L^p-Uniqueness for Infinite Dimensional Symmetric Kolmogorov Operators: The Case of Variable Diffusion Coefficients

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Abstract. The problem of strong uniqueness in L^p for infinite dimensional Kolmogorov operators is studied. The case of variable diffusion coefficients is considered. An analytic approach based on a priori estimates is employed. An application which is not covered by previous results is presented.

Key words and phrases: Infinite dimensional Kolmogorov equation, strong uniqueness, Dirichlet form, C_0-semigroup, a priori estimate.

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1 Introduction

In this paper we study the Cauchy problem for infinite dimensional symmetric Kolmogorov operators of the form

$$\mathcal{L}u = \sum_{k,j} a_{kj} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{k,j} \frac{\partial a_{kj}}{\partial x_k} \frac{\partial u}{\partial x_j} + \sum_{k,j} \beta_k^\mu a_{kj} \frac{\partial u}{\partial x_j},$$

where $u \in \mathcal{F}C_b^\infty$, i.e. the set of smooth finitely based functions on a locally convex vector space X. Here the entries of the symmetric positive definite diffusion matrix (a_{kj}) are functions on X satisfying certain conditions specified below, and β^μ is the logarithmic derivative of a given probability measure μ on X. This operator is associated with the pre-Dirichlet form

$$\mathcal{E}[u,v] = \sum_{k,j} \int_X a_{kj} \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_j} d\mu = -\int_X \mathcal{L}u \; v d\mu, \quad u, v \in \mathcal{F}C_b^\infty.$$

Let $\overline{\mathcal{L}}$ be the Friedrichs extension of $\mathcal{L}$ on $L^2(X,\mu)$. It is well-known that the semigroup $e^{t\overline{\mathcal{L}}}$ is sub-Markovian (i.e. positive and L^∞-contractive). Thus $e^{t\overline{\mathcal{L}}} \restriction L^\infty$ extends to a C_0-semigroup on every $L^p(X,\mu)$ for every $p \geq 1$, with generator $\hat{\mathcal{L}}_p$. Hence the Cauchy problem

$$\frac{\partial u}{\partial t} = \hat{\mathcal{L}}_p u, \quad u(0) = f,$$

is well-posed in L^p, and its solution is given by

$$u(t) = e^{t\hat{\mathcal{L}}_p} f.$$

However, one should realize that the above procedure is only one of the possibilities to solve the Cauchy problem related to the operator $\mathcal{L}$ with domain $\mathcal{F}C_b^\infty$. More precisely, there might be other closed extensions of $\mathcal{L}$ generating C_0-semigroups on L^p. The aim of this paper is to give geometric conditions on a_{kj} and on the "large" part of the logarithmic derivative (cf. the next section) implying that $\hat{\mathcal{L}}_p$ is the *only* such extension. We would like to mention that by a result due to W.Arendt (see [13], Theorem AII, 1.33) this uniqueness is equivalent to the property that $\mathcal{F}C_b^\infty$ is a core of the operator $\hat{\mathcal{L}}_p$.

Our main result is formulated in Theorem 2.2 in the next section to which we also refer for the precise framework. In Section 3 we present the proof of the result on strong uniqueness which is based on a-priori estimates for the first order derivatives of solutions of elliptic equations with smooth coefficients. These estimates are derived in Section 4, whereas certain auxiliary results are contained in the Appendix.

The problem we treat is refered to as the strong uniqueness problem in L^p. There are numerous publications on this problem (for the case where X is infinite dimensional, which we are most interested in, see, e.g. [1, 2, 3, 4, 10] for the case $p = 2$ and $a_{kj} = \delta_{kj}$, [5, 8] for $p \geq 1, a_{kj} = \delta_{kj}$, [7] for variable a_{kj} if $p = 2$, and [9] for arbitrary p). In [8] an approach was developed to combine the conditions on the logarithmic derivative from [3] and [10]. However, due to technical difficulties certain restrictions on the "large" part were imposed. The present paper is an extension and generalization of the main result in [8] in several directions. Firstly, we consider variable diffusion coefficients a_{kj}, and the matrix (a_{kj}) is not supposed to be uniformly bounded and uniformly elliptic. Secondly, we remove the said restrictions on the logarithmic derivative (see condition (ii(b)) of Theorem 2.2 below in comparison with condition (iv) of Theorem 1 in [8]). This was possible due to a new method of obtaining estimates for gradients of smooth approximating solutions. In comparison with [9], apart from the greater generality of the results in the present paper, we simplified the framework in order to make the conditions used more transparent. For illustration of the main result of this paper we include an application which could not be treated by previous results.

2 Framework and Main Results

Let X be a separable locally convex Hausdorff topological vector space such that its topological dual X^* contains a sequence $(l_n)_{n\in\mathbb{N}}$ of linearly independent functionals separating points. We assume that X is Souslinean, hence l_n, $n \in \mathbb{N}$, generate the Borel σ-algebra $\mathcal{B}(X)$ of X (cf. [14]).

Given $N \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$, let $UC_b^m := UC_b^m(\mathbb{R}^N)$ stand for the class of m-times differentiable functions on $\mathbb{R}^N$, whose derivatives up to order m are bounded and uniformly continuous. Now let

$$\mathcal{FC}_b^{m,u}(\mathbb{R}^N) := \{f(l_1, \ldots, l_N) : f \in UC_b^m(\mathbb{R}^N)\}.$$

¿From now on $\mathcal{FC}_b^{m.u} := \cup_N \mathcal{FC}_b^{m,u}(\mathbb{R}^N)$ and $\mathcal{FC}_b^{\infty} := \mathcal{FC}_b^{\infty,u}$.

Let μ be a probability measure on $\mathcal{B}(X)$. Suppose that $supp\ \mu = X$. For $1 \leq p \leq \infty$ let $L^p = (real)\ L^p(X, \mu)$. Since $\mathcal{B}(X) = \sigma(l_n, n \in \mathbb{N})$, the set $\mathcal{FC}_b^{\infty}$ is dense in L^p for all $p \in [1, \infty)$. Throughout the paper we use the following notation: $\|\cdot\|_p$ is the norm in L^p, $\langle \cdot, \cdot \rangle$ is the inner product in L^2, and $\langle f \rangle = \int f d\mu$.

Let $(e_k)_{k\in\mathbb{N}} \subset X$ be the unique sequence of linearly independent vectors such that $l_m(e_k) = \delta_{mk}$, $m, k \in \mathbb{N}$ and μ is differentiable along every e_k in the sense that there exist measurable functions $(\beta_k^\mu)_{k\in\mathbb{N}}$ in L^2, satisfying

$$\left\langle \frac{\partial v}{\partial e_k} \right\rangle = -\langle v, \beta_k^\mu \rangle, \quad v \in \mathcal{FC}_b^{1,u}, \, k \geq 1.$$

β_k^μ is called directional logarithmic derivative of μ along e_k. Further on we treat $(e_k)_{k\in\mathbb{N}}$ as the canonical basis in the space $\mathbb{R}^{\mathbb{N}}$ of all real sequences. Hence, we identify the linear span of $(e_k)_{k\in\mathbb{N}}$ with the space $\mathbb{R}^{fin}$ of all finite sequences. The space $\mathbb{R}^{fin}$ can be considered as the tangent space T_xX to X for all $x\in X$ in the sense that we shall take derivatives only along the elements of $\mathbb{R}^{fin}$. We introduce the spaces $(H_0,\,(\cdot,\cdot)_0)=l^2$, $(H_+,\,(\cdot,\cdot)_+)=l^2_{\gamma_k}$ and $(H_-,\,(\cdot,\cdot)_-)=l^2_{\gamma_k^{-1}}$ for a sequence $(\gamma_k)_{k\in\mathbb{N}}$ in $(0,+\infty)$ (where $l^2_{\gamma_k}:=\{h\in\mathbb{R}^{\mathbb{N}}:\sum_k h_k^2\gamma_k^2<\infty\}$ and $l^2_{\gamma_k^{-1}}$ is defined in the same way). It is obvious that H_+ and H_- are mutually dual w.r.t. the $(\cdot,\cdot)_0$-duality pairing.

For $N\in\mathbb{N}$ we define the projection $P_N:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}^{fin}$:

$$P_Nh:=\sum_{k=1}^N h_ke_k.$$

Here and below we denote the linear span of $(e_k)_{k=1}^N$ by $\mathbb{R}^N$. This notation is consistent with the definition of $\mathcal{FC}_b^{m,u}(\mathbb{R}^N)$ since for $x\in X$ we have

$$P_Nx:=\sum_{k=1}^N l_k(x)e_k.$$

For $u\in\mathcal{FC}_b^{1,u}$, $u(x)=f(P_Nx)$ let ∇u stand for the Frechet derivative of u:

$$\nabla u(x):=\sum_{k=1}^N\frac{\partial f}{\partial e_k}(P_Nx)e_k\in\mathbb{R}^N.$$

Furthermore, if $u\in\mathcal{FC}_b^{2,u}$ then D^2u stands for the second derivative of u:

$$D^2u(x):=\sum_{k,m=1}^N\frac{\partial^2 f}{\partial e_k\partial e_m}(P_Nx)e_k\otimes e_m.$$

We introduce the following notation: $\nabla_k:=\frac{\partial}{\partial e_k}$, $\nabla^2_{kj}:=\frac{\partial^2}{\partial e_k\partial e_j}$.

Let $\{a_{kj},\,k,j\in\mathbb{N}\}$ be a family of cylindric functions on X. The following conditions (A0) -(A4) on (a_{kj}) are assumed to hold throughout the paper.

(A0) For every $N\in\mathbb{N}$ the matrix $(a_{kj}(x))_{k,j=1}^N$ is symmetric and uniformly elliptic. For every $i\in\mathbb{N}$ there exists $\varepsilon_i\in(0,\infty)$, such that for μ–a.e. $x\in X$,

$$\sum_{k,j=1}^\infty a_{kj}(x)h_kh_j\ge\varepsilon_ih_i^2,\qquad\forall h=(h_k)_{k\in\mathbb{N}}\in\mathbb{R}^{fin},$$

and the completion $H_a(x)$ of $\mathbb{R}^{fin}$ with respect to the norm $\|\cdot\|_a:=(\cdot,\cdot)_a^{1/2}$, where

$$(h,g)_a(x):=\sum_{k,j=1}^\infty a_{kj}(x)\,h_k\,g_j,\qquad g,h\in\mathbb{R}^{fin},$$

embeds one–to–one and continuously into $\mathbb{R}^{\mathbb{N}}$ (the latter being equipped with the product topology).

Note that assumption (A0) is fulfilled if the infinite matrix $(a_{kj}(x))_{k,j=1}^{\infty}$ is block diagonal and each block is uniformly elliptic. By HS and $HS(a)$ we denote the spaces of Hilbert-Schmidt operators over H_0 and H_a respectively.

(A1) For every $n \in \mathbb{N}$

$$a_{kj} \in \mathcal{FC}_b^{1,u}(\mathbb{R}^{K_n}),\ k,j = 1,\dots,K_n,$$

for a sequence $(K_n)_{n\in\mathbb{N}} \subset \mathbb{N},\ K_n \nearrow \infty$.

(A2) For every $k \in \mathbb{N}$

$$\bar{c}_k := \sup_{x\in X,\, j\in\mathbb{N}} |a_{kj}|^2(x) < \infty.$$

For every $k \in \mathbb{N}$ we assume that β_k^μ can be decomposed as $\beta_k^\mu = \xi_k^\mu + \eta_k^\mu$, with ξ_k^μ and η_k^μ Borel measurable and satisfying the following conditions.

(A3) The series $\eta^\mu(x) := \sum_{k\in\mathbb{N}} \eta_k^\mu(x) e_k$ converges in $H_a(x)$ for a.a $x \in X$ and $|\eta^\mu - P_N\eta^\mu|_a \to 0$ in L^2 as $N \to \infty$.

Note that, for all $k \le N < d$, the Cauchy inequality gives

$$\left|\sum_{j=N}^{d} a_{kj}\eta_j^\mu\right| \le \Big(\sum_{l=1}^{d} \alpha_{kl}^2\Big)^{\frac{1}{2}} \Big(\sum_{l=1}^{d}\Big(\sum_{j=N}^{d} \alpha_{jl}\eta_j^\mu\Big)^2\Big)^{\frac{1}{2}} = \sqrt{a_{kk}}\,|(P_d - P_N)\eta^\mu|_a,$$

where $(\alpha_{jl})_{j,l=1}^d$ is the square root of the matrix $(a_{jl})_{j,l=1}^d$. Therefore (A3) implies that the series $\sum_{j\ge1} a_{kj}\eta_j^\mu$ converges in L^2 for all $k \in \mathbb{N}$.

(A4)The series $\xi_{a;k}^\mu := \sum_{j\in\mathbb{N}} (\nabla_j a_{kj} + a_{kj}\xi_j^\mu)$ is convergent in L^2 for all $k \in \mathbb{N}$.

The latter enables us to introduce $\xi_a^\mu(x) := \sum_{k\ge1} \xi_{a;k}^\mu(x) e_k \in \mathbb{R}^{\mathbb{N}}$. ξ_a^μ is refered to as the "large" part of the collection $(\beta_k^\mu)_{k\in\mathbb{N}}$ of the directional logarithmic derivatives of μ, since it is not a section of the "co-tangent bundle" $(H_a(x))_{x\in X}$.

For $d \in \mathbb{N}$ and $x \in X$ we introduce the quantities $\upsilon_d(x)$ and $\nu_d(x)$:

$$\upsilon_d^2(x) := \sup_{|h|_a\le1} \sum_{i,j,k,l,m,n=1}^{d} a_{kl}(x)a^{mj}(x)(\nabla_k a_{ij}(x)h_i)(\nabla_l a_{mn}(x)h_n), \tag{1}$$

and

$$\nu_d^2(x) := \sup_{|h|_0\le1} \sum_{i,j,k,l,m=1}^{d} (\gamma_l^{-2}a^{ij}(x)(\nabla_k a_{il}(x)\gamma_k h_k)(\nabla_m a_{jl}(x)\gamma_m h_m), \tag{2}$$

where $h = (h_n)_{n\in\mathbb{N}}$ and $(a^{ij})_{n,j=1}^d$ is the matrix inverse to $(a_{ij})_{n,j=1}^d$ (which exists due to (A1)). Note that if $a_{kj}(x) = \delta_{kj}\sigma_k(x)$ one has

$$\upsilon_d^2(x) = \sup_m \sum_{l=1}^d \sigma_l(x)\Big(\frac{\nabla_l\sigma_m(x)}{\sigma_m(x)}\Big)^2.$$

If one assumes, in addition, that $\sigma_k(x) = \sigma_k(x_k)$, then

$$\upsilon_d(x) = \sup_k \sigma_k^{-\frac{1}{2}}(x_k)|\sigma_k'(x_k)|$$

and

$$\nu_d(x) = \sup_k \frac{|\sigma_k'(x_k)|}{\sigma_k(x_k)}.$$

Consider the operator

$$\mathcal{L}v = \sum_{k,j\geq 1}(a_{kj}\nabla_{kj}^2 v + a_{kj}\eta_j^\mu\nabla_k v) + \sum_{k\geq 1}\xi_{a;k}^\mu\nabla_k v, \qquad v\in\mathcal{FC}_b^\infty.$$

Observe that since v is cylindric, it follows from (A1), (A3) and (A4) that $\mathcal{L}v\in L^2$ for all $v\in\mathcal{FC}_b^\infty$. Hence, the operator $\mathcal{L}$ is densely defined in L^2. Observe that the following equality holds:

$$-\langle\mathcal{L}u,\, v\rangle \;=\; \sum_{k,j\geq 1}\langle a_{kj}\nabla_k u\nabla_j v\rangle,\;\; u,\, v\in\mathcal{FC}_b^\infty. \tag{3}$$

Indeed, for $u,\, v\in\mathcal{FC}_b^{2,u}$ we have

$$-\int_X v\,\Big[\sum_{k,j=1}^\infty a_{kj}\nabla_{kj}^2 u + \sum_{k=1}^\infty \nabla_k u\Big(\sum_{j=1}^\infty(\nabla_j a_{kj} + a_{kj}\xi_j^\mu) + \sum_{j=1}^\infty a_{kj}\eta_j^\mu\Big)\Big]d\mu.$$
$$= -\sum_{k,j=1}^\infty\int_X v\,(\nabla_j + \beta_j^{mu})(a_{jk}\nabla_k u)d\mu,$$

since the sum in k is finite and the series in j converges in L^2 due to (A3) and (A4). Hence, (3) follows from the integration by parts formula. Therefore, $\mathcal{L}$ is a symmetric operator and the form

$$\mathcal{E}[u,\, v] \;=\; \sum_{k,j\geq 1}\langle a_{kj}\nabla_k u\nabla_j v\rangle,\;\; u,\, v\in\mathcal{FC}_b^\infty$$

is a closable symmetric form on L^2 whose closure $(\mathcal{E},\,\mathcal{D}(\mathcal{E}))$ is a Dirichlet form (cf. [12] or [6] for the terminology). We will not distinguish between $\mathcal{E}$ and its closure unless it leads to confusions.

It follows from the Beurling-Deny criteria that $\mathcal{E}$ is associated with a family of consistent sub-Markovian C_0-semigroups of contractions $e^{\hat{\mathcal{L}}_p t}$ on L^p, $1 \le p < \infty$. We refer the reader to [6, 12] for corresponding definitions and standard results. By construction $\hat{\mathcal{L}}_2 \supset \mathcal{L}$ (in fact $\hat{\mathcal{L}}_2$ is the Friedrichs extension of $\mathcal{L}$). Moreover, the following simple statement holds.

Lemma 2.1. *Let $s = max(2, p)$. Then $\hat{\mathcal{L}}_p \supset \mathcal{L}$ provided $|\eta^\mu|_a$, $\xi^\mu_{a;k} \in L^s$ for all $k \in \mathbb{N}$.*

Proof. Our assumptions imply that $\mathcal{L}v \in L^2 \cap L^p$ for all $v \in \mathcal{FC}^\infty_b$. Therefore,

$$\mathcal{L}v \;=\; L^p\text{-}\lim_{t\to 0} \frac{1}{t}\int_0^t e^{\hat{\mathcal{L}}_p s}\mathcal{L}v ds.$$

On the other hand,

$$\int_0^t e^{\hat{\mathcal{L}}_p s}\mathcal{L}v ds \;=\; \int_0^t e^{\hat{\mathcal{L}}_2 s}\mathcal{L}v ds \;=\; \int_0^t e^{\hat{\mathcal{L}}_2 s}\hat{\mathcal{L}}_2 v ds \;=\; e^{\hat{\mathcal{L}}_2 t}v - v \;=\; e^{\hat{\mathcal{L}}_p t}v - v.$$

Thus, $\mathcal{L}v \;=\; L^p\text{-}\lim_{t\to 0} \frac{1}{t}(e^{\hat{\mathcal{L}}_p t}v - v)$ and $\hat{\mathcal{L}}_p \supset \mathcal{L}$. □

Now we are ready to formulate the main result of the paper. Recall that conditions (A1) - (A4) are still in force.

Theorem 2.2. *Let $p \ge 1$, (K_n) be as in (A1). Set $s := p$ if $\eta^\mu \equiv 0$, and $s := max(2, p)$ otherwise. Let $\xi^\mu_{a;k} \in L^s$ for all $k \in \mathbb{N}$ and $\sup_n \| \nu_n \|_\infty < \infty$.*

Assume that

(i) there exists a sequence $\xi^n_j \in \mathcal{FC}^{1,u}_b(\mathbb{R}^{K_n})$, $j = 1, \dots, K_n$, $n \in \mathbb{N}$, such that

(a) $|\xi^n - P_{K_n}\xi^\mu_a|_- \to 0$ in L^s as $n \to \infty$;

(b) there exists a constant $c_+ \in \mathbb{R}$ independent of n such that for all x, $y \in \mathbb{R}^{K_n}$ the following inequality holds

$$\sum_{j,l=1}^{K_n} \gamma_l^2 (\nabla_l \xi^n_j)(x) y_j y_l \le c_+ \sum_{k=1}^{K_n} \gamma_k^2 y_k^2;$$

(ii) either $\eta^\mu = 0$
or

(a) $\sup_d \| \upsilon_d \|_{2p} < \infty$, $|\eta^\mu - P_N\eta^\mu|_a \to 0$ in L^{2p} as $N \to \infty$;

(b) *there exist a sequence* $(\hat{\xi}_j^m)_{m,j\in\mathbb{N}} \subset \mathcal{FC}_b^{1,u}$ *and numbers* $\varepsilon_0 \in [0,1)$ *and* $c(\varepsilon_0) \in \mathbb{R}$ *such that* $\hat{\xi}_j^m \to \xi_j^\mu$ *as* $m \to \infty$ *weakly in* L^2 *for every* $j \in \mathbb{N}$, *and for all* $n \in \mathbb{N}$ *and* $w_j \in \mathcal{FC}_b^{1,u}(\mathbb{R}^n)$, $j = 1,\dots,n$, *the following inequality holds*

$$\liminf_{m,d\to\infty} \sum_{i,k=1}^{d} \sum_{j,l=1}^{n} \langle (\nabla_k \hat{\xi}_j^m) a_{jl} w_l,\, a_{ki} w_i \rangle \le \varepsilon_0 \sum_{i,j,k,l=1}^{n} \langle a_{kj}\nabla_i w_j,\, a_{il}\nabla_k w_l\rangle + c(\varepsilon_0) \sum_{j,k=1}^{n} \langle a_{kj} w_k,\, w_j\rangle;$$

(c) $p \in \left(3 - \dfrac{3}{1+\sqrt{1+3\varepsilon_0}},\, \dfrac{2}{\varepsilon_0}\right)$.

Then the operator $\mathcal{L} \restriction \mathcal{FC}_b^\infty$ *has a unique extension which generates a* C_0-*semigroup on* L^p.

Remark. *The uniqueness result in [8] can be obtained as a particular case of Theorem 2.2 if one puts* $a_{jk}(x) \equiv \delta_{jk}$. *(Note the difference in the interval in the* L^p-*scale, which was incorrectly stated in [8], Theorem 3.) In [9] the special case of* $\xi_k^\mu = 0$, $k \in \mathbb{N}$, *was studied and strong* L^p-*uniqueness of the extension of* $\mathcal{L}$ *has been proved under weaker assumptions on the coefficients* a_{jk}, *namely, their derivatives need not be either continuous or bounded. If we confine ourselves to this situation then we can employ estimates (6) and (7) (see Proposition 3.3 below and note that in this case* $\varepsilon_0 = 0$*) and prove the uniqueness under the same assumptions as in [9].*

3 Proof of Uniqueness

Our strategy to prove the uniqueness result is as follows. We take an arbitrary extension $\hat{\mathcal{L}} \supset \mathcal{L} \restriction \mathcal{FC}_b^\infty$, which generates a C_0-semigroup on L^p. Then we take sequences $(\xi_j^m), (\eta_j^m) \subset \mathcal{FC}_b^{1,u}$ and deal with the corresponding family of Cauchy problems:

$$\begin{cases} u_t^{(m)} &= \sum_{k,j=1}^{K_m} (a_{jk}\nabla_{kj}^2 u^{(m)} + a_{kj}\eta_j^m \nabla_k u^{(m)}) + \sum_{k\ge1} \xi_k^m \nabla_k u^{(m)} \\ u^{(m)}(0) &= f, \end{cases}$$

with an arbitrary $0 \ne f \in \mathcal{FC}_b^\infty$ and (K_m) as in (A1). Then we show that $u^{(m)}(t) \to e^{\hat{\mathcal{L}}_p t} f$ strongly in L^p provided ξ_j^m, η_j^m approximate $\xi_{a;j}^\mu$, η_j^μ, $j \in \mathbb{N}$ in a proper way. This will prove strong uniqueness for the generator. The core of the proof is estimates for the gradient ∇u of the solution u to the following Cauchy problem over $\mathbb{R}^K$

$$\begin{cases} u_t & = \mathcal{L}_{\xi,\eta} u := \sum_{k,j=1}^K (a_{kj}\nabla^2_{kj} u + a_{kj}\eta_j \nabla_k u) + \sum_{k=1}^K \xi_k \nabla_k u, \quad t > 0, \\ u(0) & = f, \end{cases} \tag{4}$$

with a uniformly elliptic matrix $(a_{jk})_{j,k=1}^K$, a_{jk}, $\xi_j \in UC_b^1(\mathbb{R}^K)$, $j,k = 1,\dots,K$, $N < K$, $\eta_j \in UC_b^1(\mathbb{R}^N)$, $j = 1,\dots,N$, $\eta_j \equiv 0$, $j = N+1,\dots,K$, $f \in C_b^\infty(\mathbb{R}^K)$.

In order to obtain the required estimates we need the following result from [11].

Proposition 3.1. *([11], 3.1.9, 3.1.17, 3.1.18.)*
Set $\mathcal{D} = \{u \in \cap_{p\geq 1} W^{2,p}_{loc}(\mathbb{R}^K) : u, \mathcal{L}_{\xi,\eta} u \in C_b(\mathbb{R}^K)\}$. *Then*

(i) $\mathcal{L}_{\xi,\eta} \restriction \mathcal{D}$ *generates a positive analytic semigroup* $U(t)$ *on* C_b *, which is continuous at zero on elements from* $\bar{\mathcal{D}} = UC_b(\mathbb{R}^K)$*. In particular, problem (4) has a unique classical solution* $u \in C_b$ *(in the sense of [11], 4.1.1(iii));*

(ii) the functions $t \mapsto u(t)$ *and* $t \mapsto \mathcal{L}_{\xi,\eta} u(t)$ *are analytic* $(0,\infty) \to C_b^1(\mathbb{R}^K)$ *and* $u(t) \to f$, $\mathcal{L}_{\xi,\eta} u(t) \to \mathcal{L}_{\xi,\eta} f$ *in* $C_b^1(\mathbb{R}^K)$ *as* $t \to 0$*;*

(iii) For all $t \geq 0$ *we have* $u(t) \in \cap_{p\geq 1} W^{3,p}_{loc} \cap UC_b^2$.

By the maximum principle we have $\|u\|_\infty \leq \|f\|_\infty$.

The estimates are given in the following two propositions.

Proposition 3.2. *Let* u *be the solution to (4). Assume that there exists a constant* c_+ *independent of* x *such that for all* x, $y \in \mathbb{R}^K$ *the inequalities*

$$\sum_{j,l=1}^K \gamma_l^2 (\nabla_l \xi_j)(x) y_j y_l \leq c_+ \sum_{k=1}^K \gamma_k^2 y_k^2,$$

$$\nu_K(x) \leq c_+$$

hold (ν_K *is as in (2)). Then*

$$\|\,|\nabla u|_+\|_\infty \leq \exp(C_+ t) \|\,|\nabla f|_+\|_\infty,$$

with $C_+ = c_+ + \frac{1}{4}c_+^2 + \left(\sum_{k=1}^N \bar{c}_k\right)^{\frac{1}{2}} \|\,|\nabla\eta|_{HS}\|_\infty + c_+ \|\,|\eta|_a\|_\infty$ *and* $\eta = (\eta_1,\dots,\eta_K)$ *(recall that* $|h|_+^2 = \sum_{k\geq 1} \gamma_k^2 h_k^2$, $h \in H_+$*).*

Proposition 3.3. *Let* u *be the solution to (4). For* $3 - \frac{3}{1+\sqrt{1+3\varepsilon_0}} < p < \frac{2}{\varepsilon_0}$ *set* $s = max(p,2)$*. Let* $\sup_d \|\upsilon_d\|_{2p} < \infty$, $|\eta^\mu - P_d \eta^\mu|_a \to 0$ *in* L^{2p} *as* $d \to \infty$ *and* $\xi^\mu_{a;k} \in L^s$, $k = 1,\dots,K$*. Set* $G_p := \|\,|\eta|_a + |\eta^\mu|_a\|_{2p}^{2p} + \sup_d \|\upsilon_d\|_{2p}^{2p}$*. Let* C_+ *be as in*

Proposition 3.2. Then there exists a constant $C_{\varepsilon_0,p} > 0$, depending only on p and ε_0, such that

$$\int_0^t \| \, |\nabla u|_a\|_{2p}^{2p}(\tau)d\tau \le C_{\varepsilon_0,p}\Big[t\|f\|_\infty^{2p}\big(G_p+1\big) + \|f\|_\infty^2\| \, |\nabla f|_a\|_{2p-2}^{2p-2} + \frac{e^{sC_+t}-1}{sC_+}\|f\|_\infty^p\| \, |\nabla f|_+\|_\infty^p\| \, |P_K\xi_a^\mu - \xi|_-\|_p^p\Big]. \quad (5)$$

(Recall that $|\nabla u|_a^2 = \sum_{k,j=1}^K a_{kj}\nabla_k u \nabla_j u$.)
Furthermore, for $p = 2$,

$$\int_0^t \| \, |AD^2u|_{HS(a)}\|_2^2(\tau)d\tau \le C_{\varepsilon_0}\Big[\|f\|_\infty^2 t(G_2+1) + \| \, |\nabla f|_a\|_2^2 + \frac{e^{2C_+t}-1}{2C_+}\| \, |\nabla f|_+\|_\infty^2\| \, |P_K\xi_a^\mu - \xi|_-\|_2^2\Big]. \quad (6)$$

and, for $3 - \frac{3}{1+\sqrt{1+3\varepsilon_0}} < p < 2$,

$$\int_0^t \| \, |AD^2u|_{HS(a)}\|_p^p(\tau)d\tau \le C_{p,\varepsilon_0}\Big[t\|f\|_\infty^p(G_p+1) + \|f\|_\infty^{2-p}\| \, |\nabla f|_a\|_{2p-2}^{2p-2} + \|f\|_\infty^{p-2}\frac{e^{2C_+t}-1}{2C_+}\| \, |\nabla f|_+\|_\infty^2\| \, |P_K\xi_a^\mu - \xi|_-\|_2^2\Big], \quad (7)$$

where $|AD^2u|_{HS(a)}^2 = \sum_{i,j,k,m=1}^K a_{ij}a_{km}\nabla_{jk}^2 u \nabla_{mi}^2 u$.

We postpone the proof of Propositions 3.2 and 3.3 till the next section.

Proof of Theorem 2.2. Let $f \in \mathcal{FC}_b^\infty$. For $N \ge 1$ let $\eta_j^N \in \mathcal{FC}_b^\infty(\mathbb{R}^N)$, $j = 1, \dots, N$ satisfy $\| \, |\eta - \eta^N|_a\|_{2p} \le 1/N$, with $\eta^N := \sum_j \eta_j^N e_j$.

Let (ξ_k^n) be the sequence satisfying condition (i) of the theorem. Choose n to be such that $K_n \ge N$.

By $u^{(Nn)}$ we denote the solution to the Cauchy problem on $\mathbb{R}^{K_n}$

$$\begin{cases} u_t^{(Nn)} &= \mathcal{L}_{\xi^n,\eta^N}u^{(Nn)} =: \mathcal{L}_{Nn}u^{(Nn)}, \\ u^{(Nn)}(0) &= f \end{cases} \quad (8)$$

Let $\hat{\mathcal{L}}$ with $D(\hat{\mathcal{L}})$ stand for an arbitrary extension of $\mathcal{L} \restriction \mathcal{FC}_b^\infty$, which generates a C_0-semigroup on L^p. It is easy to show that $D(\hat{\mathcal{L}}) \supset \mathcal{FC}_b^{2,u}$ and

$$\hat{\mathcal{L}}u = \sum_{k,j\ge1}(a_{kj}\nabla_{kj}^2 u + a_{kj}\eta_j^\mu\nabla_k u) + \sum_{k\ge1}\xi_{a;k}^\mu\nabla_k u, \quad u \in \mathcal{FC}_b^{2,u}.$$

It follows from Proposition 3.1(ii)-(iii), that the function $s \mapsto e^{\hat{\mathcal{L}}(t-s)}u^{(Nn)}(s)$ is a continuously differentiable map $[0,\, t] \to L^p$. Thus we arrive at the Duhamel formula

$$u^{(Nn)}(t) - e^{t\hat{\mathcal{L}}}f = e^{(t-\tau)\hat{\mathcal{L}}}u^{(Nn)}(\tau)|_{\tau=0}^{\tau=t} = \int_0^t e^{(t-\tau)\hat{\mathcal{L}}}(\hat{\mathcal{L}} - \mathcal{L}_{Nn})u^{(Nn)}(\tau)d\tau.$$

Since $\hat{\mathcal{L}}$ is the generator of a C_0-semigroup on L^p there exist numbers $M,\, \gamma \in \mathbb{R}$, such that $\|e^{t\hat{\mathcal{L}}}\|_{L^p\to L^p} \le Me^{t\gamma}$.

Now we have

$$\|e^{t\hat{\mathcal{L}}}f - u^{(Nn)}(t)\|_p \le Me^{t\gamma}\Big[\|\,|\xi^n - P_{K_n}\xi_a^\mu|_-^2\|_s \int_0^t \|\,|\nabla u^{(Nn)}|_+\|_\infty d\tau$$
$$+ \|\,|\eta^\mu - \eta^N|_a\|_{2p}\int_0^t \|\,|\nabla u^{(Nn)}|_a\|_{2p}d\tau\Big]. \qquad (9)$$

In order to complete the proof of the theorem we need to show that

$$\lim_{N\to\infty}\limsup_{n\to\infty}\|e^{-t\hat{\mathcal{L}}}f - u^{(Nn)}(t)\|_p = 0.$$

If $\eta_k^\mu = 0$ for all k, then one can take $\eta_k^N = 0$ and the result follows from Proposition 3.2 since $\|\,|\nabla u^{(Nn)}|_+\|_\infty(t) \le e^{(c_+ + \frac{c_+^2}{4})t}\|\,|\nabla f|_+\|_\infty$.

In case $\eta_k^\mu \ne 0$ we employ Propositions 3.2 and 3.3 with the constant

$$C_+ = C_+(N) = c_+ + \frac{1}{4}c_+^2 + \Big(\sum_{k=1}^N \bar{c}_k\Big)\|\,|\nabla\eta^N|_{HS}\|_\infty + c_+\|\,|\eta^N|_a\|_\infty$$

to estimate $\|\,|\nabla u^{(Nn)}|_+\|_\infty$ and $\int_0^t \|\,|\nabla u^{(Nn)}|_a\|_{2p}ds$. Then we get

$$\limsup_{n\to\infty}\|e^{t\hat{\mathcal{L}}}f - u^{(Nn)}(t)\|_p \le C\|\,|\eta^\mu - \eta^N|_a\|_{2p}\Big[\|f\|_\infty t(\|\,|\eta^\mu|_a + |\eta^N|_a\|_{2p} + \sup_d \|\upsilon_d\|_{2p})$$
$$+ t^{\frac{2p-1}{2p}}\big(\|f\|_\infty\|\,|\nabla f|_a\|_{2p-2}^{p-1}\big)^{\frac{1}{p}}\Big].$$

Taking the limit as $N \to \infty$ we complete the proof. □

4 Proof of A-priori Estimates

Throughout this section $(a_{kj})_{k,j=1}^K$, a_{kj}, $\xi_k \in \mathcal{FC}_b^{1,u}(\mathbb{R}^K)$, $k, j = 1, \ldots, K$, $\eta_i \in \mathcal{FC}_b^{1,u}(\mathbb{R}^N)$, $i = 1, \ldots, N$ for some $N < K$, and $\eta_i \equiv 0$, $i = N+1, \ldots, K$, $f \in \mathcal{FC}_b^\infty(\mathbb{R}^K)$, $f \ne 0$; $u(t) \in \mathcal{FC}_b^{2,u}(\mathbb{R}^K)$, $t \ge 0$ is the solution to the Cauchy problem (4). Unless otherwise indicated, all the sums are from 1 to K. We also assume that the measure μ satisfies the conditions of Theorem 2.2.

We are now heading towards establishing the estimates for the derivatives of the solution to (4).

Proof of Proposition 3.2. Let us differentiate equation (4) in the direction e_k (observe that u is three times differentiable by Proposition 3.1(iii)), then multiply by $\gamma_k^2 \nabla_k u$ and sum up from 1 to K. We arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_k \gamma_k^2(\nabla_k u)^2 = &\sum_{i,j,k}(a_{ij}(\nabla^3_{kij}u)\gamma_k^2\nabla_k u + \eta_i a_{ij}\nabla^2_{kj}u\gamma_k^2\nabla_k u) + \sum_{i,k}\xi_i\nabla^2_{ik}u\gamma_k^2\nabla_k u\\ &+\sum_{i,j,k}((\nabla_k a_{ij})\nabla^2_{ij}u\gamma_k^2\nabla_k u + (\nabla_k\eta_i)a_{ij}\nabla_j u\gamma_k^2\nabla_k u)\\ &+\sum_{i,j,k}\eta_i(\nabla_k a_{ij})\nabla_j u\gamma_k^2\nabla_k u + \sum_{i,k}(\nabla_k\xi_i)\nabla_i u\gamma_k^2\nabla_k u. \qquad (10)\end{aligned}$$

Note that (10) is an equality in $C_b(\mathbb{R}^K)$ since $\nabla_k u$, $(\nabla_k \mathcal{L}_{\xi,\eta}u) \in C_b(\mathbb{R}^K)$ due to Proposition 3.1(ii).

Recall that $\sum_k \gamma_k^2(\nabla_k u)^2 = |\nabla u|_+^2$. A straightforward computation shows that

$$\frac{1}{2}\nabla_j|\nabla u|_+^2 = \sum_k \nabla^2_{kj}u\gamma_k^2(\nabla_k u),$$

$$\sum_{i,j,k} a_{ij}(\nabla^3_{kij}u)\gamma_k^2\nabla_k u = 1/2\sum_{i,j,k} a_{ij}\nabla^2_{ij}(\gamma_k^2(\nabla_k u)^2) - \sum_{i,j,k}\gamma_k^2 a_{ij}(\nabla^2_{ik}u)(\nabla^2_{jk}u).$$

Therefore one can rewrite (10) as follows

$$\frac{d}{dt}|\nabla u|_+^2(t) = \mathcal{L}_{\xi,\eta}|\nabla u|_+^2(t) + 2F(t), \qquad (11)$$

where

$$\begin{aligned}F(t) = &\sum_{i,j,k}((\nabla_k a_{ij})\eta_i\nabla_j u\gamma_k^2\nabla_k u + (\nabla_k\eta_i)a_{ij}\nabla_j u\gamma_k^2\nabla_k u) + \sum_{i,k}(\nabla_k\xi_i)\nabla_i u\gamma_k^2\nabla_k u\\ &+\sum_{i,j,k}(\nabla_k a_{ij})\nabla^2_{ij}u\gamma_k^2\nabla_k u - \sum_{i,j,k}\gamma_k^2 a_{ij}(\nabla^2_{ik}u)(\nabla^2_{jk}u).\end{aligned}$$

Observe that $|\nabla u|_+^2 \in \mathcal{D}$, where $\mathcal{D}$ is as in Proposition 3.1. Indeed, by Proposition 3.1(iii) we have $|\nabla u|_+^2 \in \cap_{p\geq 1}W^{2,p}\cap UC_b^1$. Moreover, Proposition 3.1(iii) implies that $F(t) \in C_b(\mathbb{R}^K)$ for all $t > 0$, and from Proposition 3.1(ii) we conclude that $\frac{d}{dt}|\nabla u|_+^2(t) \in C_b(\mathbb{R}^K)$ for all $t \geq 0$. Hence, (11) yields $\mathcal{L}_{\xi,\eta}|\nabla u|_+^2(t) \in C_b(\mathbb{R}^K)$, $t > 0$ and $|\nabla u|_+^2 \in \mathcal{D}$ by Proposition 3.1(i). Therefore, $|\nabla u|_+^2$ is the classical solution to the non-homogeneous problem for the operator $\mathcal{L}_{\xi,\eta} \restriction \mathcal{D}$.

Furthermore, since $t \mapsto u(t)$, $t \mapsto \nabla_k u(t)$, $t \mapsto \nabla_k \mathcal{L}_{\xi,\eta} u(t)$ are continuous functions $[0, \infty) \to C_b$, $k = 1, \ldots, K$, the function $t \mapsto \nabla^2_{km} u(t)$ is continuous. Hence, F is a continuous function $(0, \infty) \to C_b$. By [11], 4.1.2 it follows that

$$|\nabla u|^2_+(t) = U(t)|\nabla f|^2_+ + 2\int_0^t U(t-s)F(s)ds.$$

The first assumption of the proposition implies that

$$\sum_{k,j} (\nabla_k \xi_i)(\nabla_j u)\gamma_k^2 \nabla_k u \le c_+ |\nabla u|^2_+.$$

Next we estimate the terms in the expression for F, containing $\nabla_k a_{ij}$. For an arbitrary symmetric matrix $(b_{ij})_{i,j=1}^K$ and any vector $g \in \mathbb{R}^K$ the following inequality holds:

$$\begin{aligned}
&\sum_{i,j,k} (\nabla_k a_{ij}) b_{ij} \gamma_k^2 g_k \\
&\le \Big[\sum_{i,j,l} \gamma_l^2 a_{ij} b_{il} b_{lj}\Big]^{\frac12} \Big[\sum_{i,j,l} \gamma_l^{-2} a^{ij} \Big(\sum_k (\nabla_k a_{il})\gamma_k^2 g_k\Big)\Big(\sum_m (\nabla_m a_{jl})\gamma_m^2 g_m\Big)\Big]^{\frac12} \\
&\le \Big[\sum_{i,j,l} \gamma_l^2 a_{ij} b_{il} b_{lj}\Big]^{\frac12} |g|_+ \nu_K. \qquad (12)
\end{aligned}$$

In order to derive (12) we have applied the Cauchy-Schwarz inequality and used definition (2) of ν_K. From the boundedness of ν_K (the second assumption of the proposition) we conclude

$$\begin{aligned}
&\sum_{i,j,k} (\nabla_k a_{ij})(\nabla^2_{ij} u)\gamma_k^2 (\nabla_k u) \le \nu_K |\nabla u|_+ \Big(\sum_{i,j,k} \gamma_k^2 a_{ij} \nabla^2_{ik} u \nabla^2_{jk} u\Big)^{\frac12} \\
&\le (c_+^2/4)|\nabla u|^2_+ + \sum_{i,j,k} \gamma_k^2 a_{ij} \nabla^2_{ik} u \nabla^2_{jk} u
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i,j,k} (\nabla_k a_{ij})\eta_i \nabla_j u \gamma_k^2 \nabla_k u \le \nu_K |\nabla u|_+ \Big(\sum_{i,j,k} \gamma_k^2 a_{ij} \eta_i (\nabla_k u) \eta_j (\nabla_k u)\Big)^{\frac12} \\
&\le c_+ |\eta|_a |\nabla u|^2_+.
\end{aligned}$$

Thus, it remains to estimate the term in the expression for F, which contains $\nabla_k \eta_i$. Let A and $\nabla\eta$ be the operators in $\mathbb{R}^K$ associated with the matrices $(a_{ij})_{i,j=1}^K$

and $(\nabla_k\eta_i)_{i,k=1}^K$ respectively, and T be the operator defined by the diagonal matrix $(\delta_{jk}\gamma_k)_{k=1}^K$. Then one obtains

$$\sum_{i,j,k} a_{ij}(\nabla_k\eta_i)(\nabla_j u)\gamma_k^2(\nabla_k u) = (T\nabla u,\, (T(\nabla\eta)AT^{-1})T\nabla u)_0$$
$$\le |T(\nabla\eta)AT^{-1}|_{K,0}|T\nabla u|_0^2 = |(\nabla\eta)A|_{K,0}|\nabla u|_+^2,$$

where $|\cdot|_{K,0}$ stands for the operator norm $(\mathbb{R}^K,\, |\cdot|_0) \to (\mathbb{R}^K,\, |\cdot|_0)$. (Here we used the property that for any matrix W $sp(W) = sp(TWT^{-1})$.) It is well-known that, for the operator W in $\mathbb{R}^K$ associated with matrix $(w_{jk})_{j,k=1}^K$, we have $|W|_{K,0}^2 \le \sup_j \sum_k |w_{jk}|^2$. Therefore,

$$|(\nabla\eta)A|_{K,0}^2 = \sup_j \sum_k \big(\sum_{i=1}^N a_{ij}(\nabla_k\eta_i)\big)^2$$
$$\le \sup_j \left(\sum_{i,k=1}^N (\nabla_k\eta_i)^2\right)\left(\sum_{i=1}^N a_{ij}^2\right) \le \left(\sum_{i=1}^N \bar{c}_i\right)\left(\sum_{i,k=1}^N (\nabla_k\eta_i)^2\right).$$

In order to obtain the last inequality we have made use of (A2). Combining the derived estimates we arrive at

$$\sum_{i,j,k} a_{ij}(\nabla_k\eta_i)\nabla_j u\gamma_k^2\nabla_k u \le \left(\sum_{i=1}^N \bar{c}_i\right)^{\frac{1}{2}} |\nabla\eta|_{HS}|\nabla u|_+^2.$$

Since $|\nabla u|_+^2$ is non-negative and the semigroup $U(t)$ is positivity preserving and contractive we have

$$\|\,|\nabla u|_+\|_\infty^2 \le \|\,|\nabla f|_+\|_\infty^2 + 2C_+\int_0^t \|\,|\nabla u|_+\|_\infty^2(s)ds.$$

Hence the assertion follows from Gronwall's lemma. $\square$

For $d > K$ we set $\eta_j \equiv 0$, $j = K+1,\dots,d$, and introduce the quantities

$$\xi_{a;k}^{\mu,d} := \sum_{j=1}^d (a_{kj}\xi_j^\mu + \nabla_j a_{kj}),\ k = 1,\dots,K,$$

$$B_d := \sum_k (\xi_k - \xi_{a;k}^{\mu,d})\nabla_k u + \sum_{j=1}^d \sum_k a_{kj}(\eta_j - \eta_j^\mu)\nabla_k u,$$

For $p \ge 1$ we set $[\nabla u]_{\varepsilon,a}^2 := |\nabla u|_a^2 + \varepsilon^2$ with $\varepsilon > 0$. Set $\chi_\varepsilon := [\nabla u]_{\varepsilon,a}^{p-2}$. We introduce the following quantities:

$$T_{\varepsilon,a} := \|\,[\nabla u]_{\varepsilon,a}^{p-1}|\nabla u|_a\|_2^2\,,$$
$$J_{\varepsilon,a} := \|\,[\nabla u]_{\varepsilon,a}^{p-3}|\nabla|\nabla u|_a^2|_a\|_2^2\,,$$
$$I_{\varepsilon,a} := \sum_{i,j,k,l} \langle \chi_\varepsilon^2 a_{ij}\nabla_{jk}^2 u,\, a_{kl}\nabla_{li}^2 u\rangle\,.$$

Note that $I_{\varepsilon,a} = \|\chi_\varepsilon |AD^2 u|_{HS(a)}\|_2^2$ and $(p-1)^2 J_{\varepsilon,a} = 4\| \,|\nabla[\nabla u]_{\varepsilon,a}^{p-1}|_a\|_2^2$.

Lemma 4.1. *Let u be the solution to (4). Then*

$$\|\chi_\varepsilon \frac{du}{dt}\|_2^2 + \frac{1}{p-1}\frac{d}{dt}\| \,[\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2} \le 2\|\chi_\varepsilon B_d\|_2^2 + 2(p-2)^2 J_{\varepsilon,a}. \tag{13}$$

Proof. It follows from (4) that

$$\Big\langle u_t - \sum_{k,j} a_{kj}\nabla_{kj}^2 u,\, \chi_\varepsilon^2 u_t\Big\rangle = \Big\langle \sum_{k,j} a_{kj}\eta_k \nabla_j u + \sum_j \xi_j \nabla_j u,\, \chi_\varepsilon^2 u_t\Big\rangle.$$

Integration by parts yields

$$-\sum_{k,j}\langle a_{kj}\nabla_{kj}^2 u,\, \chi_\varepsilon^2 u_t\rangle = \sum_{k=1}^d\sum_j \langle a_{kj}\nabla_j u,\, (\nabla_k + \eta_k^\mu)\chi_\varepsilon^2 u_t\rangle + \sum_j \langle \xi_{a;j}^{\mu,d}\nabla_j u,\, \chi_\varepsilon^2 u_t\rangle.$$

Hence we obtain

$$\begin{aligned}
&\|\chi_\varepsilon u_t\|_2^2 + \frac{1}{2p-2}\frac{d}{dt}\| \,[\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2}\\
&= \sum_{k=1}^d\sum_j \langle a_{kj}(\eta_k - \eta_k^\mu)\nabla_j u,\, \chi_\varepsilon^2 u_t\rangle + \sum_j \langle (\xi_j - \xi_{a;j}^{\mu,d})\nabla_j u,\, \chi_\varepsilon^2 u_t\rangle - \sum_{k,j}\langle a_{kj}\nabla_j u \nabla_k \chi_\varepsilon^2,\, u_t\rangle\\
&\le \frac{1}{2}\|\chi_\varepsilon u_t\|_2^2 + \|\chi_\varepsilon B_d\|_2^2 + (p-2)^2 J_{\varepsilon,a}.
\end{aligned} \tag{14}$$

The last inequality in (14) follows from the estimate

$$4\langle \,|(\nabla u,\, \nabla\chi_\varepsilon)_a|^2\rangle \le 4\| \,[\nabla u]_{\varepsilon,a}|\nabla\chi_\varepsilon|_a\|_2^2 = (p-2)^2 J_{\varepsilon,a}, \tag{15}$$

and (14) implies the assertion. □

Lemma 4.2. *Let u be the solution to (4). Then for any $\delta > 0$ we have*

$$\|\chi_\varepsilon|\nabla u|_a\|_2^2 + \frac{\delta}{p-1}\frac{d}{dt}\| \,[\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2} \le 3\delta\|\chi_\varepsilon B_d\|_2^2 + 3\delta(p-2)^2 J_{\varepsilon,a} + \frac{3}{4\delta}\|\chi_\varepsilon u\|_2^2 \tag{16}$$

Proof. Let ψ_ε be defined by $\psi_\varepsilon := -\sum_{k=1}^d\sum_j(\nabla_k + \beta_k^\mu)(\chi_\varepsilon^2 a_{kj}\nabla_j u)$. It follows from (4) that

$$\begin{aligned}
\psi_\varepsilon &= \chi_\varepsilon^2\big[-u_t + \sum_{j=1}^d\sum_k a_{kj}(\eta_j - \eta_j^\mu)\nabla_k u + \sum_k(\xi_k - \xi_{a;k}^{\mu,d})\nabla_k u\big] - \sum_{k,j} a_{kj}(\nabla_k\chi_\varepsilon^2)\nabla_j u\\
&\equiv \chi_\varepsilon^2(B_d - u_t) - 2\chi_\varepsilon(\nabla u,\, \nabla\chi_\varepsilon)_a.
\end{aligned} \tag{17}$$

Integration by parts yields

$$\|\chi_\varepsilon|\nabla u|_a\|_2^2 = \langle u,\, \psi_\varepsilon\rangle = \langle u,\, \chi_\varepsilon^2(B_d - u_t) - 2\chi_\varepsilon(\nabla u,\, \nabla\chi_\varepsilon)_a\rangle \tag{18}$$

We estimate the RHS of (18) as follows.

$$\begin{aligned}
\langle u,\, \chi_\varepsilon^2 B_d\rangle| &\le \delta\|\chi_\varepsilon B_d\|_2^2 + \frac{1}{4\delta}\|\chi_\varepsilon u\|_2^2\,,\\
2|\langle u,\, (\nabla u,\, \nabla\chi_\varepsilon)_a\rangle| &\le \delta(p-2)^2 J_{\varepsilon,a} + \frac{1}{4\delta}\|\chi_\varepsilon u\|_2^2\,,\\
|\langle u,\, \chi_\varepsilon^2 u_t\rangle| &\le \delta\|\chi_\varepsilon u_t\|_2^2 + \frac{1}{4\delta}\|\chi_\varepsilon u\|_2^2,
\end{aligned}$$

(where we have used (15) in the second term). Applying Lemma 4.1 we complete the proof. □

We introduce the following quantities.

$$\begin{aligned}
\Upsilon_d &:= |\eta^\mu|_a + |\eta|_a + \upsilon_d,\\
\Xi_d^2 &:= |\nabla u|_+^2 \sum_{k=1}^K \gamma_k^{-2}(\xi_k - \xi_{a;k}^{\mu,d})^2,\\
\Xi^2 &:= L^1\text{-}\lim_{d\to\infty}\Xi_d^2 = |\nabla u|_+^2 \sum_{k=1}^K \gamma_k^{-2}(\xi_k - \xi_{a;k}^{\mu})^2 = |\nabla u|_+^2 |\xi - P_K\xi_a^\mu|_-^2,
\end{aligned}$$

where υ_d is as in (1) and the limit in d exists due to (A4).

Lemma 4.3. *Let $p \in (3 - \frac{3}{1+\sqrt{1+3\varepsilon_0}},\, \frac{2}{\varepsilon_0})$. Then there exist positive constants $K(\varepsilon_0,\, p)$ and $r(p)$ such that*

$$r(p)\frac{d}{dt}\|[\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2} + K(\varepsilon_0,\, p)J_{\varepsilon,a} \le\, C_{\varepsilon_0,\,p}\Big[\|\chi_\varepsilon\Xi\|_2^2 + \sup_d\|\chi_\varepsilon|\nabla u|_a\Upsilon_d\|_2^2\Big] + C_p\|\chi_\varepsilon u\|_2^2.$$

If $p < 2$ the same estimate holds for $J_{\varepsilon,a}$ replaced by $I_{\varepsilon,a}$.

Proof. It follows from (4) that

$$\langle u_t,\, \psi_\varepsilon\rangle - \sum_{k,l}\langle a_{kl}\nabla_{kl}^2 u,\, \psi_\varepsilon\rangle = \sum_{k,l}\langle a_{kl}\eta_l\nabla_l u,\, \psi_\varepsilon\rangle + \sum_l\langle\xi_l\nabla_l u,\, \psi_\varepsilon\rangle, \tag{19}$$

where the function ψ_ε was defined in Lemma 4.2. It is easy to see that the second term in the LHS of (19) equals

$$-\sum_{k=1}^d\sum_l\langle\nabla_k(a_{kl}\nabla_l u),\, \psi_\varepsilon\rangle + \sum_{k=1}^d\sum_l\langle(\nabla_k a_{kl})\nabla_l u,\, \psi_\varepsilon\rangle.$$

The key point is to evaluate the first term in the above expression. Successive integration by parts and a straightforward computation give

$$
\begin{aligned}
&\sum_{i,k=1}^{d}\sum_{j,l}\langle \nabla_k(a_{kl}\nabla_l u),\,(\nabla_i+\beta_i^{\mu})(\chi_\varepsilon^2 a_{ij}\nabla_j u)\rangle \\
&= -\sum_{i,k=1}^{d}\sum_{j,l}\langle \nabla_i\nabla_k(a_{kl}\nabla_l u),\,\chi_\varepsilon^2 a_{ij}\nabla_j u\rangle \\
&= \sum_{i,k=1}^{d}\sum_{j,l}[\langle \nabla_i(a_{kl}\nabla_l u),\,(\nabla_k+\eta_k^{\mu})(\chi_\varepsilon^2 a_{ij}\nabla_j u)\rangle + \langle \nabla_i(a_{kl}\nabla_l u),\,\xi_k^{\mu}\chi_\varepsilon^2 a_{ij}\nabla_j u\rangle] \\
&= S_1+S_2,
\end{aligned}
$$

with

$$
\begin{aligned}
S_1 &= \sum_{i,k=1}^{d}\sum_{j,l}\langle \nabla_i(a_{kl}\nabla_l u),\,(\nabla_k+\eta_k^{\mu})(\chi_\varepsilon^2 a_{ij}\nabla_j u)\rangle, \\
S_2 &= \sum_{i,k=1}^{d}\sum_{j,l}\langle \nabla_i(a_{kl}\nabla_l u),\,\xi_k^{\mu}\chi_\varepsilon^2 a_{ij}\nabla_j u\rangle.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
S_1 = I_{\varepsilon,a} &+ \sum_{i,j,k,l}\big[2\langle \chi_\varepsilon^2 a_{ij}\nabla_{kj}^2 u,\,(\nabla_i a_{kl})\nabla_l u\rangle + \langle a_{ij}\nabla_j u\nabla_k(\chi_\varepsilon^2),\,\nabla_i(a_{kl}\nabla_l u)\rangle\big] \\
&+ \sum_{i,k=1}^{d}\sum_{j,l}\big[\langle \chi_\varepsilon^2(\nabla_i a_{kj})\nabla_j u,\,(\nabla_k a_{il})\nabla_l u\rangle + \langle \chi_\varepsilon^2 a_{ij}\nabla_j u\,\eta_k^{\mu},\,\nabla_i(a_{kl}\nabla_l u)\rangle\big].
\end{aligned}
$$

We transform S_2 as follows

$$
\begin{aligned}
S_2 &= \sum_{i,k=1}^{d}\sum_{j,l}\langle \chi_\varepsilon^2\nabla_i(a_{kl}\nabla_l u),\,\xi_k^{\mu}a_{ij}\nabla_j u\rangle \\
&= \lim_m \sum_{i,k=1}^{d}\sum_{j,l}\langle \chi_\varepsilon^2\nabla_i(a_{kl}\nabla_l u),\,\hat{\xi}_k^{m}a_{ij}\nabla_j u\rangle \\
&= \lim_m \sum_{i,k=1}^{d}\sum_{j,l}\langle \hat{\xi}_k^{m}a_{kl}\nabla_l u,\,(\nabla_i+\beta_i^{\mu})\chi_\varepsilon^2 a_{ij}\nabla_j u\rangle \\
&\quad - \lim_m \sum_{i,k=1}^{d}\sum_{j,l}\langle (\nabla_i\hat{\xi}_k^{m})a_{kl}\nabla_l u,\,\chi_\varepsilon^2 a_{ij}\nabla_j u\rangle \\
&= \sum_{k=1}^{d}\sum_{l}\langle a_{kl}\xi_k^{\mu}\nabla_l u,\,\psi_\varepsilon\rangle - \lim_m \sum_{i,k=1}^{d}\sum_{j,l}\langle (\nabla_i\hat{\xi}_k^{m})a_{kl}\nabla_l u,\,\chi_\varepsilon^2 a_{ij}\nabla_j u\rangle.
\end{aligned}
$$

Thus we have

$$-\sum_{k,l}\langle a_{kl}\nabla^2_{kl}u,\,\psi_\varepsilon\rangle = S_1 + \sum_l \langle(\xi_l - \xi^{\mu,d}_{a;l})\nabla_l u,\,\psi_\varepsilon\rangle$$
$$-\lim_m \sum_{i,k=1}^d \sum_{j,l}\langle(\nabla_i\hat{\xi}^m_k)a_{kl}\nabla_l u,\,\chi^2_\varepsilon a_{ij}\nabla_j u\rangle.$$

Observe that

$$\langle u_t,\,\psi_\varepsilon\rangle = \frac{1}{2}\Big\langle \chi^2_\varepsilon \frac{d}{dt}|\nabla u|^2_a\Big\rangle = \frac{1}{2p-2}\frac{d}{dt}\|\,[\nabla u]_{\varepsilon,a}\|^{2p-2}_{2p-2}.$$

So we infer from (19) that

$$\begin{aligned}
&\frac{1}{2p-2}\frac{d}{dt}\|\,[\nabla u]_{\varepsilon,a}\|^{2p-2}_{2p-2} + I_{\varepsilon,a} + \frac{p-2}{2}J_{\varepsilon,a}\\
=&\Big[\frac{p-2}{2}J_{\varepsilon,a} - \sum_{i,j,k,l}\langle a_{ij}\nabla_j u\nabla_k(\chi^2_\varepsilon),\,\nabla_i(a_{kl}\nabla_l u)\rangle\Big]\\
&-\sum_{i,j,k,l}\big[2\langle\chi^2_\varepsilon a_{ij}\nabla^2_{kj}u,\,(\nabla_i a_{kl})\nabla_l u\rangle + \langle a_{ij}\nabla_j u\nabla_k(\chi^2_\varepsilon),\,\nabla_i(a_{kl}\nabla_l u)\rangle\big]\\
&-\sum_{i,k=1}^d\sum_{j,l}\big[\langle\chi^2_\varepsilon(\nabla_i a_{kj})\nabla_j u,\,(\nabla_k a_{il})\nabla_l u\rangle + \langle\chi^2_\varepsilon a_{ij}\eta^\mu_k\nabla_j u,\,\nabla_i(a_{kl}\nabla_l u)\rangle\big]\\
&+\lim_m\sum_{i,k=1}^d\sum_{j,l}\langle(\nabla_i\hat{\xi}^m_k)a_{kl}\nabla_l u,\,\chi^2_\varepsilon a_{ij}\nabla_j u\rangle\\
&-\sum_{k,l}\langle a_{kl}\,\eta_l\nabla_k u,\,\psi_\varepsilon\rangle - \sum_k\langle(\xi_k - \xi^{\mu,d}_{a;k})\nabla_k u,\,\psi_\varepsilon\rangle.
\end{aligned}\tag{20}$$

In order to estimate the RHS of (20) we use the following inequalities:

$$|\sum_{i,k=1}^d\sum_{j,l}(\nabla_i a_{kj})\nabla_j u(\nabla_k a_{il})\nabla_l u| \le \upsilon^2_d|\nabla u|^2_a,\tag{21}$$

$$\sum_{i,j,k,l}a_{ij}(\nabla^2_{kj}u)(\nabla_i a_{kl})\nabla_l u \le \delta|AD^2u|^2_{HS(a)} + C_\delta\upsilon^2_d|\nabla u|^2_a,\tag{22}$$

$$\begin{aligned}
&|\sum_{i,j,k,l}\langle(a_{ij}\nabla_j u)\nabla_k\chi^2_\varepsilon,\,\nabla_i(a_{kl}\nabla_l u)\rangle - \frac{p-2}{2}J_{\varepsilon,a}|\\
&\qquad\le \delta(p-2)^2J_{\varepsilon,a} + C_\delta\|\chi_\varepsilon|\nabla u|_a\upsilon_d\|^2_2,
\end{aligned}\tag{23}$$

for any positive δ. (Recall that $(h,\,g)_a := \sum_{k,j\ge1}a_{kj}h_jg_k$.) The estimates (21), (22) are immediate from the definitions and the Cauchy inequality. We postpone the proof of (23) to Appendix. It follows from assumption (ii(b)) of Theorem 2.2 that the limit in m and d in the third term from the end in (20) does not exceed

$$\begin{aligned}&\varepsilon_0 \sum_{i,j,k,l} \langle a_{li} \nabla_i (\chi_\varepsilon \nabla_j u),\, a_{jk} \nabla_k (\chi_\varepsilon \nabla_l u)\rangle + c(\varepsilon_0) \sum_{k,j} \langle \chi_\varepsilon^2 a_{kj} \nabla_k u \nabla_j u\rangle \\ \le\, &\varepsilon_0 \left(I_{\varepsilon,a} + \frac{(p-2)^2}{4} J_{\varepsilon,a} + |p-2| \sqrt{I_{\varepsilon,a} J_{\varepsilon,a}} \right) + c(\varepsilon_0) \|\chi_\varepsilon |\nabla u|_a\|_2^2. \end{aligned} \tag{24}$$

Here we used the fact that

$$\sum_{i,j,k,l} \langle a_{jk} (\nabla_i \chi_\varepsilon) \nabla_j u,\, a_{li} (\nabla_k \chi_\varepsilon) \nabla_l u\rangle = \|(\nabla u,\, \nabla \chi_\varepsilon)_a\|_2^2,$$

and applied (15).

In order to estimate the terms containing ψ_ε we use (15), (17) and the inequality

$$B_d^2 \le 2(\Xi_d^2 + \Upsilon_d^2 |\nabla u|_a^2).$$

Hence, for any positive δ we have

$$\begin{aligned}&|\sum_{k,l} \langle a_{kl}\, \eta_l \nabla_k u,\, \psi_\varepsilon\rangle + \sum_k \langle (\xi_k - \xi^\mu_{a;k}) \nabla_k u,\, \psi_\varepsilon\rangle| \\ \le\, &\delta (p-2)^2 J_{\varepsilon,a} + \delta \|\chi_\varepsilon u_t\|_2^2 + C_\delta \|\chi_\varepsilon \Xi_d\|_2^2 + C_\delta \|\chi_\varepsilon |\nabla u|_a \Upsilon_d\|_2^2\end{aligned}$$

Making use of Lemma 4.1 we arrive at

$$\begin{aligned}&|\sum_{k,l} \langle a_{kl}\, \eta_l \nabla_k u,\, \psi_\varepsilon\rangle + \sum_k \langle (\xi_k - \xi^\mu_{a;k}) \nabla_k u,\, \psi_\varepsilon\rangle| \\ \le\, &2\delta (p-2)^2 J_{\varepsilon,a} - \frac{\delta}{p-1} \frac{d}{dt} \|\,[\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2} + C_\delta \|\chi_\varepsilon \Xi_d\|_2^2 + C_\delta \|\chi_\varepsilon |\nabla u|_a \Upsilon_d\|_2^2.\end{aligned} \tag{25}$$

Combining (21)-(25) and using Lemma 4.2 (to estimate $\|\chi_\varepsilon |\nabla u|_a\|_2^2$ in (24)) we get

$$\begin{aligned}&\frac{1}{p-1} \Big(\frac{1}{2} + \delta + c(\varepsilon_0)\delta\Big) \frac{d}{dt} \|[\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2} \\ &+ (1-\varepsilon_0) I_{\varepsilon,a} + \frac{2(p-2) - \varepsilon_0 (p-2)^2}{4} J_{\varepsilon,a} - \varepsilon_0 |p-2| \sqrt{I_{\varepsilon,a} J_{\varepsilon,a}} \\ \le\, &\delta I_{\varepsilon,a} + 3\delta (1 + c(\varepsilon_0)) (p-2)^2 J_{\varepsilon,a} \\ &+ C_{\delta,\varepsilon_0} \sup_d \|\chi_\varepsilon |\nabla u|_a \Upsilon_d\|_2^2 + C_{\delta,\varepsilon_0} \lim_{d\to\infty} \|\chi_\varepsilon \Xi_d\|_2^2 + C_\delta \|\chi_\varepsilon u\|_2^2.\end{aligned} \tag{26}$$

Note that $\Xi_d \to \Xi$ as $d \to \infty$ in L^2 due to (A4). Now applying Lemma A2 (see Appendix below) we complete the proof. □

Proof of Proposition 3.3. First let $p \geq 2$. Lemma A1 states that

$$\| \, |\nabla u|_a\|_{2p}^{2p} \leq 2\|u\|_\infty^2 \| \, |\nabla u|_a^{p-2} \sum_{k=1}^d \sum_j (\nabla_k + \beta_k^\mu)(a_{kj}\nabla_j u)\|_2^2 + 2(p-1)^2 \|u\|_\infty^2 J_{0,a}.$$

(Here and below $J_{0,a} := \lim_{\varepsilon\to 0} J_{\varepsilon,a} = 4(p-1)^{-2}\| \, |\nabla|\nabla u|_a^{p-1}|_a^2\|_2^2$.) Making use of equation (4) and the maximum principle we get

$$\| \, |\nabla u|_a\|_{2p}^{2p} \leq 2\|f\|_\infty^2 \Big[\| \, |\nabla u|_a^{p-2} u_t\|_2^2 + \| |\nabla u|_a^{p-2} B_d\|_2^2\Big] + 2(p-1)^2\|f\|_\infty^2 J_{0,a}. \quad (27)$$

Observe that in Lemma 4.1 one can pass to the limit as $\varepsilon \to 0$ provided $p \geq 2$:

$$\| \, |\nabla u|_a^{p-2} u_t\|_2^2 \leq 2\| \, |\nabla u|_a^{p-2} B_d\|_2^2 + 2(p-2)^2 J_{0,a} - \frac{1}{p-1}\frac{d}{dt}\| \, |\nabla u|_a\|_{2p-2}^{2p-2}.$$

Hence, estimating $\| \, |\nabla u|_a^{p-2} u_t\|_2^2$ in (27) we arrive at the inequality

$$\| \, |\nabla u|_a\|_{2p}^{2p} \leq \, C_p\|f\|_\infty^2 (\| |\nabla u|_a^{p-2} B_d\|_2^2 + J_{0,a}) - 4\frac{\|f\|_\infty^2}{p-1}\frac{d}{dt}\| \, |\nabla u|_a\|_{2p-2}^{2p-2}. \quad (28)$$

Note that

$$\limsup_{d\to\infty} \| |\nabla u|_a^{p-2} B_d\|_2^2 \leq 2\| |\nabla u|_a^{p-2}\Xi\|_2^2 + 2\sup_d \| |\nabla u|_a^{p-1}\Upsilon_d\|_2^2.$$

It is easy to see that we can also pass to the limit as $\varepsilon \to 0$ in Lemma 4.3. Applying Lemma 4.3 to (28) we obtain

$$\begin{aligned} &\kappa_1(\varepsilon_0,\, p)\|f\|_\infty^2 \frac{d}{dt}\| \, |\nabla u|_a\|_{2p-2}^{2p-2} + \kappa_2(\varepsilon_0,\, p)\| \, |\nabla u|_a\|_{2p}^{2p} \\ &\leq C_{\varepsilon_0,p}\|f\|_\infty^2 \Big(\| |\nabla u|_a^{p-2}\Xi\|_2^2 + \| |\nabla u|_a^{p-2} u\|_2^2 + \sup_d \| |\nabla u|_a^{p-1}\Upsilon_d\|_2^2 \Big), \end{aligned} \quad (29)$$

with some positive $\kappa_1(\varepsilon_0,\, p)$ and $\kappa_2(\varepsilon_0,\, p)$. We estimate the RHS of (29) from above by

$$\delta\| \, |\nabla u|_a\|_{2p}^{2p} + C_{p,\varepsilon_0,\delta}\|f\|_\infty^p \Big(\|f\|_\infty^p + \|\Xi\|_p^p \Big) + C_{p,\varepsilon_0,\delta}\|f\|_\infty^{2p} \sup_d \|\Upsilon_d\|_{2p}^{2p},$$

for any positive δ. Choosing δ small enough we arrive at the inequality

$$\begin{aligned} &\kappa_1(\varepsilon_0,\, p)\|f\|_\infty^2 \frac{d}{dt}\| \, |\nabla u|_a\|_{2p-2}^{2p-2} + \kappa_2(\varepsilon_0,\, p)\| \, |\nabla u|_a\|_{2p}^{2p} \\ &\leq C_{p,\varepsilon_0}\|f\|_\infty^p \|\Xi\|_p^p + C_{p,\varepsilon_0}\|f\|_\infty^{2p}\Big(\sup_d \|\Upsilon_d\|_{2p}^{2p} + 1 \Big). \end{aligned} \quad (30)$$

Now we assume that $p < 2$. As in the case $p \geq 2$ we employ Lemma A.1, equation (4), the maximum principle and Lemma 4.1. Then

$$T_{\varepsilon,a} \leq \varepsilon^{2p-2}\|u\|_2^2 + 4(\|u\|_\infty^2 + \varepsilon^2)\Big[\|\chi_\varepsilon u_t\|_2^2 + \|\chi_\varepsilon B_d\|_2^2\Big] + 2(p-1)^2\|u\|_\infty^2 J_{\varepsilon,a}$$
$$\leq \varepsilon^{2p-2}\|f\|_\infty^2 + C_p(\|f\|_\infty^2 + \varepsilon^2)(\|\chi_\varepsilon B_d\|_2^2 + J_{\varepsilon,a})$$
$$- 4\frac{\|f\|_\infty^2 + \varepsilon^2}{p-1}\frac{d}{dt}\| [\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2}.$$

Setting $\varepsilon := \|f\|_\infty$, passing to the limit as $d \to \infty$ and employing Lemma 4.3 we arrive at the estimate

$$\kappa_1(\varepsilon_0, p)\|f\|_\infty^2 \frac{d}{dt}\| [\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2} + \kappa_2(\varepsilon_0, p)T_{\varepsilon,a}$$
$$\leq C_{\varepsilon_0,p}\|f\|_\infty^2\big[\|\chi_\varepsilon \Xi\|_2^2 + \sup_d \|\chi_\varepsilon|\nabla u|_a\Upsilon_d\|_2^2 + \|\chi_\varepsilon u\|_2^2 + \|f\|_\infty^{2p-2}\big]. \tag{31}$$

Observe that $\chi_\varepsilon \leq \varepsilon^{p-2}$ and $(\chi_\varepsilon|\nabla v|_a)^{p'} \leq |\nabla v|_{a,\varepsilon}^{p-1}|\nabla v|_a$. The Young inequality implies that

$$\|\chi_\varepsilon|\nabla u|_a\phi\|_2^2 \leq \delta T_{\varepsilon,a} + C_{p,\delta}\|\phi\|_{2p}^{2p}, \tag{32}$$

for all $\phi \in L^{2p}$ and any positive δ. We apply (32) to (31), choose δ small enough and obtain

$$\kappa_1(\varepsilon_0, p)\|f\|_\infty^2 \frac{d}{dt}\| [\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2} + \kappa_2(\varepsilon_0, p)T_{\varepsilon,a}$$
$$\leq C_{\varepsilon_0,p}\Big[\|f\|_\infty^{2p-2}\|\Xi\|_2^2 + \|f\|_\infty^{2p}\big(\sup_d \|\Upsilon_d\|_{2p}^{2p} + 1\big)\Big]. \tag{33}$$

In order to complete the proof of (5) we apply Proposition 3.2. to estimate $\|\Xi\|_s^s$ in (30) and (33) and integrate the derived inequalities from 0 to t.

When $p = 2$ we apply the Hölder inequality, Proposition 3.2 and (5) to (26) in order to obtain (6).

If $p < 2$ it follows from the Young inequality that

$$\||AD^2u|_{HS(a)}f\|_p^p \leq \varepsilon^{-p}\| [\nabla u]_{\varepsilon,a}\|_{2p}^{2p} + C_p\varepsilon^{2-p}I_{\varepsilon,a}. \tag{34}$$

We employ the Young and the Hölder inequalities to estimate the first term in the RHS of (34).

$$\|[\nabla u]_{\varepsilon,a}\|_{2p}^{2p} \leq T_{\varepsilon,a} + \varepsilon^2\| [\nabla u]_{\varepsilon,a}^{p-1}\|_2^2 \leq T_{\varepsilon,a} + (1/2)\|[\nabla u]_{\varepsilon,a}\|_{2p}^{2p} + C_p\varepsilon^{2p}. \tag{35}$$

We take $\varepsilon := \|f\|_\infty$. Now making successive use of (34), (35), Lemma 4.3, (26), (33) and Proposition 3.2 one completes the proof. □

5 Example

Let $X = \mathbb{R}^{\mathbb{N}}$, $H_0 = l^2$, $H_+ = l^2_{\gamma_k}$ and $H_- = l^2_{\gamma_k^{-1}}$ with $(\gamma_k)_{k\in\mathbb{N}} \subset (0, \infty)$, where $l^2_{\gamma_k}$ and $l^2_{\gamma_k^{-1}}$ are described in Section 2.

Let $(s_k)_{k\in\mathbb{N}} \subset (0, \infty)$. We define an operator S on $\mathbb{R}^{fin}$ in H_0 by setting $s_{jk} := \delta_{jk}s_k$, $j, k \in \mathbb{N}$. This operator is positive. Let μ_S stand for the Gaussian measure with correlation operator S. Recall that

$$\mu_S = \prod_{k\geq 1} e^{-\frac{x_k^2}{2s_k}} \frac{dx_k}{\sqrt{2\pi s_k}}.$$

Let μ be a probability measure on $\mathbb{R}^\infty$ given by

$$\mu := \prod_{k\geq 1} \frac{1}{m_k\Gamma(m_k/2)} x_k^{m_k} e^{-\frac{x_k^2}{2s_k}} \frac{dx_k}{\sqrt{2s_k}}.$$

It is easy to see that, for $k \in \mathbb{N}$, $\beta_k^\mu(x) = -s_k^{-1}x_k + m_k|x_k|^{-1} = \xi_k^\mu(x) + \eta_k^\mu(x)$ with $\xi_k^\mu(x) := -s_k^{-1}x_k$ and $\eta_k^\mu(x) := m_k|x_k|^{-1}$, $x \in \mathbb{R}^{\mathbb{N}}$.

For $\delta > 0$ we introduce functions

$$a_{jk}(x) := \delta_{jk}\frac{x_k^2 + \delta}{x_k^2 + 1}, \quad x \in \mathbb{R}^\infty.$$

Note that for every $N \in \mathbb{N}$ the matrix $(a_{jk})_{j,k=1}^N$ is cylindric, smooth and uniformly elliptic.

It is obvious that $\xi^\mu_{a;k} = -\dfrac{(x_k^2+\delta)x_k}{(x_k^2+1)s_k} + \dfrac{2(1-\delta)x_k}{(x_k^2+1)^2}$, $k \in \mathbb{N}$.

A straightforward computation shows that $|\eta^\mu - P_N\eta^\mu|_a \to 0$ in $L^{2p} := L^{2p}(\mathbb{R}^{\mathbb{N}}, \mu)$ as $N \to \infty$, provided $m_k = 2k + 2p - 1$, $k \in \mathbb{N}$ and numbers $(s_k)_{k\in\mathbb{N}}$ are such that

$$\sum_{k\geq 1} m_k^{\frac{2p-1}{2p}} \left(\frac{\Gamma(m_k/2 + 1 - p)}{\Gamma(m_k/2)}\right)^{\frac{1}{2p}} s_k^{\frac{m_k/2-p}{2p}} < \infty.$$

For example this condition is satisfied if we take the sequence $(s_k)_{k\in\mathbb{N}}$ to be bounded.

One can verify directly that $\upsilon_N \leq 2|1-\delta|$ and $\nu_N \leq \dfrac{|1-\delta|}{\delta}$, $N \in \mathbb{N}$.

We choose $\xi_k^n := -\dfrac{(x_k^2+\delta)x_k}{(x_k^2+1)s_k} + \dfrac{2(1-\delta)x_k}{(x_k^2+1)^2}$, $n \in \mathbb{N}$, $k = 1, \dots, n$. Then condition (i)(a) of Theorem 2.2 is satisfied. A straightforward computation shows that that if $\delta \in (0, 3] \cup [9, \infty)$ then condition (i)(b) holds for arbitrary positive $(s_k)_{k\in\mathbb{N}}$. of Theorem 2.2. It is also readily seen that the sequence

$\hat{\xi}_k^m := -s_k^{-1} x_k$, $m \in \mathbb{N}$, $k = 1, \ldots, m$ satisfies condition (ii(b)) with $\varepsilon_0 = 0$ and $c(\varepsilon_0) = 0$.

Hence, by Theorem 2.2 the operator $\mathcal{L} \restriction \mathcal{FC}_b^\infty$ is strongly unique in L^p for all $p > 3/2$.

Appendix: Auxiliary inequalities

Let $v \in \mathcal{FC}_b^{2,u}(\mathbb{R}^K)$ and quantities $T_{\varepsilon,a}$, $I_{\varepsilon,a}$ and $J_{\varepsilon,a}$ be defined as in the previous section (with v replacing the solution u of (4)). (Recall that then $\chi_\varepsilon = [\nabla v]_{\varepsilon,a}^{p-2}$). We use the same summation convention as in section 4.

Below we present several estimates which are used in the proof of Proposition 3.3. It is noteworthy that Lemma A.1 is an extension of the Gagliardo–Nirenberg inequality to the case when the matrix of coefficients is not the identity. Let us stress that the function v need not be a solution to a Cauchy problem. Lemma A.2 is an elementary statement which is needed in the proof of Lemma 4.3.

Lemma A 1. *Put $\Delta_a v := \sum_{k=1}^d \sum_j (\nabla_k + \beta_k^\mu)(a_{kj} \nabla_j v)$, $d \geq K$. Then*

$$T_{0,a} = \| \, |\nabla v|_a \|_{2p}^{2p} \leq 2 \|v\|_\infty^2 \| \, |\nabla v|_a^{p-2} \Delta_a v \|_2^2 + 2(p-1)^2 \|v\|_\infty^2 J_{0,a},$$

for $p \geq 2$ ($J_{0,a} := \lim_{\varepsilon \to 0} J_{\varepsilon,a} = 4(p-1)^{-2} \| \, |\nabla |\nabla u|_a^{p-1}|_a \|_2^2$), and

$$T_{\varepsilon,a} \leq \varepsilon^{2p-2} \|v\|_2^2 + 2(\|v\|_\infty^2 + \varepsilon^2) \|\chi_\varepsilon \Delta_a v\|_2^2 + 2(p-1)^2 \|v\|_\infty^2 J_{\varepsilon,a},$$

for $1 \leq p < 2$.

Lemma A 2. *If $3 - \frac{3}{1+\sqrt{1+3\varepsilon_0}} < p < \frac{2}{\varepsilon_0}$ then there exist positive constants $K(\varepsilon_0, p)$ and $C_{\varepsilon_0,p}$ such that for sufficiently small $\delta > 0$*

$$\begin{aligned}(1-\varepsilon_0) I_{\varepsilon,a} + \frac{2(p-2) - \varepsilon_0 (p-2)^2}{4} J_{\varepsilon,a} - \varepsilon_0 |p-2| \sqrt{I_{\varepsilon,a} J_{\varepsilon,a}} - \delta I_{\varepsilon,a} - c\delta (p-2)^2 J_{\varepsilon,a} \\ \geq K(\varepsilon_0, p) J_{\varepsilon,a} - C_{\varepsilon_0,p} \sup_d \|\chi_\varepsilon |\nabla v|_a v_d\|_2^2. \qquad (36)\end{aligned}$$

Moreover, if $p < 2$ then

$$\begin{aligned}(1-\varepsilon_0) I_{\varepsilon,a} + \frac{2(p-2) - \varepsilon_0 (p-2)^2}{4} J_{\varepsilon,a} - \varepsilon_0 |p-2| \sqrt{I_{\varepsilon,a} J_{\varepsilon,a}} - \delta I_{\varepsilon,a} - c\delta (p-2)^2 J_{\varepsilon,a} \\ \geq \hat{K}(\varepsilon_0, p) I_{\varepsilon,a} - C_{\varepsilon_0,p} \sup_d \|\chi_\varepsilon |\nabla v|_a v_d\|_2^2.\end{aligned}$$

Proof of Lemma A.1. Integration by parts yields

$$T_{\varepsilon,a} = \langle v [\nabla v]_{\varepsilon,a}^{2p-2}, \, \Delta_a v \rangle - (p-1) \langle v [\nabla v]_{\varepsilon,a}^{2p-4} (\nabla [\nabla v]_{\varepsilon,a}^2, \nabla v)_a \rangle. \qquad (37)$$

Note that $\nabla[\nabla v]^2_{\varepsilon,a} = \nabla|\nabla v|^2_a$. Therefore, $|(\nabla[\nabla v]^2_{\varepsilon,a}, \nabla v)_a| \leq |\nabla v|_a|\nabla|\nabla v|^2_a|_a$ by the Schwarz inequality. Thus, the absolute value of the last term in the RHS of (37) does not exceed

$$|p-1|\|v\|_\infty\langle[\nabla v]^{2p-4}_{\varepsilon,a}|\nabla v|_a|\nabla|\nabla v|^2_a|_a\rangle \leq \frac{1}{4}T_{\varepsilon,a} + (p-1)^2\|v\|^2_\infty J_{\varepsilon,a}. \tag{38}$$

Consider the first term in the RHS of (37). Using the Young inequality we estimate it by

$$\begin{aligned}&\langle\chi_\varepsilon\varepsilon|v|,\, \chi_\varepsilon\varepsilon|\Delta_a v|\rangle + \langle\chi_\varepsilon|\nabla v|^2_a,\, \chi_\varepsilon|\Delta_a v||v|\rangle \\ \leq &\frac{1}{4}\|\chi_\varepsilon|\nabla v|^2_a\|^2_2 + \frac{\varepsilon^2}{4}\|\chi_\varepsilon v\|^2_2 + (\|v\|^2_\infty + \varepsilon^2)\|\chi_\varepsilon\Delta_a v\|^2_2.\end{aligned} \tag{39}$$

If $p \geq 2$ then we can pass to the limit as $\varepsilon \to 0$ in (39). This yields the first assertion. If $p < 2$ we observe that $\varepsilon^2\|\chi_\varepsilon v\|^2_2 \leq \varepsilon^{2p-2}\|v\|^2_2$ and $|\nabla v|^2_a\chi_\varepsilon \leq [\nabla v]^{p-1}_{\varepsilon,a}|\nabla v|_a$ and combine (37)-(39) in order to obtain the second statement. $\square$

Proof of Lemma A.2. Let first $p \geq 2$. Setting $r := \left(\frac{I_{\varepsilon,a}}{J_{\varepsilon,a}}\right)^{\frac{1}{2}}$ we rewrite the LHS of (36) as follows

$$\frac{J_{\varepsilon,a}}{4}\big(4(1-\varepsilon_0-\delta)r^2 - 4\varepsilon_0|p-2|r + 2(p-2) - (\varepsilon_0+4c\delta)(p-2)^2\big) = \frac{J_{\varepsilon,a}}{4}F(r). \tag{40}$$

We need to find all p such that $F(r) > 0$, $r \geq 0$. A direct computation shows that if $p \in [2, \frac{2}{\varepsilon_0})$ then the discriminant of the quadratic function F is negative, provided δ is small enough.
Now we assume that $p < 2$. The following inequality holds.

$$J_{\varepsilon,a} \leq (4+\delta_1)I_{\varepsilon,a} + C_{\delta_1}\|\chi_\varepsilon|\nabla v|_a\Upsilon\|^2_2, \quad \forall\delta_1 > 0. \tag{41}$$

We give the proof of (41) below.
Making use of the Cauchy inequality and (41) we estimate the LHS of (36) from below by

$$\begin{aligned}&\Big[2(p-2) + 1 - \varepsilon_0 + 2\varepsilon_0(p-2) - \varepsilon_0(p-2)^2 - C_{\varepsilon_0,p}\delta\Big]I_{\varepsilon,a} - C_\delta\|\chi_\varepsilon|\nabla v|_a\Upsilon\|^2_2 \\ = &[G(p) - C_{\varepsilon_0,p}\delta]I_{\varepsilon,a} - C_\delta\|\chi_\varepsilon|\nabla v|_a\Upsilon\|^2_2,\end{aligned}$$

for every $\delta > 0$. It is easy to verify that $G(p) > 0$ provided $p \in (3 - \frac{3}{1+\sqrt{1+3\varepsilon_0}}, 2)$.
Hence $\hat{K}(\varepsilon_0, p) := G(p) - C_{\varepsilon_0,p}\delta > 0$ provided δ is small enough. This proves the second assertion. Inequality (36) now follows from (41).

Now we prove inequality (41). First we notice that for any $f, g, h \in \mathbb{R}^d$ the following inequality holds

$$\sum_{i,k,j,l=1}^{d} f_i\, a_{il}(\nabla_l a_{jk}) g_j h_k \leq \upsilon_d |f|_a |g|_a |h|_a. \tag{42}$$

We observe that

$$\nabla_k |\nabla v|_a^2 = \sum_{j,l\geq 1} 2(\nabla_{kj}^2 v) a_{jl} \nabla_l v + (\nabla_k a_{jl}) \nabla_j v \nabla_l v. \tag{43}$$

Hence,

$$|\nabla|\nabla v|_a^2|_a^2 \leq 2|AD^2 v|_{HS(a)}^2 |\nabla|\nabla v|_a^2|_a |\nabla v|_a + \upsilon_d |\nabla|\nabla v|_a^2|_a |\nabla v|_a^2.$$

This yields (41). ☐

Finally, we prove inequality (23). Since $\nabla[\nabla v]_{\varepsilon,a}^2 = \nabla|\nabla v|_a^2$, one gets

$$\begin{aligned}&\sum_{i,j,k,l} a_{kj} \nabla_k v \nabla_j (a_{li} \nabla_i v) \nabla_l [\nabla v]_{\varepsilon,a}^{2p-4}\\ =&(p-2)[\nabla v]_{\varepsilon,a}^{2p-6} \sum_{i,j,k,l} \left[a_{kj} \nabla_k v (\nabla_j a_{li}) \nabla_i v \nabla_l |\nabla v|_a^2 + a_{li} a_{kj} \nabla_k v (\nabla_{ji}^2 v) \nabla_l |\nabla v|_a^2\right].\end{aligned}$$

Using (43) we obtain

$$\sum_{i,j,k,l} a_{li} a_{kj} \nabla_k v \nabla_{ji}^2 v \nabla_l |\nabla v|_a^2 = \frac{1}{2} |\nabla|\nabla v|_a^2|_a^2 - \frac{1}{2} \sum_{i,j,k,l} a_{kj} (\nabla_k |\nabla v|_a^2)(\nabla_j a_{li}) \nabla_i v \nabla_l v.$$

Recall that $J_{\varepsilon,a} = 4\langle [\nabla v]_{\varepsilon,a}^{2p-6} |\nabla v|_a^2 |\nabla|\nabla v|_a|_a^2\rangle$. In order to estimate the remaining terms we employ (42):

$$|\sum_{i,j,k,l} \left[a_{kj} \nabla_k v (\nabla_j a_{li}) \nabla_i v \nabla_l |\nabla v|_a^2\right| \leq \upsilon_d |\nabla|\nabla v|_a^2|_a |\nabla v|_a^2 \leq \upsilon_d |\nabla v|_a |\nabla|\nabla v|_a^2|_a [\nabla v]_{\varepsilon,a}.$$

The last term is estimated in the same manner. This yields (23).

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