if $k, l \geq 3$, it gives rise to some restrictions on K and L for given k, l and s. They are collected in the following table, containing all possible values of k and l (up to permutation).

Table 5.

k	l	s	Restrictions on K and L
5	5	even	$K \equiv 0 \pmod 5$ or $L \equiv 0 \pmod 5$
5	3	even	$K \equiv 0 \pmod 5$ or $L \equiv 0 \pmod 3$
4	4	even	$K \equiv 0 \pmod 4$ or $L \equiv 0 \pmod 4$; $K, L \not\equiv 2 \pmod 4$
4	3	even	$K \equiv 0 \pmod 4$ or $L \equiv 0 \pmod 3$; $K \not\equiv 2 \pmod 4$
3	3	even	$K \equiv 0 \pmod 3$ or $L \equiv 0 \pmod 3$
5	3	odd	$K \not\equiv 0 \pmod 5$ and $L \not\equiv 0 \pmod 3$
4	4	odd	$K, L \not\equiv 0 \pmod 4$; $K \equiv 2 \pmod 4$ or $L \equiv 2 \pmod 4$
4	3	odd	$K \not\equiv 0 \pmod 4$ and $L \not\equiv 0 \pmod 3$
3	3	odd	$K,\ L \not\equiv 0 \pmod 3$

For $m = 2$ we did not get an apriori upper bound for s, so we restricted ourselves with the cases $s = 2, 3$. Under this restriction there are only few cases to be checked, taking into account Table 5. The result is contained in the following two propositions.

Proposition 7 *All the primitive pseudo-finite g.t.g. with* $m = 2, \ k, l \geq 3$ *and* $s = 2$ *are, up to equivalence, the following 7 groups:*

1) $\langle\, x, \ y \quad | \quad x^5 = y^3 = (\, x\, y\, x^2\, y^2\,)^2 = 1 \,\rangle$*;*

2) $\langle\, x, \ y \quad | \quad x^5 = y^3 = (\, x\, y\, x^4\, y\,)^2 = 1 \,\rangle$*;*

3) $\langle\, x, \ y \quad | \quad x^4 = y^4 = (\, x\, y\, x^2\, y^3\,)^2 = 1 \,\rangle$*;*

4) $\langle\, x, \ y \quad | \quad x^4 = y^3 = (\, x\, y\, x^2\, y^2\,)^2 = 1 \,\rangle$*;*

5) $\langle\, x, \ y \quad | \quad x^4 = y^3 = (\, x\, y\, x^3\, y\,)^2 = 1 \,\rangle$*;*

6) $\langle\, x, \ y \quad | \quad x^3 = y^3 = (\, x\, y\, x^2\, y^2\,)^2 = 1 \,\rangle$*;*

7) $\langle\, x, \ y \quad | \quad x^3 = y^3 = (\, x\, y\, x^2\, y\,)^2 = 1 \,\rangle$*.*

As it follows from the classification of finite g.t.g., the last two groups are finite, while all the others are infinite.

Remark 1. It follows from Proposition 3 and Proposition 9 below, that there are, up to equivalence, exactly two imprimitive pseudo-finite g.t.g. with $m = 2, \ k, l \geq 3$ and $s = 2$, namely, the groups

$$\langle\, x, \ y \quad | \quad x^4 = y^4 = (\, x\, y^2\, x^3\, y^2\,)^2 = 1 \,\rangle;$$

$$\langle\, x, \ y \quad | \quad x^3 = y^4 = (\, x\, y^2\, x^2\, y^2\,)^2 = 1 \,\rangle.$$

Proposition 8 *All the primitive pseudo-finite g.t.g. with* $m = 2, \ k, l \geq 3$ *and* $s = 3$ *are, up to equivalence, the following 4 groups:*

1) $\langle\, x, \ y \quad | \quad x^5 = y^3 = (\, x\, y\, x\, y\, x^4\, y^2\,)^2 = 1 \,\rangle$*;*

2) $\langle\, x, \ y \quad | \quad x^5 = y^3 = (\, x\, y\, x^2\, y^2\, x^3\, y\,)^2 = 1 \,\rangle$*;*

3) $\langle\, x, \ y \quad | \quad x^4 = y^4 = (\, x\, y\, x^3\, y^3\, x\, y^2\,)^2 = 1 \,\rangle$*;*

4) $\langle\, x, \ y \quad | \quad x^3 = y^3 = (\, x\, y\, x\, y\, x^2\, y^2\,)^2 = 1 \,\rangle$*.*

As it follows from the classification of finite g.t.g., all these groups are infinite.

Remark 2. It follows from Proposition 3 and Proposition 10 below, that there are no imprimitive pseudo-finite g.t.g. with $m = 2$, $k, l \geq 3$ and $s = 3$.

7 Case $l = m = 2$

For $l = 2$ we have

$$W(x, y) = x^{k_1}\, y\, x^{k_2}\, y\, \ldots\, x^{k_s}\, y.$$

One may assume that $k \geq 3$, otherwise $s = 1$ and the group Γ is dihedral.

Admissible transformations in this case reduce to multiplying modulo k the exponents $k_1, \ldots, k_s$ by a factor prime to k, cyclic permutations of them, and reversing their order.

If, moreover, $m = 2$, the action of the Galois group allows us to restrict the consideration to the case $u = 1$.

Set

$$\varepsilon = \varepsilon_k = e^{\frac{2\pi i}{k}}$$

and choose matrices $X,\ Y \in \mathrm{SL}_2(\mathbb{C})$ as follows:

$$X = \begin{pmatrix} e^{\frac{\pi i}{k}} & 0 \\ 0 & e^{-\frac{\pi i}{k}} \end{pmatrix} = e^{-\frac{\pi i}{k}} \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \quad Y = -i \begin{pmatrix} \tau & 1-\tau \\ 1+\tau & -\tau \end{pmatrix}.$$

Then

$$\operatorname{tr} X = 2\cos\frac{\pi}{k}, \quad \operatorname{tr} Y = 0,$$

$$\operatorname{tr} XY = 2\,\tau \sin\frac{\pi}{k}, \tag{18}$$

and

$$\operatorname{tr} W(X, Y) = e^{-\pi i\,(\frac{K}{k} + \frac{s}{2})}\, g(\tau),$$

where

$$g(\tau) = \operatorname{tr} \begin{pmatrix} \varepsilon^{k_1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau & 1-\tau \\ 1+\tau & -\tau \end{pmatrix} \begin{pmatrix} \varepsilon^{k_2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau & 1-\tau \\ 1+\tau & -\tau \end{pmatrix} \cdots$$
$$\cdots \begin{pmatrix} \varepsilon^{k_s} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau & 1-\tau \\ 1+\tau & -\tau \end{pmatrix}.$$

Note that multiplying Y by -1 affects $\operatorname{tr} X$, $\operatorname{tr} Y$ and $\operatorname{tr} XY$ in the same way as multiplying τ by -1. Obviously,

$$\operatorname{tr} W(X, -Y) = (-1)^s \operatorname{tr} W(X, Y).$$

Hence,

$$g(-\tau) = (-1)^s g(\tau),$$

so

$$g(\tau) = \begin{cases} h(\tau^2) & \text{for } s \text{ even}, \\ \tau h(\tau^2) & \text{for } s \text{ odd}, \end{cases}$$

where h is a polynomial of degree $[\frac{s}{2}]$.

As it follows from Table 3 and (18), the group Γ is pseudo-finite if and only if all the roots of the polynomial h are among the numbers indicated in the following table.

Table 6.

k	Admissible roots of h
≥ 6	0
5	$0,\ \frac{1+\sqrt{5}}{2\sqrt{5}},\ \frac{-1+\sqrt{5}}{2\sqrt{5}}$
4	$0,\ \frac{1}{2}$
3	$0,\ \frac{1}{3},\ \frac{2}{3},\ \frac{3+\sqrt{5}}{6}\ \ \frac{3-\sqrt{5}}{6}$

Let us find the polynomial h explicitly.

The polynomial $g(\tau)$ is the sum of all the products of entries of the matrices

$$\begin{pmatrix} \varepsilon^{k_1} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \tau & 1-\tau \\ 1+\tau & -\tau \end{pmatrix}, \begin{pmatrix} \varepsilon^{k_2} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \tau & 1-\tau \\ 1+\tau & -\tau \end{pmatrix}, \ldots$$
$$\ldots, \begin{pmatrix} \varepsilon^{k_s} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \tau & 1-\tau \\ 1+\tau & -\tau \end{pmatrix}, \qquad (19)$$

chosen so that the column number of the entry of each matrix equals the row number of the entry of the subsequent matrix, if considering the matrices (19) ordered cyclically.

Clearly, one can take only diagonal entries of the matrices

$$\begin{pmatrix} \varepsilon^{k_1} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \varepsilon^{k_2} & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} \varepsilon^{k_s} & 0 \\ 0 & 1 \end{pmatrix}. \tag{20}$$

Every time, when we retain the number of the diagonal entry passing to the subsequent matrix (20), we have to take an entry $\pm\tau$ of the intermediate factor $\begin{pmatrix} \tau & 1-\tau \\ 1+\tau & -\tau \end{pmatrix}$. Every time, when we switch to another number, we have to take an entry $1 \pm \tau$ of the intermediate factor.

It follows that each product has the form

$$(-1)^{s-q-r}\, \varepsilon^{\,k_{i_1}+\ldots+k_{i_s}}\, \tau^{\,s-2q}\, (1-\tau^2)^{\,q},$$

where $1 \le i_1 < \ldots < i_r \le s$ and the number q is defined as follows:

1) if the set $\{i_1, \ldots, i_r\}$ is a proper subset of $\{1, \ldots, s\}$, then q is equal to the number of its "connected components", where a connected component is a maximal subset of $\{i_1, \ldots, i_r\}$ consisting of consecutive elements of the set $\{1, \ldots, s\}$ considered cyclically ordered;

2) if $\{i_1, \ldots, i_r\} = \{1, \ldots, s\}$, then $q = 0$.

Thus , the polynomial h has the form

$$h(\sigma) = \sum_{q=0}^{[\frac{s}{2}]} (-1)^q\, h_q\, \sigma^{[\frac{s}{2}]-q}\, (1-\sigma)^{\,q}, \tag{21}$$

where

$$h_0 = \varepsilon^K + (-1)^s \tag{22}$$

and, for $q > 0$,

$$h_q = \sum_{\substack{1 \le i_1 < \ldots < i_r \le s \\ \{i_1, \ldots, i_r\} \text{ has } q \\ \text{connected components}}} (-1)^{s-r}\, \varepsilon^{k_{i_1}+\ldots+k_{i_r}}, \tag{23}$$

In particular, for $s = 2$ we have

$$h(\sigma) = (\varepsilon^K + 1)\,\sigma + (\varepsilon^{k_1} + \varepsilon^{k_2})(1-\sigma) = (\varepsilon^{k_1} - 1)(\varepsilon^{k_2} - 1)\,\sigma + \varepsilon^{k_1} + \varepsilon^{k_2}$$

Proposition 9 *All the primitive pseudo-finite g.t.g. with $l = m = 2$ and $s = 2$ are, up to equivalence, the following groups:*

1) $\langle\, x,\ y \quad | \quad x^{2n} = y^2 = (\, x\, y\, x^{n+1}\, y\,)^2 = 1\,\rangle \quad (n \geq 2)$;

2) $\langle\, x,\ y \quad | \quad x^{5} = y^2 = (\, x\, y\, x^{2}\, y\,)^2 = 1\,\rangle$;

3) $\langle\, x,\ y \quad | \quad x^{3} = y^2 = (\, x\, y\, x^{2}\, y\,)^2 = 1\,\rangle$.

The groups of the first type are pseudo-dihedral. They give rise to some imprimitive pseudo-finite g.t.g. according to the cases (I1) and (I2) of Proposition 3.

The last group is pseudo-tetrahedral. It gives rise to an imprimitive pseudo-finite g.t.g. according to the case (I5) of Proposition 3.

As it follows from the classification of finite g.t.g., the groups of the first type are infinite, while the last two groups are finite.

Proof. The root of h equals 0 if and only if $\varepsilon^{k_1} + \varepsilon^{k_2} = 0$, which means that k is even and

$$k_1 - k_2 \equiv \frac{k}{2} \pmod{k}$$

Under this condition, if Γ is primitive, at least one of the integers k_1, k_2 must be prime to k, and we may assume that it is equal to 1. In this way we obtain the groups of the first type.

In all the other cases $k \leq 5$. Only 5 such cases are to be tried. This gives the two last groups of the proposition. □

For $s = 3$ we have

$$h(\sigma) = (\varepsilon^{K} - 1)\,\sigma + (\varepsilon^{k_2+k_3} + \varepsilon^{k_3+k_1} + \varepsilon^{k_1+k_2} - \varepsilon^{k_1} - \varepsilon^{k_2} - \varepsilon^{k_3})\,(1-\sigma) =$$

$$(\varepsilon^{k_1} - 1)(\varepsilon^{k_2} - 1)\,(\varepsilon^{k_3} - 1)\,\sigma + \varepsilon^{k_2+k_3} + \varepsilon^{k_3+k_1} + \varepsilon^{k_1+k_2} - \varepsilon^{k_1} - \varepsilon^{k_2} - \varepsilon^{k_3}.$$

Proposition 10 *All the primitive pseudo-finite g.t.g. with $l = m = 2$ and $s = 3$ are, up to equivalence, the following groups:*

1) $\langle\, x,\ y \quad | \quad x^{2n} = y^2 = (\, x\, y\, x^{n+1}\, y\, x^{n+2}\, y\,)^2 = 1\,\rangle \quad (n \geq 3)$;

2) $\langle\, x,\ y \quad | \quad x^{3n} = y^2 = (x\, y\, x^{n+1}\, y\, x^{2n+1}\, y\,)^2 = 1\,\rangle \quad (\,n \geq 2\,)$;

3) $\langle\, x,\ y \quad | \quad x^{6n} = y^2 = (\, x^d\, y\, x^n\, y\, x^{5n}\, y\,)^2 = 1 \,\rangle \quad (d\,|\,6,\ (d,\,n)=1);$

4) $\langle\, x,\ y \quad | \quad x^{30} = y^2 = (\, x^2\, y\, x^3\, y\, x^{26}\, y\,)^2 = 1 \,\rangle;$

5) $\langle\, x,\ y \quad | \quad x^5 = y^2 = (\, x\, y\, x^2\, y\, x^3\, y\,)^2 = 1 \,\rangle;$

6) $\langle\, x,\ y \quad | \quad x^5 = y^2 = (\, x\, y\, x\, y\, x^4\, y\,)^2 = 1 \,\rangle;$

7) $\langle\, x,\ y \quad | \quad x^4 = y^2 = (\, x\, y\, x^2\, y\, x^3\, y\,)^2 = 1 \,\rangle;$

8) $\langle\, x,\ y \quad | \quad x^4 = y^2 = (\, x\, y\, x\, y\, x^3\, y\,)^2 = 1 \,\rangle;$

9) $\langle\, x,\ y \quad | \quad x^3 = y^2 = (\, x\, y\, x\, y\, x^2\, y\,)^2 = 1 \,\rangle.$

The groups of the first three types and the group no. 4 are pseudo-dihedral. They give rise to some imprimitive pseudo-finite g.t.g. according to the case (I1) of Proposition 3.

As it follows from the classification of finite g.t.g. the groups of the first three types and the groups nos. 4, 7 are infinite, while the groups nos. 5, 6, 8, 9 are finite.

To prove the proposition, we need some lemmas.

Lemma 3 *Let* $z_1,\, z_2,\, \ldots$ *be complex numbers with modulus* 1*. Then*

a) $z_1 + z_2 + z_3 = 0$ *if and only if* $z_1,\, z_2,\, z_3$ *divide the unit circle into equal parts;*

b) $z_1 + z_2 + z_3 + z_4 = 0$ *if and only if* $z_1,\, z_2,\, z_3,\, z_4$ *decompose into two pairs of opposite numbers.*

The lemma is easily proved by a geometrical reasoning.

Let $\varepsilon_k = e^{\frac{2\pi i}{k}}$ as above, and let φ be a Laurent polynomial with rational coefficients. The following lemma provides an algorithm for finding out if $\varphi(\varepsilon_k) = 0$.

Let p be a prime divisor of k.

Lemma 4 a) *If* $p^2\,|\,k$ *, write the polynomial* φ *in the form*

$$\varphi(z) = \varphi_0(z^p) + \varphi_1(z^p)\, z + \varphi_2(z^p)\, z^2 + \ldots + \varphi_{p-1}(z^p)\, z^{p-1},$$

where $\varphi_0,\ \varphi_1, \ldots, \varphi_{p-1}$ *are Laurent polynomials (with rational coefficients). Then* $\varphi(\varepsilon_k) = 0$ *if and only if*

$$\varphi_0(\varepsilon_{k/p}) = \varphi_1(\varepsilon_{k/p}) = \ldots = \varphi_{p-1}(\varepsilon_{k/p}) = 0. \tag{24}$$

b) *If* $p^2 \nmid k$, *write the polynomial* φ *in the form*

$$\varphi(z) = \psi_0(z^p) + \psi_1(z^p)\, z^{\frac{k}{p}} + \psi_2(z^p)\, z^{\frac{2k}{p}} + \ldots + \psi_{p-1}(z^p)\, z^{\frac{(p-1)k}{p}},$$

where $\psi_0, \psi_1, \ldots , \psi_{p-1}$ *are Laurent polynomials (with rational coefficients). Then* $\varphi(\varepsilon_k) = 0$ *if and only if*

$$\psi_0(\varepsilon_{k/p}) = \psi_1(\varepsilon_{k/p}) = \ldots = \psi_{p-1}(\varepsilon_{k/p}). \tag{25}$$

Proof. a) If $p^2 \,|\, k$, then $[\, \mathbb{Q}(\varepsilon_k) : \mathbb{Q}(\varepsilon_{k/p})\,] = p$, and $1, \varepsilon_k, \varepsilon_k^2, \ldots , \varepsilon_k^{p-1}$ constitute a basis of $\mathbb{Q}(\varepsilon_k)$ over $\mathbb{Q}(\varepsilon_{k/p})$.

b) If $p^2 \nmid k$, then $[\, \mathbb{Q}(\varepsilon_k) : \mathbb{Q}(\varepsilon_{k/p})\,] = p-1$, and $1, \varepsilon^{\frac{k}{p}}, \varepsilon^{\frac{2k}{p}}, \ldots , \varepsilon^{\frac{(p-1)k}{p}}$ linearly span $\mathbb{Q}(\varepsilon_k)$ over $\mathbb{Q}(\varepsilon_{k/p})$ with the only linear dependence
$1 + \varepsilon^{\frac{k}{p}} + \varepsilon^{\frac{2k}{p}} + \ldots + \varepsilon^{\frac{(p-1)k}{p}} = 0.$ □

The conditions (24) can be interpreted as follows. Decompose the set of exponents of the polynomial φ into congruence classes modulo p. Then the equalities (24) mean that the sum of terms of φ corresponding to each class vanishes at ε_k.

In the case b), if not all the residues modulo p are represented by the exponents of non-zero terms of φ (e.g. if the number of these terms is less then p), at least one of the polynomials $\psi_0, \psi_1, \ldots , \psi_{p-1}$ is (identically) equal to 0, and the conditions (25) turn to be equivalent to the conditions (24).

Corollary. *Assume that among the exponents of non-zero terms of* φ *there is one that is not congruent modulo* p *to any of the others. Let, moreover,* $p^2 \,|\, k$ *or the number of non-zero terms of* φ *is less than* p. *Then* $\varphi(\varepsilon_k) \neq 0$.

Proof of the proposition. The root of the polynomial h equals 0 if and only if

$$\varepsilon^{k_1} + \varepsilon^{k_2} + \varepsilon^{k_3} = \varepsilon^{k_2+k_3} + \varepsilon^{k_3+k_1} + \varepsilon^{k_1+k_2}. \tag{26}$$

Let p be a prime divisor of k. Assume that $p^2 \,|\, k$ or $p \geq 7$. Due to the preceding corollary the equality (26) can hold only if each of the integers

$$k_1,\ k_2,\ k_3,\ k_2+k_3,\ k_3+k_1,\ k_1+k_2 \tag{27}$$

is congruent modulo p to some of the others. It is easy to see that such a situation takes place only in the following cases, up to permutation of $k_1,\ k_2,\ k_3$:

1) $k_1 \equiv k_2 \pmod p$, $\quad k_3 \equiv k_1 + k_2 \pmod p$;
2) $k_1 \equiv k_2 \equiv k_3 \pmod p$;
3) $k_2 \equiv k_3 \equiv 0 \pmod p$;
4) $k_3 \equiv k_1 + k_2 \equiv 0 \pmod p$.
Consider all these cases.

Case 1. In this case, if $p \neq 2$, the decomposition of the set of integers (27) into congruence classes modulo p looks as follows:

$$\{k_1,\ k_2\} \cup \{k_3,\ k_1 + k_2\} \cup \{k_1 + k_3,\ k_2 + k_3\}.$$

By Lemma 4, the equality (26) holds only if

$$\varepsilon^{k_1} + \varepsilon^{k_2} = \varepsilon^{k_3} - \varepsilon^{k_1+k_2} = -\varepsilon^{k_1+k_3} - \varepsilon^{k_2+k_3} = 0,$$

which means that k is even and

$$k_1 - k_2 \equiv \frac{k}{2} \pmod k, \quad k_3 \equiv k_1 + k_2 \pmod k.$$

If the group Γ is primitive, at least one of the integers k_1, k_2 must be prime to k, and we may assume that it is equal to 1. This gives case 1) of the proposition.

If $p = 2$, two of the above congruence classes must glue together. It is easy to see that these are the first and the third classes. We get

$$\varepsilon^{k_1} + \varepsilon^{k_2} - \varepsilon^{k_1+k_3} - \varepsilon^{k_2+k_3} = \varepsilon^{k_3} - \varepsilon^{k_1+k_2} = 0.$$

Since

$$\varepsilon^{k_1} + \varepsilon^{k_2} - \varepsilon^{k_1+k_3} - \varepsilon^{k_2+k_3} = (\varepsilon^{k_1} + \varepsilon^{k_2})(1 - \varepsilon^{k_3})$$

and $\varepsilon^{k_3} \neq 1$, we come to the same result as above.

Case 2. The decomposition of the set (27) into congruence classes is

$$\{k_1,\ k_2,\ k_3\} \cup \{k_2 + k_3,\ k_3 + k_1,\ k_1 + k_2\},$$

whence

$$\varepsilon^{k_1} + \varepsilon^{k_2} + \varepsilon^{k_3} = -\varepsilon^{k_2+k_3} - \varepsilon^{k_3+k_1} - \varepsilon^{k_1+k_2} = 0.$$

Due to Lemma 3, it follows that k is divisible by 3 and ε^{k_1}, ε^{k_2}, ε^{k_3} divide the unit circle into equals parts. At least one of the integers k_1, k_2, k_3 must be prime to k, and we may assume that it is equal to 1. This gives case 2) of the proposition.

Case 3. The decomposition of the set (27) into congruence classes is

$$\{ k_1,\ k_1+k_2,\ k_1+k_3 \} \cup \{ k_2,\ k_3,\ k_2+k_3 \},$$

whence

$$\varepsilon^{k_1} - \varepsilon^{k_1+k_2} - \varepsilon^{k_1+k_3} = \varepsilon^{k_2} + \varepsilon^{k_3} - \varepsilon^{k_2+k_3} = 0.$$

It follows that

$$\varepsilon^{k_2} + \varepsilon^{k_3} = \varepsilon^{k_2+k_3} = 1,$$

which means that k is divisible by 6 and, up to interchanging k_2 and k_3,

$$k_2 \equiv \frac{k}{6} \pmod{k}, \quad k_3 \equiv \frac{5k}{6} \pmod{k}.$$

Multiplying $k_1,\ k_2,\ k_3$ modulo k by an integer prime to k, one may assume that $k_1 \mid k$. But, if Γ is primitive, $(k_1,\ \frac{k}{6}) = 1$, whence $k_1 \mid 6$. This gives case 3) of the proposition.

Case 4. If $k_1 \equiv k_2 \pmod{p}$, we come to Case 1. Otherwise, the decomposition of the set (27) into congruence classes is

$$\{ k_1,\ k_1+k_3 \} \cup \{ k_2,\ k_2+k_3 \} \cup \{ k_3,\ k_1+k_2 \},$$

whence

$$\varepsilon^{k_1} - \varepsilon^{k_1+k_3} = \varepsilon^{k_2} - \varepsilon^{k_2+k_3} = \varepsilon^{k_3} - \varepsilon^{k_1+k_2} = 0,$$

which is impossible.

If k has no prime divisors satisfying the above conditions, then $k \mid 30$. For all such k and all, up to admissible transformations, $k_1,\ k_2,\ k_3$ we tried the equality (26) with help of a computer. It turned out that it held, beyond the series 1) - 3), only in case 4) of the proposition.

Finally, if Γ is pseudo-finite, but the root of h does not equal 0, then $k \leq 5$ and the root of h must belong to the numbers indicated in Table 6. There are only few cases to be tried. This gives the remaining 5 cases of the proposition. □

8 Two families of pseudo-dihedral g.t.g.

Finite g.t.g. exist only for $s \leq 8$. The following propositions show that pseudo-finite g.t.g. exist for any s.

Proposition 11 *The group*

$$\Gamma(s,n,c) = \langle\, x,\ y \quad | \quad x^{sn} = y^2 = (x^c\, y\, x^{n+c}\, y\, x^{2n+c}\, y\ \ldots\ x^{(s-1)n+c}\, y)^2 = 1\,\rangle$$

is pseudo-dihedral (and thereby pseudo-finite) for any s, n, c *with* $0 < c < n$.

Proof. One has to prove that all the roots of the polynomial h (see (21)) equal 0, i.e. that $h_q = 0$ for $q = 1, \ldots, [\frac{s}{2}]$.

We have (see (23))

$$h_q = \sum_{\substack{1 \leq i_1 < \ldots < i_r \leq s \\ \{i_1, \ldots, i_r\} \text{ has } q \\ \text{connected components}}} (-1)^{s-r}\, \varepsilon^{(i_1+\ldots+i_r-r)n+cr}$$

We shall prove that the sum $h_{q,r}$ of terms of h_q with a fixed r vanishes for each r.

Consider the transformation $i \mapsto i+1$ of the set $\{1,\ 2, \ldots,\ s\}$ (where $s+1$ is taken modulo s). It does not change the number of connected components of a subset of $\{1,\ 2, \ldots,\ s\}$, so $h_{q,r}$ is invariant under this transformation. But, on the other hand, each term of $h_{q,r}$ is multiplied by $\varepsilon^{rn} \neq 1$. Hence, $h_{q,r} = 0$. □

Proposition 12 *The group*

$$\Delta(s,n,c) = \langle\ x,\ y \quad | \quad x^{2sn} = y^2 = (x^n\, y\, x^{3n}\, y\ \ldots\ x^{(s-2)n}\, y\, x^c\, y\, x^{(s+2)n}\, y\ \ldots\ x^{(2s-3)n}\, y\, x^{(2s-1)n}\, y\,)^2 = 1\ \rangle$$

is pseudo-dihedral for any s, n, c *with* s *odd and* $0 < c < 2sn$.

Proof. One has to prove that $h_q = 0$ for $q = 1, \ldots, [\frac{s}{2}]$. Obviously, $h_q = h'_q\, \varepsilon^c + h''_q$, where h'_q and h''_q do not depend on c. Hence, it suffices to prove that $h_q = 0$ for $c = 0$ (when $\varepsilon^c = 1$) and for $c = sn$ (when $\varepsilon^c = -1$).

For $c = sn$ we have

$$\Delta\,(s,\, n,\, c) = \Gamma\,(s,\, 2n,\, n),$$

so $h_q = 0$ by Proposition 11.

For $c = 0$ we have $W(x, y) = y$, so $\operatorname{tr} W(X, Y) = \operatorname{tr} Y = 0$ (identically). □

Note that the above two families cover the series 1) of Proposition 9 and the series 2) and 3) of Proposition 10. Moreover, they cover all the series that we know for $s = 4, 5$.

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