ON THE TRACE OF ALGEBRAIC INTEGERS OF SMALL HEIGHT

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Abstract. We give an upper bound for the modulus of the first non–zero trace among natural powers of an algebraic integer of small house. An upper bound for this power is obtained for Pisot and Salem numbers. Although the house of these is not at all small, the similar bounds for the first non–zero trace are also established. Finally, we give an upper bound for the trace of an algebraic number with the Mahler measure bounded above by the square root of the degree.

Key words: small house, Salem numbers, Pisot numbers, Mahler measure.

1. Introduction.

Let α be an algebraic integer of degree $d \geqslant 2$ with the minimal polynomial

$$x^d + a_1 x^{d-1} + \ldots + a_{d-1}x + a_d = (x - \alpha_1)\ldots(x - \alpha_d)$$

over the rationals. There are various inequalities between different heights of algebraic numbers. For instance, we have

$$(d+1)^{-1/2} \leqslant H(\alpha)/M(\alpha) \leqslant 2^d,$$

where $H(\alpha) = \max_{1\leqslant k\leqslant d} |a_k|$ and $M(\alpha) = \prod_{k=1}^d \max\{1, |\alpha_k|\}$ is the Mahler measure of an algebraic integer α.

Mignotte [10] was the first to show that the second inequality can be improved if the Mahler measure is small and the degree d is large. His result was only slightly strengthened in [2] and [7], and it is known to be not far from being sharp (see e.g. [1]). If, for instance, $M(\alpha) \leqslant d^\lambda$ then $H(\alpha) \leqslant d^{(1+\lambda)\sqrt{d/2}}$, where $\lambda > 0$ and d is sufficiently large (see [7]). Clearly, the Mahler measure for algebraic integers is bounded above by the dth power of the house, namely, $M(\alpha) \leqslant \overline{|\alpha|}^d$, where $\overline{|\alpha|} = \max_{1\leqslant k\leqslant d} |\alpha_k|$. The above inequality for heights can, therefore, be rewritten in terms of the "small" house. But in any case it remains subexponential in d. Indeed, it is known [16] that if α is a non–cyclotomic algebraic number of degree d then

$$M(\alpha) > 1 + \frac{2}{(\log(3d))^3}.$$

Consequently, we cannot get the inequality better than $H(\alpha) \leqslant d^{\sqrt{d/2}}$ for the height of non–cyclotomic α.

The upper bound for the modulus of the coefficient a_n in case when n is small is much better than the one for the maximal in absolute value coefficient a_ℓ, where

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$|a_\ell| = H(\alpha) = \max_{1 \leqslant k \leqslant d} |a_k|$. The reason for that is very simple. In 1950, Erdös and Turán [8] proved that the roots of the polynomial of small height are in some sense uniformly distributed: there is a "right" number of roots of such a polynomial in every angle of the complex plane with vertex at the origin. The versions of this statement with the height being replaced by the Mahler measure were later given by Mignotte [11], Bilu [4], and by the author [5], [6].

In this paper, we investigate the case when a_n is the first non–zero coefficient in the minimal polynomial. To be precise, let (for an algebraic integer α) $n = n(\alpha)$ be the smallest positive integer n so that $a_n = a_n(\alpha) \neq 0$.

Set $S_m(\alpha) = \sum_{k=1}^d \alpha_k^m$ for the sum over the m th powers of conjugates of α . From the Newton identities, we deduce that the trace of α^k is zero for every k in the range $1 \leqslant k < n = n(\alpha)$. Consequently, $S_n = -na_n$. Since $|S_n| \leqslant d\overline{|\alpha|}^n$, we obtain the inequality

$$|a_n(\alpha)| \leqslant \frac{d\overline{|\alpha|}^n}{n}.$$

Here, the equality holds for every algebraic integer of the form $\alpha = \exp\{\pi\sqrt{-1}/2^N\}$, where N is a positive integer. Indeed, the minimal polynomial for such α is $x^{2^N} + 1$, and its house is equal to 1, since α is cyclotomic. Also, $n = n(\alpha) = 2^N = d$, so that both sides in the above inequality are equal to 1. Throughout this paper, we reserve this notation $\sqrt{-1}$ for the imaginary unit in the upper half–plane. Below, we shall refer to this inequality for $|a_n|$ as to the "trivial inequality".

The trivial inequality is not sharp if α is not cyclotomic. Below we will show that for some α the quantity on the right–hand side of the trivial inequality can be replaced essentially by $\sqrt{d/n}$.

This paper is organized as follows. In the next section we give our results. In Section 3 we present some already known (or in some cases well–known) lemmas. The proofs are given in the final section.

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2. Results.

The folowing theorem strengthens the trivial inequality for small n in the case when the roots of the minimal polynomial are all in the annulus $d^{-1/d} \leqslant |z| \leqslant d^{1/d}$.

Theorem 1. *Suppose that α is an algebraic integer of degree d so that*

$$\max\{\overline{|\alpha|}, \overline{|1/\alpha|}\} \leqslant d^{1/d}.$$

If $n = n(\alpha) \leqslant d/(2\log d)$ then

$$|a_n| < 4\sqrt{\frac{d\log d}{n}}.$$

Clearly, the trivial inequality might give stronger bound than that of Theorem 1 provided that $\overline{|\alpha|}$ is very small, say, if $\overline{|\alpha|} < 1+c/d$ with some absolute positive constant c. But there are not too many of such numbers except for the cyclotomic ones. In fact, the conjecture of Schinzel and Zassenhaus [12] asserts that there are none of these for some c. Presumably (see the conjecture of Boyd [3]), it is enough to take $c = 1/3$ (even something smaller than that) to claim that there are only cyclotomic numbers for which this bound (now $\overline{|\alpha|} < 1 + 1/(3d)$) is true. With the conditions of the theorem, the trivial inequality, in general, gives only the bound $|a_n| \leqslant e^{1/2}d/n$. The latter expression is greater than $4\sqrt{(d \log d)/n}$ for all n in the range $n < (1/6)\sqrt{d/\log d}$.

The number $n(\alpha)$ is clearly bounded above by d. If α is non–cyclotomic then this inequality is strict. Therefore, $n(\alpha) \leqslant d - 1$. If, in addition, α is reciprocal (so that $1/\alpha$ is a conjugate to α) then $n(\alpha) \leqslant d/2$. In some cases $n(\alpha)$ is bounded above by a smaller quantity. The next statement is a sample of this kind of results. We recall that an algebraic integer $\theta > 1$ is called a Pisot number if all its remaining conjugates (if any) all lie strictly inside the unit circle $|z| < 1$. Also, an algebraic integer $\sigma > 1$ of degree $\geqslant 4$ is called a Salem number if it is conjugate to $1/\sigma$ and has all its remaining conjugates on the unit circle $|z| = 1$.

Proposition. *If θ is a Pisot number of degree $d \geqslant 2$ then $n(\theta) < 4 \log d$. If σ is a Salem number of degree $d \geqslant 4$ then $n(\sigma) < (1/2)(\log(3d))^4$. Furthemore, in the latter case, for every $\varepsilon > 0$ there is a $d(\varepsilon)$ so that if $d > d(\varepsilon)$ then*

$$n(\sigma) < \left(\frac{2}{9} + \varepsilon\right)\frac{(\log d)^4}{(\log \log d)^3}.$$

Although Salem numbers do not satisfy the condition of Theorem 1, for them we can get a bound on the modulus of a_n similar to that of Theorem 1.

Theorem 2. *Let σ be a Salem number of degree $d \geqslant 4$. Then*

$$|a_n| < 6\sigma\sqrt{\frac{d \log(3d)}{n}},$$

where $n = n(\sigma)$.

Which values can the trace of the Salem number attain ? It can certainly be equal to -1. An estimate for the number of Salem numbers of trace -1 was recently obtained by Smyth [15]. He showed that the cardinality of the set of those Salem numbers of degree d which have trace -1 is greater than $cd(\log \log d)^{-2}$ with some absolute positive constant c.

Recall the formulas for the trace $\mathrm{Trace}(\alpha) = S_1(\alpha)$, and for the Weil height $h(\alpha) = (1/d) \log M(\alpha)$. In our final theorem we establish an upper bound for the trace of an algebraic number of small height.

Theorem 3. *Let* $\varepsilon > 0$ *and let* α *be an algebraic number of the Mahler measure* $M(\alpha) \leqslant \sqrt{d}$. *Then, for every sufficiently large* d, *we have*

$$\frac{|\mathrm{Trace}(\alpha)|}{d} \leqslant \frac{1}{d} + h(\alpha)^{1/3-\varepsilon}.$$

Theorem 3 is, in fact, a direct corollary of the results on the discrepancy of arguments of roots of integer polynomials in the interval $[0, 2\pi)$. The power $1/3$ (and some extra power of logarithm) of the Weil height for the discrepancy was obtained, for instance, by Mignotte [11], and by the author [6].

3. Lemmas.

Lemma 1. *Let* $z_1 = \varrho_1 e^{\sqrt{-1}\varphi_1}, z_2 = \varrho_2 e^{\sqrt{-1}\varphi_2}, \dots, z_D = \varrho_D e^{\sqrt{-1}\varphi_D}$ *be* D *complex numbers in the annulus* $1/\varrho \leqslant |z| \leqslant \varrho$. *Set* $\delta(z) = \prod_{1\leqslant i<j\leqslant D} |z_i - z_j|$, *and* $\Lambda(z) = \prod_{1\leqslant i\leqslant D} |z_i|$. *Then*

$$\sum_{m=1}^{\infty} \frac{\varrho^{-2m}}{m} \Big| \sum_{k=1}^{D} e^{\sqrt{-1}m\varphi_k} \Big|^2 \leqslant -2\log\delta(z) + D^2 \log\varrho - D\log(\varrho - 1/\varrho) + (D-1)\log\Lambda(z).$$

See [5] for the proof.

Lemma 2. (Siegel [13].) *Let* θ *be a Pisot number. Then* $\theta \geqslant \theta_0$, *where* $\theta_0 = 1.32\dots$ *(which is a Pisot number itself) is the only positive root of the polynomial* $x^3 - x - 1$.

Lemma 3. *Let* σ *be a Salem number of degree* d. *Then* $\log\sigma > 2(\log(3d))^{-3}$. *Furthemore, for every* $\varepsilon > 0$ *there is a* $d(\varepsilon)$ *so that if* $d > d(\varepsilon)$ *then*

$$\log\sigma > \Big(\frac{9}{4} - \varepsilon\Big)\Big(\frac{\log\log d}{\log d}\Big)^3.$$

The bounds of Lemma 3 are in fact true for the Mahler measure of every non–cyclotomic algebraic integer. For Salem numbers, we have $M(\sigma) = \sigma$. The first inequality of Lemma 3 was proved by Voutier [16], whereas the second is due to Louboutin [9]. Also, Lemma 2 is now a consequence of a more general lower bound of Smyth [14] for the Mahler measure of the non–reciprocal algebraic numbers.

Lemma 4. *Suppose we are given* u *numbers* $y_1, \dots, y_u$ *all greater than or equal to 1. If the product of these is* m *then their sum is at most* $u - 1 + m$.

Indeed, if in the maximal sum there are numbers y and y' so that $1 < y \leqslant y'$, then on replacing them by 1 and yy' we would increase the sum, because

$$y + y' < 1 + yy'.$$

The maximal sum, therefore, contains $u - 1$ numbers equal to 1.

4. Proofs.

Proof of Theorem 1. We take $\varrho = d^{1/d}$ and apply Lemma 1 for $D = d$ and $z_k = \alpha_k$, where $1 \leqslant k \leqslant d$. Since $\Lambda(z) \leqslant \varrho^d$, $1/(\varrho - 1/\varrho) < 1/(2\log\varrho) < d$, and because of $\delta(\alpha)$ is bounded below by 1 (as the square root of the discriminant of α), the right–hand side expression in Lemma 1 is less than

$$(2d^2 - d)\log\varrho + d\log d < 3d\log d.$$

Set $L_m = \sum_{k=1}^{d}(\alpha_k/|\alpha_k|)^m$. From Lemma 1 we now obtain that

$$L_m^2 < 3md^{2m/d}d\log d$$

for every positive integer m. In particular, this inequality holds for $m = n = n(\alpha) < d/(2\log d)$, so that

$$|S_n(\alpha)| \leqslant |L_n| + d(\overline{|\alpha|}^n - 1) < \sqrt{3\mathrm{e}nd\log d} + 2(\sqrt{\mathrm{e}} - 1)n\log d,$$

as $\mathrm{e}^x - 1 \leqslant 2x(\sqrt{\mathrm{e}} - 1)$ for all x in the interval $0 < x \leqslant 1/2$. This is less than $4\sqrt{nd\log d}$, because

$$\frac{1}{2} < \Big(\frac{4 - \sqrt{3\mathrm{e}}}{2(\sqrt{\mathrm{e}} - 1)}\Big)^2.$$

This completes the proof of Theorem 1, because as it was noticed earlier we have the equality $|a_n| = |S_n(\alpha)|/n$.

Proof of Proposition. Suppose θ is a Pisot number of degree d so that $n = n(\theta) \geqslant 4\log d > 2$. Since $a_k(\theta) = 0$ for every $1 \leqslant k < n$, we obtain that $S_{n-1}(\theta) = 0$. As the other conjugates of θ are all less than 1 in absolute value, we deduce that $\theta^{n-1} < d - 1$. Thus, on applying Lemma 2, we obtain the inequality

$$(n-1)\log\theta_0 \leqslant (n-1)\log\theta < \log(d-1).$$

Consequently, as $1/\log\theta_0 < 3.6$ and because of n is an integer, we deduce that

$$n \leqslant [3.6\log(d-1)] + 1.$$

Here, $[\ldots]$ stands for the integral part. The right–hand side in the last inequality is clearly less than $4\log d$ for $d \geqslant 13$, because for such d one has $1 < 0.4\log d$. For d in the range $2 \leqslant d \leqslant 12$ one also can easily verify that the inequality $[3.6\log(d-1)] + 1 <$

$4\log d$ is true. Thus, $n < 4\log d$, contrary to our assumption. The first statement of the proposition is, therefore, established.

For the second, assume that σ is a Salem number of degree d, $d \geqslant 4$, so that $n = n(\sigma) \geqslant (1/2)(\log(3d))^4 > 19$. Then, as above, we deduce the inequality

$$(n-1)\log\sigma < \log(d-1).$$

By Lemma 3, $\log\sigma$ is greater than $2(\log(3d))^{-3}$. We obtain, therefore,

$$(1/2)(\log(3d))^4 \leqslant n \leqslant [(1/2)(\log(3d))^3 \log d] + 1.$$

The last expression is less than $(1/2)(\log(3d))^4$ for every $d \geqslant 4$, a contradiction. This proves the second statement of the proposition.

We postpone the proof of the final (third) statement in the proposition to somewhere in the proof of Theorem 2, as we need a more subtle upper bound for σ^{n-1} in terms of d.

Proof of Theorem 2. There is no loss of generality neither for Theorem 2 nor for the final statement of the proposition to assume that $\sigma < d$. Indeed, if $\sigma \geqslant d$ then

$$S_1(\sigma) > \sigma - (d-1) > 0,$$

and so $n(\sigma) = 1$. The inequality

$$|a_1(\sigma)| = |S_1(\sigma)| < \sigma + d - 1 < 2\sigma$$

is stronger than required for Theorem 2. Also, as for such σ we have $n(\sigma) = 1$, the upper bound in the third statement of the proposition is true.

On applying Lemma 3, with $\varrho = \mathrm{e}^{1/d}$, to the $D = d-2$ conjugates of σ lying on the unit circle we obtain that the right–hand side is equal to

$$-2\log\delta(\sigma) + d - 2 - d\log(\mathrm{e}^{1/d} - \mathrm{e}^{-1/d}).$$

This is less than

$$-2\log\delta(\sigma) + d + d\log(d/2).$$

The square root of the discriminant of σ is now equal to

$$\delta(\sigma)(\sigma - \frac{1}{\sigma}) \prod_{3\leqslant i\leqslant d} \Big|(\sigma - \sigma_i)(\frac{1}{\sigma} - \sigma_i)\Big|,$$

where σ_i are the conjugates of σ on the unit circle. As every term in the product is less than $2\sigma \times 2 = 4\sigma$, we obtain the inequality

$$1 < \delta(\sigma)4^{d-2}\sigma^{d-1},$$

thus $-2\log\delta(\sigma) < 2(d-2)\log 4 + 2(d-1)\log\sigma$. The right–hand side of the expression in Lemma 1 is, therefore, less than

$$2d\log 4 + 2d\log d + d + d\log(d/2) < 3d\log(3d).$$

Hence, for every positive integer m, we deduce from Lemma 1 the following inequality:

$$|S_m(\sigma) - \sigma^m - \sigma^{-m}|^2 < 3m\mathrm{e}^{2m/d} d\log(3d).$$

We refer to it below as to the "main" inequality.

We now deduce the final statement in the proposition on applying the main inequality to $m = n-1$. Clearly, $S_{n-1}(\sigma) = 0$. Also, by the second statement of the proposition, $n < 4\log d$. It follows that

$$|\sigma^{n-1}| < 4\sqrt{d}\log d$$

for all d sufficiently large. By taking logs, and by using the asymptotic lower bound for $\log\sigma$ as in Lemma 3, we obtain the final statement in the proposition as required.

In order to conclude the proof of Theorem 2, we first apply the main inequality for $m = n \leqslant d/2$. It is easy to see that

$$|S_n(\sigma)| < \sigma^n + \sigma^{-n} + 3\sqrt{nd\log(3d)}.$$

The main inequality for $m = n-1 \geqslant 1$ implies that

$$\sigma^{n-1} + \sigma^{-n+1} < 3\sqrt{(n-1)d\log(3d)},$$

and so

$$\sigma^n + \sigma^{-n} < \sigma^n + \sigma^{-n+2} < 3\sigma\sqrt{nd\log(3d)}.$$

Finally,

$$|S_n(\sigma)| < 3(\sigma+1)\sqrt{nd\log(3d)} < 6\sigma\sqrt{nd\log(3d)},$$

and Theorem 2 follows from the identity $|a_n(\sigma)| = |S_n(\sigma)|/n$.

Proof of Theorem 3. There is no loss of generality to assume that α is a non–cyclotomic, for otherwise the theorem is trivial: the trace of the cyclotomic number is either 0 or ± 1. This also explains why we do need the extra summand $1/d$ in the theorem: the Weil height of the cyclotomic number is equal to 0.

Let us fix the numbers α and $\mu = 2h(\alpha)^{2/3-4\varepsilon}$. Suppose that the conjugates of α, say $\alpha_1, \dots, \alpha_m$, all lie in the annulus $\mathrm{e}^{-\mu} \leqslant |z| \leqslant \mathrm{e}^{\mu}$, so that the remaining ones, namely $\alpha_{m+1}, \dots, \alpha_d$, are all outside the annulus Let also u and v of the latter ones be strictly outside and strictly inside the unit circle, respectively. We have

$$2\log M(\alpha) = \log M(\alpha) + \log M(1/\alpha) \geqslant u\mu + v\mu = (d-m)\mu.$$

Thus, $d - m \leqslant (2/\mu)\log M(\alpha)$.

We now combine the inequalities (10), (16) in [6] with the inequality

$$||\Psi|| = \max_{|z|\leqslant 1} \prod_{k=1}^{m} \left| z - \frac{\alpha_k}{|\alpha_k|} \right| \leqslant \exp\{dh(\alpha)^{1/3-8\varepsilon}\}$$

(the latter follows from the results of [6]). We obtain that

$$L_1(\alpha)^2 \leqslant \mathrm{e}^{2\mu}(m^2\mu - m\log(2\mu) + 4\mu m(d-m) + 2(d-1)\log M(\alpha) + 2d(d-m)h(\alpha)^{1/3-8\varepsilon}).$$

Here,

$$L_1(\alpha) = \sum_{k=1}^{m} \frac{\alpha_k}{|\alpha_k|},$$

and compared to [6] the roles of d and n now belong to m and d, respectively.

Note that $\mu > d^{-1/4}$ for d sufficiently large (see the remark after Lemma 3). Substituting μ, the upper bound for $d-m$, and the trivial bound $m \leqslant d$, we deduce that

$$L_1(\alpha)^2 < 5d^2h(\alpha)^{2/3-4\varepsilon}$$

for d sufficiently large. Note that

$$|\sum_{k=1}^{m} \alpha_k - L_1(\alpha)| \leqslant m(\mathrm{e}^{\mu} - 1) < 3dh(\alpha)^{2/3-4\varepsilon},$$

and, as $M(\alpha) \leqslant \sqrt{d}$, by Lemma 4,

$$|\sum_{k=m+1}^{d} \alpha_k| \leqslant v + u - 1 + M(\alpha) < \frac{2\log M(\alpha)}{\mu} + M(\alpha) < dh(\alpha)^{1/3}.$$

It follows that

$$|\mathrm{Trace}(\alpha)| \leqslant dh(\alpha)^{1/3} + |L_1(\alpha)| + 3dh(\alpha)^{2/3-4\varepsilon}.$$

We see that the right–hand side in the above inequality is less than

$$dh(\alpha)^{1/3} + 3dh(\alpha)^{1/3-2\varepsilon} + 3dh(\alpha)^{2/3-4\varepsilon} < dh(\alpha)^{1/3-\varepsilon}$$

for all d sufficiently large. The proof of Theorem 3 is now completed.

References

1. F. Amoroso, Sur des polynômes de petites mesures de Mahler, *C. R. Acad. Sci., Paris*, **321**, 11–14 (1995).
2. F. Amoroso, Upper bounds for the resultant and Diophantine applications, in: *Number Theory Eger (1996)*, K. Györy. A. Pethö, V.T. Sós (eds.), Walter de Gruyter, Berlin, 23–36 (1998).

3. D.W. Boyd, The maximal modulus of an algebraic integer, *Math. Comp.*, **45**, 243–249 (1985).
4. Yu. Bilu, Limit distributions of small points on algebraic tori, *Duke Math. J.*, **89**, 465–476 (1997).
5. A. Dubickas, On the distribution of roots of polynomials in sectors. I, *Liet. Matem. Rink.*, **38**, 34–58 (1998).
6. A. Dubickas, On the distribution of roots of polynomials in sectors. II, *Liet. Matem. Rink.*, **38**, 151–168 (1998).
7. A. Dubickas, On certain geometric mean of the values of a polynomial, (to appear).
8. P. Erdös and P. Turán, On the distribution of roots of polynomials, *Ann. of Math.*, **51**, 105–119 (1950).
9. R. Louboutin, Sur la mesure de Mahler d'un nombre algébrique, *C. R. Acad. Sci., Paris*, **296**, 707–708 (1983).
10. M. Mignotte, An inequality about irreducible factors of integer polynomials, *J. Number Theory*, **30**, 156–166 (1988).
11. M. Mignotte, Sur un thèoréme de M. Langevin, *Acta Arith.*, **54**, 81–86 (1989).
12. A. Schinzel and H. Zasssenhaus, A refinement of two theorems of Kronecker, *Michigan Math. J.*, **12**, 81–85 (1964).
13. C.L. Siegel, Algebraic integers whose conjugates lie in the unit circle, *Duke Math. J.*, **11**, 597–602 (1944).
14. C.J. Smyth, On the product of conjugates outside the unit circle of an algebraic integer, *Bull. London Math. Soc.*, **3**, 169–175 (1971).
15. C.J. Smyth, Salem numbers of negative trace, *Math. Comp.*, **69**, 827–838 (2000).
16. P. Voutier, An effective lower bound for the height of algebraic numbers, *Acta Arith.*, **74**, 81–95 (1996).

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