MINIMAL GENERATORS OF RINGEL-HALL ALGEBRAS OF AFFINE QUIVERS

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Abstract. Let $\mathcal{H}(A)$ be the Ringel-Hall algebra of A, where A is the path algebra of an arbitrary affine quiver. For any $\mathbf{d} \in \mathbb{N}_0^n$, we determine the number of minimal generators of $\mathcal{H}(A)$ of degree $\mathbf{d}$; and such systems of minimal generators can be explicitly written out.

1. Introduction

Let k denote the finite field with q elements and A a finite-dimensional hereditary k-algebra, with all simple $A-$modules $S(1), \cdots, S(n)$ up to isomorphism. Denote by $A-mod$ the category of finite-dimensional left A-modules, which is exactly the category of A-modules with finitely many elements. The Grothendieck group $K_0(A)$ of all finite $A-$modules modulo short exact sequences can be identified with $\mathbb{Z}^n$, such that the image of $S(i)$ in it is the $i-$th coordinate vector. For $M \in A-mod$, denote its isoclass by $[M]$, and its image in $K_0(A)$ by $\mathbf{dim}M$ which is called the dimension vector of M.

Let $\mathbb{R}$ be the field of real numbers. By definition [R2] (see also [Mac], but only for discrete valution rings), the Ringel-Hall algebra $\mathcal{H}(A)$ of A is a $\mathbb{R}$-space with basis the set of isoclasses $[M]$ of all finite modules, with multiplication given by

$$[M] \cdot [N] := \sum_{[L]} g_{M,N}^L [L]$$

where the structure constant $g_{M,N}^L$ is the number of submodules V of L such that $V \cong N$ and $L/V \cong M$. Then $\mathcal{H}(A)$ is an $\mathbb{N}_0^n$-graded associative $\mathbb{R}$-algebra with identity $[0]$, where for $\mathbf{d} \in \mathbb{N}_0^n$, $\mathcal{H}(A)_{\mathbf{d}}$ is the $\mathbb{R}$-space with basis

$$\{ \ [M] \mid \ \mathbf{dim}\, M = \mathbf{d} \ \}.$$

In particular, $\mathcal{H}(A)_0 = \mathbb{R}$. Note that here we use the untwisted multiplication in $\mathcal{H}(A)$. However, all considerations hold for the twisted one introduced in [R5].

By definition ([R3]) the composition algebra $\mathcal{C}(A)$ of A is the subalgebra of $\mathcal{H}(A)$ generated by all isoclasses of simple A-modules $[S(1)], \cdots, [S(n)]$. For $\mathbf{d} = (d_1, \cdots, d_n) \in \mathbb{N}_0^n$ with $l = d_1 + \cdots + d_n$, let $\mathcal{C}(A)_{\mathbf{d}}$ be the $\mathbb{R}$-space spanned by all monomials $[S(i_1)] \cdots [S(i_l)]$, such that the number of occurrences of i in the sequence $i_1, \cdots, i_l$ is exactly d_i for $1 \le i \le n$. Then $\mathcal{C}(A) = \bigoplus_{\mathbf{d}} \mathcal{C}(A)_{\mathbf{d}}$ is an $\mathbb{N}_0^n$-graded, $\mathbb{R}$-subalgebra of $\mathcal{H}(A)$.

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Note that there is one-one correspondence between the types of a finite quivers Q and the symmetric generalized Cartan matrices Δ. By the work of Ringel and Green (see [R3] - [R7], [G]), Ringel's composition algebra $\mathcal{C}(kQ)$ is exactly the positive part of the Drinfeld-Jimbo quantized algebra of type Δ (see [L]).

Also, given a quiver, by the work of Sevenhant and Van den Bergh one can construct the corresponding Borcherds-Cartan matrix via Ringel-Hall algebra, the corresponding Borcherds form B is exactly the symmetrization of the Ringel's form as defined in 2.1 below. From this Borcherds-Cartan matrix B one has the corresponding generalized Kac-Moody algebra (see [B]), and its quantized enveloping algebra (see [Kang]). Then by [SV] the Ringel-Hall algebra $\mathcal{H}(kQ)$ is exactly the positive part of the corresponding quantized generalized Kac-Moody algebra. It is proved in [SV] that the degrees of the real simple roots of B are exactly the coordinate vectors; and if Q is an affine quiver, the degrees of the imaginary simple roots are of form $\lambda\mathbf{n}$, where λ are positive integers, and $\mathbf{n}$ is the minimal positive imaginary root of Q. The aim of this paper is to determine the number of the imaginary simple roots of degree $\lambda\mathbf{n}$ of B when Q is an affine quiver. By [SV] this amounts to determine the number of homogeneous minimal generators of Ringel-Hall algebra $\mathcal{H}(kQ)$ at degree $\lambda\mathbf{n}$. We do this from the structure of $\mathcal{H}(kQ)$, using the representation theory of kQ. As we are informed by Shi-Lin Yang that such a work was also done by Hua and Xiao in [HX], with a completely different method. We also write out explicitly a system of minimal generators of $\mathcal{H}(kQ)$.

2. Representations of affine quivers over finite fields

The aim of this section is to recall some basic points of the representation theory of affine quivers over finite fields which is needed in this paper, from [DR], [ARS] and [R1]. Mainly, we need to have the number of homogeneous quasi-simple modules with fixed dimension vector.

2.1. Throughout this section, let k be a finite field with q elements, A the path k-algebra of an affine quiver Q, i.e. Q is of type $\widetilde{A}_n$ $(n \geq 1)$, or $\widetilde{D}_n$ $(n \geq 4)$, or $\widetilde{E}_n$ $(n = 6, 7, 8)$, with arbitrary orientation, except in the case of type $\widetilde{A}_n$ $(n \geq 1)$, we exclude an oriented cycle. Denote by $\mathbf{n}$ the minimal positive imaginary root of A. If we want to emphasize the algebra A, then denote $\mathbf{n}$ by $\mathbf{n}_A$.

Given two A-modules X, Y, define

$$\langle X, Y\rangle \ = \ \dim_k \operatorname{Hom}_A(X, Y) - \dim_k \operatorname{Ext}^1_A(X, Y). \tag{1}$$

By a simple homological argument it is clear that $\langle X, Y\rangle$ depends only on $\mathbf{dim}_A X$ and $\mathbf{dim}_A Y$, not on X and Y themselves, so, it can be bilineary extended to $\mathbb{Z}^n$, where n is the number of pairwise non-isomorphic simple A-modules. Denote by $(-,-)$ the symmetric, bilinear form on $\mathbb{Z}^n$ given by

$$(X, Y) = \langle X, Y\rangle + \langle Y, X\rangle, \tag{2}$$

and by q_A the quadratic form on $\mathbb{Z}^n$ given by $q_A(x) = \langle x, x\rangle$. Then $q_A(x)$ is positive semi-definite but not positive definite, and $\{\ z \in \mathbb{Z}^n \mid q_A(z) = 0\} = \mathbb{Z}\mathbf{n}$, see [DR] and [R1].

Let $\tau = \mathrm{DTr}$ and $\tau^{-1} = \mathrm{TrD}$ be the Auslander-Reiten translates (see, e.g. [ARS]). Then $\tau = D\operatorname{Ext}^1_A(-, A)$ and $\tau^{-1} = \operatorname{Ext}^1_A(D(A), -)$, where $D = \operatorname{Hom}_k(-, k)$. An indecomposable A-module M is said to be preprojective (resp. preinjective) provided that there exists a positive integer m such that $\tau^m(M) = 0$ (resp. $\tau^{-m}(M) = 0$), and to be regular for otherwise. An arbitrary A-module X is said to be preprojective (resp. regular, preinjective) provided that every indecomposable direct summand of X is so.

If P, R and I are respectively preprojective, regular and preinjective modules, then there holds the following nice property, which is frequently used for calculations in $\mathcal{H}(A)$:

$$\mathrm{Hom}_A(R, P) = \mathrm{Hom}_A(I, P) = \mathrm{Hom}_A(I, R) = 0 \tag{3}$$

and

$$\mathrm{Ext}^1_A(P, R) = \mathrm{Ext}^1_A(P, I) = \mathrm{Ext}^1_A(R, I) = 0. \tag{4}$$

Define the defect $\partial(M)$ of a module M to be the integer $\langle \mathbf{n}, \mathbf{dim}\, M\rangle$. Then by [DR] an indecomposable module M is preprojective (resp. regular; preinjective) if and only if $\partial(M) < 0$ (resp. $\partial(M) = 0$; $\partial(M) > 0$).

With indecomposables as vertices, and using irreducible maps between indecomposables to attach arrows, we obtain the Auslander-Reiten quiver of A, see [ARS]. Then by the work of [DR], the Auslader-Reiten quiver of A has one preprojective component, which consists of all indecomposable preprojective modules; one preinjective component, which consists of all indecomposable preinjective modules; and all other components turn out to be "tubes", which are of the form $T = \mathbb{Z}\mathbb{A}_\infty/m$, where m is called the rank of T, see [ARS, p. 287; R1, p.113]. If $m = 1$, then T is called a homogeneous tube; and if otherwise, a non-homogeneous tube.

Note that indecomposable modules in different tubes have no non-zero homomorphisms and no non-trivial extensions.

By [DR], the ranks of non-homogeneous tubes of A is completely determined by the type of Q, except in the case of type $\widetilde{A}_n$ $(n \geq 1)$. Namely, type $\widetilde{D}_n$ $(n \geq 4)$ has three non-homogeneous tubes of ranks $n-2, 2, 2$ respectively; type $\widetilde{E}_n$ $(n = 6, 7, 8)$ has three non-homogeneous tubes of ranks $2, 3, n-3$ respectively. For type $\widetilde{A}_n$ $(n \geq 1)$, by iteratedly using reflections of quivers, we can assume that Q has n_1 arrows going clockwise and n_2 arrows going anticlockwise. Then the ranks of non-homogeneous tubes of A is completely determined by the pair $(n_1, n_2), n_1, n_2 \geq 1$. Namely, if $n_1 = n_2 = 1$, then it is the Kornceker algebra and has no non-homogeneous tubes; if $n_1 > n_2 = 1$, then A has a unique non-homogeneous tube of rank n_1; if $n_1 \geq n_2 > 1$, then A has two non-homogeneous tubes of ranks n_1 and n_2, respectively.

Note that all regular modules form an extension-closed abelian subcategory of $A - mod$, the simple objects in this subcategory will be called quasi-simple modules; any indecomposable regular module M is regular uniserial, and hence M is uniquely determined by its quasi-socle and quasi-length, and also by its quasi-top and quasi-length.

An indecomposable module M is called a stone provided $\mathrm{Ext}^1_A(M, M) = 0$. Any indecomposable non-regular module is a stone; there are no stones in a homogeneous tube; and an indecomposable M in a non-homogeneous tube of rank m is a stone if and only if the quasi-length of M is less than m. Note that the endomophism algebra of a stone is always the base field k; and that the existence of a stone with a fixed dimension vector does not depend on the base field, see [HHKU].

Any indecomposable M in a tube T of rank m has the property $\tau^m(M) = M$; and $\mathbf{dim}\, M$ is a multiple of $\mathbf{n}$ if and only if either $m = 1$, or $m \geq 2$ and the quasi-length of M is a multiple of m; if E is a quasi-simple in T, then $E_i = \tau^i(E), 1 \leq i \leq m$, are the all quasi-simples in T; the dimension vectors of the all quasi-simples in non-homogeneous tubes have been listed in [DR] Tables; in particular, the sum of dimension vectors of the all quasi-simples in a non-homogeneous tube is $\mathbf{n}$.

Thus, we are interested in the number of homogeneous quasi-simples with fixed dimension vector $\lambda\mathbf{n}$, where λ is an arbitrary positive integer.

2.2. We need the perpendicular category introduced by Geigle - Lenzing in [GL], and Schofield in [S].

Let X be a stone, i.e. an indecomposable A-module M with $\mathrm{Ext}^1_A(M,M)=0$. Recall the perpendicular category $X^\perp$ is the full subcategory of $A-mod$ given by

$$X^\perp = \{\ M \in A-mod \mid \ \mathrm{Hom}_A(X,M)=0=\mathrm{Ext}^1_A(X,M)\ \}. \tag{5}$$

Then $X^\perp$ is equivalent to $B-mod$, with B again a path algebra of $n-1$ simple modules, where n is the number of simple A-modules. Note that the embedding functor $B-mod \longrightarrow A-mod$ is exact and induces the isomorphisms on both Hom and Ext.

Lemma. *Let X be an A-stone, and B the path algebra with $B-mod$ equivalent to $X^\perp$. Let $S_1,\cdots,S_m$ be the all pairwise non-isomorphic simple B-modules.*

(i) If $M\in X^\perp$ with $\mathbf{dim}_B\, M=(d_1,\cdots,d_m)$. Then

$$\mathbf{dim}_A\, M = d_1\,\mathbf{dim}_A\, S_1+\cdots+d_m\,\mathbf{dim}_A\, S_m. \tag{6}$$

In particular, if $M,N\in X^\perp$ with $\mathbf{dim}_B\, M=\mathbf{dim}_B\, N$, then $\mathbf{dim}_A\, M=\mathbf{dim}_A\, N$.

(ii) If both A and B are tame, then

$$\mathbf{n}_A = n_1\,\mathbf{dim}_A\, S_1+\cdots+n_m\,\mathbf{dim}_A\, S_m, \tag{7}$$

where $\mathbf{n}_B=(n_1,\cdots,n_m)$.

(iii) If both A and B are tame, and $M\in X^\perp$, then $\mathbf{dim}_B\, M=\lambda\mathbf{n}_B$ if and only if $\mathbf{dim}_A\, M=\lambda\mathbf{n}_A$.

Proof. (i) This follows from the definition of dimension vectors.

(ii) Since both A and B are tame, it follows that X is regular. Let $X\in T$, where T is a non-homogeneous tube of A. Choose an indecomposable regular B-module N with $\mathbf{dim}_B\, N=\mathbf{n}_B$. Then N is also indecomposable regular as an A-module, and

$$\begin{aligned} q_A(\mathbf{dim}_A\, N) &= \dim_k \mathrm{Hom}_A(N,N)-\dim_k \mathrm{Ext}^1_A(N,N)\\ &= \dim_k \mathrm{Hom}_B(N,N)-\dim_k \mathrm{Ext}^1_B(N,N)\\ &= q_B(\mathbf{dim}_B\, N)=q_B(\mathbf{n}_B)=0, \end{aligned}$$

it follows that $\mathbf{dim}_A\, N=\lambda\mathbf{n}_A$ for some positive integer λ.

Now choose an indecomposable regular A-module M with $\mathbf{dim}_A\, M=\mathbf{n}_A$ and $M\in X^\perp$. It is easy to see that such an M exists. Again we have $\mathbf{dim}_B\, M=t\mathbf{n}_B$ for some positive integer t. Then by (i) we have

$$\lambda\mathbf{n}_A=\mathbf{dim}_A\, N = n_1\,\mathbf{dim}_A\, S_1+\cdots+n_m\,\mathbf{dim}_A\, S_m,$$

and

$$\mathbf{n}_A=\mathbf{dim}_A\, M = t(n_1\,\mathbf{dim}_A\, S_1+\cdots+n_m\,\mathbf{dim}_A\, S_m)=t\lambda\mathbf{n}_A,$$

therefore $t = \lambda = 1$.

(iii) If $\mathbf{dim}_B M = \lambda \mathbf{n}_B$, then $\mathbf{dim}_A M = \lambda \mathbf{n}_A$ by (i) and (ii). Conversely, if $\mathbf{dim}_A M = \lambda \mathbf{n}_A$, then $\mathbf{dim}_B M = t\mathbf{n}_B$ for some positive integer t. Again by (i) and (ii) we get $\lambda = t$. ■

2.3. Denote by $t_\lambda(A)$ the number of homogeneous quasi-simple $A-$modules X with $\mathbf{dim}_A X = \lambda \mathbf{n}_A$. This number is of course well known, but it seems that a proof is not easy available in literature.

Let T a non-homogeneous tube of A, and E be a quasi-simple stone in T with $E^\perp$ equivalent to B-mod.

Lemma. *(i) If* $\operatorname{rank}(T) > 2$, *then* $t_\lambda(A) = t_\lambda(B)$ *for any positive integer* λ.

(ii) If $\operatorname{rank}(T) = 2$, *then*

$$t_\lambda(A) = \begin{cases} t_\lambda(B), & \lambda \neq 1; \\ \\ t_1(B) - 1, & \lambda = 1. \end{cases}$$

Proof. Note that any homogeneous quasi-simple A-module X with $\mathbf{dim}_A X = \lambda \mathbf{n}_A$ is a homogeneous quasi-simple B-module, with $\mathbf{dim}_B X = \lambda \mathbf{n}_B$ by Lemma 2.2(iii). Other possible homogeneous quasi-simple B-modules have to be in T as A-modules; however, if $\operatorname{rank}(T) = m > 2$, then those indecomposable modules in T which belong to $E^\perp$ form a non-homogeneous tube T' of B with $\operatorname{rank}(T') = m - 1$. This proves (i).

Let X be a homogeneous quasi-simple B-module with $\mathbf{dim}_B X = \lambda \mathbf{n}_B$. Then $\mathbf{dim}_A X = \lambda \mathbf{n}_A$ by Lemma 2.2(iii). Such an X is not a homogeneous quasi-simple A-module if and only if $\operatorname{rank}(T) = 2$, $\lambda = 1$, and as an indecomposable A-module X is in T with quasi-top E and quasi-length 2. In particular, such an X is unique. This completes (ii). ■

Denote by K the Kronecker k-algebra, i.e. K is the path k-algebra of the quiver

$$1\cdot \overset{\longrightarrow}{\longrightarrow} \cdot 2.$$

Corollary 2.4. *We have* $t_\lambda(A) = t_\lambda(K)$ *for* $\lambda \neq 1$; *and*

$$t_1(A) = \begin{cases} q, & A \text{ of type } \widetilde{A}_{(n_1,1)}, \quad n_1 > 1; \\ \\ q-1, & A \text{ of type } \widetilde{A}_{(n_1,n_2)}, \quad n_1, n_2 > 1; \\ \\ q-2, & A \text{ of type } \widetilde{D}_n, \text{ or } \widetilde{E}_n. \end{cases}$$

Proof. The assertion follows from Lemma 2.3, by iteratedly using perpendicular reductions. It is clear that $t_1(K) = q + 1$. Note that from type $\widetilde{A}_{(n_1,1)}$ $(n_1 > 1)$ to K, we need several times of perpendicular reductions using quasi-simple modules in the tube of rank > 2, together with one time perpendicular reduction using quasi-simple modules in the tube of rank 2; from type $\widetilde{A}_{(n_1,n_2)}$ $(n_1 \geq n_2 > 1)$ to K, we need several times of perpendicular reductions using quasi-simple modules in tubes of rank > 2, together with two times perpendicular reductions using quasi-simple modules in tubes of rank 2; and that from types $\widetilde{D}_n$ and $\widetilde{E}_n$ to K, we need several times of perpendicular reductions using quasi-simple modules in tubes of rank > 2, together with three times perpendicular reductions using quasi-simple modules in tubes of rank 2. ■

2.5. In this subsection we will determine $t_\lambda(K)$, the number of homogeneous quasi-simple K-modules with dimension vector (λ, λ), where K is the Kronecker algebra over k. It is clear that $t_1(K) = q + 1$, so, we assume $\lambda \geq 2$ in the following.

Denote by $N(q, \lambda)$ the number of monic irreducible polynomials of degree λ over the field of q elements. Then we have the well known formula due to Gauss:

$$N(q, \lambda) = \frac{1}{\lambda} \sum_{d|\lambda} \mu(\frac{\lambda}{d}) q^d, \tag{8}$$

where μ is the Möbius function, i.e. $\mu(1) = 1, \mu(n) = 0$ if n has a square factor, and $\mu(n) = (-1)^s$ if $n = p_1 \cdots p_s$, p_i distinct primes.

Lemma. *If $\lambda \geq 2$, then $t_\lambda(K) = N(q, \lambda)$.*

Proof. Recall that $K - mod$ can be identified with the category $\mathcal{K}$, whose object is a quartet $M = (k^m,\ \alpha,\ \beta,\ k^n)$, where $\alpha,\ \beta:\ k^m \longrightarrow k^n$ are k-linear maps. For two objects $M_1 = (k^{m_1},\ \alpha_1,\ \beta_1,\ k^{n_1})$ and $M_2 = (k^{m_2},\ \alpha_2,\ \beta_2,\ k^{n_2})$, the morphism set $\mathrm{Hom}_{\mathcal{K}}(M_1, M_2)$ is defined to be the set of the pairs $f = (f_1, f_2)$, where $f_1:\ k^{m_1} \longrightarrow k^{m_2}$ and $f_2:\ k^{n_1} \longrightarrow k^{n_2}$ are k-linear maps, such that (we write the composition of morphisms from left to right)

$$f_1\alpha_2 = \alpha_1 f_2, \qquad f_1\beta_2 = \beta_1 f_2.$$

Such a morphism $f = (f_1, f_2)$ is an isomorphism of K-modules if and only if both f_1 and f_2 are invertible.

Note that $M_1 \oplus M_2 = (k^{m_1} \oplus k^{m_2},\ \alpha_1 \oplus \alpha_2,\ \beta_1 \oplus \beta_2,\ k^{n_1} \oplus k^{n_2})$.

Now, let M be an indecomposable K-module of dimension vector (λ, λ). Then M is of the form $M = (k^\lambda,\ \alpha,\ \beta,\ k^\lambda)$. Let $\mathrm{rank}(\alpha) = r$. Then we have invertible $\lambda \times \lambda$ matrices g_1 and g_2 such that $g_1\alpha g_2 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where I_r is the $r \times r$ identity matrix. It easy to see that M is isomorphic to the K-module $(k^\lambda,\ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},\ g_1\beta g_2,\ k^\lambda)$, via the isomorphism (g^{-1}, g_2). It follows that we can assume that M is of the form $M = (k^\lambda,\ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},\ \beta,\ k^\lambda)$.

Case (i): $r = \lambda$. In this case, let g be the rational canonical form of β, and h be a $\lambda \times \lambda$ invertible matrix such that $h\beta h^{-1} = g$. Then it is easy to see that M is isomorphic to the K-module

$$(k^\lambda,\ I_\lambda,\ g,\ k^\lambda)$$

via the isomorphism (h^{-1}, h^{-1}). Since M is indecomposable, it follows that g has to be an indecomposable matrix, and that g has to be invertible. Thus, g is the companion matrix of a polynomial $(\phi(x))^d$ of degree λ, where $\phi(x)$ is a monic irreducible polynomial in $k[x]$, and d is a positive factor of λ, and if $\deg(\phi(x)) = 1$, then $\phi(x) \neq x$ (see, e.g. [FIS], Theorems 6.11 and 6.13). Therefore, we can identify M with the K-module given by the pair

$$(k[x]/((\phi(x))^d),\ id,\ m,\ k[x]/((\phi(x))^d)) \tag{9}$$

where $m: \ k[x]/(\phi(x)^d) \longrightarrow k[x]/(\phi(x)^d)$ is the k-map given by $m(a) = a\bar{x}$, where $\bar{x}$ is the coset $x + (\phi(x)^d)$. Since g is the companion matrix of a polynomial $(\phi(x))^d$ of degree λ, it is easy to see m coincides with g.

Using this presentation of M we can easily see that $\mathrm{End}_K\, M \cong k[x]/((\phi(x))^d)$.

In fact, and endmorphism in $\mathrm{End}_K\, M$ is of the form (f, f), where f is completely determined by $f(1)$, i.e. we have

$$f(\bar{x^i}) = f(1)\bar{x^i}, \quad i = 1, \cdots, \lambda - 1.$$

It follows that $\mathrm{End}_K\, M$ is a field if and only if $d = 1$. In this case, $\mathrm{End}_K\, M \cong k[x]/(\phi(x))$ and hence $[\mathrm{End}_K\, M : k] = \deg(\phi(x)) = \lambda$, where $\dim_K M = \lambda(1, 1)$.

Notice that if $\phi_1(x)$ and $\phi_2(x)$ are different monic irreducible polynomials of degree λ in $k[x]$, then the corresponding modules $M_1 = (k[x]/(\phi_1(x)),\ id,\ m,\ k[x]/(\phi_1(x)))$ and $M_2 = (k[x]/(\phi_2(x)),\ id,\ m,\ k[x]/(\phi_2(x)))$ are not isomorphic. In this way we have already obtained $N(q, \lambda)$ homogeneous quasi-simple K-modules of dimension vector (λ, λ). In the next case, we shall see that this is the complete list of homogeneous quasi-simples of dimension vector (λ, λ).

Case (ii): $r < \lambda$. Since $\lambda \geq 2$ and M is indecomposable, it follows that $r \geq 1$. Let M be given by the pair

$$M = (k^\lambda,\ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},\ g,\ k^\lambda)$$

where $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$ is a $\lambda \times \lambda$ matrix and g_1 is a $r \times r$ matrix.

We want to prove that $\mathrm{End}_K\, M$ is not a field, and hence M is not a homogeneous quasi-simple K-module. For this purpose, consider the pair $f = (f_1, f_2)$ of $\lambda \times \lambda$ matrices, where

$$f_1 = \begin{pmatrix} 0 & g_2 \\ 0 & g_4 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ g_3 & g_4 \end{pmatrix}.$$

Then it is easy to see $f = (f_1, f_2) \in \mathrm{End}_K\, M$, namely, there hold the following equalities:

$$\begin{pmatrix} 0 & g_2 \\ 0 & g_4 \end{pmatrix} \cdot \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = 0 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ g_3 & g_4 \end{pmatrix};$$

$$\begin{pmatrix} 0 & g_2 \\ 0 & g_4 \end{pmatrix} \cdot \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} = \begin{pmatrix} g_2g_3 & g_2g_4 \\ g_4g_3 & g_4g_4 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ g_3 & g_4 \end{pmatrix}.$$

It is clear that f is not an automorphism of $K-$module M; and since M is indecomposable, it follows that g_2 and g_3 cannot be zero simultaneously, and hence $f \neq 0$.

This completes the proof. ■

From the proof of Lemma 2.5 Case (i), and using Lemma 2.2(iii) we see the following

Corollary 2.6. *Let E be a homogeneous quasi-simple A-module of dimension vector $\lambda \mathbf{n}_A$. Then $\mathrm{End}_A\, E$ is the field with $[\mathrm{End}_K\, E : k] = \lambda$.*

By Corollary 2.4 and Lemma 2.5, we get the following

Theorem 2.7. *Let $t_\lambda(A)$ be the number of homogeneous quasi-simple A-modules of dimension vector $\lambda\mathbf{n}$. Then*

$$t_\lambda(A) = N(q,\lambda), \quad \lambda > 1;$$

and

$$t_1(A) = \begin{cases} q+1, & A \text{ of type } \widetilde{A}_{(1,1)}; \\ q, & A \text{ of type } \widetilde{A}_{(n_1,1)}, \quad n_1 > 1; \\ q-1, & A \text{ of type } \widetilde{A}_{(n_1,n_2)}, \quad n_1, n_2 > 1; \\ q-2, & A \text{ of type } \widetilde{D}_n, \text{ or } \widetilde{E}_n. \end{cases}$$

Since $N(q,\lambda) \geq 1$ for all q and λ, and since $N(q,\lambda) = 1$ if and only if $q = \lambda = 2$, we see the following

Corollary 2.8. *(i) $t_\lambda(A) = 0$ if and only if $\lambda = 1, q = 2$, and A is of type $\widetilde{D}_n$ or $\widetilde{E}_n$.*

(ii) $t_\lambda(A) = 1$ if and only if one of the following cases occurs

$$(a) \qquad \lambda = 1,\ q = 3,\ A \text{ of type } \widetilde{D}_n \text{ or } \widetilde{E}_n;$$

$$(b) \qquad \lambda = 1,\ q = 2,\ A \text{ of type } \widetilde{A}_{(n_1,n_2)},\ n_1, n_2 \geq 2;$$

$$(c) \qquad \lambda = q = 2.$$

(iii) In the remaining cases we have $t_\lambda(A) \geq 2$.

Corollary 2.9. *Let n be the number of vertices in an affine quiver Q, and $A = kQ$. For $\mathbf{d} \in \mathbb{N}_0^n$, denote by $i_{\mathbf{d}}(Q,q)$ the number of indecomposable A-modules with dimension vector $\mathbf{d}$. Then we have*

$$i_{\mathbf{d}}(Q,q) = \begin{cases} \sum_{s|\lambda} N(q,s) + n - 1, & \mathbf{d} = \lambda\mathbf{n}; \\ 1, & q_A(\mathbf{d}) = 1; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. If $\mathbf{d}$ is the dimension vector of an indecomposable A-module, then we have either $\mathbf{d} = \lambda\mathbf{n}$ for some positive integer λ, or $q_A(\mathbf{d}) = 1$; and if $q_A(\mathbf{d}) = 1$, then there is a unique indecomposable with dimension vector $\mathbf{d}$, see [K] Theorem 1, p.79. If $\mathbf{d} = \lambda\mathbf{n}$, then the formula follows from Theorem 2.7 together with by counting the number of indecomposables with dimension vector $\lambda\mathbf{n}$ in non-homogeneous tubes (of course, this need to use [DR] Tables). ■

Remark 2.10. (i) Notice that the number $n-1$ in Corollary is exactly the rank of the (symmetric) generalized Cartan matrix of Q.

(ii) Using Euler φ-function, one can rewrite the number $\sum\limits_{s|\lambda} N(q,s)$ as

$$\sum_{s|\lambda} N(q,s) = \frac{1}{\lambda} \sum_{s|\lambda} \varphi(\frac{\lambda}{s})\ q^s \tag{10}$$

where $\varphi(\lambda)$ is the number of positive integers $< \lambda$ which are relatively prime to λ.

In fact, it is well known that (see e.g. [J])

$$\varphi(\lambda) = \sum_{s|\lambda} s\ \mu(\frac{\lambda}{s}). \tag{11}$$

It follows that

$$\begin{aligned}\sum_{s|\lambda} N(q,s) &= \sum_{s|\lambda} \frac{1}{s} \sum_{d|s} \mu(\frac{s}{d})\ q^d \\ &= \frac{1}{\lambda} \sum_{s|\lambda} \frac{\lambda}{s} \sum_{d|s} \mu(\frac{s}{d})\ q^d \\ &= \frac{1}{\lambda} \sum_{d|\lambda} \Big(\sum_{s,\, \frac{s}{d} | \frac{\lambda}{d}} \frac{\lambda}{s}\ \mu(\frac{s}{d}) \Big)\ q^d \\ &\overset{\text{by}(11)}{=} \frac{1}{\lambda} \sum_{d|\lambda} \varphi(\frac{\lambda}{d})\ q^d.\end{aligned}$$

3. The space $B_{\mathbf{d}}(A)$

3.1. Let A be a finite-dimensional algebra over a finite field, with $K_0(A) = \mathbb{Z}^n$, $\mathcal{H}(A)$ and $\mathcal{C}(A)$ be the Ringel-Hall algebra and Ringel's composition algebra of A, respectively. For $\mathbf{0} \neq \mathbf{d} \in \mathbb{N}_0^n$, define

$$B_{\mathbf{d}}(A) := \sum_{x+y=\mathbf{d};x,y\neq \mathbf{0}} \mathcal{H}(A)_x \mathcal{H}(A)_y \subseteq \mathcal{H}(A)_{\mathbf{d}} \tag{1}$$

and $B_{\mathbf{0}} := \mathbb{R}[0]$. Denote by

$$B(A) = \bigoplus_{\mathbf{d}} B_{\mathbf{d}}(A).$$

Then $B(A)$ is an $\mathbb{N}_0^n$-graded, proper subalgebra of $\mathcal{H}(A)$.

The spaces $B_{\mathbf{d}}(A)$ are closely related to minimal homogeneous generators of $\mathcal{H}(A)$, hence we will make a detailed investigation on them.

Let $\mathbf{e}_i, i = 1, \cdots, n$, be the coordinate vectors. Then by definition we have $B_{\mathbf{e}_i} = 0, i = 1, \cdots, n$. Also, it is clear by definition that for $\mathbf{d} \neq \mathbf{e}_i, i = 1, \cdots, n$, there holds

$$\mathcal{C}(A)_{\mathbf{d}} \subseteqq B_{\mathbf{d}}(A) \subseteqq \mathcal{H}(A)_{\mathbf{d}}, \tag{2}$$

and

Lemma. $B_{\mathbf{d}}(A) = \mathcal{H}(A)_{\mathbf{d}}$ *for all* $\mathbf{d} \neq \mathbf{e}_i, i = 1, \cdots, n$, *if and only if* $\mathcal{C}(A) = \mathcal{H}(A)$.

In the rest of this section, let A be the path k-algebra of an affine quiver Q, where k is a finite field of q elements. Denote by $\mathbf{n}$ the minimal positive imaginary root of A, and n the number of vertices of Q.

3.2. For $\mathbf{d} \in K_0(A) = \mathbb{Z}^n$, define the following element in $\mathcal{H}(A)$:

$$r_{\mathbf{d}} := \sum_{[M]} [M], \quad \text{where } M \text{ runs over all regular modules with } \mathbf{dim}\, M = \mathbf{d}.$$

Note that this is a finite sum since k is a finite field. If there are no regular modules M with $\mathbf{dim}\, M = \mathbf{d}$, then set $r_{\mathbf{d}} := 0$. Set $r_{\mathbf{0}} := [0]$.

In [Z1] Theorem 1, we have proved that $r_{\mathbf{d}} \in \mathcal{C}(A)$ for all $\mathbf{d}$. Let $\mathcal{T}$ denote the subalgebra of $\mathcal{H}(A)$ generated by all elements $r_{\mathbf{d}}$ with $\mathbf{d} \in \mathbb{N}_0^n$. Then $\mathcal{T}$ is the $\mathbb{R}$-space spanned by all products $r_{\mathbf{d}_1} \cdots r_{\mathbf{d}_m}$, where $\mathbf{d}_1, \cdots, \mathbf{d}_m \in \mathbb{N}_0^n$, and $m \in \mathbb{N}_1$.

Let $\mathcal{P}$ and $\mathcal{I}$ denote the subalgebra of $\mathcal{H}(A)$ generated by indecomposable preprojectives and by indecomposable preinjectives, respectively. Then $\mathcal{P}$ (resp. $\mathcal{I}$) has a basis $[P]$ (resp. $[I]$), where P (resp. I) runs over all preprojective (resp. preinjective) modules. Let $\mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$ be the $\mathbb{R}$-subspace of $\mathcal{H}(A)$ spanned by all products $[P] \cdot r_{\mathbf{d}_1} \cdots r_{\mathbf{d}_m} \cdot [I]$, where P (resp. I) runs over the preprojectives (resp. preinjectives), $\mathbf{d}_1, \cdots, \mathbf{d}_m \in \mathbb{N}_0^n$, and $m \in \mathbb{N}_1$.

We need the following structure theorem proved in [Z3].

Theorem. *We have* $\mathcal{C}(A) = \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I} = \mathcal{P} \otimes_{\mathbb{R}} \mathcal{T} \otimes_{\mathbb{R}} \mathcal{I}$.

Remark. The proof of this theorem in [Z3] used the existence of a homogeneous quasi-simple with dimension vector $\mathbf{n}$, i.e. $t_1(A) \neq 0$. Anyway, if $t_1(A) = 0$, then we can replace a homogeneous quasi-simple with dimension vector $\mathbf{n}$ by a homogeneous quasi-simple with dimension vector $2\mathbf{n}$, since $t_2(A) \neq 0$ by Corollary 2.8, and then the proof in [Z3] still holds.

3.3. Terminologies: Let T be a tube of A, and M an A-module. By $M \in T$ we mean that every indecomposable direct summand of M belongs to T.

Let $x = \sum_{[M]} c_M [M] \in \mathcal{H}(A)$.

If $c_M \neq 0$, then $[M]$ is said to be a term of x with coefficient c_M.

If $[M]$ is a term of x, and M is indecomposable, then $[M]$ is said to be an indecomposable term of x.

If $[M]$ is a term of x, and $M \in T$, then $[M]$ is said to be a $T-$term of x.

Define the regular part of x to be

$$r(x) := \sum_{M \text{ is regular}} c_M [M]. \tag{3}$$

Define the T-part of x to be

$$r_T(x) := \sum_{M \in T} c_M [M]. \tag{4}$$

For $\mathbf{d} \in \mathbb{N}_0^n$, denote by $r_{\mathbf{d}}(T)$ the T-part of $r_{\mathbf{d}}$, i.e.

$$r_{\mathbf{d}}(T) := \sum_{[M]} [M], \quad \text{where } M \text{ runs over all modules in } T \text{ with } \mathbf{dim}\, M = \mathbf{d}. \tag{5}$$

Lemma 3.4. *Let $x \in \mathcal{H}(A)_{\mathbf{d}}$, where $\mathbf{d} \neq \mathbf{e}_i,\ i = 1, \cdots, n$. Then there holds*

$$x - r(x) \in B_{\mathbf{d}}(A).$$

In particular, $x \in B_{\mathbf{d}}(A)$ if and only if $r(x) \in B_{\mathbf{d}}(A)$.

Proof. Note that any term of $x - r(x)$ is of the form $[P \oplus R \oplus I]$, such that at least one of P and I is not zero, where P, R, I are respectively preprojective, regular, and preinjective. Since $[P \oplus R \oplus I] = [P] \cdot [R] \cdot [I]$, it follows that if any two of P, R and I are not zero module, then $[P \oplus R \oplus I] \in B_{\mathbf{d}}(A)$. If $R = 0 = I$, then $[P] \in \mathcal{C}(A)_{\mathbf{d}} \subsetneqq B_{\mathbf{d}}(A)$ since $\mathbf{d} \neq \mathbf{e}_i, i = 1, \cdots, n$. Similarly for the case $P = 0 = R$. Thus, $x - r(x) \in B_{\mathbf{d}}(A)$. ∎

Lemma 3.5. *Let $x \in \mathcal{H}(A)_{\mathbf{d}}$, where $\mathbf{d} \neq \mathbf{e}_i,\ i = 1, \cdots, n$. Then there holds*

(i)

$$r(x) - \sum_{\text{tube } T} r_{T}(x) \in B_{\mathbf{d}}(A).$$

In particular, $x \in B_{\mathbf{d}}(A)$ if and only if $\sum_{\text{tube } T} r_{T}(x) \in B_{\mathbf{d}}(A)$; and if $r_{T}(x) \in B_{\mathbf{d}}(A)$ for all tube T, then $x \in B_{\mathbf{d}}(A)$.

(ii)

$$\sum_{\text{tube } T} r_{\mathbf{d}}(T) \in B_{\mathbf{d}}(A).$$

Proof. Note that any term of $r(x) - \sum_{\text{tube } T} r_{T}(x)$ is of the form $[R_1 \oplus R_2]$, where both R_1 and R_2 are non-zero regular modules, such that $R_1 \in T$ for some tube T, but R_2 has no direct summands in T. Therefore $[R_1 \oplus R_2] = [R_1] \cdot [R_2]$, and hence (i) follows.

If $\mathbf{d} \neq \mathbf{e}_i, i = 1, \cdots, n$, then $r_{\mathbf{d}} \subsetneqq \mathcal{C}(A)_{\mathbf{d}} \subsetneqq B_{\mathbf{d}}(A)$, and hence (ii) follows from (i). ∎

3.6. Let X, Y be non-zero A-modules. Consider the regular part $r([X] \cdot [Y])$. Let $X = P \oplus R \oplus I,\ Y = P' \oplus R' \oplus I'$, with P, P' preprojective; R, R' regular; and I, I' preinjective. Thus

$$[X] \cdot [Y] = [P] \cdot [R] \cdot [I] \cdot [P'] \cdot [R'] \cdot [I'],$$

it follows that if $P \neq 0$, or $I' \neq 0$, then $r([X] \cdot [Y]) = 0$. Now assume $P = 0 = I'$. Then by comparing defects we see

$$r([X] \cdot [Y]) = r([R] \cdot [I] \cdot [P'] \cdot [R']) = [R] \cdot r([I] \cdot [P']) \cdot [R'].$$

While by Theorem in 3.2 we have $[I] \cdot [P'] \in \mathcal{C}(A) = \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$ and $r([I] \cdot [P']) \in \mathcal{T}$, it follows that if $R = 0 = R'$, then $r([X] \cdot [Y]) = r([I] \cdot [P'])$ is of the form

$$r([X] \cdot [Y]) = c r_{\mathbf{dim}\, X + \mathbf{dim}\, Y} + \sum_{t \geq 2} c_{\mathbf{d}_1, \cdots, \mathbf{d}_t} r_{\mathbf{d}_1} \cdots r_{\mathbf{d}_t}$$

with $\mathbf{d}_1 + \cdots + \mathbf{d}_t = \mathbf{dim}\, X + \mathbf{dim}\, Y$, and $c, c_{\mathbf{d}_1, \cdots, \mathbf{d}_t} \in \mathbb{R}$; and that if $R \neq 0$, or $R' \neq 0$, then $r([X] \cdot [Y])$ is of the form $\sum c_{M,N} [M] \cdot [N]$, where M, N are non-zero regular modules and $c_{M,N} \in \mathbb{R}$.

This proves the following

Lemma. *Let X, Y be non-zero A-modules. Then the regular part $r([X]\cdot[Y])$ is of the form*

$$r([X]\cdot[Y]) = cr_{\mathbf{dim}\,X+\mathbf{dim}\,Y} + \sum c_{M,N}[M]\cdot[N] \tag{6}$$

where M, N are non-zero regular modules with $\mathbf{dim}\,M+\mathbf{dim}\,N = \mathbf{dim}\,X+\mathbf{dim}\,Y$, and $c, c_{M,N} \in \mathbb{R}$.

Since $B_{\mathbf{e}_i} = 0$ for $i = 1, \cdots, n$, the following result gives a description of the all spaces $B_{\mathbf{d}}(A)$, $\mathbf{d} \in \mathbb{N}_0^n$. Note that the expressions (7) and (8) will be very useful in §5 and §6.

Theorem 3.7. *Let $x \in \mathcal{H}(A)_{\mathbf{d}}$, where $\mathbf{d} \neq \mathbf{e}_i$, $i = 1, \cdots, n$. Then the following are equivalent*

(i) $x \in B_{\mathbf{d}}(A)$;

(ii) $r(x) \in B_{\mathbf{d}}(A)$;

(iii) $r(x)$ *is of the following form*

$$r(x) = cr_{\mathbf{d}} + \sum c_{M,N}[M]\cdot[N] \tag{7}$$

where M, N are non-zero regular modules with $\mathbf{dim}\,M + \mathbf{dim}\,N = \mathbf{d}$, and $c, c_{M,N} \in \mathbb{R}$.

(iv) There exists a $c \in \mathbb{R}$ such that for every tube T of A, $r_T(x)$ is of the following form

$$r_T(x) = cr_{\mathbf{d}}(T) + \sum c_{M,N}[M]\cdot[N] \tag{8}$$

where M, N are non-zero modules in T with $\mathbf{dim}\,M + \mathbf{dim}\,N = \mathbf{d}$, and $c_{M,N} \in \mathbb{R}$.

In particular, if $r(x) = 0$, then $x \in B_{\mathbf{d}}(A)$.

Proof. The implication $(i) \Longrightarrow (ii)$ follows from Lemma 3.4.

$(ii) \Longrightarrow (iii)$: if $r(x) \in B_{\mathbf{d}}(A)$, then we can write

$$r(x) = \sum c_{X,Y}[X]\cdot[Y],$$

where X, Y are non-zero modules with $\mathbf{dim}\,X + \mathbf{dim}\,Y = \mathbf{d}$, and $c_{X,Y} \in \mathbb{R}$. Now, taking the regular parts from the both sides of the preceeding equality, and then using Lemma in 3.6, we see that $r(x)$ is of the following form

$$r(x) = r(r(x)) = \sum c_{X,Y}r([X]\cdot[Y]) = cr_{\mathbf{d}} + \sum c_{M,N}[M]\cdot[N],$$

where M, N are non-zero regular modules with $\mathbf{dim}\,M + \mathbf{dim}\,N = \mathbf{d}$, and $c, c_{M,N} \in \mathbb{R}$.

$(iii) \Longrightarrow (iv)$: this follows from by taking the T-parts from the both sides of (7).

$(iv) \Longrightarrow (i)$: if there exists a $c \in \mathbb{R}$ such that for every tube of A, $r_T(x)$ is of the form (8), then

$$\sum_{\text{tube } T} r_T(x) = c \sum_{\text{tube } T} r_{\mathbf{d}}(T) + y$$

with $y \in B_{\mathbf{d}}(A)$. While $\sum_{\text{tube } T} r_{\mathbf{d}}(T) \in B_{\mathbf{d}}(A)$ by Lemma 3.5 (ii), hence $r(x) \in B_{\mathbf{d}}(A)$ by Lemma 3.5(i), therefore, $x \in B_{\mathbf{d}}(A)$ by Lemma 3.4. ■

3.8. Remark. (i) As pointed out by Sevenhant and Van den Bergh in [SV], the imaginary simple roots of $\mathcal{H}(A)$ is of the form $\lambda\mathbf{n}$, where λ is a positive integer, that is to say, if $\mathbf{d} \neq \mathbf{e}_i, i = 1, \cdots, n$, and if $B_{\mathbf{d}}(A) \neq \mathcal{H}(A)_{\mathbf{d}}$, then $\mathbf{d} = \lambda\mathbf{n}$. Therefore, Theorem 3.7 is only used for $=\lambda\mathbf{n}$.

(ii) Let $x \in \mathcal{H}(A)_{\lambda\mathbf{n}}$. As pointed out in Lemma 3.5, if $r_{_T}(x) \in B_{\lambda\mathbf{n}}(A)$ for any tube T, then $x \in B_{\lambda\mathbf{n}}(A)$. But, the converse is not true.

For example, let K be the Kronecker algebra, and $N_1, \cdots, N_{q+1}$ be the all indecomposable modules of dimension vector $(1,1)$. Then $r_{_{(1,1)}} = \sum\limits_{1 \leq i \leq q+1} [N_i] \in \mathcal{C}(A)_{(1,1)} \subsetneqq B_{(1,1)}(A)$, but every $[N_i] \notin B_{(1,1)}(A)$.

4. Ringel-Hall algebras of tubes

Throughout this section, A is an arbitrary tame hereditary algebra over a finite field k, with minimal positive imaginary root $\mathbf{n}$.

4.1. Let T be an arbitrary tube of A. Denote by $\mathcal{H}(T)$ the subspace of $\mathcal{H}(A)$ with basis $\{ [M] \mid M \in T \}$. Then $\mathcal{H}(T)$ is also an $\mathbb{N}_0^n$-graded algebra with homogeneous component $\mathcal{H}(T)_{\mathbf{d}}$ being the space with basis $\{ [M] \mid M \in T, \mathbf{dim}\, M = \mathbf{d} \}$. Denote by $B_{\mathbf{d}}(T)$ the the following subspace of $B_{\mathbf{d}}(A) \cap \mathcal{H}(T)_{\mathbf{d}}$:

$$B_{\mathbf{d}}(T) = \sum_{x+y=\mathbf{d}; x,y\neq 0} \mathcal{H}(T)_x \cdot \mathcal{H}(T)_y. \tag{1}$$

Note that by Remark 3.8(ii), $x \in B_{\lambda\mathbf{n}}(A)$ does not imply $r_T(x) \in B_{\lambda\mathbf{n}}(T)$. The motivation of introducing the spaces $B_{\mathbf{d}}(T)$ is the following. Let $x \in \mathcal{H}(T)_{\lambda\mathbf{n}}$. We want to reduce the criterion of $x \in B_{\lambda\mathbf{n}}(A)$ to the one of $x \in B_{\lambda\mathbf{n}}(T)$. The advantage of this reduction is that, for the Ringel - Hall multiplication inside a tube, more combinatorial techniques could be used, and the work [DR] and [R1] can be used more efficiently; especially, in [R4] the structure of $\mathcal{H}(T)$ has been extensively studied for non-homogeneous tube T. As we will see in Theorem 5.2, this idea works.

The aim of this section is to study the Ringel-Hall algebras of tubes, for application in the next section. The main results are Proposition 4.6, Corollary 4.10 and Lemma 4.11.

4.2. Notations. We fix the following notations throughout this section.

Let T be a tube. If $\mathrm{rank}(T) = m \geq 2$, then by $E_1, \cdots, E_m$ we denote the all quasi-simples in T with $\tau(E_i) = E_{i+1}, 1 \leq i \leq m-1$, and $\tau(E_m) = E_1$; and by $E_i(j)$ the indecomposable in T with quasi-length j and quasi-top E_i. Thus $E_i = E_i(1)$; and $E_i(m), i = 1, \cdots, m$, are the all indecomposables of quasi-length m in T. Then we have

$$\mathbf{dim}\, E_1(m) + \cdots + \mathbf{dim}\, E_m(m) = g\mathbf{n} \tag{2}$$

where g is a positive integer with $1 \leq g \leq 3$. This integer g is called the tier number of A, see [DR] or [M]; in particular, if A is the path algebra of an affine quiver, then $g = 1$, see [DR] Tables.

Let $x, x_1, \cdots, x_t \in \mathcal{H}(A)$. We say that x is generated by $x_1, \cdots, x_t$, provided that x is a $\mathbb{R}$-combination of some products with all divisors being in $\{ x_1, \cdots, x_t \}$. A subset of $\mathcal{H}(A)$ is said to be generated by $x_1, \cdots, x_t$, provided that every element in it can be generated by $x_1, \cdots, x_t$.

We need the following

Lemma 4.3. *([GP]) (i) $\mathcal{H}(A)$ is generated by the isoclasses of all indecomposable A-modules.*

(ii) $\mathcal{H}(T)$ is generated by the isoclasses of all indecomposables in tube T.

Lemma 4.4. *Let T be a non-homogeneous tube of rank m, and M be an indecomposable in T with quasi-length λm, where λ is a positive integer. Let N be an arbitrary indecomposable in T with quasi-length λm, say, $N = \tau^i(M)$. Then*

$$[N] \in i[M] + B_{\lambda g\mathbf{n}}(T). \tag{3}$$

Proof. Denote by L the (unique) maximal regular submodule of M, and by E the quasi-top of M. Since $\dim_k \mathrm{Ext}^1_A(E, L) = 1 = \dim_k \mathrm{Ext}^1_A(L, E)$ and $\mathrm{Hom}_A(L, E) = 0 = \mathrm{Hom}_A(E, L)$, it follows that

$$[E] \cdot [L] = [M] + [E \oplus L]$$

and

$$[L] \cdot [E] = [\tau(M)] + [E \oplus L].$$

It follows that

$$[\tau(M)] = [M] + [L] \cdot [E] - [E] \cdot [L] \in [M] + B_{\lambda g\mathbf{n}}(T),$$

and hence the assertion follows by repeating this process. ■

Lemma 4.5. *Let T be a non-homogeneous tube with* rank *m, and $\mathbf{d} \in N_0^n$.*

(i) If $\mathbf{d} < g\mathbf{n}$, where the partial order in $\mathbb{Z}^n$ is defined pointwisely, then $\mathcal{H}(T)_{\mathbf{d}}$ is generated by $[E_1], \cdots, [E_m]$.

(ii) Let L be an indecomposable in T with quasi-length $\lambda m + l$, where λ is a positive integer, and $1 \le l \le m-1$. Let N denote the indecomposable regular submodule of L with quasi-length λm. Then

$$[L] = [L/N] \cdot [N] - [N] \cdot [L/N].$$

In particular, $[L] \in B_{\mathbf{dim}\, L}(T)$.

Proof. The first assertion follows from Lemma 4.3 and a direct calculation, see [Z3] Theorem 1.1 for a proof. Here we only prove (ii). By the Auslander-Reiten quiver we have $\dim_k \mathrm{Ext}^1_A(L/N, N) = 1$, $\mathrm{Ext}^1_A(N, L/N) = 0$, and $\mathrm{Hom}_A(N, L/N) = 0 = \mathrm{Hom}_A(L/N, N)$, it follows that

$$[L/N] \cdot [N] = [L] + [N \oplus L/N]; \qquad [N] \cdot [L/N] = [N \oplus L/N]. \quad ■$$

Proposition 4.6. *Let T be a non-homogeneous tube with* rank *m, g the tier number of A, and λ an arbitrary positive integer. Then $\mathcal{H}(T)_{\lambda g\mathbf{n}}$ is generated by*

$$[E_1],\ \cdots,\ [E_m],\ [M_1],\ \cdots,\ [M_{\lambda-1}],\ [M_\lambda],$$

where $M = M_\lambda$ is an arbitrary indecomposable in T with quasi-length λm, and M_i is the indecomposable regular submodule of M with quasi-length im, $1 \le i \le \lambda$.

In particular, we have

$$\mathcal{H}(T)_{\lambda\mathbf{n}} = \mathbb{R}[M_\lambda] + B_{\lambda\mathbf{n}}(T). \quad (4)$$

Proof. Use induction on λ. If $\lambda = 1$, then the assertion follows from Lemmas 4.4 and 4.5(i). Let $\lambda > 1$. By Lemma 4.3, $\mathcal{H}(T)_{\lambda g\mathbf{n}}$ is generated by the isoclasses indecomposable modules in T, therefore, it suffices to prove that $[N]$ can be generated by $[E_1], \cdots, [E_m], [M_1], \cdots, [M_{\lambda-1}], [M_\lambda]$, where N is an arbitrary indecomposable module in T with quasi-length λm. While by Lemma 4.4, it suffices to prove that $B_{\lambda g\mathbf{n}}(T)$ is generated by $[E_1], \cdots, [E_m], [M_1], \cdots, [M_{\lambda-1}]$. This reduce to prove $[L]$ can be generated by $[E_1], \cdots, [E_m], [M_1], \cdots, [M_{\lambda-1}]$, for any indecomposable module in T with $\mathbf{dim}\, L < \lambda g\mathbf{n}$. This follows from Lemmas 4.5, 4.4 and induction. ■

Note that we have the dual of Proposition 4.6, i.e. $\mathcal{H}(T)_{\lambda g\mathbf{n}}$ is generated by

$$[E_1],\ \cdots,\ [E_m],\ [N_1],\ \cdots,\ [N_{\lambda-1}],\ [N_\lambda],$$

where $N = N_\lambda$ is an arbitrary indecomposable in T with quasi-length λm, and N_i is the indecomposable regular quotient of N with quasi-length im, $1 \le i \le \lambda$.

4.7. The following lemma has been proved in [ZZ], see also [Z4].

Lemma. *Let T be a non-homogeneous tube of rank m, and N be an arbitrary indecomposable module in T with quasi-length m. Then*

$$[N] \notin B_{g\mathbf{n}}(T),$$

where g is the tier number of A.

Anyway, we cannot prove the following conjecture:

Conjecture. *Let T be a non-homogeneous tube of rank m, and N be an arbitrary indecomposable module in T with quasi-length λm, where λ is a positive integer. Then*

$$[N] \notin B_{\lambda g\mathbf{n}}(T).$$

We include in Appendix a proof of this conjecture in case $m = \lambda = 2$, by direct computations.

Corollary 4.8. *Let T be a non-homogeneous tube of rank m. Denote by*

$$s_1(T) = \sum [N], \quad \textit{where } N \textit{ runs over indecomposable modules in } T \textit{ with quasi-length } m.$$

Then

$$s_1(T) \notin B_{g\mathbf{n}}(T).$$

Proof. Let N be an indecomposable in T with quasi-length m. Then by Lemma 4.4 we have

$$\begin{aligned} s_1(T) &= [N] + [\tau(N)] + \cdots + [\tau^{m-1}(N)] \\ &\in (1 + 1 + \cdot + m - 1)[N] + B_{g\mathbf{n}}(T) = \frac{m^2 - m + 2}{2}[N] + B_{g\mathbf{n}}(T), \end{aligned}$$

and hence the assertion follows from Lamma 4.7. ■

4.9. Denote by Σ_m be the symmetric group of degree m. Let $\sigma = (12\cdots m) \in \Sigma_m$. For $[E_{i_1}(j_1)]\cdots[E_{i_t}(j_t)] \in \mathcal{H}(T)$, where $1 \le i_1, \cdots, i_t \le m$; $t \ge 1; j_1, \cdots, j_m \ge 1$. Define

$$\sigma([E_{i_1}(j_1)]\cdots[E_{i_t}(j_t)]) = [E_{\sigma(i_1)}(j_1)]\cdots[E_{\sigma(i_t)}(j_t)]). \tag{5}$$

We introduce the following element in $\mathcal{H}(T)_{g\mathbf{n}}$:

$$c_{g\mathbf{n}} := \sum_{1\le i,j\le m} \sigma^i([E_1]\cdots[E_{m-j}]\cdot[E_{m-j+1}(j)]). \tag{6}$$

Thus

$$\begin{aligned} c_{g\mathbf{n}} = \sum_{1\le i\le m} \sigma^i([E_1]\cdots[E_m]) \ + \ \sigma^i([E_1]\cdots[E_{m-2}]\cdot[E_{m-1}(2)]) \ + \ \cdots \\ + \sigma^i([E_1]\cdots[E_{m-j}]\cdot[E_{m-j+1}(j)]) \ + \ \cdots \ + \ \sigma^i([E_1(m)]). \end{aligned}$$

Notice that by definition we have

$$c_{g\mathbf{n}} = s_1(T) + x, \quad \text{with} \quad x \in B_{g\mathbf{n}}(T). \tag{7}$$

Recall that by $r_{g\mathbf{n}}(T)$ we denote the sum of the isoclasses of the all modules in T with dimension vector $g\mathbf{n}$, see 3.3.

Lemma. *Let T be a non-homogeneous tube of rank m. Then there holds the following*

$$c_{g\mathbf{n}} = mr_{g\mathbf{n}}(T). \tag{8}$$

Proof. Let M be an arbitrary module in T with $\mathbf{dim}\, M = g\mathbf{n}$. Since $\mathbf{dim}\, E_1, \cdots, \mathbf{dim}\, E_m$ are $\mathbb{Z}$-linear independent (see [DR] Tables, or [R1] p.146), it follows that M can be uniquely written as

$$M = E_i(v_1) \oplus E_{i+v_1}(v_2) \oplus \cdots \oplus E_{i+v_1+\cdots+v_{j-1}}(v_j) \tag{$*$}$$

with $v_1, \cdots, v_j \ge 1$, $j \ge 1$, and $v_1 + \cdots + v_j = m$. We take the low indices modulo m. Thus $E_{i+v_1+\cdots+v_{j-1}}(v_j) = E_{i-v_j}(v_j)$.

We claim that $[M]$ is a term of $c_{g\mathbf{n}}$ with coefficients m, and hence the assertion follows.

In fact, by the presentation $(*)$ of M, we can easily analyse the types of filtritions of M from the Auslander-Reiten quiver. Note that $c_{g\mathbf{n}}$ is a sum of m^2 monomials. Those monomials in which $[M]$ is a term are exactly the following monomials (this can be seen geometrically from the structure of a tube):

$$[E_i]\cdots[E_{i+v_1-1}]\cdots([E_{i-v_j}]\cdots[E_{i-1}]) = \sigma^{i-1}([E_1]\cdots[E_m]),$$
$$[E_i]\cdots[E_{i+v_1-1}]\cdots([E_{i-v_j}]\cdots[E_{i-3}]\cdot[E_{i-2}(2)]) = \sigma^{i-1}([E_1]\cdots[E_{m-2}]\cdot[E_{m-1}(2)]),$$
$$\vdots$$
$$[E_i]\cdots[E_{i+v_1-1}]\cdots([E_{i-v_j}(v_j)]) = \sigma^{i-1}([E_1]\cdots[E_{m-v_j}]\cdot[E_{m-v_j+1}(v_j)]);$$

$$[E_{i-v_j}]\cdots[E_{i-1}]\cdots([E_{i-v_j-v_{j-1}}]\cdots[E_{i-v_j-1}]) = \sigma^{i-v_j-1}([E_1]\cdots[E_m]),$$
$$[E_{i-v_j}]\cdots[E_{i-1}]\cdots([E_{i-v_j-v_{j-1}}]\cdots[E_{i-v_j-3}]\cdot[E_{i-v_j-2}(2)]) = \sigma^{i-v_j-1}([E_1]\cdots[E_{m-2}]\cdot[E_{m-1}(2)]),$$
$$\vdots$$
$$[E_{i-v_j}]\cdots[E_{i-1}]\cdots([E_{i-v_j-v_{j-1}}(v_{j-1})]) = \sigma^{i-v_j-1}([E_1]\cdots[E_{m-v_j}]\cdot[E_{m-v_{j-1}+1}(v_{j-1})]);$$

$$\vdots$$

$$\vdots$$
$$[E_{i+v_1}]\cdots[E_{i+v_1+v_2-1}]\cdots([E_i]\cdots[E_{i+v_1-1}]) = \sigma^{i+v_1-1}([E_1]\cdots[E_m]),$$
$$[E_{i+v_1}]\cdots[E_{i+v_1+v_2-1}]\cdots([E_i]\cdots[E_{i+v_1-3}]\cdot[E_{i+v_1-2}(2)]) = \sigma^{i+v_1-1}([E_1]\cdots[E_{m-2}]\cdot[E_{m-1}(2)]),$$
$$\vdots$$
$$[E_{i+v_1}]\cdots[E_{i+v_1+v_2-1}]\cdots([E_i](v_1)]) = \sigma^{i+v_1-1}([E_1]\cdots[E_{m-v_1}]\cdot[E_{m-v_1+1}(v_1)]).$$

Altother we have $v_j + v_{j-1} + \cdots + v_1 = m$ such monomials. Note that the coefficients of $[M]$ in every monomial in the preceeding list are 1, since $\mathrm{Hom}_A(E_i, E_j) = 0$ for $i \neq j$. This proves that $[M]$ is a term of $c_{g\mathbf{n}}$ with coefficients m. ■

Corollary 4.10. *Let $r_{g\mathbf{n}}(T)$ be as in 4.9. Then*

$$r_{g\mathbf{n}}(T) \notin B_{g\mathbf{n}}(T).$$

Proof. By the relation (7) in Lemma 4.9 and Corollary 4.8 we see that $c_{g\mathbf{n}} \notin B_{g\mathbf{n}}(T)$, and hence the assertion follows from Lemma in 4.9. ■

Now, we consider homogeneous tubes.

Lemma 4.11. *Let T be a homogeneous tube with quasi-simple E and $\mathbf{dim}\, E = s\mathbf{n}$, and M an indecomposable module in T, then we have*

$$\mathcal{H}(T)_{\mathbf{dim}\, M} = \mathbb{R}[M] \oplus B_{\mathbf{dim}\, M}(T). \tag{10}$$

Proof. Denote by M_i the indecomposable in T with quasi-length i. Let $\mathbf{dim}\, M = \lambda s\mathbf{n}$. Thus, $M_1 = E$ and $M = M_\lambda$.

Let $\mathbb{P}(\lambda)$ be the set of partitions of λ. A partition p of λ is denoted by $p = (\lambda_1^{n_1} \cdots \lambda_t^{n_t})$, i.e.

$$n_1\lambda_1 + \cdots + n_t\lambda_t = \lambda; \quad 0 < \lambda_1 < \cdots < \lambda_t; \quad n_1, \cdots, n_t > 0; \quad t > 0.$$

For every partition $p = (\lambda_1^{n_1} \cdots \lambda_t^{n_t}) \in \mathbb{P}(\lambda)$, set

$$[M(p)] = [M_{\lambda_1}^{n_1} \oplus \cdots \oplus M_{\lambda_t}^{n_t}] \in \mathcal{H}(T)_{\lambda s\mathbf{n}},$$

and

$$m_p = [M_{\lambda_1}]^{n_1} \cdots [M_{\lambda_t}]^{n_t} \in \mathcal{H}(T)_{\lambda s\mathbf{n}}.$$

Then

$$\{ \ [M(p)] \mid p \in \mathbb{P}(\lambda) \ \}$$

is a basis of $\mathcal{H}(T)_{\lambda s\mathbf{n}}$. Since $\mathcal{H}(T)$ is a commutative algebra (see [M] p.183, or [Z2]), and since every isoclass of a module in T is generated by isoclasses of indecomposable modules in T by Lemma 4.3, it follows that

$$B_{\lambda s\mathbf{n}}(T) = \text{the space spanned by } m_p, \ p \in \mathbb{P}(\lambda), \ p \neq (\lambda);$$

and that

$$\{ \ m_p \mid p \in \mathbb{P}(\lambda) \ \}$$

is a generating system of $\mathcal{H}(T)_{\lambda s\mathbf{n}}$and hence also a basis of $\mathcal{H}(T)_{\lambda s\mathbf{n}}$. In particular we have

$$\begin{aligned}\mathcal{H}(T)_{\mathbf{dim}\, M} &= \mathbb{R}m_{(\lambda)} \oplus \text{the space spanned by } m_p, \ p \in \mathbb{P}(\lambda), \ p \neq (\lambda) \\ &= \mathbb{R}[M] \oplus B_{\lambda s\mathbf{n}}(T). \ \blacksquare\end{aligned}$$

5. Reduction from $B_{\lambda\mathbf{n}}(A)$ to $B_{\lambda\mathbf{n}}(T)$

From now on, we keep the assumption that A is the path algebra of an affine quiver over a finite field k of q elements.

Let $x \in \mathcal{H}(T)_{\mathbf{d}}$. The aim of this section is to reduce the criterion of $x \in B_{\mathbf{d}}(A)$ to the one of $x \in B_{\mathbf{d}}(T)$, see Theorem 5.2, this is crucial for applying results in §4 to count the number of minimal generators of degree $\lambda\mathbf{n}$ of $\mathcal{H}(A)$.

Lemma 5.1. *Let E be a homogeneous quasi-simple A-module with $\mathbf{dim}\, E = \lambda\mathbf{n}$. Then*

$$[E] \notin B_{\lambda\mathbf{n}}(A).$$

.

Proof. Otherwise, by Theorem 3.7(iii) there exists non-zero regular modules M, N with $\mathbf{dim}\, M + \mathbf{dim}\, N = \lambda\mathbf{n}$, and $c, c_{M,N} \in \mathbb{R}$, such that

$$[E] = cr_{\lambda\mathbf{n}} + \sum c_{M,N}[M] \cdot [N] \tag{1}$$

Since E is quasi-simple, it follows that $[E]$ is not a term of $\sum c_{M,N}[M] \cdot [N]$; but $[E]$ is a term of $r_{\lambda\mathbf{n}}$, and hence by comparing the coefficients of $[E]$ in the both sides of (1) we get $c = 1$. This forces $t_\lambda(A) = 1$ in the sense of 2.3, i.e. E has to be the unique homogeneous quasi-simple A-module with

$\mathbf{dim}\, E = \lambda\mathbf{n}$ (otherwise, let E' be a homognenous quasi-simple with $\mathbf{dim}\, E' = \lambda\mathbf{n}$ and $E' \ncong E$. Then by comparing the coefficients of $[E']$ in the two sides of (1) we get the contradiction $c = 0$).

Now let $t_\lambda(A) = 1$. Then according to Corollary 2.8(ii) we have $\lambda = 2$ or $\lambda = 1$.

If $\lambda = 2$, then we divide it into two situations. First, if $t_1(A) \neq 0$, then let E' be a homogeneous quasi-simple module with $\mathbf{dim}\, E' = \mathbf{n}$. Let L be the (homogeneous) indecomposable with quasi-socle E' and $\mathbf{dim}\, L = 2\mathbf{n}$. Then both $[L]$ and $[E' \oplus E']$ are terms of $r_{2\mathbf{n}}$, with coefficient 1. But, since E' is a homogeneous quasi-simple, it follows that products $[M] \cdot [N]$ in the right hand of (1), such that $[L]$ or $[E' \oplus E']$ is a term of $[M] \cdot [N]$, is unique and has to be $[E'] \cdot [E']$. Note that

$$[E'] \cdot [E'] = [L] + (q + 1)[E' \oplus E'].$$

Then by comparing the coefficients of $[L]$ and $[E' \oplus L']$ in the both sides of (1) we get a contradiction

$$0 = 1 + c_{E',E'}, \quad \text{and} \quad 0 = 1 + (q + 1)c_{E',E'}.$$

Second, if $t_1(A) = 0$, then A is of type $\widetilde{D}_n$ or $\widetilde{E}_n$ by Corollary 2.8(i), and hence A has a tube T of rank 2. Taking the T-part in the both sides of (1), and noticing that if M, N are regular, then $[M] \cdot [N]$ has a $T-$term if and only if $M, N \in T$, we then get

$$r_{2\mathbf{n}}(T) = -\sum c_{M,N}[M] \cdot [N],$$

where $r_{2\mathbf{n}}(T) = \sum [N]$ with N running over all modules in T with $\mathbf{dim}\, N = 2\mathbf{n}$. This means $r_{2\mathbf{n}}(T) \in B_{2\mathbf{n}}(T)$, which contradicts Corollary 7.2 below.

If $\lambda = 1$, then A is not the Kronecker algebra since $t_1(A) = 1$, and then there exists a non-homogeneous tube T'. Taking the T'-part in the both sides of (1), and noticing that if M, N are regular, then $[M] \cdot [N]$ has a $T'-$term if and only if $M, N \in T'$, we then get

$$-r_{\mathbf{n}}(T') = \sum c_{M,N}[M] \cdot [N],$$

where $r_{\mathbf{n}}(T') = \sum [N]$ with N running over all modules in T' with $\mathbf{dim}\, N = \mathbf{n}$. This means $r_{\mathbf{n}}(T') \in B_{\mathbf{n}}(T')$, which contradicts Corollary 4.10. ■

Theorem 5.2. *For any tube T of A, and any positive integer λ, there holds*

$$B_{\lambda\mathbf{n}}(T) = B_{\lambda\mathbf{n}}(A) \cap \mathcal{H}(T)_{\lambda\mathbf{n}} \tag{2}$$

.

Proof. Let $0 \neq x \in B_{\lambda\mathbf{n}}(A) \cap \mathcal{H}(T)_{\lambda\mathbf{n}}$. Then by Theorem 3.7(iii) we have

$$x = r(x) = c r_{\lambda\mathbf{n}} + \sum c_{M,N}[M] \cdot [N] \tag{3}$$

where M, N are non-zero regular modules with $\mathbf{dim}\, M + \mathbf{dim}\, N = \lambda\mathbf{n}$, and $c, c_{M,N} \in \mathbb{R}$.

First, assume that $t_\lambda(A) \neq 0$, i.e. there exists a homogeneous quasi-simple E with $\mathbf{dim}\, E = \lambda\mathbf{n}$. It is easy to see that $[E]$ is not a term of x: otherwise, $E \in T$ and then T is a homogeneous tube; since E is the quasi-simple in T with $\mathbf{dim}\, E = \lambda\mathbf{n}$, it follows that $x = a[E]$ with $a \neq 0$. But by Lemma 5.1 we have $[E] \notin B_{\lambda\mathbf{n}}(A)$.

Thus, by comparing the coefficients of $[E]$ in the both sides of (3) we get $c = 0$. Now, taking the T-part in the both sides of (3), and noticing that if M, N are regular, then $[M] \cdot [N]$ has a $T-$term if and only if $M, N \in T$, we then get

$$x = \sum c_{_{M,N}} [M] \cdot [N],$$

and hence $x \in B_{\lambda\mathbf{n}}(T)$.

Second, if $t_\lambda(A) = 0$, then $\lambda = 1$ and A is of type $\widetilde{D}_n$ or $\widetilde{E}_n$ by Corollary 2.8(i), and hence A has a non-homogeneous tube T' such that $T' \neq T$. Taking the T'-part in the both sides of (3), we then get

$$cr_{_\mathbf{n}}(T') \in B_\mathbf{n}(T');$$

by Corollary 4.10 this forces $c = 0$, and then $x \in B_\mathbf{n}(T)$. ■

By Lemma 4.11 and Theorem 5.2 we have

Corollary 5.3. *Let T be a homogeneous tube and M an arbitrary indecomposable module in T. Then*

$$[M] \notin B_{\mathbf{dim}\, M}(A).$$

6. Homogeneous minimal generators of $\mathcal{H}(A)$

The aim of this section is to count the number of minimal generators of degree $\lambda\mathbf{n}$ of $\mathcal{H}(A)$, see Theorem 6.3; and then we can write out systems of minimal generators of the Ringel-Hall algebras of affine quivers explicitly, see Theorem 6.4.

6.1. Let λ be a positive integer.

Let $\mathbb{T}_1(\lambda)$ denote the set of homogeneous tubes T, such that the quasi-simple module of T has dimension vector $s\mathbf{n}$ with $s|\lambda$.

For $T \in \mathbb{T}_1(\lambda)$, define $N_\lambda(T)$ to be the unique indecomposable module in T with dimension vector $\lambda\mathbf{n}$. Note that the quasi-length of $N_\lambda(T)$ is $\frac{\lambda}{s}$.

Let $\mathbb{T}_2(\lambda)$ denote the set of non-homogeneous tubes T, which contains an indecomposable M with quasi-length λm, such that $[M] \notin B_{\lambda\mathbf{n}}(T)$, where m is the rank of T. Note that for $T \in \mathbb{T}_2$ we have $[N] \in B_{\lambda\mathbf{n}}(T)$, where N is an arbitrary indecomposable in T with quasi-length λm, by Lemma 4.4.

For $T \in \mathbb{T}_2(\lambda)$, define $N_\lambda(T)$ to be a fixed (but can be arbitrary) indecomposable module in T with quasi-length λm, where m is the rank of T. Note that $\mathbf{dim}\, N_\lambda(T) = \lambda\mathbf{n}$.

Set $\mathbb{T}(\lambda) = \mathbb{T}_1(\lambda) \cup \mathbb{T}_2(\lambda)$. Let $T \in \mathbb{T}(\lambda)$. Recall that we have denoted $r_{_{\lambda\mathbf{n}}}(T)$ by the sum of the isoclasses of all modules in T with dimension vector $\lambda\mathbf{n}$. Then by Proposition 4.6 and Lemma 4.11, we can write

$$r_{_{\lambda\mathbf{n}}}(T) = c_{_\lambda}(T)[N_\lambda(T)] + x \tag{1}$$

where $c_{_\lambda}(T) \in \mathbb{R}$, $x \in B_{\lambda\mathbf{n}}(T)$. Since $[N_\lambda(T)] \notin B_{\lambda\mathbf{n}}(T)$ by Lemma 4.11 and by our choice of $T \in \mathbb{T}_2$, it follows that such a number $c_{_\lambda}(T)$ is unique.

Now we introduce the following element in $\mathcal{H}(A)_{\lambda\mathbf{n}}$:

$$b_{\lambda\mathbf{n}} := \sum_{T\in\mathbb{T}(\lambda)} c_{\lambda}(T)[N_\lambda(T)]. \tag{2}$$

Note that $b_{\lambda\mathbf{n}} \neq 0$ for any positive integer. One can see this as following: take a homogeneous tube T with quasi-simple E such that $\mathbf{dim}\, E = \lambda\mathbf{n}$. By Corollary 2.8 such a tube T exists. Then $r_{\lambda\mathbf{n}}(T) = [E]$, and then by definition we have $c_{\lambda}(T) = 1$.

Lemma 6.2. *Let λ be a positive integer. Let V_λ denote the $\mathbb{R}$-space with basis the set of elements $[N_\lambda(T)]$ as defined above, where $T \in \mathbb{T}(\lambda)$. Then we have*

(i)

$$\mathcal{H}(A)_{\lambda\mathbf{n}} = V_\lambda + B_{\lambda\mathbf{n}}(A). \tag{3}$$

(ii)

$$V_\lambda \cap B_{\lambda\mathbf{n}}(A) = \mathbb{R}b_{\lambda\mathbf{n}}. \tag{4}$$

(iii)

$$\dim_{\mathbb{R}} V_\lambda = 1 + \sum_{s|\lambda} N(q,s) + |\mathbb{T}_2(\lambda)| - m \tag{5}$$

where $N(q,s)$ is the number of monic irreducible polynomials of degree s over the field of q elements, and m is the number of non-homogeneous tubes.

Proof. (i) This follows from Proposition 4.6 and Lemma 4.11.

(ii) Let $x = \sum_{T\in\mathbb{T}(\lambda)} a_T[N_\lambda(T)]$. By Theorem 3.7(iv), $x \in B_{\lambda\mathbf{n}}(A)$ if and only if there exists a $c \in \mathbb{R}$ such that for every tube T of A, $r_T(x)$ is of the following form

$$r_T(x) = cr_{\lambda\mathbf{n}}(T) + \sum c_{M,N}[M]\cdot[N]$$

where M, N are non-zero modules in T with $\mathbf{dim}\, M + \mathbf{dim}\, N = \lambda\mathbf{n}$, and $c_{M,N} \in \mathbb{R}$. Since for non-homogeneous tube T with $T \notin \mathbb{T}_2(\lambda)$ we have $\mathcal{H}(T)_{\lambda n} = B_{\lambda\mathbf{n}}(T)$, it follows from the expression (1) that $x \in B_{\lambda\mathbf{n}}(A)$ if and only if there exists a $c \in \mathbb{R}$ such that for $T \in \mathbb{T}(\lambda)$ there holds:

$$\begin{aligned} r_T(x) = a_T[N_\lambda(T)] &= cr_{\lambda\mathbf{n}}(T) + y \\ &= cc_{\lambda}(T)[N_\lambda(T)] + z, \end{aligned}$$

for some $y, z \in B_{\lambda\mathbf{n}}(T)$. Since $[N_\lambda(T)] \notin B_{\lambda\mathbf{n}}(A)$ by Corollary 5.3 and our choice of $\mathbb{T}_2(\lambda)$, it follows that $x \in B_{\lambda\mathbf{n}}(A)$ if and only if $a_T = cc_{\lambda}(T)$ for $T \in \mathbb{T}(\lambda)$, i.e. $x = cb_{\lambda\mathbf{n}}$.

(iii) Let m be the number of non-homogeneous tubes of A, and $t_s(A)$ be the number of homogeneous quasi-simples with dimension vector $s\mathbf{n}$. Then by Theorem 2.7 we have

$$\begin{aligned}
\dim_{\mathbb{R}} V_\lambda &= |\mathbb{T}_1(\lambda)| + |\mathbb{T}_2(\lambda)| \\
&= \sum_{s|\lambda} t_s(A) + |\mathbb{T}_2(\lambda)| \\
&= m + t_1(A) + \sum_{s|\lambda, s>1} N(q,s) + (|\mathbb{T}_2(\lambda)| - m) \\
&= 1 + q + \sum_{s|\lambda, s>1} N(q,s) + (|\mathbb{T}_2(\lambda)| - m) \\
&= 1 + \sum_{s|\lambda} N(q,s) + (|\mathbb{T}_2(\lambda)| - m). \quad \blacksquare
\end{aligned}$$

The following theorem determines the number of minimal generators of $\mathcal{H}(A)$ of degree $\lambda\mathbf{n}$.

Theorem 6.3. *Let $\mathbf{d} \in \mathbb{N}_0^n$. Then we have*

$$\operatorname{codim}_{\mathbb{R}} B_{\mathbf{d}}(A) = \begin{cases} 1, & \mathbf{d} = \mathbf{e}_i, \ i = 1, \cdots, n; \\ \sum\limits_{s|\lambda} N(q,s) - (m - |\mathbb{T}_2(\lambda)|), & \mathbf{d} = \lambda\mathbf{n}, \ \lambda \text{ a positive integer}; \\ 0, & \text{otherwise}, \end{cases}$$

where $N(q,s)$ is the number of monic irreducible polynomials of degree s over the field of q elements, and m is the number of non-homogeneous tubes, and $\mathbb{T}_2(\lambda)$ is defined in 6.1.

In particular, $B_{\mathbf{d}}(A) = \mathcal{H}_{\mathbf{d}}(A)$ if and only if $\mathbf{d} \neq \lambda\mathbf{n}, \neq \mathbf{e}_i, i = 1, \cdots, n$.

Proof. If $\mathbf{d} = \mathbf{e}_i, i = 1, \cdots, n$, then $B_{\mathbf{d}}(A) = 0$ by definition, and $\dim_{\mathbb{R}} \mathcal{H}(A)_{\mathbf{d}} = 1$.

If $\mathbf{d} \neq \lambda\mathbf{n}, \neq \mathbf{e}_i, i = 1, \cdots, n$, then $B_{\mathbf{d}}(A) = \mathcal{H}(A)_{\mathbf{d}}$. This is a consequence of Proposition 3.2 in [SV]. Here we would like to include an argument using Ringel-Hall multiplication: in fact we only need to prove $[M] \in B_{\mathbf{d}}(A)$ for indecomposable M in T with $\mathbf{dim}\, M = \mathbf{d}$, where T is an arbitrary non-homogeneous tube. This follows from Lemma 4.5, and the fact that the isoclasses of quasi-simples in T belong to $\mathcal{C}(A)$ (see [Z1]).

Assume $\mathbf{d} = \lambda\mathbf{n}$. Then by Lemma 6.2 we have

$$\operatorname{codim}_{\mathbb{R}} B_{\mathbf{d}}(A) = \dim_{\mathbb{R}} V_\lambda - \dim_{\mathbb{R}}(V_\lambda \cap B_{\lambda\mathbf{n}}(A)) = \sum_{s|\lambda} N(q,s) + |\mathbb{T}_2(\lambda)| - m. \quad \blacksquare$$

Remark. (i) Note that the number $m - |\mathbb{T}_2(\lambda)|$ is zero if and only if Conjecture in 4.7 is true.

(ii) For the Kronecker algebra K, this number is of course zero since $m = 0$; for type $\widetilde{A}_{(n_1,1)}$ $(n_1 \geq 2)$, $m - |\mathbb{T}_2(\lambda)|$ is 0 or 1; for type $\widetilde{A}_{(n_1,n_2)}$ $(n_1, n_2 \geq 2)$ we have $0 \leq m - |\mathbb{T}_2(\lambda)| \leq 2$; and for types $\widetilde{D}_n$ $(n \geq 4)$ and $\widetilde{E}_n$ $(n = 6, 7, 8)$ we have $0 \leq m - |\mathbb{T}_2(\lambda)| \leq 3$.

(iii) In general, $\operatorname{codim}_{\mathbb{R}} B_{\lambda\mathbf{n}}(A)$ are not divisible by q. In the notation in [SV], that is to say, the cardinality of set

$$\{ \ i \in I^{im} \ | \ \deg \ \theta_i = \lambda \mathbf{n} \ \}$$

is not divisible by q.

6.4. Let $[S(i)], i = 1, \cdots, n$, be the isoclasses of all simple A-modules. For each positive integer λ, choose an arbitrary tube $T' \in \mathbb{T}(\lambda)$ such that $c_\lambda(T') \neq 0$. Consider the subspace W_λ of V_λ with basis the set

$$\{ \ [N_\lambda(T)] \mid T \in \mathbb{T}(\lambda), \quad T \neq T'. \ \}. \tag{6}$$

Then

$$[N_\lambda(T')] \in W_\lambda + B_{\lambda \mathbf{n}}(A)$$

and

$$b_{\lambda \mathbf{n}} \notin W_\lambda.$$

and hence by Lemma 6.2(i)(ii) we have

$$\mathcal{H}(A)_{\lambda \mathbf{n}} = W_\lambda \oplus B_{\lambda \mathbf{n}}(A). \tag{7}$$

This proves the following

Theorem. *Keep the notations above, the set*

$$G = \{ \ [S(i)] \ | \ 1 \leq i \leq n \ \} \cup \bigcup_\lambda \{ \ [N_\lambda(T)] \ | \ T \in \mathbb{T}(\lambda), \ T \neq T' \ \} \tag{8}$$

is a system of minimal generators of $\mathcal{H}(A)$.

By Theorem 6.3 and §1 (10) we have

Corollary 6.5. *Let K be the Kronecker k-algebra. Then we have*

$$\operatorname{codim}_{\mathbb{R}} B_{\mathbf{d}}(K) = \begin{cases} 1, & \mathbf{d} = \mathbf{e}_i, \ \ i = 1, 2; \\ \\ \frac{1}{\lambda} \sum\limits_{s \mid \lambda} \varphi(\frac{\lambda}{s}) \ q^s, & \mathbf{d} = (\lambda, \lambda), \ \ \lambda \ \textit{a positive integer}; \\ \\ 0, & \textit{otherwise} \end{cases}$$

where φ is the Euler function.

6.6. Let $[S(1)]$ and $[S(2)]$ be the isoclasses of the two simple K-modules. For each positive integer λ, let T_λ be an fixed homogeneous tube such that the quasi-simple of T_λ has dimension vector (λ, λ). As in 6.1, let $\mathbb{T}(\lambda)$ denote the set of homogeneous tubes T, such that the quasi-simple module of T has dimension vector (s, s) with $s|\lambda$; and for $T \in \mathbb{T}(\lambda)$, let $N_\lambda(T)$ denote the (unique) indecomposable in T with $\mathbf{dim}\, N_\lambda(T) = (\lambda, \lambda)$. Then by Theorem 6.4 we have

Corollary. *The set*

$$G = \{ \ [S(1)], \ [S(2)] \ \} \cup \bigcup_\lambda \{ \ [N_\lambda(T)] \mid T \in \mathbb{T}(\lambda), \ T \neq T_\lambda \ \} \tag{9}$$

is a system of minimal generators of $\mathcal{H}(K)$.

7. Appendix

Lemma 7.1. *Let T be a tube with* $\mathrm{rank}(T) = 2$*, and L an indecomposable in T with* $\mathbf{dim}\, L = 2g\mathbf{n}$*. Then $[L] \notin B_{2g\mathbf{n}}(T)$.*

Proof. Denote by $L' = \tau L$, where τ is the Auslander-Reiten translate. Let E_1, E_2 be the quasi-simple modules in T with $\mathrm{Hom}_A(L, E_1) \neq 0$. Let N_1 (resp. M_1) denote the indecomposable in T of quasi-length 2 (resp. 3) and $\mathrm{Hom}_A(N_1, E_1) \neq 0$ (resp. $\mathrm{Hom}_A(M_1, E_1) \neq 0$). Set $N_2 = \tau N_1$ and $\tau M_1 = M_2$. Then $\dim_{\mathbb{R}} \mathcal{H}(T)_{2g\mathbf{n}} = 10$, and with a basis (we fix the following order)

$$[L],\ [L'],\ [M_1 \oplus E_2],\ [M_2 \oplus E_1],\ [N_1^2],\ [N_2^2],\ [N_1 \oplus E_1 \oplus E_2], [N_2 \oplus E_1 \oplus E_2], [E_1^2 \oplus E_2^2],\ [N_1 \oplus N_2].$$

Let $\mathbf{d}_i = 2g\mathbf{n} - \mathbf{dim}\, E_i$, $i = 1, 2$. Then by the dual of Proposition 4.6 we have

$$B_{2g\mathbf{n}}(T) = [E_1] \cdot \mathcal{H}(T)_{\mathbf{d}_1} + [E_2] \cdot \mathcal{H}(T)_{\mathbf{d}_2} + [N_1] \cdot \mathcal{H}(T)_{\mathbf{n}},$$

It follows that $B_{2g\mathbf{n}}(T)$ is spanned by the following 11 elements

$$[E_1] \cdot [M_2] = [L] + [M_2 \oplus E_1], \tag{1}$$

$$[E_2] \cdot [M_1] = [L'] + [M_1 \oplus E_2], \tag{2}$$

$$[E_1] \cdot [N_2 \oplus E_2] = [M_1 \oplus E_2] + [N_1 \oplus N_2] + [N_2 \oplus E_1 \oplus E_2], \tag{3}$$

$$[E_2] \cdot [N_1 \oplus E_1] = [M_2 \oplus E_1] + [N_1 \oplus N_2] + [N_1 \oplus E_1 \oplus E_2], \tag{4}$$

$$[E_1] \cdot [N_1 \oplus E_2] = (q+1)[N_1^2] + q[N_1 \oplus E_1 \oplus E_2], \tag{5}$$

$$[E_2] \cdot [N_2 \oplus E_1] = (q+1)[N_2^2] + q[N_2 \oplus E_1 \oplus E_2], \tag{6}$$

$$[E_1] \cdot [E_1 \oplus E_2^2] = [N_1 \oplus E_1 \oplus E_2] + (q+1)[E_1^2 \oplus E_2^2], \tag{7}$$

$$[E_2] \cdot [E_1^2 \oplus E_2] = [N_2 \oplus E_1 \oplus E_2] + (q+1)[E_1^2 \oplus E_2^2], \tag{8}$$

$$[N_1] \cdot [N_1] = [L] + (q+1)[N_1^2], \tag{9}$$

$$[N_1] \cdot [E_1 \oplus E_2] = [M_1 \oplus E_2] + q[N_1 \oplus E_1 \oplus E_2], \tag{10}$$

$$[N_1] \cdot [N_2] = (q-1)[M_1 \oplus E_2] + q[N_1 \oplus N_2]. \tag{11}$$

But by a direct calculation we know that the rank of these 11 elements is 9. It follows that $B_{2g\mathbf{n}}(T) \neq \mathcal{H}(T)_{2g\mathbf{n}}$, and $[L] \notin B_{2g\mathbf{n}}(T)$. ∎

Corollary 7.2. *Let T be a tube with* $\operatorname{rank}(T) = 2$. *Then $r_{2g\mathbf{n}}(T) \notin B_{2g\mathbf{n}}(T)$, where $r_{2g\mathbf{n}}(T)$ is the sum of the isoclasses of all modules in T with dimension vector $2g\mathbf{n}$.*

Proof. By Lemma 4.4 we have

$$[L'] \in [L] + B_{2g\mathbf{n}}(T);$$

by (1) and (2) we have

$$[M_2 \oplus E_1], \quad [M_1 \oplus E_2] \in -[L] + B_{2g\mathbf{n}}(T);$$

similarly we have

$$[N_1^2], \quad [N_2^2] \in -\frac{1}{q+1}[L] + B_{2g\mathbf{n}}(T);$$

$$[N_1 \oplus E_1 \oplus E_2], \quad [N_2 \oplus E_1 \oplus E_2] \in \frac{1}{q}[L] + B_{2g\mathbf{n}}(T);$$

$$[N_1 \oplus N_2] \in \frac{q-1}{q}[L] + B_{2g\mathbf{n}}(T);$$

$$[E_1^2 \oplus E_2^2] \in -\frac{1}{q(q+1)}[L] + B_{2g\mathbf{n}}(T).$$

It follows that

$$r_{2g\mathbf{n}}(T) \in \frac{q}{q+1}[L] + B_{2g\mathbf{n}}(T),$$

and hence $r_{2g\mathbf{n}}(T) \notin B_{2g\mathbf{n}}(T)$ by Lemma 7.1. ■

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References

[ARS] M. Auslander, I. Reiten and S. Smalϕ: Representation theory of Artin algebras, Cambridge Studies in Advanced Math. 36, Cambridge Univ. Press, 1994.

[B] R. Borcherds, Generalized Kac-Moddy algebras, J. Algebra 115(1988), 501-512.

[DR] V. Dlab and C. M. Ringel: Indecomposable representations of graphs and algebras, Memoirs Amer. Math. Soc. 173 (1976).

[FIS] S. H. Friedberg, A. J. Insel and L. E. Spence: Linear Algebra, Pretice-Hall, Inc., 1979.

[G] J. A. Green: Hall algebras, hereditary algebras and quantum groups, Inventiones Math. 120(1995), 361-377.

[GL] W. Geigle and H. Lenzing: Perpendicular categories with applications to representations and sheaves, J. Algebra 144(1991), 273-234.

[GP] J. Y. Guo and L. G. Peng: Universal PBW-basis of Hall - Ringel algebras and Hall polynomials, J. Algebra 198(1997), 339-351.

[HHKU] D. Happel, S. Hartlieb, O. Kerner and L.Unger: On perpendicular categories of stones over quiver algebras, Comment. Math. Helvetici 71(1996) 463-474.

[HX] J. Z. Hua and J. Xiao: On Ringel-Hall algebras of tame hereditary algebras, preprint.

[J] N. Jacobson: Basic algebra I, second edition, W.H.Freeman and Company, New York 1985.

[K] V. G. Kac: Infinite root systems, representations of graphs and invariant theory, Inventiones Math. 56(1980), 57-92.

[Kang] S. J. Kang: Quantum deformations of generalized Kac-Moody algebras and their modules, J. Algebra 175 (1995), 1041-1066.

[L] G. Lusztig: Introduction to quantum groups, Birkhäuser, Boston, 1993.

[Mac] I. G. Macdonald: Symmetric functions and Hall polynomials, Second Edition. Oxford University Press, Oxford. Basel. New York, 1995.

[M] R. V. Moody: Euclidean Lie algebras, Canadian J. Math. 21(1969), 1432-1454.

[R1] C. M. Ringel: Tame algebras and integral quadratic forms, Lecture Notes in Math. 1099, Springer-Verlag, Berlin, New York, 1984.

[R2] C. M. Ringel: Hall algebras, In: Topics in Algebra, Banach Center Publ. 26(1990), 433-447.

[R3] C. M. Ringel: Hall algebras and quantum groups, Inventiones Math. 101(1990), 583-592.

[R4] C. M. Ringel: The composition algebra of a cyclic quiver, Proc. London Math. Soc.(3) 66 (1993), 507-537.

[R5] C. M. Ringel: Hall algebras revisited, Israel Mathematica Conference Proceedings. Vol.7 (1993), 171-176.

[R6] C. M. Ringel: PBW-bases of quantum groups, J. reine angew. Math. 470(1996), 51-88.

[R7] C. M. Ringel: Green's Theorem on Hall algebras, Canad. Math. Soc. Conf. Proc. Vol. 19 (R. Bautista, R. Martinez-Villa and J. A. de la Peña, eds.), 185-245. Amer. Math. Soc. Providence RI 1996.

[S] A. Schofield: Semi-invariants of quivers, J. London Math. Soc.43(1991), 385-395.

[SV] B. Sevenhant and M. Van den Bergh: A relation between a conjecture of Kac and the structure of the Hall Algebra, J. Algebra.

[Z1] P. Zhang: Triangular decomposition of the composition algebra of the Kronecker algebra, J. Algebra 184(1996), 159-174.

[Z2] P. Zhang: Ringel-Hall algebras of standard homogeneous tubes, Algebra Colloquium 4:1(1997), 89-94.

[Z3] P. Zhang: Composition algebras of affine types, J. Algebra 206(1998), 505-540.

[Z4] P. Zhang: Representations as elements in affine composition algebras, to appear in Trans. Amer. Math. Soc.

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