Cohomology of quaternionic Shimura varieties at parahoric levels

by

Harry Reimann

(April 26, 2000)

Fakultät für Mathematik, Universität Bielefeld,
PF 100131, D−33501 Bielefeld
(reimann@mathematik.uni−bielefeld.de)

Introduction

In the present paper we generalize the results of [8] concerning the cohomology of the Shimura variety Sh_D associated with the multiplicative group $D^\times$ (considered as an algebraic group over $\mathbb{Q}$) of a quaternion division algebra D over a totally real number field F. For an open compact subgroup $C \subset D^\times(\mathbb{A}_f)$, $Sh_{D,C} = Sh_D/C$ is a projective variety over a number field $E(D) \subset \mathbb{C}$. The inductive limit H^i of the cohomology groups $H^i_C = \mathrm{H}^i(Sh_{D,C}(\mathbb{C}), \mathbb{C})$ is a module over the Hecke algebra $\mathcal{H}(D^\times(\mathbb{A}_f))$. We assume that D is not totally definite, i.e. $Sh_{D,C}$ has dimension $\mathsf{d} > 0$. Let $\mathcal{A}^\infty$ denote the set of isomorphism classes of automorphic representations $\pi = \pi_f \otimes \pi_\infty$ of $D^\times$ such that π_∞ has trivial central and infinitesimal characters. For a representation $\pi \in \mathcal{A}^\infty$, the $\mathbb{C}$−vector space

$$H^i(\pi) = \mathrm{Hom}_{\mathcal{H}(D^\times(\mathbb{A}_f))}(\pi_f, H^i)$$

vanishes unless π has dimension 1 and $i \leqslant 2\mathsf{d}$ is even, or π is infinite-dimensional and $i = \mathsf{d}$. Moreover, there is a canonical decomposition

$$H^i = \bigoplus_{\pi \in \mathcal{A}^\infty} H^i(\pi) \otimes \pi_f .$$

We fix rational primes $\ell \neq p$, an embedding $\mathbb{Q}_\ell \to \mathbb{C}$, and a prime $\wp$ of $E(D)$ dividing p, and denote the completion of $E(D)$ at $\wp$ by E, the corresponding Weil group by W_E, and the Weil-Deligne group $\mathbb{C} \rtimes W_E$ by W'_E. Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the algebraic closure of $\mathbb{Q}$. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of E and extend the natural embedding of $E(D)$ into E to a homomorphism $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Using the canonical isomorphisms

$$H^i_C \cong \mathrm{H}^i(Sh_{D,C} \otimes_{E(D)} \overline{\mathbb{Q}}_p, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C}$$

we obtain, for every $\pi \in \mathcal{A}^\infty$, an algebraic representation of W'_E on $H^i(\pi)$. Let

$$H(\pi) = \begin{cases} \bigoplus_{i=0}^{\mathsf{d}} H^{2i}(\pi) & \text{if } \dim \pi = 1\,; \\ H^{\mathsf{d}}(\pi) & \text{otherwise}\,; \end{cases}$$

considered as a representation of W'_E. Let, on the other hand, π^* denote the automorphic representation of $\mathrm{Gl}_2(F)$ (considered as an algebraic group over $\mathbb{Q}$) which has the same local factors as π at all places of $\mathbb{Q}$ at which D is split, and recall that there is a natural 2^{d}−dimensional representation (ϱ°, V°) of

$$\widehat{D^\times} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}|E(D)) \subseteq {}^L D^\times = \widehat{D^\times} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$$

where $\gamma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ acts as $(g_v) \mapsto (g_{\gamma^{-1}v})$ on

$$\widehat{D^\times} = \prod_{v\colon F\to\overline{\mathbb{Q}}} \mathrm{Gl}_2(\mathbb{C})\,.$$

This representation is given by

$$V^\circ = \bigotimes_{v\in\mathcal{Z}} \mathbb{C}^2 \quad \text{and} \quad \varrho^\circ\big((g_v),\gamma\big)\colon \bigotimes_v x_v \longmapsto \bigotimes_v (\det g_v)^{-1} g_v x_{\gamma^{-1}v}$$

where $\mathcal{Z}$ denotes the set of infinite places of F at which D is split. Note that we normalize the Artin reciprocity maps of local class field theory so that uniformizing parameters correspond to geometric Frobenius elements.

Conjecture 1 *For every $\pi \in \mathcal{A}^\infty$, $L(H(\pi),s) = L_\wp(\pi^*,\varrho^\circ, s-\frac{\mathsf{d}}{2})$.*

For every algebraic representation τ of W'_E, one can consider the representation of W_E on the graded module $\mathrm{gr}(\tau)$ associated with the monodromy filtration. This gives rise to a variant of Conjecture 1 concerning the semi-simple L–functions.

Conjecture 2 *For every $\pi \in \mathcal{A}^\infty$, $L^{\mathrm{ss}}(H(\pi),s) = L^{\mathrm{ss}}_\wp(\pi,\varrho^\circ, s-\frac{\mathsf{d}}{2})$.*

In the present paper we assume that p is unramified in F, and we show that Conjecture 2 holds for all $\pi \in \mathcal{A}^\infty$ such that π_p admits a non-zero Iwahori-fixed vector and that Conjecture 1 holds for these π if the monodromy filtration on H^i_C is pure of weight i for every $i \leqslant 2\mathsf{d}$ and every sufficiently small open compact subgroup $C \subset D^\times(\mathbb{A}_f)$ containing an Iwahori subgroup of $D^\times(\mathbb{Q}_p)$. This generalizes [8] where π_p had to be unramified.

§1. Semi-simple L–functions and traces

We fix an Iwahori subgroup $C_p \subset D^\times(\mathbb{Q}_p)$ and consider the O_E–scheme $\widetilde{Sh}_{D,C_p}$ constructed in [9] (§5). It is equipped with a continuous action of $D^\times(\mathbb{A}^p_f)$ and an equivariant isomorphism

$$\widetilde{Sh}_{D,C_p} \otimes_{O_E} E \cong Sh_D/C_p \otimes_{E(D)} E\,.$$

Let $\kappa \subset \overline{\kappa}$ denote the residue class fields of E and $\overline{\mathbb{Q}}_p$. For every sufficiently small open subgroup $C \subset D^\times(\mathbb{A}^p_f)$, there is a $\mathrm{Gal}(\overline{\mathbb{Q}}_p|E)$–equivariant spectral sequence

$$E_2^{i,k} = \mathrm{H}^i(\widetilde{Sh}_{D,C_p}/C \otimes_{O_E} \overline{\kappa}\,, \mathrm{H}^k(\Psi_C)) \Longrightarrow \mathrm{H}^{i+k}(Sh_{D,C_p\times C} \otimes_{E(D)} \overline{\mathbb{Q}}_p\,, \mathbb{Q}_\ell)$$

where $\Psi_C \in D^b_{ctf}\big((\widetilde{Sh}_{D,C_p}/C \otimes_{O_E} \overline{\kappa})_{et}, \mathbb{Q}_\ell\big)$ denotes the complex of vanishing cycles. These spectral sequences form an inductive system (with respect to C) which is compatible with the action of $D^\times(\mathbb{A}_f^p)$. In particular, we obtain a filtration $F^\bullet$ on $H^i_{C_p\times C}$ which is stable under the action of $W'_E\times\mathcal{H}(D^\times(\mathbb{A}_f^p)/\!/C)$. The inertia group $I_E\subset W_E$ acts trivially on the sheaves of vanishing cycles (cf. [9], Theorem 5.3 (vi)). Hence $\mathbb{C}\times I_E\subset W'_E$ acts trivially on $\mathrm{gr}^{F^\bullet}(H^i_{C_p\times C})$. The image of this module in the Grothendieck group of finite-dimensional $\mathbb{C}$–vector spaces with an action of $W'_E\times\mathcal{H}(D^\times(\mathbb{A}_f^p)/\!/C)$ does not change if $F^\bullet$ is replaced by the monodromy filtration (cf. [7], Lemma 8.2 and its proof). For every positive integer j and every algebraic representation τ of W'_E, let

$$\mathrm{Tr}^{\mathrm{ss}}(\tau,j)=\mathrm{Tr}\left(\Phi^j\;\middle|\;\mathrm{gr}(\tau)^{I_E}\right)$$

where $\Phi\in W_E$ denotes a geometric Frobenius element, and for $f\in\mathcal{H}(D^\times(\mathbb{A}_f^p))$, let

$$\mathrm{Tr}^{\mathrm{ss}}(f,j)=\sum_{i=0}^{2\mathrm{d}}(-1)^i\,\mathrm{Tr}\left(\Phi^j\times f\;\middle|\;\mathrm{gr}(H^i_{C_p\times C})\right)$$

where $C\subset D^\times(\mathbb{A}_f^p)$ is a small open subgroup such that $f\in\mathcal{H}(D^\times(\mathbb{A}_f^p)/\!/C)$. We will identify $\mathrm{Tr}^{\mathrm{ss}}(f,j)$ with the trace of a certain function in $C_c^\infty(D^\times(\mathbb{A})/F^\times(\mathbb{R}))$ acting on the space $\mathcal{A}$ of automorphic forms on $D^\times\backslash D^\times(\mathbb{A})/F^\times(\mathbb{R})$. To this end, we fix a measure on $D^\times(\mathbb{R})/F^\times(\mathbb{R})$ and a function $f_\infty\in C_c^\infty(D^\times(\mathbb{R})/F^\times(\mathbb{R}))$ such that the orbital integral $\mathrm{O}_\gamma(f_\infty)$ vanishes for every non-elliptic regular semi-simple element $\gamma\in D^\times(\mathbb{R})/F^\times(\mathbb{R})$ and such that, for every irreducible admissible representation $\pi=\bigotimes_{v\colon F\to\mathbb{R}}\pi_v$ of $D^\times(\mathbb{R})/F^\times(\mathbb{R})$,

$$\mathrm{Tr}\big(\pi(f_\infty)\big)=\prod_v m(\pi_v)\quad\text{with}\quad m(\pi_v)=\begin{cases}1 & \text{if }\dim\pi_v=1\,;\\ -1 & \text{if } v\in\mathcal{Z},\ \pi_v=\pi_0\,;\\ 0 & \text{otherwise}\,;\end{cases}$$

where π_0 denotes the unique discrete series representation of $\mathrm{PGl}_2(\mathbb{R})$ with trivial infinitesimal character (cf. [3], §3). The algebra structure of $C_c^\infty(D^\times(\mathbb{A})/F^\times(\mathbb{R}))$ and its action on $\mathcal{A}$ depend on the choice of a measure. We will use the product of the measure on $D^\times(\mathbb{R})/F^\times(\mathbb{R})$ fixed above and the measure on $D^\times(\mathbb{A}_f)$ used in the introduction.

Recall now that every irreducible admissible representation π of $D^\times(\mathbb{Q}_p)$ corresponds to a $\widehat{D^\times}$–conjugacy class of admissible homomorphisms

$$\varphi(\pi)\colon W'_{\mathbb{Q}_p}=\mathbb{C}\rtimes W_{\mathbb{Q}_p}\longrightarrow {}^L D^\times_{\mathbb{Q}_p}=\widehat{D^\times}\rtimes\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)\,,$$

and let $\sigma(\pi)=\varrho^\circ\circ\varphi(\pi)|_{W'_E}$. If π admits a non-zero Iwahori-fixed vector, $\sigma(\pi)$ can be described as follows. Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_m$ be all primes of F dividing p. For every $i\in\{1,\ldots,m\}$, there is a representation π_i of $D^\times_{\mathfrak{p}_i}$ such that $\pi=\bigotimes_{i=1}^m\pi_i$. If D is ramified at $\mathfrak{p}_i$, there is an unramified character χ_i of $\mathbb{Q}_p^\times$ such that π_i is the one-dimensional representation $\chi_i \circ \mathrm{Nm}_{D_{\mathfrak{p}_i}|\mathbb{Q}_p}$, and we define a map $g_i \colon W'_{\mathbb{Q}_p} \to \mathrm{Gl}_2(\mathbb{C})$ by

$$g_i(x,w) = \chi_i(w) \begin{pmatrix} |w|^{1/2} & x\,|w|^{1/2} \\ 0 & |w|^{-1/2} \end{pmatrix}$$

where χ_i is considered as a character of $W_{\mathbb{Q}_p}$ via local class field theory. Otherwise, we have to distinguish three cases. If there is an unramified character χ_i of $\mathbb{Q}_p^\times$ such that π_i is the special representation corresponding to the character $\chi_i \circ \mathrm{Nm}_{F_{\mathfrak{p}_i}|\mathbb{Q}_p}$, then we define g_i as above. If there is an unramified character χ_i of $\mathbb{Q}_p^\times$ such that $\pi_i = \chi_i \circ \mathrm{Nm}_{D_{\mathfrak{p}_i}|\mathbb{Q}_p}$, let

$$g_i(x,w) = \chi_i(w) \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}.$$

In the remaining case there are unramified characters $\chi_{i,1}$ and $\chi_{i,2}$ of $\mathbb{Q}_p^\times$ such that π_i is the principal series representation $\mathrm{PS}(\chi_{i,1} \circ \mathrm{Nm}_{F_{\mathfrak{p}_i}|\mathbb{Q}_p},\, \chi_{i,2} \circ \mathrm{Nm}_{F_{\mathfrak{p}_i}|\mathbb{Q}_p})$, and we set

$$g_i(x,w) = \begin{pmatrix} \chi_{i,1}(w) & 0 \\ 0 & \chi_{i,2}(w) \end{pmatrix}.$$

Using these maps and the identification of $\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}})$ with $\coprod_{i=1}^m \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \overline{\mathbb{Q}}_p)$ induced by the fixed embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$ we define a representative $\varphi(\pi)$ of the conjugacy class referred to above by

$$\varphi(\pi)(x,w) = \big((\, g_v\,)\,, w\big) \quad \text{where} \quad g_v = g_i(x,w) \text{ if } v \in \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \overline{\mathbb{Q}}_p)\,.$$

For every $i \in \{1, \dots, m\}$, let

$$\begin{aligned} &\alpha_i = q^{1/2}\chi_i(q)^{-1} && \text{and} \quad \beta_i = q^{-1/2}\chi_i(q)^{-1} && \text{if } i \in I(\pi) \text{ or } \dim(\pi_i) = 1\,; \\ &\alpha_i = \chi_{i,1}(q)^{-1} && \text{and} \quad \beta_i = \chi_{i,2}(q)^{-1} && \text{otherwise}\,; \end{aligned}$$

where $q = \#\kappa$ and $I(\pi) = \{\, i \in \{1, \dots, m\} \mid \pi_i \text{ square integrable} \,\}$, and for every subset $J \subseteq \mathcal{Z}$, let

$$c_J = \prod_{i=1}^m \alpha_i^{\#(\mathcal{Z}_i \smallsetminus J)} \beta_i^{\#(\mathcal{Z}_i \cap J)} \quad \text{where} \quad \mathcal{Z}_i = \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \overline{\mathbb{Q}}_p) \cap \mathcal{Z}\,.$$

The standard basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ gives rise to a basis of V°,

$$\{\, e_J = \textstyle\bigotimes_{v \in \mathcal{Z}} e_{J,v} \mid J \subseteq \mathcal{Z} \,\} \quad \text{where} \quad e_{J,v} = \begin{cases} e_1 & \text{if } v \in J\,; \\ e_2 & \text{otherwise}\,; \end{cases}$$

in terms of which the action of Φ defined by $\sigma(\pi)$ is given by

$$\Phi e_J = c_J e_{\Phi J} \quad \text{for all } J \subseteq \mathcal{Z}\,,$$

and the action of $(x,w) \in \mathbb{C} \times I_E$ is given by $\exp(xN)$ for the nilpotent endomorphism N of V° such that, for all $J \subseteq \mathcal{Z}$,

$$Ne_J = \sum_{v \in \mathcal{Z}_\pi \smallsetminus J} e_{J \cup \{v\}} \quad \text{where} \quad \mathcal{Z}_\pi = \bigsqcup_{i \in I(\pi)} \mathcal{Z}_i\,.$$

The monodromy filtration $F^\bullet V^\circ$ of $\sigma(\pi)$ is the Schmid filtration corresponding to N, i.e. for every integer j, $F^j V^\circ \subseteq V^\circ$ is the subspace generated by

$$\{\, e_J \mid \nu(J) \leqslant j \,\} \quad \text{where} \quad \nu(J) = \#(\mathcal{Z}_\pi \smallsetminus J) - \#(\mathcal{Z}_\pi \cap J)\,.$$

Thus, we have an isomorphism $\mathrm{gr}\big(\sigma(\pi)\big) \cong \bigotimes_{i=1}^m \sigma_i$ where σ_i is the unramified representation of W_E on the $\mathbb{C}$–vector space with basis $\{\, y_J \mid J \subseteq \mathcal{Z}_i \,\}$ such that

$$\Phi y_J = \alpha_i^{\#(\mathcal{Z}_i \smallsetminus J)} \beta_i^{\# J} y_{\Phi J} \quad \text{for all } J \subseteq \mathcal{Z}_i\,.$$

Assume now that π is tempered, i.e. for every $i \in \{1, \ldots, m\}$, either D is ramified at $\mathfrak{p}_i$ or π_i is infinite-dimensional, and the characters χ_i or $\chi_{i,1}$ and $\chi_{i,2}$, respectively, are unitary. Then, for every subset $J \subseteq \mathcal{Z}$, $|c_J| = q^{\nu(J)/2}$, i.e. for every integer j, $\mathrm{gr}_j(\sigma(\pi))$ is pure of weight j.

Proposition 1.1 *For every integer $j > 0$, there is a function $f_p^{(j)} \in \mathcal{H}(D^\times(\mathbb{Q}_p)/\!/C_p)$ such that, for every irreducible tempered representation π of $D^\times(\mathbb{Q}_p)$ and every function $f \in \mathcal{H}(D^\times(\mathbb{A}_f^p))$,*

$$\begin{aligned}
\mathrm{Tr}(\pi(f_p^{(j)})) &= q^{j\mathsf{d}/2} \dim(\pi^{C_p})\, \mathrm{Tr}^{\mathrm{ss}}(\sigma(\pi), j) \\
\mathrm{Tr}(f^{(j)} | \mathcal{A}) &= \mathrm{Tr}^{\mathrm{ss}}(f, j)
\end{aligned}$$

where $f^{(j)} = f_p^{(j)} \cdot f \cdot f_\infty \in C_c^\infty(D^\times(\mathbb{A})/F^\times(\mathbb{R}))$ and the measure on $D^\times(\mathbb{Q}_p)$ is the quotient of the measures on $D^\times(\mathbb{A}_f)$ and $D^\times(\mathbb{A}_f^p)$ used above.

The proof of this proposition will be given in the following sections. In the remainder of the present section we use it to deduce our main result.

Theorem 1.2 *For every $\pi \in \mathcal{A}^\infty$ such that $\pi_p^{C_p} \neq 0$,*

$$L^{\mathrm{ss}}(H(\pi), s) = L_\wp^{\mathrm{ss}}(\pi, \varrho^\circ, s - \tfrac{\mathsf{d}}{2})\,,$$

and if, for every $i \leqslant 2\mathsf{d}$, the monodromy filtration on $H^i(\pi)$ is pure of weight i, then

$$L(H(\pi), s) = L_\wp(\pi^*, \varrho^\circ, s - \tfrac{\mathsf{d}}{2})\,.$$

Proof. Let π^p denote the representation of $D^\times(\mathbb{A}_f^p)$ such that $\pi_f = \pi_p \otimes \pi^p$ and, for every $i \in \{1, \ldots, m\}$, let π_i denote the representation of $D_{\mathfrak{p}_i}^\times$ such that $\pi_p = \bigotimes_{i=1}^m \pi_i$. We assume that π is infinite-dimensional and that π_p is tempered since, otherwise, either π is one-dimensional or, for every finite place v of F, D_v is split and the corresponding factor π_v of π is not square-integrable (cf. [6], Corollary 3.2.3). But then π_p is unramified, and the theorem is proved in [8]. We choose a sufficiently small open subgroup $C \subset D^\times(\mathbb{A}_f^p)$ such that π is contained in the finite set $\Pi_C = \{\, \widetilde{\pi} \in \mathcal{A}^\infty \mid \widetilde{\pi}_f^{C_p \times C} \neq 0 \,\}$ and a function $f \in \mathcal{H}(D^\times(\mathbb{A}_f^p)/\!/C)$ such that $\mathrm{Tr}\big(\pi^p(f)\big) = (-1)^d$ and $\widetilde{\pi}^p(f) = 0$ for all $\widetilde{\pi} \in \Pi_C \setminus \{\pi\}$ (cf. [8], Lemma 2.5). For $f_p^{(j)} \in \mathcal{H}(D^\times(\mathbb{Q}_p)/\!/C_p)$ as in Proposition 1.1 and $f^{(j)} = f_p^{(j)} \cdot f \cdot f_\infty$ this yields

$$\mathrm{Tr}\,(\pi_p(f_p^{(j)})) = \mathrm{Tr}\,(\pi(f^{(j)})) = \mathrm{Tr}\,(f^{(j)}\,|\,\mathcal{A}) = \mathrm{Tr}^{\mathrm{ss}}(f, j) = \dim(\pi_p^{C_p})\,\mathrm{Tr}^{\mathrm{ss}}(H(\pi), j)$$

since π occurs in $\mathcal{A}$ with multiplicity 1 and H^i decomposes as pointed out in the introduction. Consequently, we have

$$\mathrm{Tr}^{\mathrm{ss}}(H(\pi), j) = q^{j\mathsf{d}/2}\,\mathrm{Tr}^{\mathrm{ss}}(\sigma(\pi_p), j)\,.$$

Together with the power series expansions

$$\log L^{\mathrm{ss}}(H(\pi), s) = \sum_{j=1}^\infty \mathrm{Tr}^{\mathrm{ss}}(H(\pi), j)\frac{q^{-js}}{j}$$
$$\log L_\wp^{\mathrm{ss}}(\pi, \varrho^\circ, s) = \sum_{j=1}^\infty \mathrm{Tr}^{\mathrm{ss}}(\sigma(\pi_p), j)\frac{q^{-js}}{j}\,.$$

this implies our first assertion.

Assume now that the purity condition of the second assertion is satisfied. Since we consider the case where π is infinite-dimensional, π^* may be replaced by π. Let τ denote the Φ–semi-simplification of $H(\pi) \otimes |\;|^{-\mathsf{d}/2}$ (cf. [4], §8). The first assertion is equivalent to

$$\mathrm{gr}\,(\tau) \cong \mathrm{gr}\,(\sigma(\pi_p))$$

since on both sides I_E acts trivially and Φ defines a semi-simple endomorphism. Moreover, since $H(\pi) = H^{\mathsf{d}}(\pi)$, each $\mathrm{gr}_j(\tau)$ is pure of weight j, and since we consider the case where π_p is tempered, the same holds for $\mathrm{gr}_j(\sigma(\pi_p))$. Hence, for all j,

$$\mathrm{gr}_j(\tau) \cong \mathrm{gr}_j(\sigma(\pi_p))\,.$$

According to [4] (Proposition 8.9) this implies that τ and $\sigma(\pi_p)$ are isomorphic which, in turn, yields the second assertion of the theorem. ∎

§2. Orbital and twisted orbital integrals

We fix a small open subgroup $C \subset D^\times(\mathbb{A}_f^p)$, an element $\xi \in D^\times(\mathbb{A}_f^p)$, and a positive integer j, and consider the geometric correspondence

$$b = (\mathrm{Frob}^j \circ b_1\,,\, b_2)\colon \widetilde{Sh}_{D,C_p}/C^\xi \otimes_{O_E} \kappa \longrightarrow \widetilde{Sh}_{D,C_p}/C \otimes_{O_E} \kappa \;\times\; \widetilde{Sh}_{D,C_p}/C \otimes_{O_E} \kappa$$

where $C^\xi = \xi C \xi^{-1} \cap C$, Frob denotes the Frobenius endomorphism over κ, b_1 is the morphism induced by the action of ξ, and b_2 is the natural projection. Let

$$u\colon (b_2)_*(\mathrm{Frob}^j \circ b_1)^* \Psi_C \longrightarrow \Psi_C$$

be the cohomological correspondence on Ψ_C with support in b which is induced by the canonical isomorphisms

$$\mathrm{Frob}^* \Psi_C \cong \Psi_C\,, \quad b_1^* \Psi_C \cong \Psi_{C^\xi} \cong b_2^* \Psi_C\,,$$

and the trace map

$$(b_2)_* b_2^* \Psi_C \longrightarrow \Psi_C\,.$$

The action of u on $\mathrm{H}^i(\widetilde{Sh}_{D,C_p}/C \otimes_{O_E} \overline{\kappa}\,, \mathrm{H}^k(\Psi_C)) \otimes_{\mathbb{Q}_\ell} \mathbb{C}$ coincides with the action of $\Phi^j \times f$ where $f \in \mathcal{H}(D^\times(\mathbb{A}_f^p)/\!/C)$ denotes $\mathrm{vol}(C)^{-1}$ times the characteristic function of the double coset $C\xi C$. Hence, using the spectral sequence of vanishing cycles we obtain

$$\mathrm{Tr}^{\mathrm{ss}}(f,j) = \sum_{i,k=0}^{\infty} (-1)^{i+k}\, \mathrm{Tr}\left(u_* \;\middle|\; \mathrm{H}^i(\widetilde{Sh}_{D,C_p}/C \otimes_{O_E} \overline{\kappa}\,, \mathrm{H}^k(\Psi_C)) \right).$$

Let Ψ_x^k denote the stalk of the sheaf $\mathrm{H}^k(\Psi_C)$ at a geometric point x. From [9] (Theorem 5.3 (iii) and Corollary 5.4) we infer that b and Ψ_C satisfy all of the conditions considered in §1 of [8]. By Theorem 1.2 in [8] we can, therefore, write

$$\mathrm{Tr}^{\mathrm{ss}}(f,j) = \sum_{\beta \in B} T(\beta) \tag{2.1}$$

where

$$B = \{\, \beta \in \widetilde{Sh}_{D,C_p}(\overline{\kappa})/C^\xi \mid \mathrm{Frob}^j(b_1(\beta)) = b_2(\beta) \,\}$$

and

$$T(\beta) = \sum_{k=0}^{\infty} (-1)^k\, \mathrm{Tr}\left(u_\beta \;\middle|\; \Psi^k_{b_2(\beta)} \right).$$

Using Theorem 5.3 (vi) in [9] we can give an explicit description of B and T. To this end, we choose a maximal order $O_{D,p} \subset D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ containing C_p and consider the biggest left ideal $X \subseteq O_{D,p}$ such that

$$C_p = \{\, d \in O_{D,p} \mid Xd = X \,\}.$$

Let $\mathcal{K}$ be the completion of the maximal unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$. The Frobenius automorphisms of $\mathcal{K}$ and its residue class field $\bar{\kappa}$ are both denoted by σ. Let $E' \subset \mathcal{K}$ be the extension of degree j of E, and set

$$C'_p = \{\, d \in O_{D,p} \otimes_{\mathbb{Z}_p} O_{E'} \mid (X \otimes_{\mathbb{Z}_p} O_{E'})d = X \otimes_{\mathbb{Z}_p} O_{E'} \,\}.$$

We will use the natural identifications $\mathrm{Hom}(F, \mathbb{R}) \cong \mathrm{Hom}(F, \mathcal{K}) \cong \mathrm{Hom}(O_F, \bar{\kappa})$. Let Y denote the set of all $y \in D^\times(E')$ such that $y^{-1} \in O_{D,p} \otimes_{\mathbb{Z}_p} O_{E'}$ and such that, for all $v \in \mathrm{Hom}(F, \mathcal{K})$,

$$\mathrm{ord}_p\big((v \otimes \mathrm{id}_{E'})(\mathrm{Nm}_{D|F}(y))\big) = \begin{cases} -1 & \text{if } v \in \mathcal{Z}; \\ \ \ 0 & \text{otherwise}. \end{cases}$$

For every $d \in Y_0 = \{\, y \in Y \mid X \otimes_{\mathbb{Z}_p} O_{E'} \subseteq (X \otimes_{\mathbb{Z}_p} O_{E'})y \,\}$, let

$$\begin{aligned}
\mathcal{Z}_1(d) &= \{\, v \in \mathcal{Z} \mid f_v(X \otimes_{\mathbb{Z}_p} O_{\mathcal{K}})d \ \subseteq O_{D,p} \otimes_{\mathbb{Z}_p} O_{\mathcal{K}} \,\} \\
\mathcal{Z}'_1(d) &= \{\, v \in \mathcal{Z} \mid f_v(X \otimes_{\mathbb{Z}_p} O_{\mathcal{K}})d^* \subseteq O_{D,p} \otimes_{\mathbb{Z}_p} O_{\mathcal{K}} \,\} \\
\mathcal{Z}_2(d) &= \{\, \varphi \in \mathrm{Hom}_{\mathbb{Z}_p}(O_{D,p}, \bar{\kappa}) \mid d^{-1} \in \ker(O_{D,p} \otimes_{\mathbb{Z}_p} O_{E'} \xrightarrow{\varphi \otimes \varrho} \bar{\kappa}) \,\}
\end{aligned}$$

where * denotes the main involution of D, $f_v \in F^\times(\mathcal{K})$ is the idempotent corresponding to $v \in \mathrm{Hom}(F, \mathcal{K})$, and $\varrho\colon O_{E'} \to \bar{\kappa}$ is the residue class map. The inclusions $\mathcal{Z}_1(d) \to \mathcal{Z}$ and $\mathcal{Z}'_1(d) \to \mathcal{Z}$ and the restriction map

$$\begin{aligned} \mathcal{Z}_2(d) &\longrightarrow \mathcal{Z} \\ \varphi &\longmapsto \varphi|_{O_F} \end{aligned}$$

induce an injection

$$\mathbb{Q}^{\mathcal{Z}} \longrightarrow \mathbb{Q}^{\mathcal{Z}_1(d)} \oplus \mathbb{Q}^{\mathcal{Z}'_1(d)} \oplus \mathbb{Q}^{\mathcal{Z}_2(d)}.$$

Let Θ_d denote the automorphism of its cokernel which is induced by the permutations $v \mapsto \sigma^r v$ of $\mathcal{Z}_1(d)$ and $\mathcal{Z}'_1(d)$ and $\varphi \mapsto \sigma^r \varphi$ of $\mathcal{Z}_2(d)$, respectively, where $r = [\,E' : \mathbb{Q}_p\,]$. Using these automorphisms we define a map

$$\begin{aligned} \Theta\colon D^\times(E') &\longrightarrow \mathbb{Q} \\ d &\longmapsto \begin{cases} \sum\limits_{k=0}^{\infty} (-1)^k p^{kr}\, \mathrm{Tr}\big(\bigwedge^k \Theta_d\big) & \text{if } d \in Y_0; \\ 0 & \text{otherwise}. \end{cases} \end{aligned}$$

In the description of the set B we will use the following notation. Let A_D be the totally definite quaternion algebra over F which has the same invariants as D at all finite places not dividing p and invariant $(\mathsf{d}_i + \delta_i)/2$ at $\mathfrak{p}_i$ where

$$\mathsf{d}_i = \#\mathcal{Z}_i \quad \text{and} \quad \delta_i = \begin{cases} 0 & \text{if } D \text{ splits at } \mathfrak{p}_i\,; \\ 1 & \text{otherwise}\,; \end{cases}$$

for all $i \in \{1, \ldots, m\}$. For every $a \in A_D^\times$, let A_a denote the commutant of $L_a = F(a)$ in A_D. The set of isomorphism classes of pairs (L, ε) where $L|F$ is a totally imaginary quadratic extension which splits D at all places not dividing p and ε is a map from the set of all primes of L dividing p into the set of non-negative integers such that $\varepsilon(\mathfrak{p}) \neq \varepsilon(\overline{\mathfrak{p}})$ for at least one $\mathfrak{p}$, and

$$\sum_{\mathfrak{p}|\mathfrak{p}_i} (\varepsilon(\mathfrak{p}) + \delta_i) = \mathsf{d}_i + \delta_i$$

for all i, is denoted by $\mathfrak{I}_D$. For $\mathfrak{t} = (L, \varepsilon) \in \mathfrak{I}_D$ and $a \in L^\times$, let $L_a = A_a = A_{\mathfrak{t}} = L$. For every $\mathfrak{t} \in \{D\} \cup \mathfrak{I}_D$, we fix an injective $F \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–algebra homomorphism $\iota_{\mathfrak{t}}\colon A_{\mathfrak{t}} \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to D \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ and a system of representatives $\mathfrak{R}(\mathfrak{t}) \subset A_{\mathfrak{t}}^\times$ for the set of conjugacy classes in $A_{\mathfrak{t}}^\times/(O_F^\times \cap C)$. Let $\mathfrak{R}'(\mathfrak{t})$ be the set of all $a \in \mathfrak{R}(\mathfrak{t})$ such that there are a $\delta \in D^\times(E')$ and an $F \otimes_{\mathbb{Q}} \mathcal{K}$–algebra homomorphism $\iota\colon A_a \otimes_{\mathbb{Q}} \mathcal{K} \to D \otimes_{\mathbb{Q}} \mathcal{K}$ which satisfy the following conditions.

(2.2) For every $i \in \{1, \ldots, m\}$ and every prime $\mathfrak{p}$ of L_a dividing $\mathfrak{p}_i$,

$$\frac{-2[\,F_{\mathfrak{p}_i} : \mathbb{Q}_p\,]\,\mathrm{ord}_{\mathfrak{p}}(a)}{r} = \begin{cases} \mathsf{d}_i + \varepsilon(\mathfrak{p}) - \varepsilon(\overline{\mathfrak{p}}) & \text{if } \mathfrak{t} = (L, \varepsilon) \in \mathfrak{I}_D \text{ and } \overline{\mathfrak{p}} \neq \mathfrak{p}\,; \\ \mathsf{d}_i\,\mathrm{ord}_{\mathfrak{p}}(p) & \text{otherwise}\,. \end{cases}$$

(2.3) ι is injective and $\iota(a) = \mathrm{Nm}_\sigma(\delta) = \delta\,\sigma(\delta) \cdots \sigma^{r-1}(\delta)$.

(2.4) For all $x \in A_a \otimes_{\mathbb{Q}} \mathcal{K}$, $\iota(\sigma(x)) = \delta\,\sigma(\iota(x))\,\delta^{-1}$.

For every $a \in \mathfrak{R}'(\mathfrak{t})$, we fix δ and ι as above and set

$$\begin{aligned} Y_p(a) &= \{\, d \in D^\times(E') \mid d^{-1}\delta\,\sigma(d) \in Y_0 \,\}/C'_p \\ Y^p(a) &= \{\, d \in D^\times(\mathbb{A}_f^p) \mid d^{-1} a d \in \xi C \,\}/C^\xi \end{aligned}$$

where a is considered as an element of $D^\times(\mathbb{A}_f^p)$ via $\iota_{\mathfrak{t}}$. Since a cannot be contained in F unless $\mathfrak{t} = D$, $\iota_{\mathfrak{t}}$ identifies $A_a^\times(\mathbb{A}_f^p)$ with $D_a^\times(\mathbb{A}_f^p)$, and ι identifies $A_a^\times(\mathbb{Q}_p)$ with $D_{\delta\sigma}^\times(\mathbb{Q}_p)$, where $D_a^\times(\mathbb{A}_f^p)$ denotes the centralizer of a in $D^\times(\mathbb{A}_f^p)$ and

$$D_{\delta\sigma}^\times(\mathbb{Q}_p) = \{\, d \in D^\times(E') \mid \delta\,\sigma(d) = d\delta \,\}\,.$$

In particular, $A_a^\times$ acts on $Y_p(a)$ and $Y^p(a)$ by left multiplication via these identifications. Using a standard argument (cf. [7], p. 60–62) we derive from Theorem 5.3 (vi) in [9] that, for a certain subset $\mathfrak{T}_D \subseteq \{D\} \cup \mathfrak{I}_D$ containing D and all $(L, \varepsilon) \in \mathfrak{I}_D$ such that L splits D, there is a bijection

$$B \cong \coprod_{\mathfrak{t} \in \mathfrak{T}_D} \coprod_{a \in \mathfrak{R}'(\mathfrak{t})} \Big(A_a^\times \big\backslash \big(Y_p(a) \times Y^p(a) \big) \Big)$$

such that T corresponds to the map induced by

$$\begin{aligned} Y_p(a) &\longrightarrow \mathbb{Q} \\ d &\longmapsto \Theta(d^{-1} \delta \sigma(d)) \end{aligned}$$

for all a.

Now we return to our trace formula (2.1) and use the description of B and T given above to rewrite it as

$$\mathrm{Tr}^{\mathrm{ss}}(f, j) = \sum_{\mathfrak{t} \in \mathfrak{T}_D} \sum_{a \in \mathfrak{R}'(\mathfrak{t})} \sum \Theta(d_1^{-1} \delta \sigma(d_1)) f'(d_2^{-1} a d_2)$$

where the third sum is extended over all

$$(d_1, d_2) \in A_a^\times \big\backslash \big(D^\times(E')/C_p' \times D^\times(\mathbb{A}_f^p)/C^\xi \big)$$

and where f' denotes the characteristic function of ξC. Let $c_1, \ldots, c_s$ be representatives for the factor space C/C^ξ. Then $c_1 \xi, \ldots, c_s \xi$ are representatives for $C\xi C/C$, and an element $d \in D^\times(\mathbb{A}_f^p)$ is contained in $C \xi C$ if and only if there is an $i \in \{1, \ldots, s\}$ such that $f'(c_i^{-1} d c_i) = 1$. This i unique, and we obtain

$$\mathrm{Tr}^{\mathrm{ss}}(f, j) = \mathrm{vol}(C) \sum_{\mathfrak{t} \in \mathfrak{T}_D} \sum_{a \in \mathfrak{R}'(\mathfrak{t})} \sum \Theta(d_1^{-1} \delta \sigma(d_1)) f(d_2^{-1} a d_2)$$

where the third sum is extended over all

$$(d_1, d_2) \in A_a^\times \big\backslash \big(D^\times(E')/C_p' \times D^\times(\mathbb{A}_f^p)/C \big) .$$

By linearity, this holds for every $f \in \mathcal{H}(D^\times(\mathbb{A}_f^p)/\!/C)$. We choose Haar measures on $D_a^\times(\mathbb{A}_f^p)$ and $D_{\delta\sigma}^\times(\mathbb{Q}_p)$ and use the identifications explained above, to get a measure on $A_a^\times(\mathbb{A}_f)$. Moreover, we fix measures on $D^\times(E')$ and $A_a^\times C_F \subset A_a^\times(\mathbb{A}_f)$ where

$$C_F = (O_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \times (F^\times(\mathbb{A}_f^p) \cap C) \subset F^\times(\mathbb{A}_f)$$

such that $\mathrm{vol}(C_p') = \mathrm{vol}(C_F) = 1$. Then we apply the argument of Kottwitz [5] (1.5) to obtain the formula

$$\mathrm{Tr}^{\mathrm{ss}}(f, j) = \sum_{\mathfrak{t} \in \mathfrak{T}_D} \sum_{a \in \mathfrak{R}'(\mathfrak{t})} \mathrm{vol}\big(A_a^\times C_F \big\backslash A_a^\times(\mathbb{A}_f) \big)\, \mathrm{TO}_{\delta\sigma}(\Theta)\, \mathrm{O}_a(f) \tag{2.5}$$

where

$$\mathrm{O}_a(f) = \int\limits_{D_a^\times(\mathbb{A}_f^p)\backslash D^\times(\mathbb{A}_f^p)} f(x^{-1}ax)\,d\overline{x}$$

$$\mathrm{TO}_{\delta\sigma}(\Theta) = \int\limits_{D_{\delta\sigma}^\times(\mathbb{Q}_p)\backslash D^\times(E')} \Theta(x^{-1}\delta\,\sigma(x))\,d\overline{x}\,.$$

Recall now that an arbitrary element $\delta \in D^\times(E')$ is called σ–semi-simple if $\mathrm{Nm}_\sigma(\delta)$ is semi-simple. We say that $\delta \in D^\times(E')$ and $\gamma \in D^\times(\mathbb{Q}_p)$ are associated if $\mathrm{Nm}_\sigma(\delta)$ and γ are conjugate in $D^\times(E')$.

Proposition 2.6 *If $\delta \in D^\times(E')$ is a σ–semi-simple element which is not associated with an element of $D^\times(\mathbb{Q}_p)$, then $\mathrm{TO}_{\delta\sigma}(\Theta) = 0$. Moreover, there is a function $f_p^{(j)} \in \mathcal{H}(D^\times(\mathbb{Q}_p)/\!/C_p)$ such that, for every irreducible tempered representation π of $D^\times(\mathbb{Q}_p)$,*

$$\mathrm{Tr}\,(\pi(f_p^{(j)})) = q^{j\mathsf{d}/2}\dim(\pi^{C_p})\,\mathrm{Tr}^{\mathrm{ss}}(\sigma(\pi),j)\,,$$

and for every semi-simple element $\gamma \in D^\times(\mathbb{Q}_p)$,

$$\xi(\gamma)\,\mathrm{O}_\gamma(f_p^{(j)}) = \begin{cases} \mathrm{TO}_{\delta\sigma}(\Theta) & \text{if } \gamma \text{ is associated with } \delta \in D^\times(E')\,; \\ 0 & \text{if there is no such } \delta\,; \end{cases}$$

where $\xi(\gamma) = (-1)^{\mathsf{d}}$ if $\gamma \in F^\times(\mathbb{Q}_p)$ and $\xi(\gamma) = 1$ otherwise, and where the measures on $D_\gamma^\times(\mathbb{Q}_p)$ and $D_{\delta\sigma}^\times(\mathbb{Q}_p)$ are chosen such that Iwahori subgroups have equal volume.

The proof of this proposition will be given in the following section. In the present section we show that a function $f_p^{(j)}$ as above satisfies the conditions in Proposition 1.1. For any function $f \in \mathcal{H}(D^\times(\mathbb{A}_f^p)/\!/C)$, the trace of the action of $f^{(j)} = f_p^{(j)} \cdot f \cdot f_\infty$ on $\mathcal{A}$ is given by the Selberg trace formula

$$\mathrm{Tr}\,(f^{(j)}\,|\,\mathcal{A}) = \sum_{\gamma\in\mathfrak{R}} \mathrm{vol}\big(D_\gamma^\times C_F\backslash D_\gamma^\times(\mathbb{A})/F^\times(\mathbb{R})\big)\cdot \int\limits_{D_\gamma^\times(\mathbb{A})\backslash D^\times(\mathbb{A})} f^{(j)}(x^{-1}\gamma x)\,d\overline{x} \tag{2.7}$$

where $\mathfrak{R} \subset D^\times$ is a system of representatives for the set of conjugacy classes in $D^\times/(O_F^\times \cap C)$, $d\overline{x}$ denotes the quotient measure obtained from the given measure on $D^\times(\mathbb{A})/F^\times(\mathbb{R})$ and a certain measure on $D_\gamma^\times(\mathbb{A})/F^\times(\mathbb{R})$, and the volume is taken using the quotient of the latter with respect to the measure on the subgroup $D_\gamma^\times C_F$ with $\mathrm{vol}(C_F) = 1$. We will establish a bijective correspondence between the non-vanishing summands in the formulas (2.5) and (2.7). To this end we recall Lemma 10.7 in [7] which can be stated as follows.

Lemma 2.8 *Let $\gamma \in D^\times \subset D^\times(\mathbb{Q}_p)$ be associated with an element of $Y \subset D^\times(E')$ and such that $L = F(\gamma) \subset D$ is either equal to F, or a totally imaginary quadratic extension of F. If $\operatorname{ord}_{\mathfrak{p}}(\gamma) = \operatorname{ord}_{\mathfrak{p}}(\gamma^*)$ for every prime $\mathfrak{p}$ of L dividing p, then there is an F–algebra homomorphism $\lambda\colon L \to A_D$, and we set $a = \lambda(\gamma)$ and $\mathfrak{t} = D$. Otherwise, let $a = \gamma$ and $\mathfrak{t} = (L, \varepsilon)$ where, for every $i \in \{1, \ldots, m\}$ and every prime $\mathfrak{p}$ of L dividing $\mathfrak{p}_i$,*

$$\varepsilon(\mathfrak{p}) = \begin{cases} \mathsf{d}_i & \text{if } \overline{\mathfrak{p}} = \mathfrak{p}\,; \\ \dfrac{-[\,F_{\mathfrak{p}_i} : \mathbb{Q}_p\,]\operatorname{ord}_{\mathfrak{p}}(\gamma)}{r} & \text{otherwise}\,. \end{cases}$$

In both cases, $\mathfrak{t}$ is contained in $\mathfrak{T}_D$, a satisfies condition (2.2)*, and there are a $\delta \in D^\times(E')$ and an $F \otimes_{\mathbb{Q}} \mathcal{K}$–algebra homomorphism $\iota\colon A_a \otimes_{\mathbb{Q}} \mathcal{K} \to D \otimes_{\mathbb{Q}} \mathcal{K}$ such that conditions* (2.3) *and* (2.4) *are satisfied.*

Consider now an element $a \in \mathfrak{R}'(\mathfrak{t})$ for some $\mathfrak{t} \in \mathfrak{T}_D$, choose δ and ι as in (2.3) and (2.4), and assume that the corresponding summand in (2.5) does not vanish. In connection with Proposition 2.6 this implies that δ is associated with an element in $D^\times(\mathbb{Q}_p)$, i.e. $\iota(a)$ is conjugate to such an element. Since either $a \in F$ or L_a is a quadratic extension of F which splits D at all places not dividing p, it follows that $\iota(a)$ is conjugate to an element in $D^\times$. Let $\gamma \in \mathfrak{R}$ be the representative of the image of such an element in the set of conjugacy classes in $D^\times/(O_F^\times \cap C)$. Then γ is uniquely determined, and there is an isomorphism $L_a \cong F(\gamma)$ of F–algebras such that $\operatorname{ord}_{\mathfrak{p}}(a) = \operatorname{ord}_{\mathfrak{p}}(\gamma)$ for all places $\mathfrak{p}$ above p. In particular, $\mathfrak{t}$ can be recovered from γ and (2.2). Recall now from [3] (§5) the equation

$$\operatorname{vol}\big(A_a^\times(\mathbb{R})/F^\times(\mathbb{R})\big) \cdot \int\limits_{D_\gamma^\times(\mathbb{R})\backslash D^\times(\mathbb{R})} f_\infty(x^{-1}\gamma x)\, d\overline{x} = \xi(\gamma)$$

where $\xi(\gamma) = (-1)^{\#\mathcal{Z}}$, $D_\gamma = D$ and $A_a = A_D$ if a is contained in F, and $\xi(\gamma) = 1$ and $D_\gamma \cong A_a$ otherwise, and where the measures on $A_a^\times(\mathbb{R})/F^\times(\mathbb{R})$ and $D_\gamma^\times(\mathbb{R})/F^\times(\mathbb{R})$ are related (cf. [7], Remark 10.5). Moreover, we have

$$\operatorname{vol}\big(D_\gamma^\times C_F\backslash D_\gamma^\times(\mathbb{A})/F^\times(\mathbb{R})\big) = \operatorname{vol}\big(A_a^\times C_F\backslash A_a^\times(\mathbb{A}_f)\big) \cdot \operatorname{vol}\big(A_a^\times(\mathbb{R})/F^\times(\mathbb{R})\big)$$

where all the measures are chosen as indicated in the preceding considerations. Together with Proposition 2.6 this implies that the summand corresponding to γ in (2.7) coincides with the contribution of a to (2.5).

Consider, on the other hand, an arbitrary $\gamma \in \mathfrak{R}$ which contributes a non-zero summand to (2.7). In particular, the orbital integrals of f_∞ and $f_p^{(j)}$ do not vanish. By the choice of f_∞ this requires that either γ is contained in F or $F(\gamma)$ is a totally imaginary quadratic extension of F, and by Proposition 2.6 γ is associated with an element $\delta \in D^\times(E')$ for which $\mathrm{TO}_{\delta\sigma}(\Theta) \neq 0$. Since Θ has support in Y, the σ-conjugacy class of δ must contain an element of Y. Hence, we may apply Lemma 2.8 to obtain $\mathfrak{t} \in \mathfrak{T}_D$ and an element in $A_{\mathfrak{t}}^\times$. We replace the latter with the representative $a \in \mathfrak{R}(\mathfrak{t})$ of its image in the set of conjugacy classes in $A_{\mathfrak{t}}^\times/(O_F^\times \cap C)$. Since every element of $(O_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ can be represented as $\mathrm{Nm}_\sigma(z)$ for a certain $z \in F^\times(E')$, a is contained in $\mathfrak{R}'(\mathfrak{t})$. Obviously, the map $\gamma \mapsto (\mathfrak{t}, a)$ obtained this way is inverse to the map $(\mathfrak{t}, a) \mapsto \gamma$ constructed above, and we conclude that $\mathrm{Tr}(f^{(j)} | \mathcal{A}) = \mathrm{Tr}^{\mathrm{ss}}(f, j)$ as required.

§3. Base change

In this section we will prove Proposition 2.6. Obviously, Θ is a product of certain functions defined on the various factors of the decomposition

$$D^\times(E') \cong \prod_{i=1}^{m} D_{\mathfrak{p}_i}^\times(E')\,,$$

and we can apply Lemma 7.6 of [7] to compute these functions explicitly. The assertion of Proposition 2.6 can be decomposed into several independent statements each of which is concerned with one of the groups $D_{\mathfrak{p}_i}^\times$ over $\mathbb{Q}_p$. For all i such that $D_{\mathfrak{p}_i}$ is ramified, we can apply Proposition 9.4 and the proof of Lemma 11.4 in [7], but if $D_{\mathfrak{p}_i}$ is split, the situation is different from that considered in [7]. In order to treat this case now, we replace our notation as follows. Let F' be a finite unramified extension of $\mathbb{Q}_p$ which is generated by two subfields E' and F containing $\mathbb{Q}_p$, and let $\mathcal{Z}$ be a subset of $\mathrm{Hom}_{\mathbb{Q}_p}(F, F')$ which is invariant with respect to the natural action of $\mathrm{Gal}(F'|E')$. In particular, there is an integer n such that $\#\mathcal{Z} = n \cdot [F' : E']$. Consider the Iwahori subgroup

$$C = \{\, d \in \mathrm{M}_2(O_F) \mid Xd = X \,\} \subset \mathrm{Gl}_2(F) \quad \text{where} \quad X = \mathrm{M}_2(O_F)\eta\,,\ \eta = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$$

and, similarly, $X' \subset \mathrm{M}_2(F')$ and $C' \subset \mathrm{Gl}_2(F')$. In the sequel, whenever we need a measure on a p-adic group, we choose it such that an Iwahori subgroup has volume 1. Let V' and W' denote the characteristic functions of the subsets $C'\eta^{-1}$ and $C'\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)C'$ of $\mathrm{Gl}_2(F')$, respectively, and set

$$\Theta' = W'V' + (1 - q')V' + V'W' \in \mathcal{H}(\mathrm{Gl}_2(F')/\!/C')$$

where q' is the number of elements in the residue class field of F'. The support of this function is the set

$$Y' = \{\, d \in \mathrm{Gl}_2(F') \mid d^{-1} \in \mathrm{M}_2(O_{F'}),\ \mathrm{ord}_p(\det(d)) = -1,\ X' \subset X'd \,\}\,,$$

and for every $d \in Y'$,

$$\Theta'(d) = \begin{cases} 1 - q' & \text{if } X'd = \mathrm{M}_2(O_{F'}) = X'd^* \,; \\ 1 & \text{otherwise}\,. \end{cases}$$

Let σ be the Frobenius automorphism of F', and for every integer i, let

$$\beta_i = \mathrm{Gl}_2(\mathrm{id} \otimes \sigma^i) \colon \ \mathrm{Gl}_2(F \otimes_{\mathbb{Q}_p} E') \longrightarrow \mathrm{Gl}_2(F')\,.$$

Set $E = E' \cap F$ and $e = [\,E : \mathbb{Q}_p\,]$, and consider the function

$$\Theta = \prod_{i=0}^{e-1} \Theta_i \circ \beta_i \colon \ \mathrm{Gl}_2(F \otimes_{\mathbb{Q}_p} E') \longrightarrow \mathbb{Q}$$

where Θ_i is the characteristic function of $C' \subset \mathrm{Gl}_2(F')$ if $\sigma^i \mathcal{Z} \cap \mathrm{Hom}_E(F\,, F') = \varnothing$, and $\Theta_i = \Theta'$ otherwise.

To every irreducible admissible representation π of $\mathrm{Gl}_2(F)$ with $\pi^C \neq 0$ we attach an endomorphism Φ_π of the $\mathbb{C}-$vector space with basis $\{\, y_J \mid J \subseteq \mathcal{Z} \,\}$ as follows. Fix unramified characters χ_1 and χ_2 of $\mathbb{Q}_p^\times$ such that π is a constituent of $\mathrm{PS}\,(\chi_1 \circ \mathrm{Nm}_{F|\mathbb{Q}_p}, \chi_2 \circ \mathrm{Nm}_{F|\mathbb{Q}_p})$, and set

$$\Phi_\pi\, y_J = \mu_1(p)^{-\#(\mathcal{Z} \smallsetminus J)} \mu_2(p)^{-\#J} y_{\Phi J} \quad \text{for all } J \subseteq \mathcal{Z}$$

where $\mu_j = \chi_j \circ \mathrm{Nm}_{E'|\mathbb{Q}_p}$ and $\Phi = \sigma^{-[\,E':\mathbb{Q}_p\,]}$. The trace of this endomorphism is given by

$$\mathrm{Tr}\,(\Phi_\pi) = \sum_{\substack{J \subseteq \mathcal{Z} \\ \Phi J = J}} \mu_1(p)^{-\#(\mathcal{Z} \smallsetminus J)} \mu_2(p)^{-\#J} = \big(\nu_1(p)^{-1} + \nu_2(p)^{-1}\big)^n$$

where $\nu_j = \chi_j \circ \mathrm{Nm}_{F'|\mathbb{Q}_p}$. In order to prove Proposition 2.6 we have to find a function $f \in \mathcal{H}(\mathrm{Gl}_2(F) /\!/ C)$ such that, for every irreducible tempered representation π of $\mathrm{Gl}_2(F)$ with $\pi^C \neq 0$,

$$\mathrm{Tr}\,(\pi(f)) = {q'}^{\,n/2} \dim(\pi^C)\, \mathrm{Tr}\,(\Phi_\pi)\,,$$

and for every semi-simple element $\gamma \in \mathrm{Gl}_2(F)$,

$$\text{(3.1)} \qquad \xi(\gamma)\, \mathrm{O}_\gamma(f) = \begin{cases} \mathrm{TO}_{\delta\sigma}(\Theta) & \text{if } \gamma \text{ is associated with } \delta \in \mathrm{Gl}_2(F \otimes_{\mathbb{Q}_p} E')\,; \\ 0 & \text{if there is no such } \delta\,; \end{cases}$$

where $\xi(\gamma) = (-1)^{\#\mathcal{Z}}$ if $\gamma \in F^\times$ and $\xi(\gamma) = 1$ otherwise.

There is a unique generator $\tau \in \mathrm{Gal}(F'|F)$ such that $\sigma^{-e}\tau \in \mathrm{Gal}(F'|E')$, and a simple calculation shows that, for every $\sigma-$semi-simple $\delta \in \mathrm{Gl}_2(F \otimes_{\mathbb{Q}_p} E')$,

$$\mathrm{TO}_{\delta\sigma}(\Theta) = \mathrm{TO}_{\delta'\tau}(\widetilde{\Theta})$$

where $\delta' = \beta_0(\delta)\beta_1(\delta)\cdots\beta_{e-1}(\delta) \in \mathrm{Gl}_2(F')$ and $\widetilde{\Theta} \in \mathcal{H}(\mathrm{Gl}_2(F')/\!/C')$ denotes the $n-$th power of Θ' (cf. [7], p. 73). Moreover, δ is associated with an element $\gamma \in \mathrm{Gl}_2(F)$ if and only if $\mathrm{Nm}_\tau(\delta') = \delta'\tau(\delta')\cdots\tau^{[F':F]-1}(\delta') = \beta_0\big(\mathrm{Nm}_\sigma(\delta)\big)$ and γ are conjugate in $\mathrm{Gl}_2(F')$. Such a γ always exists, and we claim that $\xi(\gamma)$ coincides with

$$e(\delta') = \begin{cases} (-1)^{\mathrm{ord}_p(\det(\delta'))} & \text{if } \mathrm{Nm}_\tau(\delta') \in F^\times\,; \\ 1 & \text{otherwise}\,; \end{cases}$$

if $\mathrm{TO}_{\delta'\tau}(\widetilde{\Theta}) \neq 0$. In fact, since every element d of the support of $\widetilde{\Theta}$ satisfies the equation $\mathrm{ord}_p\big(\det(d)\big) = -n$, $\mathrm{TO}_{\delta'\tau}(\widetilde{\Theta}) = 0$ if $\xi(\gamma) \neq e(\delta')^{[F':E']}$. On the other hand, if $e(\delta') = -1$, then γ is contained in $F^\times$ and

$$[\,F' : F\,]\,\mathrm{ord}_p\big(\det(\delta')\big) = 2\,\mathrm{ord}_p(\gamma)\,,$$

hence $[\,F' : F\,]$ is even. Since $[\,F' : E'\,]$ and $[\,F' : F\,]$ are relatively prime, our claim follows. Hence a function $f \in \mathcal{H}(\mathrm{Gl}_2(F))$ satisfies condition (3.1) for every semi-simple element $\gamma \in \mathrm{Gl}_2(F)$ if and only if f and $\widetilde{\Theta}$ are associated in the sense of [1] (Chapter 1, §3) which, in turn, holds if and only if, for every irreducible tempered representation π of $\mathrm{Gl}_2(F)$,

$$\mathrm{Tr}\,(\Pi(\widetilde{\Theta})I_\tau) = \mathrm{Tr}\,(\pi(f))$$

where Π denotes the base change lift of π to $\mathrm{Gl}_2(F')$ and $I_\tau \colon \Pi \to \Pi^\tau$ is the canonical intertwining operator (cf. [1], Chapter 1, §6, and [7], p. 76–77). There are the following three possibilities:

- ▷ $\pi^C = 0$ and $\Pi^{C'} = 0$;
- ▷ $\dim(\pi^C) = 1$ and there is an unramified unitary character μ of $F^\times$ such that π is isomorphic to the quotient of $\mathrm{PS}\,(\mu_1, \mu_2)$ with $\mu_1 = \mu|\ |_F^{-1/2}$ and $\mu_2 = \mu|\ |_F^{1/2}$ by a one-dimensional subrepresentation isomorphic to $\mu \circ \det$ and Π is isomorphic to the quotient of $\mathrm{PS}\,(\nu_1, \nu_2)$ with $\nu_j = \mu_j \circ \mathrm{Nm}_{F'|F}$ by a one-dimensional subrepresentation isomorphic to $\mu \circ \mathrm{Nm}_{F'|F} \circ \det$;
- ▷ $\dim(\pi^C) = 2$ and there are unramified unitary characters μ_1 and μ_2 of $F^\times$ such that $\pi \cong \mathrm{PS}\,(\mu_1, \mu_2)$ and $\Pi \cong \mathrm{PS}\,(\nu_1, \nu_2)$ with $\nu_j = \mu_j \circ \mathrm{Nm}_{F'|F}$.

Note that, since $\widetilde{\Theta}$ is $C'-$bi-invariant, $\mathrm{Tr}\,(\Pi(\widetilde{\Theta})I_\tau) = 0$ if $\Pi^{C'} = 0$. Similarly, $\mathrm{Tr}\,(\pi(f)) = 0$ if f is $C-$bi-invariant and $\pi^C = 0$.

Let now $f \in \mathcal{H}(\mathrm{Gl}_2(F)/\!/C)$ be the $n-$th power of

$$f' = (WV + (1-q)V)^{[F':F]} + (VW)^{[F':F]}$$

where V and W are the characteristic functions of the subsets $C\eta^{-1}$ and $C\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)C$ of $\mathrm{Gl}_2(F)$, respectively, and q is the number of elements in the residue class field of F. Consider an unramified character μ of $F^\times$, and set $\widetilde{\mu}=\mu\circ\det$, $\nu=\mu\circ\mathrm{Nm}_{F'|F}$, and $\widetilde{\nu}=\nu\circ\det$. Then we find

$$\widetilde{\mu}(V)=\mu(p)^{-1}\quad\text{and}\quad\widetilde{\mu}(W)=\mathrm{vol}\bigl(C\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)C\bigr)=q\,,$$

hence

$$\widetilde{\mu}(f)=\bigl(\mu(p)^{-[F':F]}+(q\mu(p)^{-1})^{[F':F]}\bigr)^n=\bigl(\nu(p)^{-1}+q'\nu(p)^{-1}\bigr)^n\,.$$

Similarly, we have

$$\widetilde{\nu}(V')=\nu(p)^{-1}\quad\text{and}\quad\widetilde{\nu}(W')=q'\,,$$

hence

$$\widetilde{\nu}(\widetilde{\Theta})=\bigl(\nu(p)^{-1}+q'\nu(p)^{-1}\bigr)^n\,.$$

Together with the following lemma this implies that f has the required properties.

Lemma 3.2 *Let $\pi=\mathrm{PS}(\mu_1,\mu_2)$ for unramified characters μ_1 and μ_2 of $F^\times$, and let $\Pi=\mathrm{PS}(\nu_1,\nu_2)$ with $\nu_1=\mu_1\circ\mathrm{Nm}_{F'|F}$ and $\nu_2=\mu_2\circ\mathrm{Nm}_{F'|F}$. Then*

$$\mathrm{Tr}(\Pi(\widetilde{\Theta})I_\tau)=\mathrm{Tr}(\pi(f))=2q'^{\,n/2}\bigl(\nu_1(p)^{-1}+\nu_2(p)^{-1}\bigr)^n$$

where $I_\tau\colon\Pi\to\Pi^\tau$ is the canonical intertwining operator.

Proof. According to [3] (Theorem 3.13), $\Pi^{C'}$ admits a basis on which I_τ acts trivially and with respect to which

$$\Pi^{C'}(V')=\begin{pmatrix}0 & q'^{\,1/2}\nu_1(p)^{-1}\\ q'^{\,-1/2}\nu_2(p)^{-1} & 0\end{pmatrix}\quad\text{and}\quad\Pi^{C'}(W')=\begin{pmatrix}0 & q'\\ 1 & q'-1\end{pmatrix}.$$

This implies that Θ' acts on $\Pi^{C'}$ as multiplication by $\alpha=q'^{\,1/2}\bigl(\nu_1(p)^{-1}+\nu_2(p)^{-1}\bigr)$. Similarly, there is a basis of π^C with respect to which

$$\pi^C(V)=\begin{pmatrix}0 & q^{1/2}\mu_1(p)^{-1}\\ q^{-1/2}\mu_2(p)^{-1} & 0\end{pmatrix}\quad\text{and}\quad\pi^C(W)=\begin{pmatrix}0 & q\\ 1 & q-1\end{pmatrix},$$

hence

$$\pi^C(WV+(1-q)V)=\begin{pmatrix}q^{1/2}\mu_2(p)^{-1} & (1-q)q^{1/2}\mu_1(p)^{-1}\\ 0 & q^{1/2}\mu_1(p)^{-1}\end{pmatrix}$$

and

$$\pi^C(VW)=\begin{pmatrix}q^{1/2}\mu_1(p)^{-1} & (q-1)q^{1/2}\mu_1(p)^{-1}\\ 0 & q^{1/2}\mu_2(p)^{-1}\end{pmatrix}.$$

Thus f' acts on π^C as multiplication by $\bigl(q^{1/2}\mu_1(p)^{-1}\bigr)^{[F':F]}+\bigl(q^{1/2}\mu_2(p)^{-1}\bigr)^{[F':F]}=\alpha$, and we obtain $\mathrm{Tr}(\Pi(\widetilde{\Theta})I_\tau)=\mathrm{Tr}(\pi(f))=2\alpha^n$. ∎

References

[1] J. Arthur, L. Clozel: *Simple algebras, base change, and the advanced theory of the trace formula.* Annals of Mathematics Studies **120**, Princeton University Press (1989).

[2] W. Casselman: *An assortment of results on representations of* $\mathrm{Gl}_2(k)$. Modular functions of one variable II, Lecture Notes in Mathematics **349**, Springer (1973), 1–54.

[3] W. Casselman: *The Hasse-Weil* ζ*−function of some moduli varieties of dimension greater than one.* Automorphic Forms, Representations and L−functions, Proceedings of Symposia in Pure Mathematics **33**, part 2, American Mathematical Society (1979), 141–163.

[4] P. Deligne: *Les constantes des équations fonctionnelles des fonctions* L. Modular functions of one variable II, Lecture Notes in Mathematics **349**, Springer (1973), 501–597.

[5] R.E. Kottwitz: *Shimura varieties and twisted orbital integrals.* Mathematische Annalen **269** (1984), 287–300.

[6] M. Ohta: *On the Zeta Function of an Abelian Scheme over the Shimura Curve II.* Galois Groups and their Representations, Advanced Studies in Pure Mathematics **2** (1983), 37–54.

[7] H. Reimann: *The semi-simple zeta function of quaternionic Shimura varieties.* Lecture Notes in Mathematics **1657**, Springer (1997).

[8] H. Reimann: *On the zeta function of quaternionic Shimura varieties.* Mathematische Annalen **317** (2000), 41–55.

[9] H. Reimann: *Reduction of Shimura varieties at parahoric levels.* Preprint **99 − 139**, Universität Bielefeld, Sonderforschungsbereich 343 (1999).