CLEAVING FUNCTORS AND CONTROLLED WILD ALGEBRAS

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Abstract. We show that the cleaving functors introduced in [BGRS] as a tool for proving infinite representation type of finite-dimensional algebras can also be used to establish controlled wildness. The main application is that an algebra is controlled wild if there is an indecomposable projective module with a Loewy factor having a homogeneous direct summand which is of length at least 3. As a second application we derive Han's covering criterion.

1. Introduction and Summary of Contents

1.1. **Variations of Wildness.** A finite-dimensional associative algebra A over an algebraically closed field k is said to be *wild* if there is a faithful exact functor H : $\text{mod}-k\{X,Y\} \to \text{mod}-A$ which preserves indecomposability and isomorphism classes. We use the notation $\text{mod}-B$ for the full subcategory of finite-dimensional modules inside the category $\text{Mod}-B$ of right modules over an associative k-algebra B. A wild algebra is called *strictly wild* provided the functor H can be chosen to be full.

Ringel proposed a notion of wildness which lies between common and strict wildness. A finite-dimensional algebra A is said to be *controlled wild* by a full additive subcategory $\mathcal{C}$ of $\text{mod}-A$ if there is a faithful exact functor $H : \text{mod}-k\{X,Y\} \to \text{mod}-A$ such that $\text{Hom}_A(HM, HN) = H\,\text{Hom}_{k\{X,Y\}}(M,N) \oplus \text{Hom}_A(HM,HN)_{\mathcal{C}}$ and $\text{Hom}_A(HM,HN)_{\mathcal{C}} \subseteq \text{rad}_A(HM,HN)$ holds for all M, N in $\text{mod}-k\{X,Y\}$. We use the notation $\text{Hom}_A(-,-)_{\mathcal{C}}$ for the ideal of $\text{mod}-A$ formed by all homomorphisms factoring through an object in $\mathcal{C}$ and rad_A for the Jacobson radical of $\text{mod}-A$ which is the ideal generated by all non-isomorphisms between indecomposable modules.

As it is shown in [Hn] using arguments from [Si], in all these variations of wildness the functor H can be chosen to be of the form $- \otimes_{k\{X,Y\}} W$ for a $k\{X,Y\}$-A-bimodule W which is finitely generated free as $k\{X,Y\}$-module.

Obviously, any strictly wild algebra is controlled wild by the subcategory formed by the zero module. Moreover, any controlled wild algebra is wild (see [Hn]). It is well-known that a local wild algebra (and there are plenty of them) is never strictly wild. On the other hand, it is conjectured that any wild algebra is controlled wild. This paper may serve as support for this conjecture.

1.2. **Cleaving Functors.** In order to introduce cleaving functors, a more categorical point of view is appropriate. For a skeletally small k-linear category $\mathcal{S}$ we denote by $\text{Mod}-\mathcal{S}$ the category of all k-linear contravariant functors from $\mathcal{S} \to \text{Mod}-k$ where $\text{Mod}-k$ is the category of all vector spaces over k. The functors in $\text{Mod}-\mathcal{S}$ will be called

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right modules over $\mathcal{S}$. If we consider an associative algebra A as a category with one object in the obvious way, then this definition of right modules is consistent with the usual one.

A module in Mod$-\mathcal{S}$ is called finitely generated projective if it is a direct summand of a module of the shape $\bigoplus_{i=1}^{n}\mathcal{S}(-,x_i)$ where $x_1,\dots,x_n$ are objects of $\mathcal{S}$. We denote by proj$-\mathcal{S}$ the full subcategory of Mod$-\mathcal{S}$ given by the finitely generated projective modules. A module M in Mod$-\mathcal{S}$ is said to be finitely presented if there is an exact sequence $P_1 \longrightarrow P_0 \longrightarrow M \to 0$ such that P_1 and P_0 are finitely generated projective. By $\mathrm{mod}_{\mathrm{fp}}-\mathcal{S}$ we denote the skeletally small full subcategory of Mod$-\mathcal{S}$ given by the finitely presented modules.

Let $\mathcal{S}$, $\mathcal{T}$ be two skeletally small k-linear categories and $F : \mathcal{S} \to \mathcal{T}$ a k-linear functor. The restriction functor $F^{\bullet} : \mathrm{Mod}-\mathcal{T} \to \mathrm{Mod}-\mathcal{S}$, $M \mapsto MF$ has a left adjoint F^{λ} which is up to natural isomorphism well-defined by requiring that $F^{\lambda} : \mathrm{Mod}-\mathcal{S} \to \mathrm{Mod}-\mathcal{T}$ is a right exact, coproduct preserving functor such that $F^{\lambda}\mathcal{S}(-,x) = \mathcal{T}(-,Fx)$ for all objects x of $\mathcal{S}$. The functor F is called *cleaving* if the canonical natural transformation from the identity of Mod$-\mathcal{S}$ to $F^{\bullet}F^{\lambda}$ is a section. Cleaving functors where introduced in [BGRS] as an elementary tool for showing that algebras are representation-infinite. It was observed in [Pe] that cleaving functors are also helpful for detecting wild representation type. We will refine these arguments in order to deal with controlled wildness.

For our purposes we will need a characterisation of cleaving functors from [BGRS]. We observe that $F\mathcal{S}(-,-)$ is a subbifunctor of the bifunctor $\mathcal{T}(F-,F-) : \mathcal{S}^{op} \times \mathcal{S} \to \mathrm{Mod}-k$. In [BGRS] it is shown that a functor $F : \mathcal{S} \to \mathcal{T}$ is cleaving if and only if F is faithful and there is subbifunctor U of $\mathcal{T}(F-,F-)$ such that $F\mathcal{S}(-,-) \oplus U = \mathcal{T}(F-,F-)$. A subbifunctor U with this property is called a *cleavage* of F.

The key result of our paper says that, if $F : \mathcal{S} \to \mathcal{T}$ is a cleaving functor, then $F^{\lambda} : \mathrm{mod}_{\mathrm{fp}}-\mathcal{S} \to \mathrm{mod}_{\mathrm{fp}}-\mathcal{T}$ is a cleaving functor as well. This assertion will be proved in section 2 as Proposition 2.2 by constructing a cleavage $\mathrm{Hom}_{\mathcal{T}}(-,-)_U$ for F^{λ} from a cleavage U for F.

For our applications it will be convenient to use the language of spectroids and aggregates which was introduced in [GR]. We will recall the relevant notation in section 3. In particular, we will see that finite spectroids are practically the same as finite-dimensional algebras. After these preparations in Theorem 3.2 we will present a sufficient condition to obtain controlled wildness of a finite spectroid $\mathcal{T}$ from a cleaving functor $F : \mathcal{S} \to \mathcal{T}$ starting in a strictly wild locally bounded spectroid $\mathcal{S}$. This theorem may be considered as the main result of our paper but it has the drawback to depend on clever choices of subcategories $\mathcal{W}$ of mod$-\mathcal{S}$ and $\mathcal{C}$ of mod$-\mathcal{T}$ satisfying certain conditions. We will finish section 3 by giving some hints how these conditions can be checked.

1.3. **Applications.** As mentioned before, in order to apply Theorem 3.2, one has to choose suitable subcategories $\mathcal{W}$ and $\mathcal{C}$. In section 4 we will give two applications where this is possible. As main application, which was actually the motivation for this paper, we will establish that an algebra B is strictly wild if there is an indecomposable projective module with a Loewy factor having a homogeneous direct summand which is of length at least 3. As a second application, we will reprove Han's covering criterion (see [Hn]) using our approach.

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2. Lifting Cleaving to Finitely Presented Modules

2.1. **Projective Presentations.** For a skeletally small k-linear category $\mathcal{S}$ we denote by $\operatorname{add}\mathcal{S}$ a minimal additive k-linear category containing $\mathcal{S}$ in which idempotents split. The category $\operatorname{add}\mathcal{S}$ will be skeletally small again and the Yoneda functor $\operatorname{add}\mathcal{S} \to \operatorname{proj}-\mathcal{S}$, $x \mapsto \operatorname{add}\mathcal{S}(-,x)$ will be an equivalence. (Indeed, this shows that $\operatorname{proj}-\mathcal{S}$ may be chosen as a model for $\operatorname{add}\mathcal{S}$.)

A functor $F : \mathcal{S} \to \mathcal{T}$ of skeletally small k-linear categories extends uniquely to a functor $\operatorname{add}\mathcal{S} \to \operatorname{add}\mathcal{T}$. We will denote this extended functor by F as well. If F is cleaving then the extended functor $\operatorname{add}\mathcal{S} \to \operatorname{add}\mathcal{T}$ will be cleaving as well because any cleavage U can also be extended uniquely. We will also keep the name U for the extended cleavage.

For a skeletally small additive k-linear category $\mathcal{A}$ with splitting idempotents let us introduce the category $\operatorname{mat}\mathcal{A}$ whose objects are morphisms $a : x_1 \to x_0$ in $\mathcal{A}$. The space of morphism $\operatorname{mat}\mathcal{A}(a,a')$ from $a : x_1 \to x_0$ to $a' : x'_1 \to x'_0$ is the set of pairs (h_1,h_0) of maps $h_1 : x_1 \to x'_1$, $h_0 : x_0 \to x'_0$ in $\mathcal{A}$ satisfying $a'h_1 = h_0a$. The composition is defined componentwise. Clearly, $\operatorname{mat}\mathcal{A}$ is a skeletally small k-category again.

If $\mathcal{A} = \operatorname{add}\mathcal{S}$, then we obtain a full and dense functor $\Psi_{\mathcal{S}} : \operatorname{mat}\mathcal{A} \to \operatorname{mod}-\mathcal{S}$ which sends an object $a : x_1 \to x_0$ of $\operatorname{mat}\mathcal{A}$ to the cokernel of the map $\mathcal{A}(-,a) : \mathcal{A}(-,x_1) \to \mathcal{A}(-,x_0)$. Usually, $\operatorname{mat}\mathcal{A}$ is called the category of *projective presentations* for $\mathcal{S}$.

2.2. **Construction of the Cleavage of the Left Adjoint.** For a k-linear functor $F : \mathcal{S} \to \mathcal{T}$ of skeletally small k-categories we put $\mathcal{A} := \operatorname{add}\mathcal{S}$ and $\mathcal{B} := \operatorname{add}\mathcal{T}$. The induced functor $F : \mathcal{A} \to \mathcal{B}$ yields a functor $F^\beta : \operatorname{mat}\mathcal{A} \to \operatorname{mat}\mathcal{B}$ which sends an object a to Fa and a morphism (h_1,h_0) to (Fh_1,Fh_0). Because of the defining properties of the left adjoint of F we obtain $F^\lambda\Psi_{\mathcal{S}} = \Psi_{\mathcal{T}}F^\beta : \operatorname{mat}\mathcal{A} \to \operatorname{mod}_{\mathrm{fp}}-\mathcal{B}$. We use this equation to prove the key result of this paper:

Proposition. *If $F : \mathcal{S} \to \mathcal{T}$ is a cleaving functor of skeletally small k-linear categories, then $F^\lambda : \operatorname{mod}_{\mathrm{fp}}-\mathcal{S} \to \operatorname{mod}_{\mathrm{fp}}-\mathcal{T}$ is cleaving as well.*

Proof. If F is cleaving with cleavage U, then for a and a' in $\operatorname{mat}\mathcal{A}$ we denote by $\operatorname{mat}U(a,a')$ the set of all (u_1,u_0) in $\operatorname{mat}\mathcal{B}(F^\beta a, F^\beta a')$ such that u_1 lies in $U(x_1,x'_1)$ and u_0 lies in $U(x_0,x'_0)$. It is rather obvious that $\operatorname{mat}U$ is a cleavage for the faithful functor F^β. Thus we have already shown that F^β is cleaving.

For M, M' in $\operatorname{mod}-\mathcal{S}$ we choose a, a' in $\operatorname{mat}\mathcal{A}$ such that $\Psi_{\mathcal{S}}a = M$ and $\Psi_{\mathcal{S}}a' = M'$. Now we define $\operatorname{Hom}_{\mathcal{T}}(M,M')_U$ as $\Psi_{\mathcal{T}}\operatorname{mat}U(a,a')$ and observe that $\operatorname{Hom}_{\mathcal{T}}(-,-)_U$ is a well-defined subbifunctor of the bifunctor $\operatorname{Hom}_{\mathcal{T}}(F^\lambda-,F^\lambda-) : (\operatorname{mod}-\mathcal{S})^{op} \times \operatorname{mod}-\mathcal{S} \to \operatorname{Mod}-k$ because $\operatorname{mat}U(-,-)$ is a subbifunctor of $\operatorname{mat}\mathcal{B}(F^\beta-,F^\beta-) : (\operatorname{mat}\mathcal{A})^{op} \times \operatorname{mat}\mathcal{A} \to \operatorname{Mod}k$. It remains to show that F^λ is faithful and $\operatorname{Hom}_{\mathcal{T}}(F^\lambda M, F^\lambda M') = F^\lambda \operatorname{Hom}_{\mathcal{S}}(M,M') \oplus \operatorname{Hom}_{\mathcal{T}}(M,M')_U$.

For the first assertion we consider (h_1,h_0) in $\operatorname{mat}\mathcal{A}(a,a')$ such that $\Psi_{\mathcal{T}}F^\beta(h_1,h_0) = 0$. Hence there is $g : Fx_0 \to Fx'_1$ satisfying $Fh_0 = F(a')g$. Because F is cleaving, we can write g as $g = Ff + u$ where f in $\mathcal{A}(x_0,x'_1)$ and u in $U(x_0,x'_1)$. Since $F(a')u$ belongs to U, from the equation $Fh_0 = F(a'f) + F(a')u$ follows that $h_0 = fa'$ using that F is faithful. Hence $\Psi_{\mathcal{S}}(h_1,h_0) = 0$.

Because $\Psi_{\mathcal{T}}$ is full, for the second assertion it suffices to prove that the sum is direct. Assume that $\Psi_{\mathcal{T}}F^\beta(h_1,h_0) = \Psi_{\mathcal{T}}(u_1,u_0)$ for (h_1,h_0) in $\operatorname{mat}\mathcal{A}(a,a')$ and (u_1,u_0) in $\operatorname{mat}U(a,a')$. We obtain the existence of a map g in $\mathcal{B}(Fx_0,Fx'_1)$ such that $Fh_0 - u_0 =$

$F(a')g$. Writing g as $g = Ff + u$ with f in $\mathcal{A}(x_0, x_1')$ and u in $U(x_0, x_1')$, we get $F(a'f) - Fh_0 = -F(a')u - u_0$. Hence $F(a'f - h_0) = 0$ and therefore $\Psi_{\mathcal{T}} F^{\beta}(h_1, h_0) = 0$. □

3. Cleaving Functors for Spectroids

3.1. **Spectroids and Aggregates.** It will be convenient to pass from finite-dimensional algebras to spectroids. The first step is that we can always assume that the algebra A under consideration is basic because any finite-dimensional algebra is Morita equivalent to a basic algebra and Morita equivalent algebras have equivalent module categories.

A *spectroid* $\mathcal{S}$ is a small k-linear category with finite-dimensional morphism spaces such that for each object x of $\mathcal{S}$ the endomorphism algebra $\mathcal{S}(x, x)$ is local and pairwise different objects are non-isomorphic. Given a basic finite-dimensional algebra A, we can write the unit element of A as a sum $\sum_{x=1}^{n} e_x$ of mutually orthogonal primitive idempotents and obtain a spectroid $\mathcal{S}_A$ which has the set $\{1, \dots, n\}$ as set of objects and the space $e_y A e_x$ as set $\mathcal{S}_A(x, y)$ of morphisms from x to y. The composition is given by multiplication. The spectroid $\mathcal{S}_A$ is finite meaning that it has a finite set of objects. It is obvious that $\mathrm{Mod}{-}A$ can be identified with $\mathrm{Mod}{-}\mathcal{S}_A$ by sending a module M to the functor which maps the object x of $\mathcal{S}_A$ to Me_x.

For a spectroid $\mathcal{S}$ we will denote by $\mathrm{mod}{-}\mathcal{S}$ the full subcategory of $\mathrm{Mod}{-}\mathcal{S}$ formed by the finite-dimensional modules M (i.e. $\sum_{x\in\mathcal{S}} \dim_k M(x) < \infty$). Clearly for $\mathcal{S}_A$ the identification of $\mathrm{Mod}{-}A$ with $\mathrm{Mod}{-}\mathcal{S}_A$ yields an identification of $\mathrm{mod}{-}A$ with $\mathrm{mod}{-}\mathcal{S}_A$.

Since we also want to derive Han's covering criterion using our set-up, we will have to consider a slight generalisation of finite spectroids, namely the locally bounded spectroids. A spectroid $\mathcal{S}$ is said to be *locally bounded* if $\sum_{y\in\mathcal{S}} \dim_k \mathcal{S}(x, y) + \sum_{y\in\mathcal{S}} \dim_k \mathcal{S}(y, x) < \infty$ for all objects x of $\mathcal{S}$. For a locally bounded spectroid $\mathcal{S}$ the categories $\mathrm{mod}{-}\mathcal{S}$ and $\mathrm{mod}_{\mathrm{fp}}{-}\mathcal{S}$ coincide.

An *aggregate* is a skeletally small additive k-linear category with finite-dimensional morphism spaces such that all idempotents split. For a spectroid $\mathcal{S}$ the categories $\mathrm{add}\,\mathcal{S}$ and $\mathrm{mod}_{\mathrm{fp}}{-}\mathcal{S}$ are aggregates. The objects of $\mathrm{add}\,\mathcal{S}$ are just finite direct sums $\bigoplus_{i=1}^{n} x_i$ of objects $x_1, \dots, x_n$ of $\mathcal{S}$ because the category formed by these objects has already splitting idempotents.

For a locally bounded spectroid $\mathcal{S}$ we put $\mathcal{A} := \mathrm{add}\,\mathcal{S}$ and analyse more precisely the full and dense functor $\Psi_{\mathcal{S}} : \mathrm{mat}\mathcal{A} \to \mathrm{mod}{-}\mathcal{S}$. The kernel of this functor is the ideal of $\mathrm{mat}\mathcal{A}$ formed by the morphisms which factor through an object of the shape $(x \xrightarrow{\mathrm{id}_x} x) \oplus (y \to 0)$ where x, y are objects of $\mathcal{A}$. If we define $\mathrm{mmat}\mathcal{A}$ as the full subcategory of $\mathrm{mat}\mathcal{A}$ formed by the objects which do not admit a direct summand of the shape $(x \xrightarrow{\mathrm{id}_x} x) \oplus (y \to 0)$, then $\Psi_{\mathcal{S}} : \mathrm{mmat}\mathcal{A} \to \mathrm{mod}{-}\mathcal{S}$ is a representation equivalence. The objects in $\mathrm{mmat}\mathcal{A}$ are called *minimal projective presentations.*

3.2. **A Sufficient Condition for Controlled Wildness.** Let us now formulate a condition which produces a controlled wild spectroid $\mathcal{T}$ from a cleaving functor $F : \mathcal{S} \to \mathcal{T}$ starting in a strictly wild spectroid $\mathcal{S}$.

Theorem. *Let $F : \mathcal{S} \to \mathcal{T}$ be a cleaving functor of locally bounded spectroids with cleavage U where $\mathcal{S}$ is strictly wild. Thus there exists a fully faithful exact functor $H : \mathrm{mod}{-}k\{X, Y\} \to \mathrm{mod}{-}\mathcal{S}$.*
Assume that we find a full subcategory $\mathcal{W}$ of $\mathrm{mod}{-}\mathcal{S}$ which contains $H(\mathrm{mod}{-}k\{X, Y\})$ and a full additive subcategory $\mathcal{C}$ of $\mathrm{mod}{-}\mathcal{T}$ satisfying the following conditions:

(a) $F^\lambda : \mathcal{W} \to \text{mod}-\mathcal{T}$ *is exact.*
(b) $F^\lambda : \mathcal{W} \to \text{mod}-\mathcal{T}$ *preserves indecomposability.*
(c) For all M, N *in* $\mathcal{W}$ *the equation* $\text{Hom}_\mathcal{T}(M,N)_U = \text{Hom}_\mathcal{T}(F^\lambda M, F^\lambda N)_\mathcal{C}$ *holds.*
Then $\mathcal{T}$ *is controlled wild by* $\mathcal{C}$*.*

Proof. If we consider the composition $F^\lambda H : \text{mod}-k\{X,Y\} \to \text{mod}-\mathcal{T}$, then all what is left to show is that $\text{Hom}_\mathcal{T}(F^\lambda M, F^\lambda N)_\mathcal{C} \subseteq \text{rad}_\mathcal{T}(F^\lambda M, F^\lambda N)$ for all M, N in $\mathcal{W}$. We may assume that M, N are indecomposable which by (b) also forces $F^\lambda M$ and $F^\lambda N$ to be indecomposable. If we assume that $\text{Hom}_\mathcal{T}(F^\lambda M, F^\lambda N)_\mathcal{C} \not\subseteq \text{rad}_\mathcal{T}(F^\lambda M, F^\lambda N)$, then there is an isomorphism α in $\text{Hom}_\mathcal{T}(F^\lambda M, F^\lambda N)_\mathcal{C}$. Because $\text{Hom}_\mathcal{T}(-,-)_\mathcal{C}$ is an ideal of mod$-\mathcal{T}$, the identity $\text{id}_{F^\lambda M} = F^\lambda(\text{id}_M)$ has to lie in $\text{Hom}_\mathcal{T}(F^\lambda M, F^\lambda M)_\mathcal{C} = \text{Hom}_\mathcal{T}(M,M)_U$. Because F^λ is cleaving, this implies $\text{id}_M = 0$, a contradiction. □

3.3. **An Exact Structure for the Projective Presentations.** The difficulty in the preceeding Theorem is to find the appropriate subcategories $\mathcal{W}$ and $\mathcal{C}$ such that the assertions are satisfied. If $\mathcal{S}$ is a skeletally small k-category and $\mathcal{A} := \text{add}\,\mathcal{S}$, then mat$\mathcal{A}$ becomes an exact category (see [GR]) with the componentwise split exact structure. This means that $((f_1, f_0),(g_1,g_0))$ is an exact pair if (f_i, g_i) is a split exact pair for $i = 0,1$. Usually the functor $\Psi_\mathcal{S}$ will send exact pairs only to right exact sequences in mod$-\mathcal{S}$. Nevertheless, we will see in the next section that this exact structure on mat$\mathcal{A}$ may be helpful for establishing condition (a) of Theorem 3.2.

3.4. **Preserving Indecomposability.** Now we will present a lemma which is helpful to establish (b) of Theorem 3.2 in some situations.

If A is a k-algebra and W is a subspace of A, then for all non-negative integers i we denote by $W^{(i)}$ the subspace of A spanned by all elements of W^i where W^i is the set of all product $w_1 \cdots w_n$ such that $w_1, \dots, w_n$ are elements of W. Further we put $W^{(\infty)} := \sum_{i \in \mathbb{N}} W^{(i)}$.

Lemma. *Let* B *be a finite-dimensional* k*-algebra,* A *a subalgebra of* B*, and* W *a* A*-*A*-subbimodule of* B *satisfying* $B = A \oplus W$ *and* $W^m = 0$ *for some natural number* m*. Then the following assertions hold:*
(i) For all $i \in \mathbb{N}$ *the subspace* $W^{(i)}$ *is a* A*-*A*-subbimodule of* B*.*
(ii) $W^{(\infty)}$ *is a nilpotent ideal of* B*.*
(iii) $B/W^{(\infty)}$ *is a factor algebra of* A*.*
(iv) If A *is a local algebra, then* B *is a local algebra.*

Proof. Parts (i) and (ii) are clear. For (iii) we observe that the composition of the canonical inclusion $A \to B$ with the projection $B \to B/W^{(\infty)}$ is surjective. Now (iv) follows immediately from (ii) and (iii). □

4. Applications

4.1. **Algebras with Big Loewy Factors.** We will need the path category $\mathcal{S}$ of the 3-Kronecker quiver

$$\begin{tikzcd} x \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & y \end{tikzcd}$$

as domain for a cleaving functor and put $\mathcal{A} := \text{add}\,\mathcal{S}$. Since this quiver admits no oriented cycle, its path category $\mathcal{S}$ is actually a spectroid. We consider the full subaggregate $\mathcal{W}$ of mod$-\mathcal{S}$ formed by the modules having no simple direct summand and the subaggregate mmat$_\mathcal{W}\mathcal{A}$ of mmat$\mathcal{A}$ such that $\Psi_\mathcal{S}\text{mmat}_\mathcal{W}\mathcal{A} = \mathcal{W}$. If $a : x_1 \to x_0$ is in mmat$_\mathcal{W}\mathcal{A}$, then

a is a monomorphism, x_1 is of the shape x^m and x_0 of the shape y^n. It follows that $\Psi_{\mathcal{S}} : \mathrm{mmat}_{\mathcal{W}}\mathcal{A} \to \mathcal{W}$ is an exact equivalence.

Let us now state the result which was the original motivation for this paper. In classical language it says that an algebra is controlled wild provided there is an indecomposable projective module with a Loewy factor having a homogeneous direct summand of length at least 3. Passing to the language of spectroids this reads as follows:

Theorem. *Let $\mathcal{T}$ be a finite spectroid with radical $\mathcal{J} = \mathrm{rad}\,\mathcal{T}$ and assume that there are objects s, t of $\mathcal{T}$ and there is a natural number n such that $\dim_k(\mathcal{J}^n/\mathcal{J}^{n+1})(s,t) \geq 3$. Then $\mathcal{T}$ is controlled wild by $\mathrm{mod}{-}\mathcal{T}/\mathcal{J}^n$.*

Proof. We may assume that $\mathcal{J}^{n+1} = 0$ and $\dim_k \mathcal{J}^n(s,t) = 3$. Now we choose a basis a_1, a_2, a_3 of $\mathcal{J}^n(s,t)$. In order to construct a cleaving $F : \mathcal{S} \to \mathcal{T}$ we consider the path category $\mathcal{S}$ of the 3-Kronecker quiver and denote the arrows from x to y by α_1, α_2, α_3. Now we define $F : \mathcal{S} \to \mathcal{T}$ by $F(x) := s$, $F(y) := t$ and $F(\alpha_i) := a_i$ for $i = 1,2,3$. A cleaving U for F is given by $U(x,x) := \mathcal{J}(s,s)$, $U(y,y) := \mathcal{J}(t,t)$, $U(y,x) := \mathcal{T}(t,s)$ and $U(x,y)$ a complement of $\mathcal{J}^n(s,t)$ in $\mathcal{T}(s,t)$.

Let us put $\mathcal{B} := \mathrm{add}\,\mathcal{T}$. As subcategory $\mathcal{W}$ needed to apply Theorem 3.2 we use the $\mathcal{S}$-modules without simple direct summands as considered above. Because $\Psi_{\mathcal{S}} : \mathrm{mmat}_{\mathcal{W}}\mathcal{A} \to \mathcal{W}$ is an exact equivalence, we obtain $F^\lambda = \Psi_{\mathcal{T}} F^\beta \Psi_{\mathcal{S}}^{-1} : \mathcal{W} \to \mathrm{mod}{-}\mathcal{T}$. In order to show that this functor is exact, we have to prove that exact pairs in $\mathrm{mmat}_{\mathcal{W}}\mathcal{A}$ are mapped to exact sequences in $\mathrm{mod}{-}\mathcal{T}$ by $\Psi_{\mathcal{S}} F^\beta$. Because of $\mathcal{J}^{n+1} = 0$ we see that for any map $a : x_1 \to x_0$ in $\mathrm{mat}_{\mathcal{W}}\mathcal{A}$ the map $\mathcal{B}(-, F(a)) : \mathcal{B}(-, Fx_1) \to \mathcal{B}(-, Fx_0)$ has kernel $\mathrm{rad}\,\mathcal{B}(-, Fx_1)$. It follows that the kernel sequence induced from the diafram of projective modules associated with the image of an exact pair in $\mathrm{mmat}_{\mathcal{W}}\mathcal{A}$ under F^β is exact. The snake lemma yields the assertion. Thus we have checked condition (a) of Theorem 3.2.

Let us turn to condition (b). If $a : x_1 \to x_0$ is an object of $\mathrm{mmat}_{\mathcal{W}}\mathcal{A}$ such that $\Psi_{\mathcal{S}}(a)$ is indecomposable, then the endomorphism algebra $\mathrm{mat}\mathcal{A}(a,a)$ is local. It suffices to prove that $\mathrm{mat}\mathcal{B}(F^\beta a, F^\beta a)$ is local as well. Because F is cleaving, we know that $\mathrm{mat}\mathcal{B}(F^\beta a, F^\beta a) = F^\beta(\mathrm{mat}\mathcal{A}(a,a)) \oplus \mathrm{mat}U(a,a)$. where $F^\beta(\mathrm{mat}\mathcal{A}(a,a))$ is a subalgebra isomorphic to the local algebra $\mathrm{mat}\mathcal{A}(a,a)$. Since $U(x,x) := \mathcal{J}(s,s)$ and $U(y,y) := \mathcal{J}(t,t)$, we get that $\mathrm{mat}U(a,a)^m = 0$ for some $m \in \mathbb{N}$ and therefore Lemma 3.4 applies.

It remains to establish condition (c) saying that $\mathrm{Hom}_{\mathcal{T}}(M,N)_U$ and $\mathrm{Hom}_{\mathcal{T}}(F^\lambda M, F^\lambda N)_{\mathrm{mod}-\mathcal{T}/\mathcal{J}^n}$ coincide for all M, N in $\mathcal{W}$. Because of $U(y,y) = \mathcal{J}(t,t)$ and $\mathcal{J}^{n+1} = 0$ it is obvious that the image of a map in $\mathrm{Hom}_{\mathcal{T}}(M,N)_U$ is annihilated by $\mathcal{J}^n$. The other inclusion will be proved in the lemma below. ☐

Lemma. *Let a, a' be in $\mathrm{mmat}_{\mathcal{W}}\mathcal{A}$, (h_1, h_0) be a non-zero morphism in $\mathrm{mat}\mathcal{A}(a,a')$ and (u_1, u_0) in $\mathrm{mat}U(a,a')$. Then $\mathcal{J}^n \,\mathrm{Im}\,\Psi_{\mathcal{T}}(F(h_1,h_0) + (u_1,u_0)) \neq 0$.*

Proof. We put $f_i := F(h_i) + u_i$ for $i = 0,1$. For $a : x_1 \to x_0$ and $a' : x_1' \to x_0'$ we consider the exact sequences $\mathcal{B}(-, Fx_1) \xrightarrow{\mathcal{B}(-,Fa)} \mathcal{B}(-, Fx_0) \xrightarrow{\pi} \Psi_{\mathcal{T}} Fa \to 0$ and $\mathcal{B}(-, Fx_1') \xrightarrow{\mathcal{B}(-,Fa')} \mathcal{B}(-, Fx_0') \xrightarrow{\pi'} \Psi_{\mathcal{T}} Fa' \to 0$.

Let us assume $\mathcal{J}^n \,\mathrm{Im}\,\Psi_{\mathcal{T}}(F(h_1,h_0) + (u_1,u_0)) = 0$. Because π is an epimorphism we obtain $\pi'\mathcal{B}(-, f_0)(\mathcal{J}^n(-, Fx_0)) = 0$, thus $\pi_s'\mathcal{B}(-, f_0)\mathcal{J}^n(s, Fx_0)) = 0$ by looking at the object s of $\mathcal{T}$. Now we observe $\mathcal{B}(-, u_0)\mathcal{J}^n(s, Fx_0) = u_0\mathcal{J}^n(s, Fx_0) = 0$ using $u_0 \in \mathcal{J}(Fx_0, Fx_0')$ because $U(y,y) = \mathcal{J}(t,t)$. Thus from $\pi_s'\mathcal{B}(-, Fh_0)\mathcal{J}^n(s, Fx_0)) = 0$ follows that $\mathcal{B}(-, Fh_0)\mathcal{J}(s, Fx_0) \subseteq \mathcal{B}(-, Fa')\mathcal{B}(s, Fx_1')$. Using $\mathcal{B}(s, Fx_1') = F\mathcal{A}(x, x_1') \oplus U(x, x_1')$

and $U(x,x_1') \subseteq \mathcal{J}(s,Fx_1')$ (because of $U(x,x) = \mathcal{J}(s,s)$), from $\mathcal{B}(-,Fa')U(x,x_1') = Fa'U(x,x_1') = 0$ follows the inclusion $Fh_0\mathcal{J}^n(Fx,Fx_0) \subseteq Fa'F\mathcal{A}(x,x_1')$. By the construction of F we can replace $\mathcal{J}^n(Fx,Fx_0)$ by $F\mathcal{A}(x,x_0)$ and arrive at $F(h_0\mathcal{A}(x,x_0)) \subseteq F(a'\mathcal{A}(x,x_1'))$.

Now we use that F is faithful to obtain $h_0\mathcal{A}(x,x_0) \subseteq a'\mathcal{A}(x,x_1')$ which shows that $\Psi_\mathcal{S}(h_1,h_0)_x = 0$. Therefore $\Psi_\mathcal{S}(h_1,h_0)$ has to factor through a semisimple injective $\mathcal{S}$-module. From $(h_1,h_0) \neq 0$ follows that $\Psi_\mathcal{S}(h_1,h_0) \neq 0$ forcing $\Psi_\mathcal{S}a'$ to have a non-trivial simple injective summand, a contradiction to the choice of $\mathcal{W}$. □

4.2. **Han's Covering Criterion.** Let us pass to a second example. We consider a k-linear functor $F : \mathcal{S} \to \mathcal{T}$ of locally bounded spectroids and a group G of k-linear automorphism of F satisfying $Fg = F$ for all g in G. The functor F is called *Galois covering* with Galois group G if the following conditions are satisfied (see [BG], [Ga]):

(i) F is dense.
(ii) For each object s of $\mathcal{T}$ the group G acts transitively on $F^{-1}s$.
(iii) The group G acts freely on the objects of $\mathcal{S}$, i.e. for each object x of $\mathcal{S}$ and element g of G the equation $gx = x$ implies $g = e$ where e is the neutral element of G.
(iv) F is a covering functor, i.e. for any two two objects s, t of $\mathcal{T}$, and x, y of $\mathcal{S}$ with x in $F^{-1}s$ and y in $F^{-1}t$ the canonical maps $F : \coprod_{g\in G}\mathcal{S}(x,gy) \to \mathcal{T}(s,t)$ and $F : \coprod_{g\in G}\mathcal{S}(gx,y) \to \mathcal{T}(s,t)$ are isomorphisms.

We recall that G acts also on $\mathrm{mod}{-}\mathcal{S}$. In fact, any element g of G yields an exact autoequivalence of $\mathrm{mod}{-}\mathcal{S}$ sending a module M to ${}^gM := Mg^{-1}$. Moreover, the equation $\mathrm{Hom}_\mathcal{T}(F^\lambda M, F^\lambda N) = \bigoplus_{g\in G} F^\lambda({}^gM, N)$ holds for all M, N in $\mathrm{mod}{-}\mathcal{S}$.

Lemma. *Let $F : \mathcal{S} \to \mathcal{T}$ be a Galois covering with Galois group G. Then the following assertions hold:*
(a) F is cleaving with cleavage $U(x,y) = \bigoplus_{g\neq e} F\mathcal{S}(gx,y)$.
(b) $F^\lambda : \mathrm{mod}{-}\mathcal{S} \to \mathrm{mod}{-}\mathcal{T}$ is exact.
(c) For all M, N in $\mathrm{mod}{-}\mathcal{S}$ the equation $\mathrm{Hom}_\mathcal{T}(M,N)_U = \bigoplus_{g\neq e} F^\lambda \mathrm{Hom}_\mathcal{S}(M^g, N)$ holds.
(d) If G is torsion free, then $F^\lambda : \mathrm{mod}{-}\mathcal{S} \to \mathrm{mod}{-}\mathcal{T}$ preserves indecomposability.

Proof. Let us put again $\mathcal{A} := \mathrm{add}\,\mathcal{S}$ and $\mathcal{B} := \mathrm{add}\,\mathcal{T}$. Part (a) is obvious and (b) follows because F^λ coincides with the right adjoint of $F^\bullet$ on $\mathrm{mod}{-}\mathcal{S}$.

For showing (c) for all $g \in G$ and any two objects $a : x_1 \to x_0$, $a' : x_1' \to x_0'$ of $\mathrm{mat}\mathcal{A}$ we define $\mathrm{mat}^g\mathcal{B}(Fa,Fa')$ as the set of all (Fv_1,Fv_0) in $\mathrm{mat}\mathcal{B}(Fa,Fa')$ such that $v_i \in \mathcal{A}({}^gx_i, x_i')$. It is clearly sufficient to establish $\Psi_\mathcal{T}\mathrm{mat}^g\mathcal{B}(Fa,Fa') = F^\lambda \mathrm{Hom}_\mathcal{S}({}^g\Psi_\mathcal{S}a, \Psi_\mathcal{S}a')$. Since F is faithful, $\mathrm{mat}^g\mathcal{B}(Fa,Fa')$ is the same as $F^\beta\mathrm{mat}\mathcal{A}({}^ga,a')$ which is contained in the right side because ${}^g\Psi_\mathcal{S}a = \Psi({}^ga)$. The other inclusion is now obvious.

Part (d) is proved in [Ga] using the Krull-Remak-Schmidt-Azumaya Theorem referring to an infinite decomposition of an infinite-dimensional module. We will apply our Lemma 3.4 to give an independent prove where no infinite-dimensional situations occur. We know from (a) and (c) that for an indecomposable module M in $\mathrm{mod}{-}\mathcal{S}$ the endomorphism algebra $\mathrm{End}_\mathcal{T}(F^\lambda M)$ can be written as $F^\lambda \mathrm{End}_\mathcal{S}(M) \oplus \bigoplus_{g\neq e} F^\lambda \mathrm{Hom}_\mathcal{S}({}^gM, M)$ where $F^\lambda \mathrm{End}_\mathcal{S}(M)$ is a local subalgebra. Hence we only need to show that the subspace $W := \bigoplus_{g\neq e} F^\lambda \mathrm{Hom}_\mathcal{S}({}^gM, M)$ satisfies $W^m = 0$ for some positive integer m. The module M has some length n which is inherited to all the modules ${}^gM$. We put $m := 2^n - 1$. In order to prove that for any sequence $w_1, \ldots, w_m$ of maps in W the product $w_m \cdots w_1$ is zero, we may assume without loss of generality that $w_i = F^\lambda f_i$ where $f_i \in \mathrm{Hom}_{\mathcal{S}}({}^{g_i}M, M)$ with $g_i \neq e$. Since the group G is torsion free, it acts freely on the indecomposable modules in mod$-\mathcal{S}$ which shows that ${}^{g_i}M \not\cong M$ and therefore none of the f_i is an isomorphism. This implies that also the map ${}^{g_m \cdots g_{i+1}}f_i$ is not an isomorphism and by the Harada-Sai Lemma we obtain $f_m {}^{g_m} f_{m-1} \cdots {}^{g_m \cdots g_2} f_1 = 0$ which by application of F^λ yields $w_m \cdots w_1 = 0$. □

Theorem (Han)**.** *Let $F : \mathcal{S} \to \mathcal{T}$ be a Galois covering with torsion free Galois group G. If there is a finite minimal strictly wild factor spectroid $\mathcal{R}$ of $\mathcal{S}$, then $\mathcal{T}$ is contolled wild by* $\mathcal{C} := \mathrm{add}\, F^\lambda \cup_{g \neq e} (\mathrm{mod}{-}\mathcal{R} \cap \mathrm{mod}{-}{}^{g}\mathcal{R})$.

Proof. We recall from [Hn] that one can find a fully faithful exact functor H : mod$-k\{X, Y\} \to$ mod$-\mathcal{S}$ whose image is contained in an abelian full subcategory $\mathcal{W}$ of mod $\mathcal{S}$ consisting of sincere $\mathcal{R}$-modules. This allows to prove (see also [Hn]) $\bigoplus_{g \neq e} F^\lambda \mathrm{Hom}_{\mathcal{S}}(M^g, N) = \mathrm{Hom}_{\mathcal{T}}(F^\lambda M, F^\lambda N)_{\mathcal{C}}$ for all M, N in $\mathcal{W}$. The rest follows from Theorem 3.2 using the Lemma above. □

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