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Diffusions on "simple" configuration spaces

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Dedicated to Sergio Albeverio on the occasion of his 60th birthday.

Abstract. The purpose of this paper is to develop general techniques to prove that diffusions associated with Dirichlet forms on multiple configuration spaces $\bar{\Gamma}_E$ (i.e., particle spaces with possibly several particles at the same point) live on the so-called "simple" configuration space Γ_E (i.e., particle spaces with at most one particle at a point). Here E is a metric space. Our results can be considered as a completion of our previous paper [**24**] where we constructed diffusions on multiple configuration spaces $\bar{\Gamma}_E$ using Dirichlet forms theory. Indeed, all processes constructed in [**24**] live in fact on Γ_E as follows from the results of this paper. We discuss in particular detail the case where E is the free loop space over $\mathbb{R}^d$, where $\mathbb{R}^d$ is equipped with a non-trivial Riemannian metric. This case is of relevance for continuous systems in quantum statistical physics.

Contents

1. Introduction

In [**24**], among other things, the authors extended the work initiated in [**28, 37**], providing a complete proof that for a large class of Dirichlet forms with square field

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operator there exist associated diffusions on the multiple configuration space $\bar{\Gamma}_E$ (i.e. the particle space over E with possibly several particles at the same point; see Sect. 2 below for the exact definition of $\bar{\Gamma}_E$), with E being an arbitrary complete separable metric space.

In particular, the authors discussed two typical examples arising from statistical mechanics: First, the case where $E = X$ with X being a complete Riemannian manifold, and second, the case $E = \mathcal{L}(\mathbb{R}^d)$ with $\mathcal{L}(\mathbb{R}^d)$ being the free loop space over $\mathbb{R}^d$, equipped with a general elliptic metric.

The argument used in [**24**] is to show that if $\bar{\Gamma}_E$ is given a suitable completely metrizable topology (which is the vague topology in case E is locally compact, cf. [**24**, Sect. 3] and Sect. 2 below), then under very general assumptions said Dirichlet forms are quasi-regular on $\bar{\Gamma}_E$ in the sense of [**23, 7**]. Therefore, by the main result in [**23, 7**] (see also [**25**]), one obtains the corresponding diffusions on $\bar{\Gamma}_E$. For a detailed discussion, see Sect. 4 of [**24**].

However, since the above methods rely on the completeness of the metric space $\bar{\Gamma}_E$ one can only locate the diffusion on the multiple configuration space $\bar{\Gamma}_E$ instead of the usual (from the point of view of physics more natural) "simple" configuration space Γ_E (i.e. the particle space over E with at most one particle at a point), which in general is not complete. However, it has been proved in [**33**] that, if $E = \mathbb{R}^d$, then under mild conditions the set $\bar{\Gamma}_E \setminus \Gamma_E$ is exceptional in the sense of Dirichlet forms (see [**23**]), and hence the corresponding diffusion lives in fact on the "simple" configuration space Γ_E.

The present paper aims at extending the results in [**33**]. We shall develop a general method based on the idea in [**33**] to show that for any given complete separable metric space E as base space, no matter if it is finite or infinite dimensional, the set $\bar{\Gamma}_E \setminus \Gamma_E$ is exceptional with respect to the Dirichlet forms discussed in [**24**] and, consequently, the diffusions constructed there are in fact living on the "simple" configuration space Γ_E.

In particular, for E being a Riemannian manifold or the free loop space as in the situation of the two main examples discussed in [**24**], we can locate the corresponding diffusions on Γ_E, as required in models of statistical mechanics.

The organization of this paper is as follows:

In Section 2 we review the topology and the metric ρ introduced in [**24**] on $\bar{\Gamma}_E$.

In Section 3 we present a condition for a probability measure μ on the Borel σ–field $\mathcal{B}(\bar{\Gamma}_E)$ (w.r.t the ρ—topology) on $\bar{\Gamma}_E$, implying that $\mu(\bar{\Gamma}_E \setminus \Gamma_E) = 0$ (cf. Theorem 3.1 below).

In Section 4 we prove our main result, i.e. that under quite weak, easy to check conditions $\bar{\Gamma}_E \setminus \Gamma_E$ is even exceptional w.r.t. the corresponding Dirichlet form defined in [**24**] (see Theorem 4.3 below).

The application to the free loop space mentioned above is discussed in the concluding Section 5.

2. $\{E_k\}$–vague topology for the multiple configuration space $\bar{\Gamma}_E$

For the convenience of the reader, we start with a review of the topology on $\bar{\Gamma}_E$ introduced in [**24**]. On the way we also fix some notations for later.

In what follows, let (E, ρ) be a separable metric space with Borel σ–field $\mathcal{B}(E)$. For a set $A \subset E$ and any $\varepsilon > 0$, we set

$$A^\varepsilon := \{x \in E \,|\, \rho(x, A) < \varepsilon\}.$$

We call $\{E_k\}_{k\geq 1}$ an *exhausting sequence*, if $\{E_k\}$ is an increasing sequence of open sets in E, such that $\bigcup_{k\geq 1} E_k = E$. $\{E_k\}_{k\geq 1}$ will be called a *well-exhausting sequence* if, in addition, there exists a sequence $\{\delta_k\}_{k\geq 1}$ of strictly positive numbers such that

$$E_k^{\delta_k} \subset E_{k+1} \quad \forall k \in \mathbb{N}. \tag{2.1}$$

Let $\{E_k\}_{k\geq 1}$ be an exhausting sequence. We shall write $\gamma \in \mathcal{M}(\{E_k\})$ if γ is a positive measure on $\mathcal{B}(E)$ and $\gamma(E_k) < \infty$ for all $k \in \mathbb{N}$.

In general, for any exhausting sequence $\{A_k\}_{k\geq 1}$, we can always find a well-exhausting sequence $\{E_k\}_{k\geq 1}$, such that $\mathcal{M}(\{A_k\}) \subset \mathcal{M}(\{E_k\})$ (cf. [**24**, Lemma 3.4]). In particular, if $E_k = E$ for all $k \geq 1$, we write $\mathcal{M}(E)$ instead of $\mathcal{M}(\{E_k\})$. That is, $\mathcal{M}(E)$ is the family of all finite positive measures on $\mathcal{B}(E)$. For any $\mathcal{B}(E)$–measurable function f on E and any positive measure γ on $\mathcal{B}(E)$ we write $\bar{f}(\gamma) := \int_E f\,d\gamma$, provided the integral makes sense. For $A \in \mathcal{B}(E)$ and $\varepsilon > 0$ we set

$$g_{A,\varepsilon}(x) := \frac{1}{1+\varepsilon}\big(\varepsilon - \rho(x,A) \wedge \varepsilon\big), \quad x \in E, \tag{2.2}$$

and define for $\gamma_1, \gamma_2 \in \mathcal{M}(E)$

$$\bar{\rho}_0(\gamma_1,\gamma_2) := \sup\Big\{\big|\bar{g}_{A,\varepsilon}(\gamma_1) - \bar{g}_{A,\varepsilon}(\gamma_2)\big| \,\Big|\, A \in \mathcal{B}(E), \varepsilon > 0\Big\} \tag{2.3}$$

Then $\bar{\rho}_0$ is a separable metric on $\mathcal{M}(E)$ and the $\bar{\rho}_0$–topology coincides with the topology of weak convergence on $\mathcal{M}(E)$. Moreover, $\bar{\rho}_0$ is complete on $\mathcal{M}(E)$ if ρ is complete on E (cf. [**24**, Thm. 3.2]).

In what follows, we fix a well exhausting sequence $\{E_k\}_{k\geq 1}$. For example, $E_k = \{x \in E \,|\, \rho(x,x_0) < k\}$ for some fixed point $x_0 \in E$. We shall write $f \in C_0(\{E_k\})$ if $f \in C_b(E)$ ($C_b(E) :=$ the set of all bounded continuous functions on E) and $\mathrm{supp}[f] \subset E_k$ for some $k \in \mathbb{N}$. Note that if $f \in C_0(\{E_k\})$, then $\bar{f}(\gamma)$ is well-defined for all $\gamma \in \mathcal{M}(\{E_k\})$. Let $\gamma, \gamma_n \in \mathcal{M}(\{E_k\})$, $n \geq 1$. We shall say that $(\gamma_n)_{n\geq 1}$ *converges to* γ $\{E_k\}$*–vaguely*, if

$$\big|\bar{f}(\gamma_n) - \bar{f}(\gamma)\big| \longrightarrow 0 \quad \forall f \in C_0(\{E_k\}). \tag{2.4}$$

If E is locally compact and E_k is relatively compact for all k, then $\mathcal{M}(\{E_k\})$ is exactly the family of all positive Radon measures on E and $\{E_k\}$–vague convergence coincides with the usual vague convergence for Radon measures. In this case, there is a complete separable metric (Kallenberg metric) on $\mathcal{M}(\{E_k\})$ which induces the topology of vague convergence. Unfortunately, the Kallenberg metric does not work very well for non-locally compact spaces E.

Nevertheless, we proved in [**24**] that there always exists a separable metric $\bar{\rho}$ on $\mathcal{M}(\{E_k\})$, inducing the topology of $\{E_k\}$–vague convergence. We now give a short description of $\bar{\rho}$.

Let $\{\delta_k\}_{k\geq 1}$ be a sequence of positive numbers so that (2.1) holds. We set (cf. (2.2)) $\phi_k(x) = \frac{1+\delta_k}{\delta_k} g_{E_k,\delta_k}(x)$ and define for $\gamma_1, \gamma_2 \in \mathcal{M}(\{E_k\})$,

$$\bar{\rho}(\gamma_1,\gamma_2) = \sup_{k\geq 1} 2^{-k}\Big(1 \wedge \bar{\rho}_0\big(\phi_k\cdot\gamma_1, \phi_k\cdot\gamma_2\big)\Big). \tag{2.5}$$

THEOREM 2.1. *(cf. [**24**, Thm. 3.6]) $\big(\mathcal{M}(\{E_k\}), \bar{\rho}\big)$ is a separable metric space. The induced topology is the topology of $\{E_k\}$–vague convergence. Moreover, $\bar{\rho}$ is a complete metric on $\mathcal{M}(\{E_k\})$ if ρ is complete on E.*

Set $\bar{\mathbb{N}} = \mathbb{N} \cup \{0, \infty\}$ and denote by $\bar{\Gamma}_E := \bar{\Gamma}_E(\{E_k\})$ the set of elements $\gamma \in \mathcal{M}(\{E_k\})$ for which $\gamma(A) \in \bar{\mathbb{N}}$ for all $A \in \mathcal{B}(E)$. Denote by Γ_E the subset of $\bar{\Gamma}_E$ consisting of all γ such that $\gamma(\{x\}) \leq 1$ for all $x \in E$. Then we have for all $\gamma \in \bar{\Gamma}_E$ that

$$\gamma = \sum_{x \in \operatorname{supp} \gamma} \gamma(\{x\}) \varepsilon_x,$$

where ε_x denotes the Dirac measure in x and $\operatorname{supp} \gamma := \{x \in E \,|\, \gamma(\{x\}) > 0\}$ is countable. Similarly, for all $\gamma \in \Gamma_E$ we have

$$\gamma = \sum_{x \in \operatorname{supp} \gamma} \varepsilon_x.$$

$\bar{\Gamma}_E$ is called the *multiple configuration space over* E (relative to $\{E_k\}$), while Γ_E is the (simple) *configuration space over* E. It is known that $\bar{\Gamma}_E$ is a closed subset of $\mathcal{M}(\{E_k\})$ in the ρ–topology (cf. [**24**, Prop. 3.9]) and Γ_E is a G_δ subset of $\bar{\Gamma}_E$ with respect to the $\bar{\rho}$–topology (cf. [**24**, Prop. 3.10]).

3. Measures not charging $\bar{\Gamma}_E \setminus \Gamma_E$

Let $\mathcal{B}(\bar{\Gamma}_E)$ be the Borel σ–field of $\bar{\Gamma}_E$ (w.r.t. $\bar{\rho}$) and μ be a probability measure on $(\bar{\Gamma}_E, \mathcal{B}(\bar{\Gamma}_E))$. We always assume that

$$(\mu.1) \qquad \int_{\bar{\Gamma}_E} \gamma(E_k)\, \mu(d\gamma) < \infty \quad \forall k \in \mathbb{N}$$

Note that if we define

$$(3.1) \qquad \sigma^\mu(A) := \int_{\bar{\Gamma}_E} \gamma(A)\, \mu(d\gamma), \quad a \in \mathcal{B}(E),$$

then $(\mu.1)$ is equivalent to saying that $\sigma^\mu \in \mathcal{M}(\{E_k\})$. Below we fix a measure $\sigma \in \mathcal{M}(\{E_k\})$. For the discussion of closability of pre-Dirichlet forms on $\bar{\Gamma}_E$, in [**24**] we imposed the following assumption $(\mu.\sigma)$ on μ with respect to σ:

$(\mu.\sigma)$ There exists a $\mathcal{B}(\bar{\Gamma}_E) \otimes \mathcal{B}(E)$–measurable function $\rho : \bar{\Gamma}_E \times E \longrightarrow \mathbb{R}_+$ such that for all $\mathcal{B}(\bar{\Gamma}_E) \otimes \mathcal{B}(E)$–measurable functions $h : \bar{\Gamma}_E \times E \longrightarrow \mathbb{R}_+$

$$\int_{\bar{\Gamma}_E} \int_E h(\gamma, x)\, \gamma(dx)\, \mu(d\gamma) = \int_{\bar{\Gamma}_E} \int_E h(\gamma + \varepsilon_x, x) \rho(\gamma, x)\, \sigma(dx)\, \mu(d\gamma).$$

There are many examples for which conditions $(\mu.1)$ and $(\mu.\sigma)$ are satisfied. Here we mention three situations which have been discussed in [**24**].

1. Poisson measures:
   Let $\sigma \in \mathcal{M}(\{E_k\})$ and $\mu := \pi_\sigma$ be the Poisson measure on $(\bar{\Gamma}_E, \mathcal{B}(\bar{\Gamma}_E))$ with intensity σ, i.e. π_σ is the unique probability measure on $(\bar{\Gamma}_E, \mathcal{B}(\bar{\Gamma}_E))$ such that

$$(3.2) \qquad \int e^{\bar{f}(\gamma)}\, \pi_\sigma(d\gamma) = \exp\left\{ \int \left(e^f - 1\right) d\sigma \right\} \quad \forall f \in C_0(\{E_k\}).$$

   Then it is well-known that $\mu := \pi_\sigma$ satisfies conditions $(\mu.1)$ (since $\sigma^\mu = \sigma$) and $(\mu.\sigma)$ with $\rho(\gamma, x) = 1$ for all $x \in E$, $\gamma \in \bar{\Gamma}_E$. Condition $(\mu.\sigma)$ is just the so-called Mecke identity (cf. [**26**, Satz 3.1]) in this case.

2. Mixed Poisson measures:
Mixed Poisson measures on $\bar{\Gamma}_E$ are defined by

$$\mu = \int_{\mathbb{R}_+} \pi_{z\sigma}\, \lambda(dz), \tag{3.3}$$

where $\pi_{z\sigma}$ is a Poisson measure with intensity $z\sigma$ and λ is a probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that

$$\int_{\mathbb{R}_+} z\, \lambda(dz) < \infty. \tag{3.4}$$

It is easy to check that a mixed Poisson measure μ satisfies $(\mu.1)$ and $(\mu.\sigma)$.
3. Gibbs measures:
A more general class of measures satisfying $(\mu.1)$ and $(\mu.\sigma)$ are Gibbs measures, for which we refer to [**12**, Sect. 6] and [**30**, Sect. 6] for the definition and a detailed discussion. Here we only mention that the above mixed Poisson measures are in fact special canonical Gibbs measures with zero potentials. In quantum statistical mechanics (mixed) Poisson measures are equilibrium measures for random particles that act independently (“free case”) while Gibbs measures are equilibrium measures for random particles that interact via potential functions (“interaction case”).

For our purpose, we would like to point out that in the situations of (1) and (2), and in some cases of situation (3),[1] the measure μ on $(\bar{\Gamma}_E, \mathcal{B}(\bar{\Gamma}_E))$ has the following property $(\mu.\sigma.1)$ relative to the “downstairs” measure σ.

$(\mu.\sigma.1)$**:** For each $k \in \mathbb{N}$, there exists a constant C_k, such that

$$\int_{\{\gamma:\gamma(B)\geq 2\}} \mu(d\gamma) \leq C_k \sigma(B)^2 \quad \forall B \in \mathcal{B}(E), B \subset E_k.$$

The remainder of this section is devoted to proving the following result:

THEOREM 3.1. *If $(E, \mathcal{B}(E))$ is a measurable Lusin space (e.g. (E,ρ) is complete) and $\sigma(\{x\}) = 0$ for all $x \in E$, then $(\mu.\sigma.1)$ implies that $\mu(\bar{\Gamma}_E \setminus \Gamma_E) = 0$.*

Before proving the above theorem, we introduce more concepts and notations, also to be used in subsequent sections.

Let A be a Borel subset of E. We say that $\mathcal{F}(A)$ is a *finite Borel covering* of A (abbreviated FBC) if $\mathcal{F}(A)$ is a finite family of Borel subsets of A such that A is the union of these subsets. The number of non-empty Borel subsets in $\mathcal{F}(A)$ is denoted by $|\mathcal{F}(A)|$.

Let $\{\mathcal{F}_n(A)\}_{n\in\mathbb{N}}$ be a sequence of FBC of A. We shall write $B \in \{\mathcal{F}_n(A)\}$ if for any $n \in \mathbb{N}$ there exists a Borel set $B_n \in \mathcal{F}_n(A)$ such that $B \subset B_n$.

Let $\mathcal{F}_1(A)$ and $\mathcal{F}_2(A)$ be two FBCs of A. We write $\mathcal{F}_1(A) \prec \mathcal{F}_2(A)$ if $|\mathcal{F}_1(A)| \leq |\mathcal{F}_2(A)|$ and for each $B_2 \in \mathcal{F}_2(A)$ there exists $B_1 \in \mathcal{F}_1(A)$ such that $B_2 \subset B_1$.

For any measure ν we denote the associated outer measure by ν^*.

As we shall see, the following lemma implies Theorem 3.1.

LEMMA 3.2. *Let A be a Borel subset of E_k and $\{\mathcal{F}_n(A)\}_{k\in\mathbb{N}}$ a sequence of FBCs of A. Suppose that the following conditions hold:*

1. $|\mathcal{F}_n(A)| \longrightarrow \infty$ *as* $n \to \infty$;

[1] e.g. for the class of grand canonical Gibbs measures which appear in classical statistical mechanics of continuous systems, such as Ruelle measures, see [**34, 30**] and references therein

2. *There exists a constant* α, *such that*

$$|\mathcal{F}_n(A)|\sigma(B) \leq \alpha \quad \forall B \in \mathcal{F}_n(A). \tag{3.5}$$

Then $(\mu.\sigma.1)$ *implies*

$$\mu^*\Big\{\gamma \in \bar{\Gamma}_E \,\Big|\, \gamma^*(B) \geq 2 \textit{ for some } B \in \big\{\mathcal{F}_n(A)\big\}\Big\} = 0. \tag{3.6}$$

PROOF. For any $N \in \mathbb{N}$, we have by $(\mu.\sigma.1)$ and (3.5)

$$\begin{aligned}
&\mu^*\Big\{\gamma \in \bar{\Gamma}_E \,\Big|\, \gamma^*(B) \geq 2 \text{ for some } B \in \big\{\mathcal{F}_n(A)\big\}\Big\} \\
&\qquad\qquad \leq \mu\Big\{\gamma \in \bar{\Gamma}_E \,\Big|\, \gamma(B_{Nj}) \geq 2 \text{ for some } B_{Nj} \in \mathcal{F}_N(A)\Big\} \\
&\leq \sum_{j=1}^{|\mathcal{F}_N(A)|} \mu\big\{\gamma \in \bar{\Gamma}_E \,\big|\, \gamma(B_{Nj}) \geq 2\big\} \leq C_k \sum_{j=1}^{|\mathcal{F}_N(A)|} \sigma(B_{Nj})^2 \leq \alpha^2 C_k \big|\mathcal{F}_N(A)\big|^{-1},
\end{aligned}$$

where $\mathcal{F}_N(A) = \big\{B_{Nj}\big\}_{1\leq j\leq|\mathcal{F}_N(A)|}$. Hence, the assertion follows from assumption (1). $\square$

PROOF OF THEOREM 3.1. We only need to show that for any $k \in \mathbb{N}$

$$\mu^*\big\{\gamma \in \bar{\Gamma}_E \,\big|\, \gamma(\{x\}) \geq 2 \text{ for some } x \in E_k\big\} = 0 \tag{3.7}$$

Since E_k is a Borel subset of a Polish space, we can find a Borel isomorphism $J : E_k \longrightarrow S \subset [0,1]$. Let m be the image measure of $\sigma\big|_{E_k}$ on $\big(S, \mathcal{B}(S)\big)$. Then m is generated by a right continuous increasing function $F : [0,1] \longrightarrow [0,\infty[$ with $F(0) = 0$. Due to the fact that $\sigma(\{x\}) = 0$ for all $x \in E$, we have that $F(t)$ is continuous in t. For any $n \in \mathbb{N}$, we can find $0 = t_0 < t_1 < t_2 < \cdots < t_{2^n} = 1$, such that $F(t_j) - F(t_{j-1}) = 2^{-n}\sigma(E_k)$ for all $1 \leq j \leq 2^n$. Let $B_{nj} := J^{-1}\big(S \cap [t_{j-1}, t_j]\big)$ and $\mathcal{F}_n(E_k) = \big\{B_{nj} \,\big|\, 1 \leq j \leq 2^n\big\}$. Then $\big\{\mathcal{F}_n(E_k)\big\}_{n\in\mathbb{N}}$ satisfies 3.2(1) and (3.5), and (3.7) follows from Lemma 3.2. $\square$

REMARK 3.3. We emphasize that in case μ is the Poisson measure or in case $E = \mathbb{R}^d$ and μ is a Gibbs measure, the assertion of Theorem 3.1 is well-known (see e.g. [**4**] and references therein).

4. Dirichlet forms with exceptional set $\bar{\Gamma}_E \setminus \Gamma_E$

Let μ and σ be as before. In this section we assume that a quasi-regular Dirichlet form $\big(\mathcal{E}^\Gamma, D(\mathcal{E}^\Gamma)\big)$ is given on $L^2(\bar{\Gamma}_E, \mu)$. Since our main concern are the Dirichlet forms discussed in [**24**], we always assume that $\big(\mathcal{E}^\Gamma, D(\mathcal{E}^\Gamma)\big)$ is of square field operator type. More precisely, we assume that there is a positive definite symmetric bilinear map (square field operator) $S^\Gamma : D(\mathcal{E}^\Gamma) \times D(\mathcal{E}^\Gamma) \longrightarrow L^1(\bar{\Gamma}_E, \mu)$ such that

$$\mathcal{E}^\Gamma(F,G) = \int S^\Gamma(F,G)\,\mu(d\gamma) \quad \forall F,G \in D(\mathcal{E}^\Gamma) \tag{4.1}$$

and the *chain rule* holds, i.e. if $F_1, \dots, F_N, G_1, \dots, G_N \in D(\mathcal{E}^\Gamma)$, $\varphi, \psi \in C_b^\infty(\mathbb{R}^N)$, then for all $\gamma \in \bar{\Gamma}_E$

$$S^\Gamma\big(\varphi(F_1, \dots, F_N), \psi(G_1, \dots, G_N)\big)(\gamma) = \sum_{i,j=1}^{N} \partial_i \varphi(F_1, \dots, F_N)\, \partial_j \psi(G_1, \dots, G_N)\, S^\Gamma(F_i, G_j)(\gamma) \tag{4.2}$$

Below, we shall denote $S^\Gamma(F, F)$ by $S^\Gamma(F)$ for $F \in D(\mathcal{E}^\Gamma)$.

REMARK 4.1.

1. We emphasize that all the Dirichlet forms discussed in [**24**] satisfy (4.1) and (4.2).
2. The square field operator S^Γ with (4.1) and (4.2) has also the following properties which will be used later. For a proof, we refer to [**32**].
   (a) $F, G \in D(\mathcal{E}^\Gamma)$ implies $F \vee G, F \wedge G \in D(\mathcal{E}^\Gamma)$ and

$$\begin{aligned} S^\Gamma(F \vee G) &\le S^\Gamma(F) \vee S^\Gamma(G) \quad \mu\text{–a.e.} \\ S^\Gamma(F \wedge G) &\le S^\Gamma(F) \vee S^\Gamma(G) \quad \mu\text{–a.e.} \end{aligned} \tag{4.3}$$

   (b) (strong local property) If $F \in D(\mathcal{E}^\Gamma)$ and F is constant μ–a.e. on an open set $U \subset \bar{\Gamma}_E$, then

$$S^\Gamma(F) = 0 \quad \mu\text{–a.e. on } U. \tag{4.4}$$

3. With the above assumption, by [**23**, Thms. IV.3.5 and V.1.11] we know that (cf. [**24**, Thm. 4.13]) there exists a conservative (strong Markov) diffusion process

$$\mathbb{M} = \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, (\theta_t)_{t \ge 0}, (X_t)_{t \ge 0}, (P_\gamma)_{\gamma \in \bar{\Gamma}_E}\big)$$

on $\bar{\Gamma}_E$ (cf. [**15**]), which is properly associated with $\big(\mathcal{E}^\Gamma, D(\mathcal{E}^\Gamma)\big)$, i.e. for all ($\mu$–versions of) $F \in L^2(\bar{\Gamma}_E, \mu)$ and all $t > 0$, the function

$$\gamma \longmapsto p_t F(\gamma) := \int_\Omega F(X_t)\, dP_\gamma, \quad \gamma \in \bar{\Gamma}_E, \tag{4.5}$$

is an $\mathcal{E}^\Gamma$–quasi-continuous μ–version of $e^{tH_\mu^\Gamma} F$, where H_μ^Γ is the generator of $\big(\mathcal{E}^\Gamma, D(\mathcal{E}^\Gamma)\big)$ (cf. [**23**, Sect. I.2]). $\mathbb{M}$ is unique up to μ–equivalence (cf. [**23**, Sect. IV.6]). In particular, $\mathbb{M}$ is μ–symmetric (i.e. $\int G\, p_t F\, d\mu = \int F\, p_t G\, d\mu$ for all $F, G : \bar{\Gamma}_E \longrightarrow \mathbb{R}_+$, $\mathcal{B}(\bar{\Gamma}_E)$–measurable) and has μ as an invariant measure.

We now impose some further restrictions on μ (relative to σ) and $\big(\mathcal{E}^\Gamma, D(\mathcal{E}^\Gamma)\big)$ as follows.

($\mu.\sigma.2$): For each $k \in \mathbb{N}$, there exists a constant C_k, such that

$$\int_{\{\gamma : \gamma(B) \ge 2\}} \gamma(B)\, \mu(d\gamma) \le C_k \sigma(B)^2 \quad \forall B \in \mathcal{B}(E), B \subset E_k.$$

($\mathcal{E}^\Gamma.1$): Let $A \in \mathcal{B}(E)$. If $A^{2\varepsilon} \subset E_k^{\delta_k}$ for some $k \in \mathbb{N}$ and $\varepsilon > 0$, then $\bar{g}_{A,\varepsilon} \in D(\mathcal{E}^\Gamma)$ and

$$S^\Gamma(\bar{g}_{A,\varepsilon})(\gamma) \le \int_{A^{2\varepsilon}} \xi_k(x)^2\, \gamma(dx) \quad \mu\text{–a.e.},$$

where ξ_k is a measurable function on E, such that

$$M_k := \sup_{x \in E_k^{\delta_k}} \xi_k(x)^2 < \infty.$$

REMARK 4.2.

1. Clearly, all Poisson measures satisfy $(\mu.\sigma.2)$. If (3.4) is strengthened by

$$\int_{\mathbb{R}_+} z^2 \, \lambda(dz) < \infty, \tag{4.6}$$

then the mixed Poisson measures in Situation (2) of Sect. 3 satisfy $(\mu.\sigma.2)$. Also, in some cases of Situation (3) of Sect.3, e.g. in the case of Ruelle measures, $(\mu.\sigma.2)$ is fulfilled.
2. If (E, ρ) is complete and if $\big(\mathcal{E}^\Gamma, D(\mathcal{E}^\Gamma)\big)$ satisfies Condition $(\bar{Q})$, specified in Sect. 4 of [**24**], then it follows from [**24**, Cond. $(S.1)$ (in Sect. 1), Lem. 4.10 and Prop. 4.6], that $\big(\mathcal{E}^\Gamma, D(\mathcal{E}^\Gamma)\big)$ satisfies $(\mathcal{E}^\Gamma.1)$. We refer to [**24**] and Sect. 5 below for concrete examples.

THEOREM 4.3. *Let A be a Borel subset of E_k and $\big\{\mathcal{F}_n(A)\big\}_{n\in\mathbb{N}}$ be a sequence of FBCs of A. Suppose that*

$$\mathcal{F}_n(A) \prec \mathcal{F}_{n+1}(A) \text{ for all } n \in \mathbb{N} \text{ and } \big|\mathcal{F}_n(A)\big| \longrightarrow \infty \text{ as } n \to \infty \tag{4.7}$$

and that there exists a decreasing sequence of numbers $\{\varepsilon(n)\}_{n\in\mathbb{N}} \subset \,]0, \frac{\delta_k}{2}[$ and a constant $\alpha > 0$, such that

$$\inf_{n\in\mathbb{N}} \varepsilon(n)^2 \big|\mathcal{F}_n(A)\big| > 0, \tag{4.8}$$

$$\big|\mathcal{F}_n(A)\big| \sigma\big(B^{2\varepsilon(n)}\big) \leq \alpha \quad \forall B \in \mathcal{F}_n(A). \tag{4.9}$$

Then, under conditions $(\mu.\sigma.2)$ and $(\mathcal{E}^\Gamma.1)$, we have that

$$\Big\{\gamma \in \bar{\Gamma}_E \,\Big|\, \gamma^*(B) \geq 2 \text{ for some } B \in \big\{\mathcal{F}_n(A)\big\}\Big\}$$

is $\mathcal{E}^\Gamma$–exceptional.

PROOF. Let ψ be a smooth increasing function on $\mathbb{R}$, such that $\psi(t) = 0$ for all $t \leq \frac{3}{2}$ and $\psi(t) = 1$ for all $t \geq 2$. Let $\mathcal{F}_n(A) = \big\{B_{nj} \,\big|\, 1 \leq j \leq \big|\mathcal{F}_n(A)\big|\big\}$. For each $n \in \mathbb{N}$ we define

$$\varphi_{nj}(\gamma) := \big(\varepsilon(n)^{-1} + 1\big)\bar{g}_{B_{nj},\varepsilon(n)}(\gamma), \quad B_{nj} \in \mathcal{F}_n(A),$$

and

$$u_{nj}(\gamma) := \psi\big(\varphi_{nj}(\gamma)\big), \quad 1 \leq j \leq \big|\mathcal{F}_n(A)\big|,$$

$$u_n(\gamma) := \bigvee_{j=1}^{|\mathcal{F}_n(A)|} u_{nj}(\gamma).$$

By Theorem 2.1 φ_{nj} is continuous in γ. Hence, u_{nj} is continuous in γ. We note that $u_{nj}(\gamma) = 1$ if $\gamma(B_{nj}) \geq 2$ and $u_{nj}(\gamma) = 0$ if $\varphi_{nj}(\gamma) < \frac{3}{2}$. Thus, by (4.4) and $(\mathcal{E}^\Gamma.1)$ we have

$$\begin{aligned} S^\Gamma\big(u_{nj}\big)(\gamma) &\leq M_k \|\psi'\|_\infty \big(\varepsilon(n)^{-1} + 1\big)^2 \gamma\big(B_{nj}^{2\varepsilon(n)}\big) 1_{\{\varphi_{nj} \geq \frac{3}{2}\}}(\gamma) \\ &\leq M_k \|\psi'\|_\infty \big(\varepsilon(n)^{-1} + 1\big)^2 \gamma\big(B_{nj}^{2\varepsilon(n)}\big) 1_{\{\gamma : \gamma(B_{nj}^{2\varepsilon(n)}) \geq 2\}}(\gamma) \quad \mu\text{–a.e.} \end{aligned}$$

Furthermore, $u_n(\gamma)$ is continuous in γ and

$$u_n(\gamma) = 1 \quad \text{if} \quad \sup_j \gamma(B_{nj}) \geq 2, \tag{4.10}$$

$$u_n(\gamma) = 0 \quad \text{if} \quad \sup_j \varphi_{nj}(\gamma) < \tfrac{3}{2}. \tag{4.11}$$

Moreover, by (4.3) we have

$$\begin{aligned} S^\Gamma(u_n) &\leq M_k\big(\varepsilon(n)^{-1}+1\big)^2 \sup_j \Big[\gamma\big(B_{nj}^{2\varepsilon(n)}\big)1_{\{\gamma:\gamma(B_{nj}^{2\varepsilon(n)})\geq 2\}}(\gamma)\Big] \\ &\leq M_k\big(\varepsilon(n)^{-1}+1\big)^2 \sum_{j=1}^{|\mathfrak{F}_n(A)|} \gamma\big(B_{nj}^{2\varepsilon(n)}\big)1_{\{\gamma:\gamma(B_{nj}^{2\varepsilon(n)})\geq 2\}}(\gamma) \quad \mu\text{–a.e.} \end{aligned}$$

Therefore, by (4.1) and $(\mu.\sigma.2)$ we have

$$\mathcal{E}^\Gamma(u_n,u_n) \leq M_k\big(\varepsilon(n)^{-1}+1\big)^2 \sum_{j=1}^{|\mathfrak{F}_n(A)|} C_k\sigma\big(B_{nj}^{2\varepsilon(n)}\big)^2,$$

which, together with (4.8) and (4.9), implies that

$$\sup_{n\in\mathbb{N}} \mathcal{E}^\Gamma(u_n,u_n) < \infty. \tag{4.12}$$

By (4.7) and the fact that $\{\varepsilon(n)\}_{n\geq 1}$ is decreasing, it is easy to see that for every $\gamma \in \bar\Gamma_E$ the limit $u_\infty(\gamma) := \lim_{n\to\infty} u_n(\gamma)$ exists. Then by a standard argument (cf. [**32**]), we conclude from (4.12) that u_∞ is $\mathcal{E}^\Gamma$–quasi-continuous. On the other hand, since

$$\bar\Gamma_E \setminus \big\{\sup_j \varphi_{nj} < \tfrac{3}{2}\big\} \subset \bigcup_{j=1}^{|\mathfrak{F}_n(A)|} \big\{\gamma \,\big|\, \gamma\big(B_{nj}^{2\varepsilon(n)}\big) \geq 2\big\},$$

we obtain by (4.11) and $(\mu.\sigma.2)$

$$\int_{\bar\Gamma_E} u_n(\gamma)\,\mu(d\gamma) \leq \sum_{j=1}^{|\mathfrak{F}_n(A)|} \int_{\{\gamma:\gamma(B_{nj}^{2\varepsilon(n)})\geq 2\}} u_n(\gamma)\,\mu(d\gamma) \leq \sum_{j=1}^{|\mathfrak{F}_n(A)|} C_k\sigma\big(B_{nj}^{2\varepsilon(n)}\big)^2.$$

Thus, by (4.9) and (4.7), we conclude that $u_n \longrightarrow 0$ in $L^2(\bar\Gamma_E,\mu)$. Therefore, by quasi-regularity we must have $u_\infty = 0$ $\mathcal{E}^\Gamma$–q.e. In particular,

$$\big\{\gamma \in \bar\Gamma_E \,\big|\, u_\infty(\gamma) = 1\big\} \quad \text{is } \mathcal{E}^\Gamma\text{–exceptional}. \tag{4.13}$$

Now let $\gamma \in \bar\Gamma_E$ be such that $\gamma^*(B) \geq 2$ for some $B \in \{\mathfrak{F}_n(A)\}$. Then for any $n \in \mathbb{N}$, there exists $B_n \in \mathfrak{F}_n(A)$, such that $\gamma(B_n) \geq \gamma^*(B) \geq 2$ and hence, by (4.10) $u_\infty(\gamma) = u_n(\gamma) = 1$. Therefore,

$$\Big\{\gamma \in \bar\Gamma_E \,\Big|\, \gamma^*(B) \geq 2 \text{ for some } B \in \{\mathfrak{F}_n(A)\}\Big\} \subset \{u_\infty = 1\}$$

and the desired conclusion follows from (4.13). □

COROLLARY 4.4. *Suppose that for each $k \in \mathbb{N}$, there exists a sequence $\{\mathfrak{F}_n(E_k)\}_{n\in\mathbb{N}}$ of FBCs of E_k satisfying* (4.7)–(4.9). *Then under conditions* $(\mu.\sigma.2)$ *and* $(\mathcal{E}^\Gamma.1)$ *$\bar\Gamma_E \setminus \Gamma_E$ is $\mathcal{E}^\Gamma$–exceptional and hence, by* [**23**, Prop. IV.5.30] *the diffusion process $\mathbb{M}$, specified in Remark 4.1(3), lives on the simple configuration space Γ_E.*

PROOF. Note that in the situation of Theorem 4.3, if $x \in A$, then $\{x\} \in \{\mathcal{F}_n(A)\}$. Hence, the assertion of Theorem 4.3 implies in particular that $A \cap (\bar{\Gamma}_E \setminus \Gamma_E)$ is $\mathcal{E}^\Gamma$–exceptional. Consequently, in the situation of this corollary we have that

$$\bar{\Gamma}_E \setminus \Gamma_E = \bigcup_{k \in \mathbb{N}} E_k \cap (\bar{\Gamma}_E \setminus \Gamma_E)$$

is $\mathcal{E}^\Gamma$–exceptional. □

5. The free loop space as base space E

In this section, we discuss in detail the construction of diffusions on the "simple" configuration space over the free loop space. For the convenience of the reader, we start by recalling the framework of [**6**] (see also [**24**]).

Let $g = (g_{ij})$ be a uniformly elliptic Riemannian metric with bounded derivatives over $\mathbb{R}^d$ and

$$\Delta_g := (\det g)^{-\frac{1}{2}} \sum_{i=1}^d \frac{\partial}{\partial x_i}\Big[(\det g)^{\frac{1}{2}} g^{ij} \frac{\partial}{\partial x_j}\Big]$$

the corresponding Laplacian. Let $p_t(x, y)$, $x, y \in \mathbb{R}^d$, $t \geq 0$, be the associated heat kernel with respect to the Riemannian volume element. Let $W(\mathbb{R}^d)$ denote the set of all continuous paths $\omega : [0,1] \longrightarrow \mathbb{R}^d$ and let $\mathcal{L}(\mathbb{R}^d) := \{\omega \in W(\mathbb{R}^d) \mid \omega(0) = \omega(1)\}$, i.e. $\mathcal{L}(\mathbb{R}^d)$ is the free loop space over $\mathbb{R}^d$. Let P_1^x be the law of the bridge defined on $\{\omega \in \mathcal{L}(\mathbb{R}^d) \mid \omega(0) = \omega(1) = x\}$, coming from the diffusion on $\mathbb{R}^d$ generated by Δ_g and let

$$\sigma := \int P_1^x p_1(x,x)\, dx \tag{5.1}$$

be the *Bismut/Høegh–Krohn measure* on $\mathcal{L}(\mathbb{R}^d)$, which is σ–finite, but not finite. We consider $\mathcal{L}(\mathbb{R}^d)$ equipped with the Borel σ–algebra $\mathcal{B}(\mathcal{L}(\mathbb{R}^d))$ coming from the uniform norm $\|\cdot\|_\infty$ on $\mathcal{L}(\mathbb{R}^d)$, which makes it a Banach space.

The tangent space $T_\omega \mathcal{L}(\mathbb{R}^d)$ at a loop $\omega \in \mathcal{L}(\mathbb{R}^d)$ was introduced in [**19**] as the space of *periodical* vector fields $X_t(\omega) = \tau_t(\omega) h(t)$, $t \in [0,1]$, along ω. Here, τ denotes the stochastic parallel transport with respect to σ associated with the Levi–Civita connection of $(\mathbb{R}^d, g)$ and h belongs to the linear space H_ω consisting of all absolutely continuous maps $h : [0,1] \longrightarrow T_{\omega(0)}\mathbb{R}^d \equiv \mathbb{R}^d$, such that

$$(h,h)_{\omega(0)} := \int_0^1 g_{\omega(0)}\big(h'(s), h'(s)\big)\, ds + \int_0^1 \big|h(s)\big|^2\, ds < \infty \tag{5.2}$$

and satisfying the holonomy condition $\tau_1(\omega)h(1) = h(0)$ (cf. [**19**] for details). Note that if we consider $\mathcal{L}(\mathbb{R}^d)$ as continuous maps from S^1 to $\mathbb{R}^d$, this notion is invariant under rotations of S^1 and (5.2) induces an inner product on $T_\omega \mathcal{L}(\mathbb{R}^d)$, which turns it into a Hilbert space.

Below, we shall also need the Hilbert space $\widetilde{T}_\omega \mathcal{L}(\mathbb{R}^d)$ ($\supset T_\omega \mathcal{L}(\mathbb{R}^d)$) with inner product $(\ ,\ )_H$, which is constructed analogously, but without the holonomy condition, i.e. H_ω is replaced by H, which denotes the linear space of all absolutely continuous maps $h : [0,1] \longrightarrow T_{\omega(0)}\mathbb{R}^d \equiv \mathbb{R}^d$ satisfying (5.2). We note that by the uniform ellipticity of g, H is indeed independent of $\omega \in \mathcal{L}(\mathbb{R}^d)$ and the norms $\|\cdot\|_{\omega(0)} := (\ ,\ )_{\omega(0)}^{1/2}$, $\omega \in \mathcal{L}(\mathbb{R}^d)$, are all equivalent.

Let $\mathcal{F}C_0^\infty$ denote the linear span of the set of all functions $u : \mathcal{L}(\mathbb{R}^d) \longrightarrow \mathbb{R}$, such that there exist $k \in \mathbb{N}$, $f \in C_0^\infty\big((\mathbb{R}^d)^k\big)$, $t_1, \dots, t_k \in [0,1]$ with

$$u(\omega) = f\big(\omega(t_1), \dots, \omega(t_k)\big), \quad \omega \in \mathcal{L}(\mathbb{R}^d). \tag{5.3}$$

Note that $\mathcal{F}C_0^\infty$ is dense in $L^2(\sigma) := (\text{real})L^2\big(\mathcal{L}(\mathbb{R}^d), \sigma\big)$. Let $\mathcal{F}C^\infty$, $\mathcal{F}C_b^\infty$ be defined correspondingly with $C^\infty\big((\mathbb{R}^d)^k\big)$, $C_b^\infty\big((\mathbb{R}^d)^k\big)$ replacing $C_0^\infty\big((\mathbb{R}^d)^k\big)$.

We define the directional derivative of $u \in \mathcal{F}C^\infty$, u as in (5.3) at $\omega \in \mathcal{L}(\mathbb{R}^d)$ with respect to $X(\omega) \in \widetilde{T}_\omega \mathcal{L}(\mathbb{R}^d)$ by

$$\begin{aligned} \partial_h u(\omega) := \partial_X u(\omega) &:= \sum_{i=1}^k d_i f\big(\omega(t_1), \dots, \omega(t_k)\big) X_{t_i}(\omega) \\ &= \sum_{i=1}^k g_{\omega(t_i)}\Big(\nabla_i f\big(\omega(t_1), \dots, \omega(t_k)\big), \tau_{t_i}(\omega) h(t_i)\Big), \end{aligned} \tag{5.4}$$

where $h \in H$ with $X(\omega) = \big(\tau_t(\omega) h(t)\big)_{t \in [0,1]}$ and ∇_i resp. d_i denotes the gradient (with respect to g) resp. the differential relative to the i'th coordinate of f. We extend ∂_h to all of $\mathcal{F}C^\infty$ by linearity. Note that if we consider u as a function on $W(\mathbb{R}^d)$, then

$$\partial_X u(\omega) = \frac{d}{ds} u\big(\omega + sX(\omega)\big)\Big|_{s=0}, \quad \omega \in \mathcal{L}(\mathbb{R}^d). \tag{5.5}$$

Hence, $\partial_X u$ is well-defined by (5.4) (i.e. independent of the special representation of u).

Let for $u \in \mathcal{F}C^\infty$ and $\omega \in \mathcal{L}(\mathbb{R}^d)$, $\widetilde{D}u(\omega)$ be the unique element in H, such that $\big(\widetilde{D}u(\omega), h\big)_{\omega(0)} = \partial_h u(\omega)$ for all $h \in H$ and let $Du(\omega)$ be its othogonal projection (w.r.t. $(\ ,\ )_{\omega(0)}$) onto H_ω.

Since H is separable and consists of continuous functions on $[0,1]$, it follows by the construction of the stochastic parallel transport that there exists $N \in \mathcal{B}\big(\mathcal{L}(\mathbb{R}^d)\big)$ with $\sigma(N) = 0$, such that both $Du(\omega)$ and $\widetilde{D}u(\omega)$ are defined for all $\omega \in N^c$ and all $u \in \mathcal{F}C^\infty$. Moreover, the map $\omega \longmapsto Du(\omega)$ is measurable (cf. [**24**, Sect. 2.4.2] for a detailed proof).

For $u \in \mathcal{F}C^\infty$, we have that for all $\omega \in N^c$

$$\big\|Du(\omega)\big\|_{\omega(0)} \leq \big\|\widetilde{D}u(\omega)\big\|_{\omega(0)} \tag{5.6}$$

and if $u(\omega) = f\big(\omega(s_1), \dots, \omega(s_k)\big)$, then for all $\omega \in N^c$

$$\widetilde{D}u(\omega)(s) = \sum_{i=1}^k G(s, s_i) \tau_{s_i}(\omega)^{-1} \nabla_i f\big(\omega(s_1), \dots, \omega(s_k)\big), \tag{5.7}$$

where G is Green's function of $-\frac{d^2}{dt^2} + 1$ with Neumann boundary conditions on $[0,1]$, i.e.

$$G(s,t) = \frac{e}{2(e^2-1)}\big(e^{t+s-1} + e^{1-(t+s)} + e^{|t-s|-1} + e^{1-|t-s|}\big). \tag{5.8}$$

For $u, v \in \mathcal{F}C^\infty$ define

$$S(u,v)(\omega) := \big(Du(\omega), Dv(\omega)\big)_{\omega(0)}, \quad \omega \in \mathcal{L}(\mathbb{R}^d). \tag{5.9}$$

We now set $E := \mathcal{L}(\mathbb{R}^d)$, equipped with the metric ρ induced by the Banach norm $\|\cdot\|_\infty$. For $k \geq 1$ let U_k be the open hypercube $]-\frac{k}{2}, \frac{k}{2}[^d$ of $\mathbb{R}^d$, and define open subsets of E by

$$E_k := \{\omega \in E \,|\, \omega(0) \in U_k\}. \tag{5.10}$$

Then $\{E_k\}_{k\geq 1}$ is a well exhausting sequence with $\delta_k = \frac{1}{2}$ for all k (cf. (2.1)). Let $\bar{\Gamma}_E := \bar{\Gamma}_E(\{E_k\})$ be as specified in Sect. 2 and let μ be a Poisson measure or a mixed Poisson measure on $\bar{\Gamma}_E$, specified by (3.3) and (3.4) with σ being the Bismut measure as specified in (5.1).

Define $\mathcal{D}$ to be the linear span of the set of all functions $u : E \longrightarrow \mathbb{R}$ of the type specified in (5.3), but with $t_1 = 0$. We now lift S to the following space $\mathcal{D}^\Gamma := \mathcal{F}^\Gamma C_b^\infty(\mathcal{D})$ of functions on the configuration space $\bar{\Gamma}_E$:

$$\begin{aligned} \mathcal{D}^\Gamma &:= \mathcal{F}^\Gamma C_b^\infty(\mathcal{D}) \\ &:= \Big\{ g\big(\langle f_1, \cdot\rangle, \dots, \langle f_n, \cdot\rangle\big) \,\Big|\, n \in \mathbb{N}, f_1, \dots, f_n \in \mathcal{D}, g \in C_b^\infty(\mathbb{R}^n)\Big\} \end{aligned} \tag{5.11}$$

We define for $F = g_F\big(\langle f_1, \cdot\rangle, \dots, \langle f_n, \cdot\rangle\big), G = g_G\big(\langle g_1, \cdot\rangle, \dots, \langle g_m, \cdot\rangle\big) \in \mathcal{D}^\Gamma$ and $\gamma \in \bar{\Gamma}_E$

$$\begin{aligned} &S^\Gamma\big(F, G\big)(\gamma) \\ &:= \sum_{i=1}^n \sum_{j=1}^m \partial_i g_F\big(\langle f_1, \gamma\rangle, \dots, \langle f_n, \gamma\rangle\big)\, \partial_j g_G\big(\langle g_1, \gamma\rangle, \dots, \langle g_m, \gamma\rangle\big)\, \big\langle S(f_i, g_i), \gamma\big\rangle \end{aligned} \tag{5.12}$$

and $S^\Gamma(F) := S^\Gamma(F, F)$. It has been proved in [**24**] that S^Γ is well-defined on $\mathcal{D}^\Gamma \times \mathcal{D}^\Gamma$, i.e. $S^\Gamma(F, G)$ is independent of the representations of F, G used on the right hand side of (5.12). See Sect. 1.3 of [**24**], where the well-definedness was discussed in a general context.

We now define $\mathcal{E}^\Gamma(F, G)$ for $F, G \in \mathcal{D}^\Gamma$ by

$$\mathcal{E}^\Gamma(F, G) := \int_{\bar{\Gamma}_E} S^\Gamma(F, G)(\gamma)\, \mu(d\gamma). \tag{5.13}$$

FACT. *$\big(\mathcal{E}^\Gamma, \mathcal{D}^\Gamma\big)$ is closable on $L^2\big(\bar{\Gamma}_E, \mu\big)$, and the closure $\big(\mathcal{E}^\Gamma, D(\mathcal{E}^\Gamma)\big)$ is a quasi-regular, local and conservative Dirichlet form on $L^2\big(\bar{\Gamma}_E, \mu\big)$. Therefore, there exists a conservative diffusion process*

$$\mathbb{M} = \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_\gamma)_{\gamma \in \bar{\Gamma}_E}\big) \tag{5.14}$$

on $\bar{\Gamma}_E$.

We refer to [**24**] for the detailed proof of the above fact. Applying Theorem 4.3, we have the following result.

THEOREM 5.1. *Suppose that $d \geq 2$ and μ is a Poisson measure or a mixed Poisson measure satisfying* (4.6). *Then $\bar{\Gamma}_E \setminus \Gamma_E$ is $\mathcal{E}^\Gamma$-exceptional. Therefore, the diffusion process $\mathbb{M}$, as specified in* (5.14), *can be constructed in such a way that it lives on Γ_E. More precisely, there is a version of $\mathbb{M}$ (up to μ-equivalence, cf.* [**23**, Sect. IV.6]*), such that*

$$P_\gamma\big\{X_t \in \Gamma_E \text{ for all } t \geq 0\big\} = 1 \quad \forall \gamma \in \Gamma_E. \tag{5.15}$$

PROOF. The last assertion is a direct consequence of the first one (cf. [**23**, Prop. IV.5.30]). Therefore, we shall concentrate on showing that $\bar{\Gamma}_E \setminus \Gamma_E$ is $\mathcal{E}^\Gamma$–exceptional.

Note that in our situation ($\mu.\sigma.2$) and ($\mathcal{E}^\Gamma.1$) are satisfied by Remark 4.2.

For fixed $k \in \mathbb{N}$, we divide $U_k =]-\frac{k}{2}, \frac{k}{2}[^d$ into 2^{dn} hypercubes of equal size, i.e. the length of each side of the hypercubes is equal to $\frac{k}{2^n}$. Denote the corresponding *closed* hypercubes by A_{nj}, $1 \le j \le 2^{dn}$ and define a finite Borel covering $\mathfrak{F}_n(E_k)$ of E_k by $\mathfrak{F}_n(E_k) = \{B_{nj} \mid 1 \le j \le 2^{dn}\}$ with $B_{nj} = \{\omega \in E_k \mid \omega(0) \in A_{nj}\}$, $1 \le j \le 2^{dn}$. Clearly, $\{\mathfrak{F}_n(E_k)\}_{n\in\mathbb{N}}$ is a sequence of FBCs of E_k satisfying (4.7). Next, we take $\varepsilon(n) := 2^{-n}$, $n \in \mathbb{N}$. Then (4.8) holds, because $d \ge 2$. Let

$$\alpha_k := \sup_{x \in U_{k+1}} \left|\det g\right|(x)\, p_1(x,x).$$

Then one can easily calculate from (5.1) that

$$\sigma\big(B_{nj}^{2\varepsilon(n)}\big) \le \alpha_k \big(k2^{-n} + 2^{-n+2}\big)^d \le \alpha_k 2^{-dn}(k+4)^d$$

and consequently, (4.9) is satisfied. Since $k \in \mathbb{N}$ is arbitrary, it follows from Corollary 4.4 that $\bar{\Gamma}_E \setminus \Gamma_E$ is $\mathcal{E}^\Gamma$–exceptional. □

We have in fact proved the following stronger result.

PROPOSITION 5.2. *In the situation of Theorem 5.1, we have that*

$$\{\gamma \in \bar{\Gamma}_E \mid \gamma(\mathcal{L}_x(\mathbb{R}^d)) \ge 2 \textit{ for some } x \in \mathbb{R}^d\} \quad \textit{is } \mathcal{E}^\Gamma\textit{–exceptional},$$

where $\mathcal{L}_x(\mathbb{R}^d) = \{\omega \in \mathcal{L}(\mathbb{R}^d) \mid \omega(0) = x\}$.

PROOF. By Theorem 4.3 and the above proof it follows that

$$\left\{\gamma \in \bar{\Gamma}_E \,\middle|\, \gamma^*(B) \ge 2 \text{ for some } B \in \{\mathfrak{F}_n(E_k)\}\right\} \quad \text{is } \mathcal{E}^\Gamma\text{–exceptional}. \tag{5.16}$$

Realizing that $\mathcal{L}_x(\mathbb{R}^d) \in \{\mathfrak{F}_n(E_k)\}$ for all $x \in E_k$, we obtain the assertion. □

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