EXACT FACTORS OF EXACT CATEGORIES

P. DRÄXLER AND Ø. SOLBERG

Abstract. In the representation theory of algebras situations occur where one has to transport an exact structure $\mathcal{E}$ on a category $\mathcal{A}$ to a factor category by a relation $\mathcal{R}$. We characterize when this is possible and discuss how almost split pairs are transferred to the factor category. As an illustration, we give applications for the category of finitely presented functors on modules over an artin algebra and for the category of regular modules over a wild hereditary algebra.

Introduction

In [4] it was studied how one can form new exact categories by looking at subfunctors of the functor Ext. The motivating examples for this investigations came from the representation theory of algebras where certain reduction functors only happen to be exact when looking at the appropriate exact structure. In fact, in some examples in addition to choosing the correct exact structure one also had to pass to a factor category by a relation. For this purpose, in [4, Proposition 1.11] a sufficient condition for a relation $\mathcal{R}$ of an exact category $\mathcal{A}$ was given which ensures that the factor category $\mathcal{A}/\mathcal{R}$ together with the appropriate induced exact structure becomes exact again.

It is the aim of the first section of this paper to prove a necessary and sufficient condition for $\mathcal{A}/\mathcal{R}$ to be exact. Moreover, we discuss how our conditions simplify if the relation $\mathcal{R}$ is generated by projective and injective objects. Finally, we study when $\mathcal{A}/\mathcal{R}$ becomes even abelian.

The modern representation theory of algebras was strongly pushed forward by the detection of Auslander and Reiten that almost split sequences exist in the category of finitely generated modules over artin algebras. Meanwhile it turned out that almost split pairs exist in many more exact categories. Therefore we devote the second section to showing how almost split pairs are pushed down to exact factor categories. As a preparation we investigate how projective and injective objects are transferred.

In the third section we use our results to reprove that the category $\operatorname{mod}(\operatorname{mod}\Lambda)$ of finitely presented functors from the category $\operatorname{mod}\Lambda$ to the category of abelian groups is abelian and has almost split sequences where Λ is an artin algebra and $\operatorname{mod}\Lambda$ is the category of finitely generated Λ-modules. Of course, this is well-known but our intention is to illustrate that there are natural examples where our settings and results from the first two sections apply. In fact, categories of the shape $\mathcal{A}/\mathcal{R}$ appear frequently by looking at the kernel $\mathcal{R}$ of a functor $\mathcal{A} \to \mathcal{B}$ where

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$\mathcal{B}$ is another category. This is also the case in our example of the third section where $\mathcal{B} = \operatorname{mod}(\operatorname{mod}\Lambda)$ and $\mathcal{A} = \operatorname{mod} T(\Lambda)$ with $T(\Lambda)$ being the triangular matrix algebra $\left(\begin{smallmatrix}\Lambda & 0\\ \Lambda & \Lambda\end{smallmatrix}\right)$ over our given artin algebra Λ. It was shown in [1, Theorem 1.1] that there is a full and dense functor $\alpha\colon \operatorname{mod} T(\Lambda) \to \operatorname{mod}(\operatorname{mod}\Lambda)$ and we look at the relation $\mathcal{R}$ given by the kernel of α.

The final fourth section deals with a very recent application in the representation theory of wild hereditary algebras which was brought to our attention by O. Kerner. For a wild hereditary algebra H with more then 2 simple modules there exist appropriate quasi-simple regular modules X such that the right perpendicular category $X^{\perp}$ can be identified with a module category over another wild hereditary algebra C. If we denote by $\operatorname{reg} H$ and $\operatorname{reg} C$ the respective subcategories of regular modules, then in [3] a full and faithful functor $F\colon \operatorname{reg} C \to \operatorname{reg} H$ is introduced whose kernel is the relation $\mathcal{R}$ of maps factoring through the orbit under the Auslander-Reiten translation of a quasi-simple C-module. In [7] an additive subfunctor of the restriction of the bifunctor Ext^1_C to $\operatorname{reg} C$ is introduced which is mapped by F onto the restriction of Ext^1_H to $\operatorname{reg} H$. We use the results of [4] to see that this subfunctor gives rise to an exact structure on $\operatorname{reg} C$ and then the results of section 1 to push this exact structure down to $\operatorname{reg} C/\mathcal{R}$ which is equivalent as an exact category to $\operatorname{reg} H$ equipped with the usual exact sequences.

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1. Conditions for the existence of exact factors

This section is mainly devoted to finding necessary and sufficient conditions for a factor of an exact category by a relation again to be an exact category. To make this precise and for the convenience of the reader we recall the definition of an exact category from [5].

Let $\mathcal{A}$ be an additive category with split idempotents. A pair (i, d) of composable morphisms $X \xrightarrow{i} Y \xrightarrow{d} Z$ in $\mathcal{A}$ is called *exact* if i is a kernel of d and d is a cokernel of i.

Let $\mathcal{E}$ be a class of exact pairs $X \xrightarrow{i} Y \xrightarrow{d} Z$ which is closed under isomorphisms. The morphisms i and d appearing in a pair (i, d) in $\mathcal{E}$ are called an *inflation* and a *deflation* of $\mathcal{E}$, respectively. The class $\mathcal{E}$ is said to be an *exact structure* on $\mathcal{A}$ and $(\mathcal{A}, \mathcal{E})$ an *exact category* if the following axioms are satisfied:

E1 The composition of two deflations is a deflation.
E2 For each f in $\mathcal{A}(Z', Z)$ and each deflation d in $\mathcal{A}(Y, Z)$, there is some Y' in $\mathcal{A}$, an f' in $\mathcal{A}(Y', Y)$ and a deflation $d'\colon Y' \to Z'$ such that $df' = fd'$.
E3 Identities are deflations. If de is a deflation, then so is d.
E3$^{\text{op}}$ Identities are inflations. If ji is a inflation, then so is i.

Let $(\mathcal{A}, \mathcal{E})$ be an exact category. In the appendix of [4] Keller showed that the above axioms are equivalent to a set of axioms involving existence of pullback and pushout diagrams. More precisely, for each f in $\mathcal{A}(Z', Z)$ and each deflation d in

$\mathcal{A}(Y,Z)$, there is a pullback diagram

$$\begin{tikzcd} Y' \arrow[r, "d'"] \arrow[d, "f'"] & Z' \arrow[d, "f"] \\ Y \arrow[r, "d"] & Z \end{tikzcd}$$

where d' is a deflation. Dually, for each morphism f in $\mathcal{A}(X,X')$ and each inflation i in $\mathcal{A}(X,Y)$, there is a pushout diagram. These preliminaries enables us to formulate the main result of this section.

Let $(\mathcal{A},\mathcal{E})$ be an exact category, and let $\mathcal{R}$ be a relation on $\mathcal{A}$. Denote by $\mathcal{A}/\mathcal{R}$ the factor category whose objects are the same as those of $\mathcal{A}$ and morphisms given by

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{R}}(A,B) = \mathrm{Hom}_{\mathcal{A}}(A,B)/\mathcal{R}(A,B).$$

The pair of maps $\{(\overline{i},\overline{d}) \mid (i,d) \in \mathcal{E}\}$ is not necessarily closed under isomorphisms. We close this class of pairs under isomorphisms in the following way. A pair $(\overline{i'},\overline{d'})$ is isomorphic to $(\overline{i},\overline{d})$ if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\overline{i}"] \arrow[d, "\overline{f}"] & B \arrow[r, "\overline{d}"] \arrow[d, "\overline{g}"] & C \arrow[d, "\overline{h}"] \\ A' \arrow[r, "\overline{i'}"] & B' \arrow[r, "\overline{d'}"] & C' \end{tikzcd}$$

where $\overline{f}$, $\overline{g}$ and $\overline{h}$ are isomorphisms in $\mathcal{A}/\mathcal{R}$. Denote the closure of $\{(\overline{i},\overline{d}) \mid (i,d) \in \mathcal{E}\}$ under isomorphisms by $\mathcal{E}/\mathcal{R}$. In this section we characterize when $\mathcal{E}/\mathcal{R}$ induces an exact structure on $\mathcal{A}/\mathcal{R}$. Special cases of relations are discussed, in particular when the relation $\mathcal{R}$ is given by all morphisms factoring through objects in a given subcategory. We also investigate when $\mathcal{A}/\mathcal{R}$ is even abelian using the characterization (see [6]) when an exact category is abelian.

Let $(\mathcal{A},\mathcal{E})$ be an exact category, and let $\mathcal{R}$ be a relation on $\mathcal{A}$. A morphism $f\colon A \to A'$ is called an $\mathcal{R}$-*isomorphism* provided $\overline{f}$ is an isomorphism in $\mathcal{A}/\mathcal{R}$.

Then, if $(\overline{i'},\overline{d'})$ is in $\mathcal{E}/\mathcal{R}$, there exist $\mathcal{R}$-isomorphisms α, β, β' and δ, and an exact pair (i,d) in $\mathcal{E}$ such that $i' = \beta i\alpha + r$ and $d' = \delta d\beta' + r'$ for some r and r' in $\mathcal{R}$ and where $\overline{\beta'} = \overline{\beta}^{-1}$. The maps i' and d' are then called an $\mathcal{R}$-*inflation* and an $\mathcal{R}$-*deflation*, respectively. Furthermore, it is easy to see that (i',d') is in $\mathcal{E}/\mathcal{R}$ if and only if there exists $\mathcal{R}$-isomorphisms a, b, b' and c, and an exact pair (i,d) in $\mathcal{E}$ such that $i = bi'a + r$ and $d = cd'b' + r'$ for some r and r' in $\mathcal{R}$ and where $\overline{b'} = \overline{b}^{-1}$.

In general the class $\mathcal{E}/\mathcal{R}$ does not induce an exact structure on $\mathcal{A}/\mathcal{R}$. In the next result we characterize when this happens.

Theorem 1.1. *Let $\mathcal{A}$, $\mathcal{E}$ and $\mathcal{R}$ be as above. The class of pairs $\mathcal{E}/\mathcal{R}$ is an exact structure on $\mathcal{A}/\mathcal{R}$ if and only if the following hold:*

(I) *If i is an inflation in $\mathcal{A}$, then*
  (i) *if ig is in $\mathcal{R}$, then g is in $\mathcal{R}$.*
  (ii) *if gi is in $\mathcal{R}$, then $gi = ri$ for some r in $\mathcal{R}$.*

(II) *If d is a deflation in $\mathcal{A}$, then*
  (i) *if sd is in $\mathcal{R}$, then s is in $\mathcal{R}$.*
  (ii) *if ds is in $\mathcal{R}$, then $ds = dr$ for some r in $\mathcal{R}$.*

(III) (i) *If ij is an $\mathcal{R}$-inflation in $\mathcal{A}$, then i is an $\mathcal{R}$-inflation in $\mathcal{A}$.*

(ii) *If de is an $\mathcal{R}$-deflation in $\mathcal{A}$, then d is an $\mathcal{R}$-deflation in $\mathcal{A}$.*

(IV) *If $d\colon B \to C$ is a deflation in $\mathcal{A}$ and ϕ is an $\mathcal{R}$-isomorphism, then in the pullback diagram* $\begin{array}{ccc} B'' & \xrightarrow{d'} & B' \\ {\scriptstyle \phi'}\downarrow & & \downarrow{\scriptstyle \phi} \\ B & \xrightarrow{d} & C \end{array}$ *, the morphism ϕ' is an $\mathcal{R}$-isomorphism.*

Proof. Assume that $\mathcal{E}/\mathcal{R}$ is an exact structure on $\mathcal{A}/\mathcal{R}$. Let (i, d) be in $\mathcal{E}$. Then by definition $(\overline{i}, \overline{d})$ is in $\mathcal{E}/\mathcal{R}$. If $\overline{i}\overline{g} = 0$ for some morphism g, then $\overline{g} = 0$. This is equivalent to having the following. If ig is in $\mathcal{R}$, then g is in $\mathcal{R}$.

Similarly we show that if sd is in $\mathcal{R}$, then s is in $\mathcal{R}$.

Suppose that $\overline{d}\overline{s} = 0$ for some morphism s. Then there exists a unique morphism $\overline{t}$ such that $\overline{s} = \overline{i}\overline{t}$. This implies that if ds is in $\mathcal{R}$, then there exists a morphism t such that $s = it + r$ for some r in $\mathcal{R}$. It follows that $ds = dit + dr = dr$ for some r in $\mathcal{R}$.

Dual arguments show that (I)(ii) is satisfied. This proves that the properties (I) and (II) are satisfied.

By assumption if $\overline{d}\overline{e} = \overline{de}$ is a deflation in $\mathcal{E}/\mathcal{R}$, then $\overline{d}$ is a deflation in $\mathcal{E}/\mathcal{R}$. A morphism $\overline{d'}$ is a deflation in $\mathcal{E}/\mathcal{R}$ if and only if there exist $\mathcal{R}$-isomorphisms ϕ and ψ and an element r in $\mathcal{R}$ such that $\phi d'\psi + r$ is deflation in $\mathcal{E}$. The property (III)(i) follows directly from this. The property (III)(ii) is shown by the dual arguments.

Let $d\colon B \to C$ be a deflation in $\mathcal{A}$, and let $\phi\colon B' \to C$ be an $\mathcal{R}$-isomorphism. In the pullback diagram

$$\begin{array}{ccc} B'' & \xrightarrow{\overline{d'}} & B' \\ {\scriptstyle \overline{\phi'}}\downarrow & & \downarrow{\scriptstyle \overline{\phi}} \\ B & \xrightarrow{\overline{d}} & C \end{array}$$

in $\mathcal{A}/\mathcal{R}$, the morphism $\overline{\phi'}$ is an isomorphism. If

$$\begin{array}{ccc} B''' & \xrightarrow{d''} & B' \\ {\scriptstyle \phi''}\downarrow & & \downarrow{\scriptstyle \phi} \\ B & \xrightarrow{d} & C \end{array}$$

is the pullback in $\mathcal{A}$, then

$$\begin{array}{ccc} B''' & \xrightarrow{\overline{d''}} & B' \\ {\scriptstyle \overline{\phi''}}\downarrow & & \downarrow{\scriptstyle \overline{\phi}} \\ B & \xrightarrow{\overline{d}} & C \end{array}$$

is also a pullback in $\mathcal{A}/\mathcal{R}$. It follows from this that $\overline{\phi''}$ is an isomorphism, hence ϕ'' is an $\mathcal{R}$-isomorphism. This shows that all the conditions (I)–(IV) are satisfied.

Conversely, assume that the relation $\mathcal{R}$ satisfies the conditions (I)–(IV). Let $(\overline{i'}, \overline{d'})$ be in $\mathcal{E}/\mathcal{R}$. First we show that $(\overline{i'}, \overline{d'})$ is a kernel-cokernel pair.

We have that there exists a diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, shift left, "f"] & B \arrow[r, "d"] \arrow[d, shift left, "g"] & C \arrow[d, shift left, "h"] \\
A' \arrow[r, "i'"] \arrow[u, shift left, "a"] & B' \arrow[r, "d'"] \arrow[u, shift left, "b"] & C' \arrow[u, shift left, "c"]
\end{tikzcd}$$

which induces a commutative diagram in $\mathcal{A}/\mathcal{R}$, where (i,d) is in $\mathcal{E}$ and all the vertical maps are $\mathcal{R}$-isomorphisms with $\overline{a}=\overline{f}^{-1}$, $\overline{b}=\overline{g}^{-1}$ and $\overline{c}=\overline{h}^{-1}$.

Assume that $\overline{i'}\overline{t}=0$ for some morphism t. Hence $\overline{g}\overline{i}\overline{a}\overline{t}=0$ and therefore $\overline{i}\overline{a}\overline{t}=0$, since $\overline{g}$ is an isomorphism. Property (I)(i) implies that at is in $\mathcal{R}$. It follows that fat is in $\mathcal{R}$, so that $\overline{fat}=\overline{f}\overline{a}\overline{t}=\overline{t}=0$. This shows that i' is a monomorphism in $\mathcal{A}/\mathcal{R}$.

Assume that $\overline{d'}\overline{s}=0$, hence $\overline{hdbs}=0$. Since $\overline{h}$ is an isomorphism, $\overline{dbs}=0$ and dbs is in $\mathcal{R}$. Property (II)(ii) implies that $dbs=dr$ for some r in $\mathcal{R}$. Hence $d(bs-r)=0$, and therefore there exists a morphism t such that $bs-r=it$. Then $gbs-gr=git$, so that $\overline{s}=\overline{gbs}-\overline{gr}=\overline{git}=\overline{g}\overline{i}\overline{a}\overline{ft}=\overline{i'ft}$ and $\overline{i'}$ is a kernel of $\overline{d'}$ in $\mathcal{A}/R$.

This shows that $\overline{i'}$ is the kernel of $\overline{d'}$ in $\mathcal{A}/\mathcal{R}$. Similarly using the properties (II)(i) and (I)(ii) we show that $\overline{d'}$ is the cokernel of $\overline{i'}$ in $\mathcal{A}/\mathcal{R}$.

The class $\mathcal{E}/\mathcal{R}$ is therefore a class of exact pairs which is closed under isomorphisms.

Let $\overline{d'}\colon B\to B'$ and $\overline{d''}\colon B'\to C$ be two deflations in $\mathcal{E}/\mathcal{R}$. Then we have a diagram as follows

$$\begin{tikzcd}
 & Y \arrow[rr, "d_0"] \arrow[dl, "c"'] & & \tilde{B}' \arrow[r, "\tilde{d}''"] \arrow[dd, shift left, "\phi"] & \tilde{C} \arrow[dd, "\psi"] \\
\tilde{B} \arrow[rr, "\tilde{d}'"] \arrow[dr, shift left, "a"] & & \hat{B}' \arrow[dr, shift left, "b"] & & \\
 & B \arrow[rr, "d'"] \arrow[ul, shift left, "a'"] & & B' \arrow[r, "d''"] \arrow[uu, shift left, "\phi'"] \arrow[ul, shift left, "b'"] & C
\end{tikzcd}$$

where a, a', b, b', ϕ, ϕ' and ψ are $\mathcal{R}$-isomorphisms, $\overline{a'}=\overline{a}^{-1}$, $\overline{b'}=\overline{b}^{-1}$, $\overline{\phi'}=\overline{\phi}^{-1}$ and the diagram commute in $\mathcal{A}/\mathcal{R}$. We construct the morphisms $d_0\colon Y\to\tilde{B}'$ and $c\colon Y\to\tilde{B}$ as the pullback of the maps $\tilde{d}'\colon B\tilde{\to}\hat{B}'$ and $b'\phi\colon\tilde{B}'\to\hat{B}'$. The morphism d_0 is a deflation in $\mathcal{A}$ and by property (IV) the morphism c is an $\mathcal{R}$-isomorphism. This shows that we have the following commutative diagram in $\mathcal{A}/\mathcal{R}$.

$$\begin{tikzcd}
Y \arrow[r, "\overline{\tilde{d}''d_0}"] \arrow[d, "\overline{ac}"'] & \tilde{C} \arrow[d, "\overline{\psi}"] \\
B \arrow[r, "\overline{d''d'}"] & C
\end{tikzcd}$$

where $\overline{ac}$ and $\overline{\psi}$ are isomorphisms and $\tilde{d}''d_0$ is a deflation in $\mathcal{A}$. Hence $\overline{d''d'}$ is a deflation in $\mathcal{A}/\mathcal{R}$, and the property E1 is satisfied.

Next we prove that the property E2 is satisfied. Let $\overline{d}\colon Y\to Z$ be a deflation in $\mathcal{A}/\mathcal{R}$, and let $\overline{f}\colon Z'\to Z$ be a morphism in $\mathcal{A}/\mathcal{R}$. Then we have the following diagram

$$\begin{tikzcd}
 & Y_1^1 \arrow[rr, "\tilde{d}'"] \arrow[dl, "\tilde{f}'"'] & & Z' \arrow[d, equal] \arrow[dl, "\phi f"] \\
Y_1 \arrow[rr, "\tilde{d}"] \arrow[d, shift left, "\psi'"] & & Z_1 \arrow[d, shift left, "\phi'"] & Z' \arrow[dl, "f"] \\
Y \arrow[rr, "d"] \arrow[u, shift left, "\psi"] & & Z \arrow[u, shift left, "\phi"] &
\end{tikzcd}$$

where ψ, ψ', ϕ and ϕ' are $\mathcal{R}$-isomorphisms with $\overline{\psi'} = \overline{\psi}^{-1}$ and $\overline{\phi'} = \overline{\phi}^{-1}$ and the front square diagram is commutative in $\mathcal{A}/\mathcal{R}$. The morphisms $\tilde{d}'\colon Y_1^1 \to Z'$ and $\tilde{f}'\colon Y_1^1 \to Y_1$ are constructed as the pullback of the morphisms $\tilde{d}$ and ϕf. Then $\tilde{d}'$ is a deflation in $\mathcal{A}$, and we get the following commutative diagram in $\mathcal{A}/\mathcal{R}$

$$\begin{tikzcd}
Y_1^1 \arrow[r, "\overline{\tilde{d}'}"'] \arrow[d, "\overline{\psi'}\overline{\tilde{f}'}"'] & Z' \arrow[d, "\overline{f}"] \\
Y \arrow[r, "\overline{d}"] & Z
\end{tikzcd}$$

where $\overline{\tilde{d}'}$ is a deflation in $\mathcal{A}/\mathcal{R}$. This shows that the property E2 is satisfied for $\mathcal{E}/\mathcal{R}$.

The condition (III) is a direct reformulation of the properties E3 and E3$^{\mathrm{op}}$, so that we have shown that $\mathcal{E}/\mathcal{R}$ is an exact structure on $\mathcal{A}/\mathcal{R}$. □

Now we discuss the properties (I)–(IV) for the special case when $\mathcal{R}(A,B) = \mathcal{R}_{\mathcal{X}}(A,B) = \{f \in \operatorname{Hom}_{\mathcal{A}}(A,B) \mid f = vu, u\colon A \to X, v\colon X \to B, X \in \mathcal{X}\}$ for a subcategory $\mathcal{X}$ in $\mathcal{A}$. We show that the condition (III) is always satisfied in this case.

Proposition 1.2. *Let $(\mathcal{A},\mathcal{E})$ be an exact category. Assume that $\mathcal{R} = \mathcal{R}_{\mathcal{X}}$ for a subcategory $\mathcal{X}$ in $\mathcal{A}$. Then the property* (III) *is satisfied.*

Proof. Suppose $\phi de\psi + vu\colon B' \to C$ is a deflation in $\mathcal{A}$, where ϕ and ψ are $\mathcal{R}$-isomorphism, and $u\colon B' \to X$ and $v\colon X \to C$ with X in $\mathcal{X}$. Since $(\phi d, v)\left(\begin{smallmatrix} e\psi \\ u \end{smallmatrix}\right) = \phi de\psi + vu$, it follows that $(\phi d, v)\colon B \oplus X \to C$ is a deflation in $\mathcal{A}$. Then we have the following commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\phi d"] \arrow[d, "a=\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & C \arrow[d, equal] \\
B \oplus X \arrow[r, "(\phi d, v)"] & C
\end{tikzcd}$$

Define $a' = (1,0)\colon B \oplus X \to B$. Then $a'a = \mathrm{id}_B$ and $aa' = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)(10) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\colon B \oplus X \to B \oplus X$, hence $\overline{a'}\overline{a} = \overline{\mathrm{id}_B}$ and $\overline{a}\overline{a'} = \overline{\mathrm{id}_{B\oplus X}}$. Therefore a and a' are $\mathcal{R}$-isomorphisms and $\overline{\phi d} = \overline{(\phi d, v)}\overline{a}$, so that $\overline{\phi d}\overline{a'} = \overline{(\phi d, v)}$. This shows that $(\phi d, v) = \phi da' + r$ for some r in $\mathcal{R}$, where ϕ and a' are $\mathcal{R}$-isomorphisms. This shows that the property (III)(ii) is satisfied. The property (III)(i) is proved by dual arguments. This completes the proof. □

When $\mathcal{R} = \mathcal{R}_{\mathcal{X}}$ for a subcategory $\mathcal{X}$ of $\mathcal{A}$ consisting of injective or projective objects we see next that this is sufficient for one of the properties (I)(ii) or (II)(ii) to hold.

Proposition 1.3. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Assume that $\mathcal{R} = \mathcal{R}_{\mathcal{X}}$ for a subcategory $\mathcal{X}$ in $\mathcal{A}$.*

(a) *If $\mathcal{X}$ consists of injective objects, then property* (I)(ii) *is satisfied.*
(b) *If $\mathcal{X}$ consists of projective objects, then property* (II)(ii) *is satisfied.*

Proof. Assume that ti is in $\mathcal{R}$ for an inflation i in $\mathcal{A}$. Then we have that following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "u"'] & B \arrow[d, "t"] \arrow[dl, dashed, "u'"'] \\
X \arrow[r, "v"] & Z
\end{tikzcd}$$

with X in $\mathcal{X}$. Since i is an inflation and X is injective, there exists a morphism $u' \colon B \to X$ such that $u'i = u$. Hence $ti = vu = vu'i$ where $r = vu'$ is in $\mathcal{R}$, and therefore condition (I)(ii) is satisfied. The proof of the statement in (b) is dual. $\square$

Next we discuss property (IV).

Proposition 1.4. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category, and let $\mathcal{R}$ be a relation in $\mathcal{A}$.*

(a) *Assume that $\mathcal{R} = \mathcal{R}_{\mathcal{X}}$ for a subcategory $\mathcal{X}$ in $\mathcal{A}$ consisting of projective objects. Then the property* (IV) *is satisfied.*
(b) *Assume that $\mathrm{id}_A + r$ is an isomorphism for all objects A in $\mathcal{A}$ and all r in $\mathcal{R}$. Then the property* (IV) *is satisfied.*

Proof. (a) Let $d \colon B \to C$ be a deflation in $\mathcal{A}$, and let $\phi \colon C' \to C$ be an R-isomorphism. Let $\phi' \colon C \to C'$ be such that $\overline{\phi'} = \overline{\phi}^{-1}$. Consider the following pullback diagrams

$$\begin{tikzcd}
A \arrow[r, "i''"] \arrow[d, equal] & B'' \arrow[r, "d''"] \arrow[d, "\psi'"] & C \arrow[d, "\phi'"] \arrow[dr, "u"] & \\
A \arrow[r, "i'"] \arrow[d, equal] & B' \arrow[r, "d'"] \arrow[d, "\psi"] & C' \arrow[d, "\phi"] & X \arrow[dl, "v"] \arrow[dll, dashed, "v'"'] \\
A \arrow[r, "i"] & B \arrow[r, "d"] & C &
\end{tikzcd}$$

We have that $\phi\phi' = \mathrm{id}_C + vu$ with X in $\mathcal{X}$. Since X is a projective object, there exists a morphism $v' \colon X \to B$ such that $dv' = v$. This implies that $\phi\phi' d'' = d'' + dv'ud'' = \phi d'\psi' = d\psi\psi'$, hence $d(\psi\psi' - v'ud'') = d''$. Moreover $i = \psi i' = \psi\psi' i'' = (\psi\psi' - v'ud'')i''$, since $d''i'' = 0$. This shows that we have the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i''"] \arrow[d, equal] & B'' \arrow[r, "d''"] \arrow[d, "\psi\psi' - v'ud''"] & C \arrow[d, equal] \\
A \arrow[r, "i"] & B \arrow[r, "d"] & C
\end{tikzcd}$$

It follows from this that $\psi\psi' - v'ud''$ is an isomorphism in $\mathcal{A}$ where $v'ud''$ is in $\mathcal{R}$. Hence $\overline{\psi\psi'}$ is an isomorphism in $\mathcal{A}/\mathcal{R}$. Similarly one can prove that $\overline{\psi'\psi''}$ is an isomorphism in $\mathcal{A}/\mathcal{R}$ for some morphism ψ''. Therefore $\overline{\psi'}$ is an isomorphism in $\mathcal{A}/\mathcal{R}$, and consequently $\overline{\psi}$ is an isomorphism in $\mathcal{A}/\mathcal{R}$. This proves that ψ is an $\mathcal{R}$-isomorphism.

(b) Assume that $\mathrm{id}_A + r$ is an isomorphism for all objects A in $\mathcal{A}$ and all r in $\mathcal{R}$. Suppose that $\phi\colon A \to A'$ is an $\mathcal{R}$-isomorphism. Then there exists a morphism $\phi'\colon A' \to A$ such that $\phi'\phi = \mathrm{id}_A + r$ and $\phi\phi' = \mathrm{id}_{A'} + r'$ for some elements r and r' in $\mathcal{R}$. By our assumption the morphisms $\phi'\phi$ and $\phi\phi'$ are isomorphisms. It follows that ϕ is an isomorphism and therefore a morphism ϕ is an $\mathcal{R}$-isomorphism if and only if ϕ is an isomorphism. The claim in (b) follows directly from this fact. □

The property in statement (b) above appeared in the sufficient conditions in [4] for $(\mathcal{A}/\mathcal{R}, \mathcal{E}/\mathcal{R})$ to be an exact category. The application in section 3 show that this condition is not necessary as identity morphisms are in $\mathcal{R}$ in this case. The two previous propositions have the following immediate consequence.

Proposition 1.5. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Assume that $\mathcal{R} = \mathcal{R}_{\mathcal{X}}$ for a subcategory $\mathcal{X}$ in $\mathcal{A}$ consisting of projective injective objects only. Then $\mathcal{E}/\mathcal{R}$ is an exact structure on $\mathcal{A}/\mathcal{R}$ if and only if $\mathcal{R}$ satisfies the properties* (I)(i) *and* (II)(i).

By [6] an exact category $(\mathcal{A}, \mathcal{E})$ is an abelian category if and only if every morphism in $\mathcal{A}$ can be written as a composition of a deflation with an inflation. Hence we have the following.

Proposition 1.6. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category, and let $\mathcal{R}$ be a relation on $\mathcal{A}$ such that $\mathcal{A}/\mathcal{R}$ is an exact category with respect to $\mathcal{E}/\mathcal{R}$. Then $\mathcal{A}/\mathcal{R}$ is an abelian category if and only if for every morphism f in $\mathcal{A}$ there exists an $\mathcal{R}$-deflation d and an $\mathcal{R}$-inflation i such that $f - id$ lies in $\mathcal{R}$.*

2. Almost split pairs

In this section we discuss when the existence of almost split pairs in an exact category $(\mathcal{A}, \mathcal{E})$ is transferred to a factor category $\mathcal{A}/\mathcal{R}$ for some relation $\mathcal{R}$ in $\mathcal{A}$. As we allow identity morphisms to be in $\mathcal{R}$, it should not come as a surprise that the existence of almost split pairs in a factor $\mathcal{A}/\mathcal{R}$ can not be transferred back to $\mathcal{A}$ in general.

Throughout this section we restrict ourselves to Krull-Schmidt categories, that is, additive categories where each object is a finite direct sum of indecomposable objects with local endomorphism rings. Thus the indecomposable objects coincide with those having local endomorphism rings.

A morphism $g\colon Y \to Z$ in a Krull-Schmidt category $\mathcal{A}$ is called *right almost split* if it is not a retraction and for any non-retraction $t\colon A \to Z$ there exists a morphism $s\colon A \to Y$ such that $t = gs$. Given a right almost split map $g\colon Y \to Z$ the set $g\mathcal{A}(Z, Y)$ is the unique maximal proper right ideal of the ring $\mathcal{A}(Z, Z)$ and consequently Z has a local endomorphism ring. In particular, Z has to be indecomposable. We say that $\mathcal{A}$ has *right almost split morphisms* if for all indecomposable objects Z there exists a right almost split morphism ending in Z. Dually we define *left almost split* morphisms. We say that $\mathcal{A}$ has *almost split morphisms* if $\mathcal{A}$ has right and left almost split morphisms.

A morphism $g\colon Y \to Z$ is called *right minimal* if every endomorphism $s\colon Y \to Y$ with the property that $g = gs$, is an isomorphism. Minimal right almost split morphisms ending in an object Z are essentially unique. Namely, if $g\colon Y \to Z$ and $g'\colon Y' \to Z$ are right minimal almost split morphisms, then there is an isomorphism $s\colon Y' \to Y$ satisfying $g' = gs$.

It is shown in [4, Proposition 2.2] that if right (left) almost split morphisms exist, then minimal right (left) almost split morphisms exist. An exact pair $X \xrightarrow{f} Y \xrightarrow{g} Z$

in $\mathcal{E}$ is an *almost split pair* if one of the following three equivalent conditions is satisfied: (i) f is minimal left almost split, (ii) g is minimal right almost split and (iii) f is left almost split and g is right almost split [4, Proposition 2.3]. The exact category $(\mathcal{A}, \mathcal{E})$ is said to have *almost split pairs* if $\mathcal{A}$ has almost split morphisms and moreover for all indecomposable non-projective objects Z there exists an almost split pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ and dually for all indecomposable non-injective objects X there exists an almost split pair $X \xrightarrow{f} Y \xrightarrow{g} Z$. Recall that an object P in $\mathcal{A}$ is called $\mathcal{E}$-projective if each exact pair $X \xrightarrow{i} Y \xrightarrow{d} P$ in $\mathcal{E}$ splits. The $\mathcal{E}$-injective objects are defined dually. For simplicity we say projective and injective objects. Moreover, $\mathcal{E}$-projective and $\mathcal{E}$-injective objects have the same characterization in terms of exactness of Hom-functors as for module categories. By $\mathcal{P}(\mathcal{E})$ and $\mathcal{I}(\mathcal{E})$ we denote the full subcategories of $\mathcal{A}$ consisting of the projective and injective objects respectively.

Now we show that when $(\mathcal{A}, \mathcal{E})$ has almost split pairs and $\mathcal{R}$ is a relation on $\mathcal{A}$ satisfying the conditions (I)–(IV), then the exact factor category $(\mathcal{A}/\mathcal{R}, \mathcal{E}/\mathcal{R})$ has almost split pairs too.

Proposition 2.1. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category and let $\mathcal{R}$ be a relation on $\mathcal{A}$ satisfying conditions* (I)–(IV).

(a) *Then $\mathcal{P}(\mathcal{E}/\mathcal{R}) = \mathcal{P}(\mathcal{E})$ and $\mathcal{I}(\mathcal{E}/\mathcal{R}) = \mathcal{I}(\mathcal{E})$.*
(b) *If a morphism $f\colon A \to B$ is left almost split in $\mathcal{A}$ and id_A does not belong to $\mathcal{R}$, then $\overline{f}\colon A \to B$ is left almost split in $\mathcal{A}/\mathcal{R}$. Dually, if a morphism $g\colon B \to C$ is right almost split in $\mathcal{A}$ and id_C does not belong to $\mathcal{R}$, then $\overline{g}\colon B \to C$ is right almost split in $\mathcal{A}/\mathcal{R}$.*
(c) *Let (f, g) be an almost split pair in $\mathcal{A}$ such that $f\colon A \to B$ and $g\colon B \to C$. Then id_A does not lie in $\mathcal{R}$ if and only if id_C does not lie in $\mathcal{R}$. Moreover, if id_A does not belong to $\mathcal{R}$, then $(\overline{f}, \overline{g})$ is an almost split pair in $\mathcal{A}/\mathcal{R}$.*
(d) *If $\mathcal{A}$ has almost split morphisms (almost split pairs), then $\mathcal{A}/\mathcal{R}$ does.*

Proof. (a) Since every exact pair $(\overline{i}, \overline{d})$ in $\mathcal{E}/\mathcal{R}$ is isomorphic to the residue class of an exact pair in $\mathcal{E}$, it follows easily that $\mathcal{P}(\mathcal{E})$ is contained in $\mathcal{P}(\mathcal{E}/\mathcal{R})$.

Let X be indecomposable in $\mathcal{P}(\mathcal{E}/\mathcal{R})$. In particular, id_X is not in $\mathcal{R}$. Let $\eta\colon A \xrightarrow{i} B \xrightarrow{d} X$ be an exact pair in $\mathcal{E}$. In $\mathcal{A}/\mathcal{R}$ the object X is projective, so that there exists a morphism $g\colon X \to B$ such that $\overline{dg} = \overline{\mathrm{id}_X}$. Hence $dg - \mathrm{id}_X = r$ for some r in $\mathcal{R}$. If r is an isomorphism, then id_X would be in $\mathcal{R}$. Therefore r is not an isomorphism and r is in the radical of $\mathrm{End}_{\mathcal{A}}(X)$, since it is a local ring. Consequently, $\mathrm{id}_X + r$ is an isomorphism and η is a split exact pair. This shows that X is in $\mathcal{P}(\mathcal{E})$.

(b) Since any lifting of a non-retraction (non-section) in $\mathcal{A}/\mathcal{R}$ to $\mathcal{A}$ is a non-retraction (non-section), the proof of (b) is immediate.

(c) If id_A belongs to $\mathcal{R}$, then $\overline{f} = 0$ and $\overline{g}$ becomes a retraction in $\mathcal{A}/\mathcal{R}$. The assumption that id_C does not also lie in $\mathcal{R}$ implies that g is a retraction in $\mathcal{A}$, a contradiction. The rest of (c) is now a consequence of (b).

The assertion (d) follows directly from (a) to (c). □

3. Finitely presented functors

We want to look at an example where the theory developed in the previous section applies. Let Λ be an artin algebra. By $\operatorname{mod}\Lambda$ we denote the category of all finitely generated left Λ-modules and by $\operatorname{mod}(\operatorname{mod}\Lambda)$ the category of finitely presented contravariant functors from $\operatorname{mod}\Lambda$ to abelian groups. This functor category carries the exact structure given by the pointwise exact sequences which is well-known to be abelian. The aim of this section is to realize this abelian category as factor of another exact category by applying our results of section 1. The exact category we need will be introduced next.

Let $T(\Lambda)$ be the lower triangular matrix algebra $\left(\begin{smallmatrix}\Lambda & 0\\ \Lambda & \Lambda\end{smallmatrix}\right)$ which is again an artin algebra. We regard the modules X in $\operatorname{mod} T(\Lambda)$ as triples $X = (X_1, f_X, X_2)$ where X_1, X_2 are in $\operatorname{mod}\Lambda$ and f_X is a homomorphism in $\operatorname{Hom}_\Lambda(X_1, X_2)$. Thus a $T(\Lambda)$-homomorphism $u\colon X \to Y$ is a pair $u = (u_1, u_2)$ of Λ-homomorphisms $u_1\colon X_1 \to Y_1$ and $u_2\colon X_2 \to Y_2$ such that $u_2 f_X = f_Y u_1$. Following [1] we consider the functor $\alpha\colon \operatorname{mod} T(\Lambda) \to \operatorname{mod}(\operatorname{mod}\Lambda)$ which sends a $T(\Lambda)$-module X to the cokernel of the map $\operatorname{Hom}_\Lambda(-, f_X)\colon \operatorname{Hom}_\Lambda(-, X_1) \to \operatorname{Hom}_\Lambda(-, X_2)$. The following proposition is slightly more precise than the statement used in [1, Theorem 1.1].

Proposition 3.1. *The functor $\alpha\colon \operatorname{mod} T(\Lambda) \to \operatorname{mod}(\operatorname{mod}\Lambda)$ is full and dense. Its kernel is the relation $\mathcal{R}_{\mathcal{X}}$ where $\mathcal{X}$ is the full subcategory formed by the modules $(U, 0, 0) \oplus (V, \mathrm{id}_V, V)$ such that U and V are modules in $\operatorname{mod}\Lambda$.*

Proof. Only the description of the kernel of α is not given in [1]. It is obvious that α maps the objects $(U, 0, 0)$ and (V, id, V) to 0.

Let us consider an arbitrary homomorphism $u\colon X \to Y$ such that $\alpha(u) = 0$. Looking at the commutative diagram below we obtain a homomorphism $h\colon X_2 \to Y_1$ such that $u_2 = f_Y h$.

$$\begin{tikzcd}
\operatorname{Hom}_\Lambda(-, X_1) \arrow[r, "\operatorname{Hom}(-,u_1)"] \arrow[d, "\operatorname{Hom}(-,f_X)"'] & \operatorname{Hom}_\Lambda(-, Y_1) \arrow[d, "\operatorname{Hom}(-,f_Y)"] \\
\operatorname{Hom}_\Lambda(-, X_2) \arrow[r, "\operatorname{Hom}(-,u_2)"] \arrow[ur, dashed, "\operatorname{Hom}(-,h)"] \arrow[d] & \operatorname{Hom}_\Lambda(-, Y_2) \arrow[d] \\
\alpha(X) \arrow[r, "\alpha(u)=0"] \arrow[d] & \alpha(Y) \arrow[d] \\
0 & 0
\end{tikzcd}$$

Because of $f_Y(u_1 - h f_X) = 0$ there is $g\colon X_1 \to \operatorname{Ker} f_Y$ such that $f'_Y g = u_1 - h f_X$ where $f'_Y\colon \operatorname{Ker} f_Y \to Y_1$ is the canonical inclusion of the kernel of f_Y. The commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "\left(\begin{smallmatrix} g \\ f_X \end{smallmatrix}\right)"] \arrow[d, "f_X"] & \operatorname{Ker} f_Y \oplus X_2 \arrow[r, "(f'_Y\ h)"] \arrow[d, "(0\ 1)"] & Y_1 \arrow[d, "f_Y"] \\
X_2 \arrow[r, equal] & X_2 \arrow[r, "u_2"] & Y_2
\end{tikzcd}$$

shows that u factors through the object $(\operatorname{Ker} f_Y, 0, 0) \oplus (X_2, \mathrm{id}, X_2)$. $\square$

By Proposition 3.1 we can identify $\operatorname{mod}(\operatorname{mod}\Lambda)$ with $\operatorname{mod} T(\Lambda)/\mathcal{R}_{\mathcal{X}}$. We want to find an exact structure on $\operatorname{mod} T(\Lambda)$ such that the images of the exact pairs of this structure under α are precisely the exact sequences in $\operatorname{mod}(\operatorname{mod}\Lambda)$. Of course the usual exact sequences in $\operatorname{mod} T(\Lambda)$ yield too many pairs for this job.

The ideal $\underline{a} = \left(\begin{smallmatrix}\Lambda & 0\\ \Lambda & 0\end{smallmatrix}\right)$ of $T(\Lambda)$ satisfies $\underline{a}^2 = \underline{a}$ and $T(\Lambda)/\underline{a} \cong \Lambda$. For $X = (X_1, f_X, X_2)$ in $\operatorname{mod} T(\Lambda)$ the module $G(X) = \operatorname{ann}_{\underline{a}}(X)$ is $(0, 0, X_2)$ and the module $G'(X) = X/G(X)$ is $(X_1, 0, 0)$. Note that $G'(X)$ is a module over the factor algebra $T(\Lambda)/\underline{b}$ where the ideal $\underline{b}$ is given by $\underline{b} = G(\Lambda) = \left(\begin{smallmatrix}0 & 0\\ \Lambda & \Lambda\end{smallmatrix}\right)$ and therefore $T(\Lambda)/\underline{b} \cong \Lambda$.

Let us consider another ideal of $T(\Lambda)$ namely $\underline{c} = \left(\begin{smallmatrix}0 & 0\\ \Lambda & 0\end{smallmatrix}\right)$ satisfying $\underline{c}^2 = 0$ and $T(\Lambda)/\underline{c} \cong \Lambda \times \Lambda$. For X in $\operatorname{mod} T(\Lambda)$ we obtain $H(X) = \operatorname{ann}_{\underline{c}}(X) = (\operatorname{Ker} f_X, 0, X_2) = (\operatorname{Ker} f_X, 0, 0) \oplus (0, 0, X_2)$.

If L is one of the functors G, G' or H, then as in [4] we denote by F_L the subfunctor of $\operatorname{Ext}_\Lambda(-,-)$ given by all short exact sequences which are mapped to split exact sequences by L. We observe that $F_H \subseteq F_G$.

[4, Proposition 3.3] shows that F_G, $F_{G'}$, and F_H are closed subfunctors of $\operatorname{Ext}_\Lambda(-,-)$. Consequently, by [4, Corollary 1.5] the subfunctor $F = F_G \cap F_{G'} \cap F_H = F_{G'} \cap F_H$ is closed. Finally, by [4, Proposition 1.4] the set $\mathcal{E}_F$ of pairs (f, g) such that $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is an element of the subfunctor F, provides an exact structure on $\operatorname{mod} T(\Lambda)$.

Exact pairs in $\mathcal{E}_F$ are mapped to pointwise exact sequences in $\operatorname{mod}(\operatorname{mod}\Lambda)$ by α. That all exact sequences of $\operatorname{mod}(\operatorname{mod}\Lambda)$ are obtained in this way, is an immediate application of the Horseshoe Lemma. Hence we arrive at the following result.

Proposition 3.2. *The images of the exact pairs in $\mathcal{E}_F$ under α are precisely the short exact sequences of the exact structure on* $\operatorname{mod}(\operatorname{mod}\Lambda)$ *given by the pointwise exact sequences.*

Proposition 3.2 implies that the factor category $\operatorname{mod} T(\Lambda)/\mathcal{R}_{\mathcal{X}}$ with exact structure $\mathcal{E}_F/\mathcal{R}_{\mathcal{X}}$ can be identified with the abelian category $\operatorname{mod}(\operatorname{mod}\Lambda)$, where $\mathcal{X}$ is the full subcategory formed by the modules $(U, 0, 0) \oplus (V, \mathrm{id}_V, V)$ such that U and V are modules in $\operatorname{mod}\Lambda$. By Theorem 1.1 $\mathcal{R}_{\mathcal{X}}$ has to satisfy the conditions (I) to (IV) and in addition the condition from Proposition 1.6 warranting being abelian. As an illustration we want to check all the conditions directly. For preparation we calculate the $\mathcal{E}_F$-projective and $\mathcal{E}_F$-injective objects in $\operatorname{mod} T(\Lambda)$. We use the duality functor D, the respective transpose functors $\operatorname{Tr}_\Lambda$, $\operatorname{Tr}_{T(\Lambda)}$ and the functor $-^\star = \operatorname{Hom}_\Lambda(-, \Lambda)$ in the formulation and the proof of the following lemma.

Lemma 3.3. *If* $P_1 \xrightarrow{h} P_0 \longrightarrow U \longrightarrow 0$ *is a minimal projective presentation of a module U in* $\operatorname{mod}\Lambda$, *then* $D\operatorname{Tr}_{T(\Lambda)}(0, 0, U) = (D\operatorname{Tr}_\Lambda U, \mathrm{id}, D\operatorname{Tr}_\Lambda U)$, $D\operatorname{Tr}_{T(\Lambda)}(U, \mathrm{id}, U) = (D\operatorname{Tr}_\Lambda U, 0, 0)$, *and* $D\operatorname{Tr}_{T(\Lambda)}(U, 0, 0) = (D(P_1^\star), D(h^\star), D(P_0^\star))$.

Proof. The minimal projective presentation of $(0, 0, U)$ over $T(\Lambda)$ looks as follows.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
P_1 \arrow[r, "h"] & P_0 \arrow[r] & U \arrow[r] & 0
\end{tikzcd}$$

By applying $\operatorname{Hom}_{T(\Lambda)}(-,T(\Lambda))$ to the left square and adding cokernels we obtain the following commutative diagram.

$$\begin{tikzcd}
0 & \operatorname{Tr}_\Lambda U \arrow[l] \arrow[d, equal] & P_1^\star \arrow[l] \arrow[d, equal] & P_0^\star \arrow[l, "h^\star"'] \arrow[d, equal] \\
0 & \operatorname{Tr}_\Lambda U \arrow[l] & P_1^\star \arrow[l] & P_0^\star \arrow[l, "h^\star"']
\end{tikzcd}$$

The left column of this diagram describes $\operatorname{Tr}_{T(\Lambda)}(0,0,U)$. Application of D yields the first equality. The second is proved in a very similar way. For the third we observe that the minimal projective presentation of $(U,0,0)$ over $T(\Lambda)$ has the following shape.

$$\begin{tikzcd}
P_1 \arrow[r, "h"] \arrow[d, "(0\ 1)"] & P_0 \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & 0 \\
P_0 \oplus P_1 \arrow[r, "\binom{1}{h}"] & P_0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Application of $\operatorname{Hom}_{T(\Lambda)}(-,T(\Lambda))$ to the left square and adding cokernels gives the subsequent commutative diagram.

$$\begin{tikzcd}
0 & P_1^\star \arrow[l] & P_0^\star \oplus P_1^\star \arrow[l, "(h^\star\ -1)"'] & P_0^\star \arrow[l, "\binom{1}{h^\star}"'] \\
0 & P_0^\star \arrow[l] \arrow[u, "h^\star"'] \arrow[r, equal] & P_0^\star \arrow[u, "\binom{1}{0}"'] & 0 \arrow[l] \arrow[u]
\end{tikzcd}$$

Hence, the left column describes $\operatorname{Tr}_{T(\Lambda)}(U,0,0)$ and dualizing with D gives the promised result. □

Proposition 3.4. *The $\mathcal{E}_F$-projective and $\mathcal{E}_F$-injective modules in* $\operatorname{mod} T(\Lambda)$ *are given as follows:*

(a) *The indecomposable $\mathcal{E}_F$-projective modules in* $\operatorname{mod} T(\Lambda)$ *are the modules* $(U,0,0)$, $(0,0,U)$, (U,id_U,U) *for U in* $\operatorname{mod}\Lambda$ *indecomposable.*

(b) *The indecomposable $\mathcal{E}_F$-injective modules in* $\operatorname{mod} T(\Lambda)$ *are the modules* $(U,0,0)$, (U,id_U,U), $D\operatorname{Tr}_{T(\Lambda)}(U,0,0)$ *for U in* $\operatorname{mod}\Lambda$ *indecomposable.*

Proof. We use the notation introduced in [4]. From [4, Proposition 3.11] we derive $F_{G'} = F^{\mathcal{X}}$ where $\mathcal{X} = \operatorname{add}(\{(U,0,0) \mid U \in \operatorname{mod}\Lambda\} \cup \{D\operatorname{Tr}_{T(\Lambda)}(0,0,\Lambda)\})$. By [4, Corollary 2.5] $F^{\mathcal{X}}$ coincides with with $F_{\operatorname{Tr}_{T(\Lambda)} D\mathcal{X}}$. Hence by [4, Proposition 3.10] and Lemma 3.3 we arrive at $F = F_{\mathcal{Y}}$ with

$$\mathcal{Y} = \operatorname{add}(\{(U,0,0)\} \cup \{(0,0,U)\} \cup \{\operatorname{Tr}_{T(\Lambda)} D(U,0,0)\})$$

where U is always running through the modules in $\operatorname{mod}\Lambda$. We use [2, Proposition 1.10] to compute the indecomposable $\mathcal{E}_F$-projective modules. Clearly, we have the modules $(U,0,0)$ and $(0,0,U)$ for U in $\operatorname{mod}\Lambda$ indecomposable. From Lemma 3.3 we get the modules (U,id,U) for U in $\operatorname{mod}\Lambda$ indecomposable not projective. To this we have to add all indecomposable projective $T(\Lambda)$-modules. This provides also the (U,id,U) for U in $\operatorname{mod}\Lambda$ indecomposable projective.

[4, Proposition 2.6] allows to calculate the indecomposable $\mathcal{E}_F$-injective modules by forming the $D\operatorname{Tr}_{T(\Lambda)}$-shifts of the indecomposable $\mathcal{E}_F$-projective modules and adding the indecomposable injective $T(\Lambda)$-modules.

Besides the modules $D\operatorname{Tr}_{T(\Lambda)}(U,0,0)$ for U in $\operatorname{mod}\Lambda$ indecomposable we obtain the modules (U,id,U) and $(U,0,0)$ where U is non-injective using again Lemma 3.3. But the adding of the injective modules $T(\Lambda)$-modules provides just the modules of this shape where U is injective. $\square$

We see that $\mathcal{X}$ consists of objects which are $\mathcal{E}_F$-projective and $\mathcal{E}_F$-injective. Hence by Proposition 1.5 we only need to convince ourselves that conditions (I)(i) and (II)(i) are satisfied in order to obtain that $\mathcal{E}_F/R_{\mathcal{X}}$ is an exact structure on $\operatorname{mod}T(\Lambda)/\mathcal{R}_{\mathcal{X}}$. By duality we only look at (I)(i). Let ig be a morphism in $\mathcal{R}_{\mathcal{X}}$ where $i\colon X\to Y$ is an inflation and $g\colon A\to X$ a morphism in $\operatorname{mod}T(\Lambda)$. Thus there is an Λ-homomorphism $s\colon Y_1\to A_2$ such that $f_Y=i_2g_2s$. Hence $i_2f_X=f_Yi_1=i_2g_2si_1$ and therefore $f_X=g_2si_1$ because i_2 is an Λ-monomorphism. This shows that g is in $\mathcal{R}$ and that property (I) (i) is satisfied.

Now we can apply Proposition 2.1 to reprove various results about the functor category $\operatorname{mod}(\operatorname{mod}\Lambda)$.

Corollary 3.5. *The projective objects in* $\operatorname{mod}(\operatorname{mod}\Lambda)$ *are the functors* $\operatorname{Hom}_\Lambda(-,U)$ *and the injective objects are the functors* $D\operatorname{Hom}_\Lambda(U,-)$ *for* U *in* $\operatorname{mod}\Lambda$. *Moreover,* $\operatorname{mod}(\operatorname{mod}\Lambda)$ *has almost split pairs.*

Proof. In order to get the injective objects, we have to calculate $\alpha(D\operatorname{Tr}_{T(\Lambda)}(U,0,0))$. By the last equality of the lemma we have to compute the cokernel of

$$\operatorname{Hom}_\Lambda(-,D(h^\star))\cong D(\operatorname{Hom}_\Lambda(h,-))$$

which is $D\operatorname{Hom}_\Lambda(U,-)$. $\square$

It remains to use Proposition 1.6 in order to see that the exact structure $\mathcal{E}_F/\mathcal{R}_{\mathcal{X}}$ is abelian. We look at a homomorphism $u\colon A\to B$ in $\operatorname{mod}T(\Lambda)$. Let us denote by $f_A'\colon A_0\to A_1$ a kernel map of f_A and by $f_B'\colon B_0\to B_1$ a kernel map of f_B. Now we consider the following pullback diagram.

$$\begin{tikzcd}
B_0 \arrow[r, equal] \arrow[d, "h'"'] & B_0 \arrow[d, "f_B'"] \\
Z \arrow[r, "b"] \arrow[d, "h"'] & B_1 \arrow[d, "f_B"] \\
A_2 \arrow[r, "u_2"] & B_2
\end{tikzcd}$$

In particular, there is a homomorphism $a_1\colon A_1\to Z$ such that $ba_1=u_1$ and $ha_1=f_A$. Because of $ha_1f_A'=f_Af_A'=0$ there is $a_0\colon A_0\to B_0$ such that $a_1f_A'=h'a_0$. By the following diagram we present a product id such that d is a deflation and i is an inflation with respect to $\mathcal{E}_F$. There are obvious $\mathcal{R}_{\mathcal{X}}$-isomorphisms s and t such that $u=tids$.

$$\begin{tikzcd}
A_0 \oplus B_0 \arrow[r, "(a_0\ 1)"] \arrow[d, "\left(\begin{smallmatrix} f'_A & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] & B_0 \arrow[r, "\left(\begin{smallmatrix} 1 \\ h' \end{smallmatrix}\right)"] \arrow[d, "h'"'] & B_0 \oplus Z \arrow[d, "\left(\begin{smallmatrix} f'_B & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \\
A_1 \oplus Z \oplus B_0 \arrow[r, "(a_1\ 1\ h')"] \arrow[d, "\left(\begin{smallmatrix} f_A & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)"'] & Z \arrow[r, "\left(\begin{smallmatrix} b \\ h \\ 1 \end{smallmatrix}\right)"] \arrow[d, "h"'] & B_1 \oplus A_2 \oplus Z \arrow[d, "\left(\begin{smallmatrix} f_B & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)"] \\
A_2 \oplus Z \arrow[r, "(1\ h)"] & A_2 \arrow[r, "\left(\begin{smallmatrix} u_2 \\ 1 \end{smallmatrix}\right)"] & B_2 \oplus A_2
\end{tikzcd}$$

4. Regular modules over wild hereditary algebra

Recently we learned form O. Kerner that the results of this paper and [4] have an application in his study of regular modules over wild hereditary algebras. We give some outline of the situation and refer to [7] for details.

Let k be a field and Q a finite connected quiver without oriented cycles whose underlying graph is neither Dynkin nor extended Dynkin. Then the path algebra $H = kQ$ is hereditary and of wild representation type. In particular, most of the indecomposable H-modules are regular, i.e. they lie in $\mathbb{A}_\infty$-components of the Auslander-Reiten quiver of H. We denote by $\operatorname{reg} H$ the full subcategory of $\operatorname{mod} H$ given by all H-modules whose indecomposable direct summands are regular. Recall, that an indecomposable regular H-module is called quasi-simple if the Auslander-Reiten sequence $0 \longrightarrow \tau_H X \longrightarrow Z \longrightarrow X \longrightarrow 0$ ending in X has an indecomposable middle term Z. In other words this means that X sits at the border of an $\mathbb{A}_\infty$-component. If Q has n vertices and $n > 2$, then there exists a quasi-simple module X satisfying $\operatorname{Ext}^1_H(X, X) = 0$. For such a module X its right perpendicular category $X^\perp$ consisting of all H-modules M satisfying $\operatorname{Hom}_H(X, M) = 0 = \operatorname{Ext}^1_H(X, M)$, is an exact extension-closed abelian full subcategory of $\operatorname{mod} H$ which is equivalent to $\operatorname{mod} C$ where $C = kQ'$ is again a wild hereditary algebra whose connected quiver Q' has $n - 1$ vertices. Moreover, if we identify $X^\perp$ with $\operatorname{mod} C$, then Z is a quasi-simple module in $\operatorname{reg} C$.

In [3] a full and dense functor $F\colon \operatorname{reg} C \to \operatorname{reg} H$ is constructed whose kernel is the relation $\mathcal{R}_\mathcal{X}$ consisting of the morphisms factoring through a module in $\mathcal{X} = \operatorname{add}\{\tau_C^i Z \mid i \in \mathbb{Z}\}$. If one defines for U and W in $\operatorname{reg} C$ the set $E(W, U)$ as the set of all short exact sequences $0 \longrightarrow U \xrightarrow{f} V \longrightarrow W \xrightarrow{g} 0$ such that $\operatorname{Hom}_C(\bar{Z}, g)$ (or equivalently $\operatorname{Hom}_C(f, \bar{Z})$) is surjective, for all objects $\bar{Z}$ in $\mathcal{X}$, then E is an additive subfunctor of the bifunctor $\operatorname{Ext}^1_C\colon (\operatorname{reg} C)^{op} \times \operatorname{reg} C \to \operatorname{mod} k$ which by [4, Proposition 1.7] is closed. Hence, by [4, Proposition 1.4] the category $\operatorname{reg} C$ carries the exact structure $\mathcal{E}_E$. By the definition of E the modules in $\mathcal{X}$ are projective and injective with respect to $\mathcal{E}_E$. From Proposition 1.5 we derive that we only have to check (I)(i) and (II)(i) in order to ensure that $\mathcal{E}_E/\mathcal{R}_\mathcal{X}$ becomes an exact structure for $\operatorname{reg} C/\mathcal{R}_\mathcal{X}$. But this is clear because the functor F maps inflations to monomorphism and deflations to epimorphisms. Thus [7, Theorem 2] can be interpreted as saying that the functor F induces an equivalence of exact categories from $\operatorname{reg} C/\mathcal{R}_\mathcal{X}$ equipped with the exact structure $\mathcal{E}_E/\mathcal{R}_\mathcal{X}$ to $\operatorname{reg} H$ equipped with the usual exact structure inherited from $\operatorname{mod} H$.

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Fakultät für Mathematik, Universität Bielefeld, Postfach 100 131, D–33501 Bielefeld, Germany

E-mail address: draexler@mathematik.uni-bielefeld.de

Institutt for matematiske fag, NTNU, Lade, N–7491 Trondheim, Norway

E-mail address: oyvinso@math.ntnu.no