J. Math. Pures Appl. 00, 0 (2000) 1–45

S0021-7824(00)01187-9/FLA

ELLIPTIC EQUATIONS FOR INVARIANT MEASURES ON FINITE AND INFINITE DIMENSIONAL MANIFOLDS

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Received 30 June 2000

ABSTRACT. – We obtain sufficient conditions in terms of Lyapunov functions for the existence of invariant measures for diffusions on finite-dimensional manifolds and prove some regularity results for such measures. These results are extended to countable products of finite-dimensional manifolds. We introduce and study a new concept of weak elliptic equations for measures on infinite-dimensional manifolds. Then we apply our results to Gibbs distributions in the case where the single spin spaces are Riemannian manifolds. In particular, we obtain some a priori estimates for such Gibbs distributions and prove a general existence result applicable to a wide class of models. We also apply our techniques to prove absolute continuity of invariant measures on the infinite-dimensional torus, improving a recent result of A.F. Ramirez. Furthermore, we obtain a new result concerning the question whether invariant measures are Gibbsian. © 2000 Éditions scientifiques et médicales Elsevier SAS

Keywords: Invariant measures, Diffusions on manifolds, Elliptic equations for measures, Lyapunov functions, Gibbs distributions, Logarithmic gradients

AMS classification: Primary: 58G32, 58B99, 82B21; Secondary: 53C21, 53C80, 58G03, 60J60, 82B05

1. Introduction

Let M be a complete connected Riemannian manifold of dimension $d \geqslant 2$ without boundary and let λ_M be the Riemannian volume element on M. Let us consider the elliptic operator

$$L_Z\varphi := (\Delta + Z)\varphi := \Delta\varphi + \langle Z, \nabla\varphi\rangle,$$

where Δ is the Laplacian and Z is a measurable vector field on M. We say that a Radon measure μ on M satisfies the weak elliptic equation

$$L_Z^*\mu = 0 \tag{1.1}$$

if $|Z| \in L^1_{\mathrm{loc}}(\mu)$ and

$$\int_M L_Z f \,\mathrm{d}\mu = 0, \quad \forall f \in C_0^\infty(M), \tag{1.2}$$

where $C_0^\infty(M)$ is the space of all infinitely differentiable compactly supported functions on M. Equation (1.1) is satisfied for invariant measures of a diffusion process with drift $Z/2$. In this work, we study the global behavior of solutions of equation (1.1) and obtain some sufficient conditions for their existence. Then we apply our results to Gibbs distributions in the case where the single spin spaces are Riemannian manifolds. In particular, we obtain some a priori estimates for such Gibbs distributions and prove an existence result that applies to a wide class of models.

We recall that it has been shown in [12] and [15] that in the case $M=\mathbb{R}^d$, one has $\mu = p\,\mathrm{d}x$ with $\sqrt{p}\in H^{2,1}(\mathbb{R}^d,\mathrm{d}x)$ provided that a finite nonnegative measure μ satisfies (1.1) with $|Z|\in L^2(\mu)$. In addition,

$$\int_{\mathbb{R}^d} \left|\nabla\sqrt{p}\right|^2 \mathrm{d}x \leqslant \frac{1}{4}\int_{\mathbb{R}^d} |Z|^2\,\mathrm{d}\mu. \tag{1.3}$$

Local Sobolev regularity results for arbitrary solutions of (1.1) have been obtained in [11,12] and [13].

Concerning the existence and local regularity of solutions, the following result has been proved in [16] and [11,13], respectively. Recall that a function V on a topological space is said to be compact if the sets $\{V\leqslant c\}$, $c\in\mathbb{R}^1$, have compact closures.

THEOREM 1.1. – *Assume that there exists $\alpha>d$ such that $|Z|\in L^{\alpha}_{\mathrm{loc}}(\lambda_M)$.*

(i) *If there exists a compact function $V\in C^2(M)$ with $\lim_{r\to\infty}\sup_{\{V\geqslant r\}} L_Z V=-\infty$, where we set as usual $\sup\emptyset=-\infty$, then there exists a probability measure μ solving* (1.1).

(ii) *Any Radon measure μ solving* (1.1) *admits a continuous density $p\in W^{\alpha,1}_{\mathrm{loc}}(\lambda_M)$ with respect to λ_M. If μ is nonnegative and not identically zero, then p is strictly positive.*

In the case of a Riemannian manifold, it is natural to construct the function V by using the Riemannian distance function, which is related to various geometrical properties of the manifold. Unfortunately, the square of the distance function may fail to be smooth whenever the cut locus is nonempty, so that the above result from [16] is no longer applicable to such V. Certainly, there is no problem if the manifold possesses a pole (i.e., a point o such that $\exp_o : T_oM\to M$ is a diffeomorphism). Let us fix $o\in M$ and let $\varrho(x):=\varrho(x,o)$ be the Riemannian distance between x and o. The following is a direct consequence of Theorem 1.1.

COROLLARY 1.2. – *Assume that o is a pole and $|Z|\in L^{\alpha}_{\mathrm{loc}}(\lambda_M)$ for some $\alpha>d$. Suppose that there exists $F\in C^2[0,\infty)$ such that*

$$\lim_{r\to\infty} F(r)=\infty \quad \textit{and} \quad \lim_{\varrho\to\infty}\left[F'(\varrho)L_Z\varrho+F''(\varrho)\right]=-\infty. \tag{1.4}$$

Then the assertion of Theorem 1.1(i) *is valid. Condition* (1.4), *in particular, holds for $F(r)=r^2(\log(r+1))$ provided*

$$\lim_{\varrho\to\infty}(\varrho L_Z\varrho+1)\log(\varrho+1)=-\infty. \tag{1.5}$$

The first goal of this paper is to extend (1.3) and Corollary 1.2 to general finite-dimensional Riemannian manifolds. Our second objective is to consider infinite products of Riemannian manifolds. In particular, we introduce and study a new concept of a weak elliptic equation for measures on infinite-dimensional manifolds. Applications to Gibbs distributions on lattices of manifolds are obtained. In the case where $M=\mathbb{R}^d$, the results in this paper (announced in [19]) extend the results from [3] and [4] (for further development in the flat case, see also [17]).

The principal results in this work can be summarized as follows:

(1) (Cf. Theorem 2.2) If M is a finite-dimensional Riemannian manifold with Ricci curvature bounded below and positive injectivity radius and if a probability measure μ on M satisfies (1.1)

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with $|Z| \in L^2(\mu)$, then (1.3) holds. In particular, this is true if μ is an invariant probability of a diffusion process with drift Z such that $|Z| \in L^2(\mu)$.

(2) (Cf. Theorem 3.1) Corollary 1.2 is valid for general Riemannian manifolds.

(3) (Cf. Theorem 4.2 and Theorem 4.3) An analogue of (1.3) is valid in the infinite-dimensional case when M is replaced by a countable product of finite-dimensional Riemannian manifolds.

(4) (Cf. Propositions 5.2, 5.3 and 6.8, Theorem 5.5) Analogues of Theorem 1.1(i) and Corollary 1.2 are valid for a countable product of finite-dimensional Riemannian manifolds. In particular, the corresponding results enable us to construct infinite volume Gibbs measures for a broad class of lattice models with Riemannian manifolds as state spaces.

(5) (Cf. Theorem 8.3) A priori estimates are obtained for probability measures solving equation (1.1) in infinite dimensions; these estimates hold, e.g., for the above mentioned Gibbs measures.

In addition, as an application of our methods, in Section 7 we, for example, extend some results on finite range vector fields obtained by R. Holley and D. Stroock [35], J. Fritz [28,29] (cf. Theorem 7.8), and A. Ramirez [42] (cf. Theorem 7.4). In particular, the previously known fact that in dimensions one and two every stationary measure for the stochastic system associated with a Gibbs measure is also Gibbsian is extended to considerably more general state spaces (non-compact Riemannian manifolds) and more general interactions.

It would be interesting to study the objects considered in this work in the case of other infinite-dimensional manifolds such as loop spaces or more general manifolds of mappings. In particular, existence and properly defined regularity of solutions of the equation $L_Z^*\mu = 0$ as well as the non-uniqueness phenomena are important problems.

Finally, we would like to draw attention to Theorem 3.4 below which we obtain as a consequence of the above mentioned Theorem 3.1 and which extends a recent result by A.-B. Cruzeiro and P. Malliavin [24], proved, however, by completely different means (cf. Corollary 3.6 below).

2. Regularity of solutions

The class of all C^k-vector fields on M is denoted by $\mathrm{Vec}^k(M)$, $k = 0, 1, \ldots, \infty$. The sub-indices 0 and b distinguish the fields with compact supports and bounded derivatives, respectively. Let $\mathcal{P}(M)$ be the set of all Borel probability measures on M. We shall define the Sobolev space $H^{2,1}(\lambda_M)$ as the closure of $C_0^\infty(M)$ with respect to the norm $\|\cdot\|_{H^{2,1}}$ given by

$$\|\psi\|_{H^{2,1}}^2 = \int_M |\psi|^2 \, \mathrm{d}\lambda_M + \int_M |\nabla\psi|^2 \, \mathrm{d}\lambda_M.$$

There exists a nonpositive self-adjoint operator Δ with domain $\mathcal{D}(\Delta) \subset H^{2,1}(\lambda_M)$ such that

$$-\int_M \psi \Delta\varphi \, \mathrm{d}\lambda_M = \int_M \langle\nabla\psi, \nabla\varphi\rangle \, \mathrm{d}\lambda_M, \quad \forall \psi \in H^{2,1}(\lambda_M),\ \varphi \in \mathcal{D}(\Delta).$$

Let us put $H^{2,2}(\lambda_M) := \mathcal{D}(\Delta)$. It is known that Δ on $C_0^\infty(M)$ is the usual Laplace–Beltrami operator on M (defined locally in terms of the metric tensor, see [46]). In addition, $H^{2,1}(\lambda_M)$ coincides with the Sobolev class $W^{2,1}(\lambda_M)$ of all functions $f \in L^2(\lambda_M)$ such that f belongs to $W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d)$ in local charts and $|\nabla f| \in L^2(\lambda_M)$ (see[46,8]). We recall that $W^{p,1}_{\mathrm{loc}}(\mathbb{R}^d)$ is the class of all functions that are locally integrable of order p together with their generalized partial derivatives of the first order. The class $H^{2,2}(\lambda_M)$ coincides with the collection of all $f \in L^2(\lambda_M)$

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such that Δf in the distribution sense belongs to $L^2(\lambda_M)$ (see [46]). Let $W^{p,1}_{\mathrm{loc}}(\lambda_M)$, $p \geqslant 1$, be the class of all functions f on M that belong to $W^{p,1}_{\mathrm{loc}}(\mathbb{R}^d)$ in local charts. We refer to [46] concerning the definition of the heat semigroup $(P_t)_{t\geqslant 0}$ on $L^2(\lambda_M)$; its characteristic property is that $\partial_t P_t\psi = \Delta P_t\psi$ for all $\psi \in C_0^\infty(M)$.

We denote the space of all functions f on M that are locally λ_M-integrable of order p by $L^p_{\mathrm{loc}}(\lambda_M)$.

Let us set

$$\widehat{C}_b^\infty(M) = \Big\{ f \in C^\infty(M):\ \sup_M \big[\big|\Delta^n f\big| + \big|\nabla \Delta^n f\big|\big] < \infty,\ \forall n \geqslant 0 \Big\}.$$

Given a nonnegative Borel measure μ on a Riemannian manifold M, we shall denote by $L^2(\mu, \mathrm{Vec}(M))$ the Hilbert space of all μ-square integrable vector fields on M with its natural inner product

$$(X, Y)_2 = \int \langle X, Y\rangle\, \mathrm{d}\mu.$$

Let $\Gamma(\mu)$ be the closure of the set $\{\nabla\psi,\ \psi \in C_0^\infty(M)\}$ in $L^2(\mu, \mathrm{Vec}(M))$.

For the rest of this section we fix a Borel-measurable vector field Z on M.

LEMMA 2.1. – *Let $\mu \in \mathcal{P}(M)$ have a density p such that $\sqrt{p} \in W^{2,1}_{\mathrm{loc}}(\lambda_M)$. Suppose that $f \in W^{1,1}_{\mathrm{loc}}(\lambda_M)$ and that $|\nabla f| \in L^2(\mu)$. Then:*

(i) *$\nabla f \in \Gamma(\mu)$.*

(ii) *If μ satisfies* (1.2) *and if the set $\{|\nabla f| \neq 0\}$ is relatively compact, then*

$$\int_M \left\langle \nabla f, Z - \frac{\nabla p}{p}\right\rangle \mathrm{d}\mu = 0.$$

Proof. – (i) Let $\theta_r \in C_b^\infty(\mathbb{R}^1)$, $r \in \mathbb{N}$, be such that $\theta_r(t) = t$ if $|t| \leqslant r$ and $\sup|\theta_r'| \leqslant 2$. By considering compositions $\theta_r \circ f$, one reduces the claim to bounded f. Moreover, in the case when f is bounded, by considering products ζf with $\zeta \in C_0^\infty(M)$ such that $0 \leqslant \zeta \leqslant 1$, $|\nabla\zeta| \leqslant 2$, and $\zeta = 1$ on a big ball V (such a function exists for every ball V, see [30] or [46]), we reduce the claim to the case where $f = 0$ outside a compact set K. Let $\zeta_j \in C_0^\infty(M)$, $j \leqslant m$, be a finite collection of functions with supports in local charts such that $\sum_j \zeta_j = 1$ on K. Then it suffices to prove our assertion for each $\zeta_j f$. Hence we may assume that $M = \mathbb{R}^d$ (with a possibly different Riemannian metric, however) and that f has a compact support in $\mathbb{R}^d$. Moreover, the condition that f has a compact support enables us to consider $\mathbb{R}^d$ with the standard inner product. Then it remains to refer to [43] (where the desired result was established in the proof of Theorem 3.1) or to [20, Theorem 2.7].

(ii) Since the set $\{|\nabla f| \neq 0\}$ is relatively compact and because of (i), we can find $f_n \in C_0^\infty(M)$, $n \in \mathbb{N}$, such that the set $\bigcup_{n=1}^\infty \{f_n \neq 0\}$ is relatively compact and $|\nabla f_n - \nabla f| \to 0$ in $L^2(\mu)$ as $n \to \infty$. Therefore, integrating by parts we obtain by (1.2)

$$\int_M \left\langle \nabla f, Z - \frac{\nabla p}{p}\right\rangle \mathrm{d}\mu = \lim_{n\to\infty} \int_M \left\langle \nabla f_n, Z - \frac{\nabla p}{p}\right\rangle \mathrm{d}\mu = \lim_{n\to\infty} \int_M L_Z f_n\, \mathrm{d}\mu = 0. \qquad \square$$

THEOREM 2.2. – *Assume that the heat semigroup $(P_t)_{t\geqslant 0}$ on M sends $L^1(\lambda_M)$ to $L^\infty(\lambda_M)$ and satisfies the following condition: there is a function $C : [0,1] \to \mathbb{R}_+$ with $\lim_{t\to 0} C(t) = 1$*

such that

$$|\nabla P_t\varphi|^2 \leqslant C(t)P_t\big(|\nabla\varphi|^2\big), \quad \forall\varphi\in C_0^\infty(M),\ \forall t\in[0,1]. \tag{2.1}$$

Let $\mu\in\mathcal{P}(M)$ *be such that* $|Z|\in L^2(\mu)$ *and*

$$\int_M L_Z f\,\mathrm{d}\mu = 0, \quad \forall f\in\widehat{C}_b^\infty(M). \tag{2.2}$$

Then $\mu = p\,\lambda_M$, *where* $\sqrt{p}\in H^{2,1}(\lambda_M)$ *and*

$$\int_M \frac{|\nabla p|^2}{p}\,\mathrm{d}\lambda_M \leqslant \int_M |Z|^2\,\mathrm{d}\mu. \tag{2.3}$$

In addition, $\nabla p/p$, *where we set* $\nabla p/p = 0$ *on* $\{p=0\}$, *is the orthogonal projection of* Z *to* $\Gamma(\mu)$ *in* $L^2(\mu,\mathrm{Vec}(M))$.

Proof. – It follows by Theorem 1(ii) and Remark 4(iii) in [11] that the measure μ has a nonnegative density $p\in W^{1,1}_{\mathrm{loc}}(\lambda_M)$. Let

$$f_\varepsilon(x) := P_\varepsilon p(x).$$

Since $p\in L^1(\lambda_M)$ and $f_\varepsilon = P_{\varepsilon/2}P_{\varepsilon/2}p\in P_{\varepsilon/2}\mathcal{D}(\Delta)$, one has $f_\varepsilon\in H^{2,1}(\lambda_M)$ (recall that P_t sends $L^1(\lambda_M)$ to $L^1(\lambda_M)\cap L^\infty(\Lambda_M)$). For every $\varphi\in C_0^\infty(M)$, we have $P_\varepsilon\varphi\in\widehat{C}_b^\infty(M)$, hence

$$\begin{aligned}\int_M \langle\nabla\varphi,\nabla f_\varepsilon\rangle\,\mathrm{d}\lambda_M &= -\int_M \Delta\varphi\, f_\varepsilon\,\mathrm{d}\lambda_M = -\int_M P_\varepsilon(\Delta\varphi)p\,\mathrm{d}\lambda_M\\ &= -\int_M \Delta(P_\varepsilon\varphi)\,\mathrm{d}\mu = \int_M \big\langle Z,\nabla(P_\varepsilon\varphi)\big\rangle\,\mathrm{d}\mu.\end{aligned} \tag{2.4}$$

Since $C_0^\infty(M)$ is dense in $H^{2,1}(\lambda_M)$, we obtain

$$\int_M \langle\nabla\varphi,\nabla f_\varepsilon\rangle\,\mathrm{d}\lambda_M = \int_M \big\langle Z,\nabla(P_\varepsilon\varphi)\big\rangle\,\mathrm{d}\mu, \quad \forall\varphi\in H^{2,1}(\lambda_M). \tag{2.5}$$

Indeed, let $\{\varphi_j\}\subset C_0^\infty(M)$ converge to φ in $H^{2,1}(\lambda_M)$. Then by (2.1)

$$\begin{aligned}\int_M \big|\nabla(P_\varepsilon\varphi)-\nabla(P_\varepsilon\varphi_j)\big|^2\,\mathrm{d}\mu &\leqslant C(\varepsilon)\int_M P_\varepsilon|\nabla\varphi-\nabla\varphi_j|^2 p\,\mathrm{d}\lambda_M\\ &\leqslant C(\varepsilon)\|P_\varepsilon p\|_{L^\infty}\int_M |\nabla\varphi-\nabla\varphi_j|^2\,\mathrm{d}\lambda_M,\end{aligned}$$

whence (2.5) follows by the Cauchy inequality. Let $J := \int_M |Z|^2\,\mathrm{d}\mu$. We obtain from (2.5) applied to $\varphi := \log(f_\varepsilon+\delta)-\log\delta\in H^{2,1}(\lambda_M)$ with $\delta>0$ that

$$\int_M \frac{|\nabla f_\varepsilon|^2}{f_\varepsilon+\delta}\,\mathrm{d}\lambda_M = \int_M \langle Z,\nabla P_\varepsilon\varphi\rangle\,\mathrm{d}\mu$$

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$$\leqslant \sqrt{J\int_M |\nabla P_\varepsilon\varphi|^2\,\mathrm{d}\mu} \leqslant \sqrt{C(\varepsilon)J\int_M P_\varepsilon|\nabla\varphi|^2\,\mathrm{d}\mu}$$
$$= \sqrt{C(\varepsilon)J\int_M |\nabla\varphi|^2 P_\varepsilon p\,\mathrm{d}\lambda_M} \leqslant \sqrt{C(\varepsilon)J\int_M \frac{|\nabla f_\varepsilon|^2}{f_\varepsilon+\delta}\,\mathrm{d}\lambda_M}.$$

Therefore,

$$\int_M \frac{|\nabla f_\varepsilon|^2}{f_\varepsilon+\delta}\,\mathrm{d}\lambda_M \leqslant C(\varepsilon)\int_M |Z|^2\,\mathrm{d}\mu.$$

By letting $\delta\to 0$ we obtain

$$\int_M \frac{|\nabla f_\varepsilon|^2}{f_\varepsilon}\,\mathrm{d}\lambda_M \le C(\varepsilon)\int_M |Z|^2\,\mathrm{d}\mu. \tag{2.6}$$

Set $g_n=\sqrt{f_{1/n}}$. By (2.6), the sequence $\{g_n\}$ is bounded in $H^{2,1}(\lambda_M)$. Since the embedding $W^{2,1}(U;\lambda_M)\subset L^2(U;\lambda_M)$ is compact for every ball U with compact closure in M, there exists a subsequence $\{g_{n_k}\}$ that converges λ_M-a.e. and strongly in $L^2(U;\lambda_M)$ and weakly in $W^{2,1}(U;\lambda_M)$ to a function in $W^{2,1}(U;\lambda_M)$. Since the measures $f_{1/n}\lambda_M$ converge weakly to μ as $n\to\infty$, we obtain that $g_{n_k}\to\sqrt{p}$ λ_M-a.e. and that $\sqrt{p}\in H^{2,1}(\lambda_M)$. Estimate (2.3) follows from (2.6), since $C(\varepsilon)\to 1$ as $\varepsilon\to 0$. Finally, $\nabla p/p\in\Gamma(\mu)$ by Lemma 2.1. Indeed, for every $n\in\mathbb{N}$ one has $\psi_n:=2\log(\sqrt{p}+n^{-1})\in W^{2,1}_{\mathrm{loc}}(\lambda_M)$. Clearly, $|\nabla\psi_n-2\nabla\sqrt{p}/\sqrt{p}|\to 0$ in $L^2(\mu)$. It remains to note that $\nabla p/p=2\nabla\sqrt{p}/\sqrt{p}$ μ-a.e. and that $\nabla\psi_n\in\Gamma(\mu)$ by Lemma 2.1. Therefore, $\nabla p/p$ is the orthogonal projection of Z in $L^2(\mu,\mathrm{Vec}(M))$, since $Z-\nabla p/p$ is orthogonal in $L^2(\mu,\mathrm{Vec}(M))$ to every $\nabla\psi$, $\psi\in C_0^\infty(M)$, hence to $\Gamma(\mu)$. □

It is worth noting that unlike Theorem 1.1, Theorem 2.2 is not valid locally: it can happen that $L_Z^*\mu=0$ with $|Z|\in L^2_{\mathrm{loc}}(\mu)$, but the density of μ is not in $L^2_{\mathrm{loc}}(\lambda_M)$ (see an example in [11]).

COROLLARY 2.3. – *Assume that the Ricci curvature of M is bounded below and that*

$$\inf_x \lambda_M\bigl(B(x,r)\bigr)>0,\quad \forall r>0, \tag{2.7}$$

where $B(x,r)$ is the closed geodesic ball with center x and radius r. Let $\mu\in\mathcal{P}(M)$ be such that $|Z|\in L^1(\mu)$ and

$$\int L_Z f\,\mathrm{d}\mu\leqslant 0,\quad \forall f\in C_0^\infty(M),\ f\geqslant 0.$$

Then

$$\int L_Z f\,\mathrm{d}\mu=0,\quad \forall f\in C^\infty(M)\ \textit{with}\ \sup|f|<\infty,\ |L_Z f|+|\nabla f|\in L^1(\mu). \tag{2.8}$$

If, in addition, $|Z|\in L^2(\mu)$, then (2.3) *holds for* $p:=\mathrm{d}\mu/\mathrm{d}\lambda_M$.

Proof. – Since the Ricci curvature is bounded below, $\Delta\varrho$ is bounded above in the distribution sense outside any neighborhood of o. By the Greene–Wu approximation theorem (see [32, Theorem 3.2 and its Corollary 1]), there exists a nonnegative smooth compact function V with

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$|\nabla V| \leqslant 1$ and $\Delta V \leqslant 1$. Let $h \in C^\infty(\mathbb{R}^1)$ be such that $0 \leqslant h \leqslant 1$, $h(r) = 1$ for $r \leqslant 0$, $h(r) = 0$ for $r \geqslant 1$ and $-2 \leqslant h' \leqslant 0$. Let $f \in C^\infty(M)$ be bounded and nonnegative with $|L_Z f| + |\nabla f| \in L^1(\mu)$. For every $n \geqslant 1$, we set $f_n(x) := fh(V(x) - n)$. Then $f_n \geqslant 0$ and $f_n \in C_0^\infty(M)$. In addition,

$$\begin{aligned}
L_Z f_n &= h(V-n)L_Z f + f\big(L_Z h(V-n)\big) + 2\big\langle \nabla f, \nabla\big(h(V-n)\big)\big\rangle \\
&= h(V-n)L_Z f + fh'(V-n)\Delta V + fh''(V-n)|\nabla V|^2 + fh'(V-n)\langle Z, \nabla V\rangle \\
&\quad + 2\big\langle \nabla f, \nabla\big(h(V-n)\big)\big\rangle \\
&\geqslant h(V-n)L_Z f + fh'(V-n) + fh''(V-n)|\nabla V|^2 + fh'(V-n)\langle Z, \nabla V\rangle \\
&\quad + 2\big\langle \nabla f, \nabla\big(h(V-n)\big)\big\rangle.
\end{aligned}$$

Letting

$$S_n = fh'(V-n) + fh''(V-n)|\nabla V|^2 + fh'(V-n)\langle Z, \nabla V\rangle + 2h'(V-n)\big\langle \nabla f, \nabla V\big\rangle,$$

we obtain

$$0 \geqslant \int L_Z f_n \,\mathrm{d}\mu \geqslant \int h(V-n)L_Z f \,\mathrm{d}\mu + \int S_n \,\mathrm{d}\mu \to \int L_Z f \,\mathrm{d}\mu$$

as $n \to \infty$. Thus, we arrive at the estimate $\int L_Z f \,\mathrm{d}\mu \leqslant 0$. Clearly, the same is true for every bounded $f \in C^\infty(M)$ with $|L_Z f| + |\nabla f| \in L^1(\mu)$, since $f + \sup|f| \geqslant 0$. Replacing f by $-f$, we obtain (2.8).

Since the Ricci curvature is bounded below, there exists $K \geqslant 0$ such that $|\nabla P_t \varphi| \leqslant \exp[Kt]P_t|\nabla\varphi|$ for all $t > 0$ and $\varphi \in C_0^\infty$ (see, e.g., [9]), hence (2.1) holds. Next, by the Li–Yau heat kernel upper bound (see [41]), we obtain from (2.7) that P_t sends $L^1(\lambda_M)$ to $L^\infty(\lambda_M)$. Obviously, (2.8) implies (2.2), hence (2.3) holds by Theorem 2.2, provided $|Z| \in L^2(\mu)$. □

Remark 2.4. – By the proof of Corollary 2.3, we conclude that if $\mu \in \mathcal{P}(M)$ is such that $L_Z^*\mu = 0$ and there exists a compact function $V \in C^2(M)$ such that $|\nabla V|$ and $L_Z V$ are bounded above, then (2.8) holds. One only has to realize that we have the following estimate for $L_Z f_n$ rather than that in the proof:

$$\begin{aligned}
L_Z f_n &= h(V-n)L_Z f + fh'(V-n)L_Z V + fh''(V-n)|\nabla V|^2 + 2\big\langle \nabla f, \nabla h(V-n)\big\rangle \\
&\geqslant h(V-n)L_Z f + 2\big\langle \nabla f, \nabla h(V-n)\big\rangle - C1_{\{V \geqslant n\}} := h(V-n)L_Z f + S_n,
\end{aligned}$$

for some $C > 0$.

We note that if the injectivity radius of the manifold is positive (see [31, Ch. III]), then, according to Croke [23], one has $\lambda_M(B(x,r)) \geqslant cr^d$ for some $c > 0$ and all $r \in [0,1]$. Combining this with the Li–Yau heat kernel bound, we have $\|P_t\|_{1\to\infty} \leqslant c't^{-d/2}$ for some $c' > 0$ and all $t \in [0,1]$ provided that the Ricci curvature is bounded below. Hence (see, e.g., Davies [26, Corollary 2.4.3]) the Sobolev inequality holds with dimension $n \in [d,\infty) \cap (2,\infty)$. By Corollary 2.3, we have $\sqrt{p} \in H^{2,1}(\lambda_M)$ and $p \in L^{n/(n-2)}(\lambda_M)$ for $n \in [d,\infty) \cap (2,\infty)$.

COROLLARY 2.5. – *Suppose that the hypotheses of Theorem* 2.2 *are fulfilled, but with* (2.1) *replaced by the stronger condition that*

$$|\nabla P_t \varphi| \leqslant C(t)P_t\big(|\nabla\varphi|\big), \quad \forall \varphi \in C_0^\infty(M),\ t \in [0,1], \tag{2.9}$$

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where $C:[0,1]\to\mathbb{R}_+$ *with* $\lim_{t\to 0} C(t)=1$. *Assume, in addition,* $|Z|\in L^2(\lambda_M)$. *Then*

$$\int_M \frac{|\nabla p|^2}{p^2}\,\mathrm{d}\lambda_M \le \int_M |Z|^2\,\mathrm{d}\lambda_M. \tag{2.10}$$

In particular, $\log p\in W^{2,1}_{\mathrm{loc}}(\lambda_M)$.

Proof. – Let us apply (2.5) to $\varphi=(f_\varepsilon+\delta)^{-1}-\delta^{-1}\in H^{2,1}(\lambda_M)$ for $\delta>0$. By (2.9) we obtain that

$$\begin{aligned}\int_M \frac{|\nabla f_\varepsilon|^2}{(f_\varepsilon+\delta)^2}\,\mathrm{d}\lambda_M &\leqslant C(\varepsilon)\int_M |Z|P_\varepsilon\biggl(\frac{|\nabla f_\varepsilon|}{(f_\varepsilon+\delta)^2}\biggr)p\,\mathrm{d}\lambda_M\\ &= C(\varepsilon)\int_M \frac{P_\varepsilon(|Z|p)}{f_\varepsilon+\delta}\cdot\frac{|\nabla f_\varepsilon|}{f_\varepsilon+\delta}\,\mathrm{d}\lambda_M\\ &\leqslant C(\varepsilon)\biggl(\int_M\biggl|\frac{P_\varepsilon(|Z|p)}{f_\varepsilon+\delta}\biggr|^2\mathrm{d}\lambda_M\biggr)^{1/2}\biggl(\int_M\frac{|\nabla f_\varepsilon|^2}{(f_\varepsilon+\delta)^2}\,\mathrm{d}\lambda_M\biggr)^{1/2}.\end{aligned}\tag{2.11}$$

Using that

$$P_\varepsilon\bigl(|Z|p\bigr)=P_\varepsilon\biggl(|Z|\sqrt{p+1}\frac{p}{\sqrt{p+1}}\biggr)\leqslant\bigl(P_\varepsilon\bigl(|Z|^2(p+1)\bigr)\bigr)^{1/2}\biggl(P_\varepsilon\frac{p^2}{p+1}\biggr)^{1/2},$$

we conclude that

$$\int_M \frac{|\nabla f_\varepsilon|^2}{(f_\varepsilon+\delta)^2}\,\mathrm{d}\lambda_M\leqslant C(\varepsilon)^2\int_M P_\varepsilon\bigl(|Z|^2(p+1)\bigr)\frac{P_\varepsilon(p^2/(p+1))}{(P_\varepsilon p+\delta)^2}\,\mathrm{d}\lambda_M. \tag{2.12}$$

Since $P_\varepsilon(|Z|^2(p+1))\to|Z|^2(p+1)$ in $L^1(\lambda_M)$ as $\varepsilon\to 0$ and $P_\varepsilon(p^2/(p+1))\leqslant P_\varepsilon p$, we can let $\varepsilon\to 0$ in (2.12) to obtain that

$$\limsup_{\varepsilon\to 0}\int_M\frac{|\nabla f_\varepsilon|^2}{(f_\varepsilon+\delta)^2}\,\mathrm{d}\lambda_M\leqslant\int_M|Z|^2(p+1)\frac{p^2}{(p+\delta)^2(p+1)}\,\mathrm{d}\lambda_M\leqslant\int_M|Z|^2\,\mathrm{d}\lambda_M.$$

By standard arguments this implies that $\log(p+\delta)-\log\delta\in H^{2,1}(\lambda_M)$ and that

$$\int_M\frac{|\nabla p|^2}{(p+\delta)^2}\mathrm{d}\lambda_M\leqslant\int_M|Z|^2\mathrm{d}\lambda_M.$$

By letting $\delta\to 0$ we obtain (2.10).

In order to prove the last claim, let U be a bounded geodesic ball. Then there exists $C>0$ such that

$$\int_U\biggl|f-\int_U f\,\mathrm{d}\lambda_U\biggr|^2\mathrm{d}\lambda_U\leqslant C\int_U|\nabla f|^2\,\mathrm{d}\lambda_U,\quad\forall f\in H^{2,1}(\lambda_U),$$

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where $\lambda_U(A)=\lambda_M(A\cap U)/\lambda_M(U)$. Let $f_n=|\log(\min(p,n)+n^{-1})|$. Then (2.10) yields that the functions $f_n-\int_U f_n\,\mathrm{d}\lambda_U$ are uniformly bounded in $L^2(\lambda_U)$. Since $f_n\to|\log p|$ on the set $\{p>0\}$, by Fatou's theorem we obtain that the sequence $\{\int_U f_n\,\mathrm{d}\lambda_U\}$ is bounded. Hence $\sup_n\|f_n\|_{L^2(\lambda_U)}<\infty$ and $|\log p|\in L^2(\lambda_U)$. Therefore $\log p\in W^{2,1}_{\mathrm{loc}}(\lambda_M)$. $\square$

Note that the previous corollary implies, in particular, that $\log p\in W^{2,1}(\lambda_M)$ if M is compact. We recall that the hypotheses of this corollary are fulfilled if M is compact and $|Z|\in L^\alpha(\lambda_M)$ with $\alpha>\dim M$.

Remark 2.6. – Let μ be a Borel probability measure on a complete Riemannian manifold M such that $|Z|\in L^\alpha_{\mathrm{loc}}(\lambda_M)$ with $\alpha>d$ and L_Z is symmetric on $L^2(\mu)$ with domain $C_0^\infty(M)$ (which is equivalent to $Z=\nabla p/p$, where p is the density of μ). Then the operator $H_\mu\varphi:=\Delta\varphi+\langle Z,\nabla\varphi\rangle$ with domain $C_0^\infty(M)$ is essentially self-adjoint on $L^2(\mu)$. This follows from Theorem 1.1 in the same manner as in the case $M=\mathbb{R}^d$ considered in [11] (see also [46] for the case $Z=0$ and [7] for the case where Z is locally Lipschitzian).

3. Existence results in finite dimensions

Theorem 2.13 in [16] is a general result on existence of invariant measures on Riemannian manifolds. But as already pointed out in [16], the required condition, i.e. the existence of Lyapunov functions, is not always easy to check. As pointed out in the introduction, we now prove existence of invariant measures under conditions which are easier to verify in applications.

We recall that $x\in\mathrm{cut}(o)$, the cut locus of o, provided there is a unit vector $V\in T_oM$ such that $t=\varrho(o,\exp[tV])$ if and only if $t\in[0,\varrho(o,x)]$ (see, e.g., [8,31]).

THEOREM 3.1. – *Assume that Z is a measurable vector field on M such that $|Z|\in L^\alpha_{\mathrm{loc}}(\lambda_M)$, where $\alpha>d$. Suppose that there exists a function $F\in C^2[0,\infty)$ such that*

$$\lim_{r\to\infty}F(r)=\infty \quad \textit{and} \quad \lim_{r\to\infty}\sup_{\{\varrho\geqslant r\}\setminus\mathrm{cut}(o)}\bigl[F'(\varrho)L_Z\varrho+F''(\varrho)\bigr]=-\infty, \tag{3.1}$$

where once again we set $\sup\emptyset=-\infty$. Then there exists a probability measure μ which solves equation (1.1) *and has a density $p\in W^{\alpha,1}_{\mathrm{loc}}(\lambda_M)$.*

Proof. – We assume that M is noncompact because the result for the compact case is covered by Theorem 2.3 in [16]. We observe that

$$L_Z(F\circ\varrho)=F''(\varrho)\langle\nabla\varrho,\nabla\varrho\rangle+F'(\varrho)L_{Z\varrho}=F''(\varrho)+F'(\varrho)L_{Z\varrho}.$$

Let us note that condition (3.1) may be fulfilled even if $\sup_{\{\varrho\geqslant r\}\setminus\mathrm{cut}(o)}[F'(\varrho)L_{Z\varrho}+F''(\varrho)]$ is not bounded above in r from some interval $(0,T]$. However, we can always choose F satisfying (3.1) in such a way that $F\geqslant0$ and $F=0$ on some interval $[0,\tau]$ such that the function $\sup_{\{\varrho\geqslant r\}\setminus\mathrm{cut}(o)}[F'(\varrho)L_{Z\varrho}+F''(\varrho)]$ is bounded above in $r\in[0,+\infty)$. Indeed, let $\tau>0$ be such that $F''(\varrho)+F'(\varrho)L_{Z\varrho}\leqslant0$ if $\varrho\geqslant\tau$ and let $m=\max_{s\leqslant\tau}F(s)$. Let $\psi\in C^2(\mathbb{R}^1)$ be such that $\psi(s)=0$ if $s\leqslant m$, $\psi(s)=s$ if $s\geqslant m+1$ and $\psi'(s)\geqslant0$. Then the new function $F_0:=\psi(F)$ has the desired properties, since by the above observation, one has

$$L_Z(\psi\circ F\circ\varrho)=\psi'\bigl(F(\varrho)\bigr)\bigl[F''(\varrho)+F'(\varrho)L_{Z\varrho}\bigr]+\psi''\bigl(F(\varrho)\bigr)\bigl(F'(\varrho)\bigr)^2$$

and $\psi\circ F\circ\varrho=0$ if $\varrho\leqslant\tau$. In particular, by changing F, we may assume that $F\circ\varrho$ is C^2 outside $\mathrm{cut}(o)$. By (3.1), there exists $r_0>0$ such that $F'(r)>0$ for $r>r_0$. Indeed, let $r_0>0$ be such

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that $F''(\varrho) + F'(\varrho)L_Z\varrho < -1$ if $\varrho \geqslant r_0$. If $F'(r) = 0$ for some $r > r_0$, then by (3.1) $F''(r) < 0$ (note that $\{\varrho(x)\colon x \notin \mathrm{cut}(o)\} = [0,\infty)$). There is $r_1 > r$ with $F(r_1) > F(r)$. Then the function F on $[r, r_1]$ attains its minimum at some $r_{\min} \in (r, r_1)$, which is impossible, since $F''(r_{\min}) < 0$. Let $B_l = B(o,l)$, $l \geqslant 1$, be the closed ball of radius l around o. We denote by B_l^o the interior of $B_l = B(o,l)$ and by B_l^c the complement of B_l. By the proof of Theorem 2.3 in [16] it follows that there exists $\mu_l \in \mathcal{P}(M)$ with density p_l such that $p_l = 0$ on B_l^c, $p_l \in H^{\alpha,1}(B_l;\lambda_M)$, and

$$\int L_Z f\,\mathrm{d}\mu_l = 0, \quad \forall f \in C^\infty(M) \text{ with } \operatorname{supp} f \subset B_l^o. \tag{3.2}$$

Let us take an increasing function $G \in C[0,\infty)$ such that

$$\lim_{r\to\infty} G(r) = +\infty \quad \text{and} \quad L_Z(F\circ\varrho) = F''(\varrho) + F'(\varrho)L_Z\varrho \leqslant -G\circ\varrho$$

outside $\mathrm{cut}(o)$. For fixed l and every $\varepsilon > 0$, let $h_\varepsilon \in C^\infty(\mathbb{R}^1)$ be such that $0 \leqslant h'_\varepsilon \leqslant 1$, $h''_\varepsilon \leqslant 0$, $h_\varepsilon(r) = r$ for $r \leqslant F(l-\varepsilon)$ and $h_\varepsilon(r) = F(l - 3\varepsilon/4)$ for $r \geqslant F(l-\varepsilon/2)$. Then

$$L_Z(h_\varepsilon\circ F\circ\varrho) \leqslant -h'_\varepsilon(F\circ\varrho)G\circ\varrho \tag{3.3}$$

outside $\mathrm{cut}(o)$. Since $h_\varepsilon\circ F\circ\varrho \in W^{1,1}_{\mathrm{loc}}(\lambda_M)$ and is constant outside $B_{l-\varepsilon/2}$, we can take a sequence $\{f_n\} \subset C^\infty(M)$ such that $\operatorname{supp} f_n \subset B_l^o$, $\sup_{n\geqslant 1}\|\nabla f_n\|_\infty < \infty$, and $\lim_{n\to\infty}|\nabla f_n - \nabla(h_\varepsilon\circ F\circ\varrho)| = 0$ λ_M-a.e. Then, by the integration by parts formula, (3.2) yields

$$\begin{aligned}\int_{B_l}\big\langle\nabla(h_\varepsilon\circ F\circ\varrho), p_lZ - \nabla p_l\big\rangle\mathrm{d}\lambda_M &= \lim_{n\to\infty}\int_{B_l}\langle\nabla f_n, p_lZ - \nabla p_l\rangle\,\mathrm{d}\lambda_M\\ &= \lim_{n\to\infty}\int_{B_l} L_Z f_n\, p_l\,\mathrm{d}\lambda_M = 0.\end{aligned} \tag{3.4}$$

According to an observation of Cheeger and Gromoll [21], we can take a sequence of closed smooth domains D_m such that $D_m \subset B_l^o\setminus\mathrm{cut}(o)$, $D_m \uparrow B_l^o\setminus\mathrm{cut}(o)$ and $\langle\nabla\varrho, N_m\rangle \geqslant 0$, where N_m denotes the outward unit normal vector field of ∂D_m. Then we have the estimate $\langle\nabla(h_\varepsilon\circ F\circ\varrho), N_m\rangle \geqslant 0$. By the integration by parts formula, (3.3) and (3.4) imply

$$\begin{aligned}&\int_{D_m} h'_\varepsilon(F\circ\varrho)G\circ\varrho\,\mathrm{d}\mu_l\\ &\quad\leqslant -\int_{D_m} L_Z(h_\varepsilon\circ F\circ\varrho)\,\mathrm{d}\mu_l\\ &\quad\leqslant -\int_{D_m}\big\langle\nabla(h_\varepsilon\circ F\circ\varrho), p_lZ - \nabla p_l\big\rangle\mathrm{d}\lambda_M \leqslant c_l\int_{B_l\setminus D_m}|p_lZ - \nabla p_l|\,\mathrm{d}\lambda_M,\end{aligned} \tag{3.5}$$

where $c_l = \sup_{r\in[0,l]}|F'(r)|$. By first letting $\varepsilon\downarrow 0$ and then $m\uparrow\infty$, we obtain

$$\int G\circ\varrho\,\mathrm{d}\mu_l \leqslant 0. \tag{3.6}$$

This yields that for every $\varepsilon > 0$, there exists a compact ball B such that $\mu_l(M\setminus B) \leqslant \varepsilon$ for all l. Hence the sequence $\{\mu_l\}$ is relatively weakly compact. Let $\mu \in \mathcal{P}(M)$ be its cluster point in

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the weak topology. It is known (see [16]) that, for every $n \geqslant 1$, the sequence $\{p_l\colon\ l \geqslant n+1\}$ is bounded in $W^{\alpha,1}(B_n;\lambda_M)$. Therefore, there exists $p \in W^{\alpha,1}_{\mathrm{loc}}(\lambda_M)$ such that $\mu = p\lambda_M$. It is readily seen that $L^*\mu = 0$. $\square$

Since $G \circ \varrho$ is continuous and bounded below, the estimate (3.6) also holds for μ in place of μ_l. We shall now show that such an estimate for probability measures solving (1.1) is valid in a more general situation.

PROPOSITION 3.2. – *Suppose that μ is a probability measure solving equation* (1.1), *where Z is a measurable vector field on M such that $|Z| \in L^{\alpha}_{\mathrm{loc}}(\lambda_M)$ with $\alpha > d$. Assume that there exist a nondecreasing function $F \in C^2[0,\infty)$ with $\lim_{r\to\infty} F(r) = +\infty$ and a nonnegative function $G \in C(M)$ such that, for some $c > 0$, one has*

$$L_Z(F\circ\varrho) = F'\circ\varrho\, L_Z\varrho + F''\circ\varrho \leqslant c - G$$

outside $\mathrm{cut}(o)$. *Then*

$$\int_M G\,\mathrm{d}\mu \leqslant c. \tag{3.7}$$

The same is true if there exists a function $V \in C^2(M)$ such that $L_Z V \leqslant c - G$ and $\{V \leqslant k\} \cap \{|\nabla V| > 0\}$ is relatively compact for each $k \in \mathbb{N}$.

Proof. – We know that $\mu = p\lambda_M$, where $p \in W^{\alpha,1}_{\mathrm{loc}}(\lambda_M)$. Now we can employ the arguments used above to obtain (3.6). Namely, for fixed $k \in \mathbb{N}$, let $h_k \in C^\infty(\mathbb{R}^1)$ be such that $0 \leqslant h_k{}' \leqslant 1$, $h_k{}'' \leqslant 0$, $h_k(r) = r$ for $r \leqslant k$ and $h_k(r) = k+1$ for $r \geqslant k+2$. Then one has

$$\begin{aligned} L_Z(h_k\circ F\circ\varrho) &= h_k''\big(F(\varrho)\big)\big[F'(\varrho)\big]^2 + h_k'\big(F(\varrho)\big)L_Z(F\circ\varrho) \\ &\leqslant c h_k'\big(F(\varrho)\big) - h_k'\big(F(\varrho)\big)G \end{aligned} \tag{3.8}$$

outside $\mathrm{cut}(o)$. Since $h_k\circ F\circ\varrho \in W^{1,1}_{\mathrm{loc}}(\lambda_M)$ and is constant outside B_{l_k}, where l_k is such that $F(r) \geqslant k+2$ if $r \geqslant l_k$, we obtain by Lemma 2.1(ii) that

$$\int_{B_{l_k}} \big\langle \nabla(h_k\circ F\circ\varrho),\, pZ - \nabla p\big\rangle\,\mathrm{d}\lambda_M = 0. \tag{3.9}$$

As in the proof of Theorem 3.1 we choose closed smooth domains D_m such that $D_m \subset B^o_{l_k} \setminus \mathrm{cut}(o)$, $D_m \uparrow B^o_{l_k} \setminus \mathrm{cut}(o)$ and $\langle \nabla\varrho, N_m\rangle \geqslant 0$, where N_m denotes the outward unit normal vector field of ∂D_m, we obtain from the integration by parts formula, (3.8) and (3.9) that

$$\begin{aligned} \int_{D_m} h_k'(F\circ\varrho)G\,\mathrm{d}\mu &\leqslant c\int_{D_m} h_k'(F\circ\varrho)\,\mathrm{d}\mu - \int_{D_m} L_Z(h_k\circ F\circ\varrho)\,\mathrm{d}\mu \\ &\leqslant c - \int_{D_m} \big\langle \nabla(h_k\circ F\circ\varrho),\, pZ - \nabla p\big\rangle\,\mathrm{d}\lambda_M \\ &\leqslant c + c_{l_k}\int_{B_{l_k}\setminus D_m} |pZ - \nabla p|\,\mathrm{d}\lambda_M, \end{aligned}$$

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where $c_{l_k} = \sup_{r\in[0,l_k]} |F'(r)|$. By letting $m \uparrow \infty$ we obtain

$$\int\limits_{B_{l_k}} h_k'(F\circ\varrho) G \,\mathrm{d}\mu \leqslant c,$$

which by Fatou's lemma yields (3.7) by letting $k\uparrow\infty$, since $h_k'(F\circ\varrho)\to 1$. To prove the last assertion we integrate by parts and use Lemma 2.1(ii) to obtain the equality $\int L_Z(h_k\circ V)\,\mathrm{d}\mu = 0$. It only remains to apply the estimate $L_Z(h_k\circ V)\leqslant h_k'(V)(c-G)$ and let $k\to\infty$. □

Example 3.3. – Suppose that $\mathrm{Ric}\geqslant -k$, $k\geqslant 0$. If $|Z|\in L^{\alpha}_{\mathrm{loc}}(\lambda_M)$ with $\alpha > d$ and $\lim_{r\to\infty}\sup_{\{\varrho\geqslant r\}\setminus \mathrm{cut}(o)}\langle Z,\nabla\varrho\rangle < -\sqrt{k(d-1)}$, then the assertion of Theorem 3.1 is valid. It suffices to take $F(r)=r^2$ and note that $\lim_{r\to\infty}\sup_{\{\varrho\geqslant r\}\setminus \mathrm{cut}(o)}\Delta\varrho\leqslant\sqrt{k(d-1)}$.

We are now going to present a curvature condition for the existence and uniqueness of invariant measures. Given a C^1 vector field Z, we set

$$k(r) = \inf_{\{\varrho\geqslant r\}\setminus \mathrm{cut}(o)} \big\{\mathrm{Ric}(\nabla\varrho,\nabla\varrho) - \langle\nabla_{\nabla\varrho}Z,\nabla\varrho\rangle\big\}, \tag{3.10}$$

$$\overline{k}(r) = \inf_{\{\varrho\geqslant r\}\setminus \mathrm{cut}(o)} \big\{-\langle\nabla_{\nabla\varrho}Z,\nabla\varrho\rangle\big\}. \tag{3.11}$$

THEOREM 3.4. – *Assume that Z is a C^1 vector field such that*

$$\int\limits_0^\infty k(r)\,\mathrm{d}r = \infty. \tag{3.12}$$

Then there exists a probability measure μ that satisfies (1.1) *with respect to $C_0^\infty(M)$. If the Ricci curvature is bounded below, then the same is true provided that* (3.12) *holds for $\overline{k}$ in place of k.*

Proof. – Fix $x\notin \mathrm{cut}(o)$ and let $l:[0,\varrho(x)]\to M$ be the minimal geodesic from o to x. Denote the unit tangent vector field along $l=(l_s)_{s\in[0,\varrho(x)]}$ by $\mathcal{T}=(\mathcal{T}_s)_{s\in[0,\varrho(x)]}$. Let $\{U_i\}_{i=1}^{d-1}$ be parallel vector fields along l such that $\{\mathcal{T}, U_i\colon i=1,\ldots,d-1\}$ is an orthonormal basis at each point of l. Finally, let $\{J_i\}_{i=1}^{d-1}$ be Jacobi fields along l with $J_i(0)=0$ and $J_i(\varrho)=U_i(\varrho)$, where and in what follows we simply denote $\varrho(x)$ by ϱ. We have (see, e.g., the second variation formula of the distance in [22]) that

$$\Delta\varrho = \sum_{i=1}^{d-1}\int\limits_0^{\varrho}\big(|\nabla_{\mathcal{T}}J_i|^2 - \langle R(J_i,\mathcal{T})\mathcal{T},J_i\rangle\big)\,\mathrm{d}s,$$

where the integral is taken along l over the length parameter s, and $\nabla_\bullet$ is the Levi-Civita connection. Let $h\in C^1[0,\varrho]$ be such that $h(0)=0, h(\varrho)=1$. By the index lemma (see [8, Theorem 1.51] or [22]), we obtain

$$\Delta\varrho \leqslant \sum_{i=1}^{d-1}\int\limits_0^{\varrho}\big(|\nabla_{\mathcal{T}}(hU_i)|^2 - \langle R(hU_i,\mathcal{T})\mathcal{T},hU_i\rangle\big)\,\mathrm{d}s$$

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$$= (d-1)\int_0^{\varrho} (h')^2\,\mathrm{d}s - \int_0^{\varrho} h^2 \mathrm{Ric}(\mathcal{T},\mathcal{T})\,\mathrm{d}s.$$

Noting that

$$Z\varrho = \langle \nabla\varrho, Z\rangle = \langle \mathcal{T}_{\varrho}, Z\rangle = \int_0^{\varrho} \frac{\mathrm{d}}{\mathrm{d}s}\langle h^2\mathcal{T}, Z\rangle\,\mathrm{d}s = \int_0^{\varrho} \big[\big(h^2\big)'\langle \mathcal{T}, Z\rangle + h^2\langle \nabla_{\mathcal{T}} Z, \mathcal{T}\rangle\big]\,\mathrm{d}s,$$

we obtain

$$L_Z\varrho \leqslant (d-1)\int_0^{\varrho} (h')^2\,\mathrm{d}s - \int_0^{\varrho} h^2 k\,\mathrm{d}s + \int_0^{\varrho} (h^2)'\langle \mathcal{T}, Z\rangle\,\mathrm{d}s.$$

Let h be a smooth function such that $h(0)=0$, $0\leqslant h\leqslant 1$, $h(r)=1$ for $r\geqslant 1$ and $|h'|\leqslant 2$. We have that outside $\mathrm{cut}(o)\cup\{\varrho<1\}$

$$L_Z\varrho \leqslant 4(d-1) + 4\sup_{B(o,1)}|Z| - \int_1^{\varrho} k(r)\,\mathrm{d}r + \sup_{[0,1}|k|.$$

By combining this with (3.12), we see that $L_Z\varrho\to-\infty$ as $\varrho\to\infty$ (outside $\mathrm{cut}(o)$). We also have that outside $\mathrm{cut}(o)$

$$L_Z\varrho \leqslant \Delta\varrho + \big|Z(o)\big| + \int_0^{\varrho} \langle \nabla_{\mathcal{T}} Z, \mathcal{T}\rangle\,\mathrm{d}s \leqslant \Delta\varrho + |Z(o)| - \int_0^{\varrho} \overline{k}(r)\,\mathrm{d}r.$$

But if (3.12) holds for $\overline{k}$, this tends to $-\infty$ as $\varrho\to\infty$ provided the Ricci curvature is bounded below, hence $\Delta\varrho$ is bounded above outside a neighborhood of o and $\mathrm{cut}(o)$. Hence Theorem 3.1 applies with $F(r)=r$. □

Remark 3.5. – (i) Suppose that in Theorem 3.1 or in Theorem 3.4 one has $Z=\nabla W$, where $W\in W^{\alpha,1}_{\mathrm{loc}}(\lambda_M)$ and $\alpha>d$. Then the function $\exp W$ is λ_M-integrable, which is verified by the aid of (3.6). Namely, in the proof of Theorem 3.1, we take $p_l=\mathrm{const}(l)\exp W$ on B_l. Then, by (3.6), the sequence of measures μ_l is uniformly tight on M, whence the desired integrability follows. Hence we can find a normalization constant c_0 such that the probability measure $\mu:=c_0\exp W\,\mathrm{d}\lambda_M$ solves the equation $L_Z^*\mu=0$.

(ii) It is seen from the above proof that we have used in fact the following weaker condition instead of (3.12): assuming that the Ricci curvature of M is bounded below so that one has $\Delta\varrho\leqslant C$ outside a neighborhood of o and $\mathrm{cut}(o)$, where $C\in(0,+\infty)$, it suffices to have the estimate

$$\int_0^{\infty} \overline{k}(r)\,\mathrm{d}r > C + \big|Z(o)\big|. \tag{3.13}$$

In the general case, it suffices to have the estimate

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$$\int_0^\infty k(r)\,\mathrm{d}r > 4(d-1) + 4\sup_{B(o,1)}|Z| + \sup_{[0,1]}|k|. \tag{3.14}$$

In both cases, by the same reasoning as at the end of the above proof, we can apply Theorem 3.1 with $F(r)=r^2$. For example, if $W\in C^2(M)$ is such that its second derivative is nondegenerate outside o and $\nabla W(o)=0$, then the field $Z=\alpha\nabla W$ satisfies (3.13) for any sufficiently big constant α. In this case, the probability measure μ with the density const. $\exp(-\alpha W)$ is a solution of $L_Z^*\mu=0$.

(iii) It is worth noting that if $Z=\nabla W$ for some $W\in C^2(M)$, (3.12) implies the Poincaré inequality (see [47]) and furthermore the super-Poincaré inequality (see [48]), since it implies that $L_Z\varrho\to-\infty$ as shown in the proof of Theorem 3.4.

We note that the existence and uniqueness of invariant measures for L_Z-diffusion processes have been proved by Cruzeiro and Malliavin in [24] under some conditions including that $\inf k>0$. The proof of Theorem 3.4 enables us to improve their result as follows.

COROLLARY 3.6. – *If Z is C^1 and either* (3.12) *holds, or the Ricci curvature is bounded below and* (3.12) *holds for $\overline{k}$ in place of k, then the L_Z-diffusion process is ergodic and has a unique invariant probability measure.*

Proof. – By the proof of Theorem 3.4, either of our conditions implies that $L_Z\varrho^2\leqslant c-G(\varrho)$ outside $\mathrm{cut}(o)$ for some $c>0$ and a positive function $G\in C[0,\infty)$ such that $G(r)\uparrow\infty$ as $r\uparrow\infty$. Let $(x_t)_{t\geqslant 0}$ be the L_Z-diffusion process with $x_0=o$. By Itô's formula for $\varrho(x_t)$ (cf. [36]) we have

$$\mathrm{d}\varrho^2(x_t) = 2\sqrt{2}\varrho(x_t)\,\mathrm{d}b_t + 1_{\{x_t\notin\mathrm{cut}(o)\}}\big[L_Z\varrho^2(x_t)\big]\,\mathrm{d}t - \mathrm{d}L_t$$

for some increasing process L_t and a Brownian motion b_t on $\mathbb{R}$. Let $\tau_n:=\inf\{t\geqslant 0\colon \varrho(x_t)\geqslant n\}$. We obtain

$$n^2P(\tau_n\leqslant t)\leqslant \mathbb{E}\varrho^2(x_{t\wedge\tau_n})\leqslant \mathbb{E}\int_0^{t\wedge\tau_n}\big[c-G\big(\varrho(x_s)\big)\big]\,\mathrm{d}s\leqslant ct. \tag{3.15}$$

Therefore, $P(\tau_\infty\leqslant t)\leqslant P(\tau_n\leqslant t)\leqslant ct/n^2$ for any $t,n>0$. Hence $\tau_\infty=\infty$ a.s. By letting $n\uparrow\infty$, the first inequality in (3.15) yields that

$$\mathbb{E}\int_0^t G\big(\varrho(x_s)\big)\,\mathrm{d}s\leqslant ct,\quad t>0. \tag{3.16}$$

Let $\nu_t(\cdot)=\frac{1}{t}\int_0^t P(x_s\in\cdot)\,\mathrm{d}s$. We conclude from (3.16) that $(\nu_n)_{n\geqslant 1}$ is tight and hence has a subsequence which converges weakly to some $\mu\in\mathcal{P}(M)$. Then it is easy to check that μ is an invariant measure of the process (i.e., an invariant measure for the corresponding semigroup).

To prove the ergodicity and uniqueness, we shall show that, for any $\nu\in\mathcal{P}(M)$, letting $\nu T_t(\cdot)=\int P_t(y,\cdot)\,\nu(\mathrm{d}y)$, where $P_t(y,\cdot)$ is the distribution at time t of the L_Z-diffusion process starting from y, we have $\lim_{t\to\infty}\nu T_t(B)=\mu(B)$ for every Borel set B. This follows by a theorem of Doob (see, e.g., [25, Theorem 4.2.1]). Moreover, one has even $\nu T_t\to\mu$ in the total variation norm (see [44]). In order to apply the result cited, it suffices to note that the semigroup

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$(T_t)_{t\geqslant 0}$ is strongly Feller and stochastically continuous (i.e., for every x and every ball B of positive radius centered at x, one has $\lim_{t\to 0} P_t(x,B)=1$), and the transition probabilities have continuous strictly positive densities. All these properties follow from the condition that Z is continuously differentiable. □

Let us single out an important special case of equation (1.1). Let μ be any Borel measure on M with density $p\in W^{1,1}_{\mathrm{loc}}(\lambda_M)$. Set $Z=\beta^\mu:=\nabla p/p$. The vector field β^μ is called the logarithmic gradient of μ. Clearly, $|Z|\in L^1_{\mathrm{loc}}(\mu)$ and, by the integration by parts formula, (1.1) is satisfied with respect to the class $C_0^\infty(M)$. This example corresponds to symmetric diffusions on M. It is easily verified (see, e.g., [15]) that a probability measure μ is uniquely determined by its logarithmic gradient β^μ provided that $|\beta^\mu|\in L^1(\lambda_M)$.

Remark 3.7. – The uniqueness problem for equation (1.1) will be discussed in a separate paper. The case $M=\mathbb{R}^d$ has been studied in [1,13,18]. In general, there is no uniqueness for probability measures satisfying (1.1) even if Z is smooth (see [18]). By a modification of the methods employed in [18] and [45] we shall prove in a forthcoming paper that in the situation of Theorem 3.1 there exists exactly one probability measure μ such that $L_Z^*\mu=0$. In addition, there is a unique strongly continuous Markovian semigroup $(T_t^\mu)_{t\geqslant 0}$ on $L^1(\mu)$ such that its generator L^μ coincides with L_Z on $C_0^\infty(M)$. Moreover, μ is a unique invariant probability for $(T_t^\mu)_{t\geqslant 0}$ on the space of all bounded Borel functions on M. Finally, there exists a Markov process in M (in the sense explained in [45]) with the transition semigroup given by $(T_t^\mu)_{t\geqslant 0}$. If the drift Z is continuous, then such a process can be constructed as a limit of usual diffusions generated by L_Z in compact regions exhausting M.

4. The infinite product case: regularity

Let S be a countable set, e.g., let $S=\mathbb{N}$ be the set of natural numbers. For each $i\in S$, let M^i be a complete connected finite-dimensional Riemannian manifold. For $\Lambda\subset S$, let $M^\Lambda=\Pi_{i\in\Lambda}M^i$ be equipped with the product Borel σ-field $\mathcal{B}_\Lambda$. We denote the distance function on M^i from a fixed point o_i by ϱ_i. Let λ_Λ be the Riemannian volume element on M^Λ. For every $x\in M^S$ and $\Lambda\subset S$, let $x_\Lambda\in M^\Lambda$ be the natural projection of x. For any $\mu\in\mathcal{P}(M^S)$ and $\Lambda\subset S$, let $\mu_{\Lambda^c}(\mathrm{d}y_{\Lambda^c}|x_\Lambda)$ be the regular conditional probability of μ on M^{Λ^c} given $\mathcal{B}_\Lambda\times M^{\Lambda^c}:=\{B\times M^{\Lambda^c}\mid B\in\mathcal{B}_\Lambda\}$, i.e., for every bounded Borel function f on M^S, one has

$$\int f\,\mathrm{d}\mu=\int_{M^\Lambda}\int_{M^{\Lambda^c}} f(x_\Lambda\times y_{\Lambda^c})\,\mu_{\Lambda^c}(\mathrm{d}y_{\Lambda^c}|x_\Lambda)\,\mu_\Lambda(\mathrm{d}x_\Lambda),$$

where μ_Λ is the marginal distribution of μ on M^Λ.

We set

$$\mathcal{F}C_0^\infty:=\bigcup_{\Lambda\subset S,\ \Lambda\text{ is finite}} C_0^\infty(M^\Lambda).$$

Here and below, we regard a function on M^Λ as a cylindrical function on M^S in the natural way. Replacing $C_0^\infty(M^\Lambda)$ by $C_b^\infty(M^\Lambda)$, one obtains the class $\mathcal{F}C_b^\infty$. Note that $\mathcal{F}C_b^\infty$ is a linear space, but $\mathcal{F}C_0^\infty$ is not.

Let $Z=(Z_i)_{i\in S}$ be a collection of Borel maps on M^S such that $Z_i(x)\in T_{x_i}M^i$ and let

$$L_Z=\Delta+Z:=\sum_{i\in S}L_i,\quad L_i:=\Delta_i+Z_i, \tag{4.1}$$

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where Δ_i is the Laplacian on M^i, and the sense in which the sum is understood will be explained later. In particular, $L\psi$ makes sense if $\psi \in \mathcal{FC}_0^\infty$. Moreover, let $\nabla := (\nabla_i)_{i\in S}$, where ∇_i denotes the gradient on M^i, denote the gradient on M^S. We set $|\nabla f|^2 = \sum_{i\in S} |\nabla_i f|^2$ for $f : M^S \to \mathbb{R}$ provided the right-hand side exists.

DEFINITION 4.1. – *Let $\mathcal{K}$ be a certain class of bounded $\mathcal{B}$-measurable functions on M^S. We shall say that a Radon measure μ on M^S satisfies the elliptic equation*

$$L_Z^* \mu = 0 \tag{4.2}$$

with respect to the class $\mathcal{K}$ if, for every $\psi \in \mathcal{K}$, one has $L_i\psi \in L^1(\mu)$ and

$$\sum_{i\in S} \int_{M^S} L_i\psi \,\mathrm{d}\mu = 0. \tag{4.3}$$

Here ∇_i denotes the gradient on M^i.

For example, one can consider (4.2) with respect to $\mathcal{K} = \mathcal{FC}_0^\infty$ (then the series becomes a finite sum). Another possibility is to consider the class $\mathcal{K}_c(M^S)$ of all bounded Borel functions ψ on M^S such that $x_i \mapsto \psi(x_1, \ldots, x_i, \ldots)$ is a smooth compactly supported function on M^i for all $i \in S$.

One of the motivations for the study of equation (4.2) is that, as will be explained below, Gibbs distributions on M^S satisfy this equation under very broad assumptions.

Suppose first that μ is a Borel probability measure on M^S such that $L_Z^*\mu = 0$ with respect to the class $\mathcal{FC}_0^\infty$ and that $|Z_i| \in L^1(\mu)$ for all i. Given a finite set $\Lambda = \{s_1, \ldots, s_n\} \subset S$, we denote by Z_Λ^μ the conditional expectation of $Z_\Lambda := (Z_{s_1}, \ldots, Z_{s_n})$ with respect to μ and the σ-field generated by the natural projection from M^S to M^Λ. Note that μ-a.e.

$$Z_\Lambda^\mu(x) = \int_{M^{\Lambda^c}} Z_\Lambda(x_\Lambda \times y_{\Lambda^c})\, \mu_{\Lambda^c}(\mathrm{d}y_{\Lambda^c}|x_\Lambda).$$

From now on Z_Λ^μ will always denote this particular μ-version. In particular, $Z_\Lambda^\mu(x) = Z_\Lambda^\mu(x_\Lambda)$ for all $x \in M^S$. Let

$$L_\Lambda^\mu = \Delta_\Lambda + Z_\Lambda^\mu := \sum_{i\in\Lambda} \Delta_i + Z_\Lambda^\mu.$$

It is readily verified that one has $(L_\Lambda^\mu)^*\mu_\Lambda = 0$ with respect to $C_0^\infty(M^\Lambda)$. The following result about the regularity of marginal distributions then follows immediately, by Theorem 1 and Remark 4(iii) in [11]. Let ∇_Λ denote the gradient on M^Λ.

THEOREM 4.2. – *Suppose that $L_Z^*\mu = 0$ with respect to $\mathcal{FC}_0^\infty$.*

(i) *Let $|Z_\Lambda^\mu| \in L^\alpha_{\mathrm{loc}}(\lambda_\Lambda)$ for some $\alpha > 1$, then $p_\Lambda(x_\Lambda) = \mathrm{d}\mu_\Lambda/\mathrm{d}\lambda_\Lambda$ exists and $p_\Lambda \in W^{\gamma,1}_{\mathrm{loc}}(\lambda_\Lambda)$ for every $\gamma \in [1, \dim M^\Lambda/(\dim M^\Lambda - \alpha + 1))$. Moreover, if $\alpha > \dim M^\Lambda$, then $p_\Lambda \in W^{\alpha,1}_{\mathrm{loc}}(\lambda_\Lambda)$ and there exists a continuous strictly positive version of p_Λ.*

(ii) *Under the assumptions of Corollary* 2.3, *if $|Z_\Lambda^\mu| \in L^2(\mu_\Lambda)$ for every finite set $\Lambda \subset S$, then $\mathrm{d}\mu_\Lambda = \varphi_\Lambda^2 \mathrm{d}\lambda_\Lambda$ with $\varphi_\Lambda \in H^{2,1}(\lambda_\Lambda)$ and*

$$\int_{M^\Lambda} |\nabla_\Lambda \varphi_\Lambda|^2 \,\mathrm{d}\lambda_\Lambda \leqslant \frac{1}{4} \int_{M^\Lambda} |Z_\Lambda^\mu|^2 \,\mathrm{d}\mu_\Lambda \leqslant \frac{1}{4} \int_{M^S} |Z|^2 \,\mathrm{d}\mu.$$

Our next result deals with the regularity of invariant measures with respect to a fixed probability measure as in [1,12] and [15]. To this end, let $W_i \in W^{1,1}_{\mathrm{loc}}(\lambda_i)$ be such that $\eta_i(\mathrm{d}x) := \exp(W_i)\,\mathrm{d}\lambda_i \in \mathcal{P}(M^i)$, and the following logarithmic Sobolev inequality holds for some $\alpha > 0$ and all $i \in S$:

$$\int f^2 \log f^2\,\mathrm{d}\eta_i \leqslant \alpha \int |\nabla f|^2\,\mathrm{d}\eta_i, \quad \forall f \in C_0^\infty(M^i) \text{ with } \int f^2\,\mathrm{d}\eta_i = 1. \tag{4.4}$$

We set $\eta = \bigotimes_{i\in S}\eta_i$, $\eta_\Lambda = \bigotimes_{i\in\Lambda}\eta_i$. The Sobolev class $H^{2,1}(\eta)$ is defined as the completion of the linear span of $\mathcal{FC}_0^\infty$ with respect to the Sobolev norm $\|\cdot\|_{H^{2,1}}$ given by

$$\|f\|^2_{H^{2,1}} = \int |f|^2\,\mathrm{d}\eta + \sum_{i\in S}\int |\nabla_i f|^2\,\mathrm{d}\eta.$$

In the same way we define $H^{2,1}(\eta_\Lambda)$. By [5] indeed the associated quadratic forms with respect to η and η_Λ are closable on $L^2(\eta)$ and $L^2(\eta_\Lambda)$, respectively. We shall assume that the measure η satisfies the following condition: for every finite set Λ, one has

$$H^{2,1}(\eta_\Lambda) = \left\{f \in L^2(\eta_\Lambda)\colon f \in W^{2,1}_{\mathrm{loc}}(\eta_\Lambda), |\nabla f| \in L^2(\eta_\Lambda)\right\}, \tag{4.5}$$

where $W^{2,1}_{\mathrm{loc}}(\eta_\Lambda)$ is the class of all functions $f \in L^2_{\mathrm{loc}}(\eta_\Lambda)$ such that, in every local chart, f has a modification which is absolutely continuous on almost all lines parallel to the coordinate lines and the corresponding partial derivatives are locally in $L^2(\eta_\Lambda)$ (then ∇f is defined by means of these partial derivatives).

One can verify that (4.5) is, e.g., fulfilled if η_Λ has a density ϱ such that, for every compact set K, $0 < c_1(K) \leqslant \varrho \leqslant c_2(K) < \infty$.

Let $Y_i := Z_i - \nabla_i W_i$, $Y := (Y_i)_{i\in S}$.

THEOREM 4.3. – *Suppose that the M^i's satisfy the hypotheses in Corollary* 2.3 *and that* (4.4), (4.5) *hold. If $\mu \in \mathcal{P}(M^S)$ is such that $L_Z^*\mu = 0$ with respect to the class $\mathcal{FC}_0^\infty$, where $|Z_i| \in L^2(\mu)$ for every $i \in S$ and*

$$|Y| := \left(\sum_{i\in S}|Z_i - \nabla_i W_i|^2\right)^{1/2} \in L^2(\mu),$$

then $\mu = p\eta$ with $\sqrt{p} \in H^{2,1}(\eta)$.

Proof. – We may assume that $S = \mathbb{N}$. For any finite set $\Lambda \subset \mathbb{N}$, we obtain by the above results that $\mathrm{d}\mu_\Lambda = f_\Lambda \mathrm{d}\lambda_\Lambda$ with $\sqrt{f_\Lambda} \in H^{2,1}(\lambda_\Lambda)$. Since $|\nabla_i W_i| = |Z_i - Y_i| \in L^2(\mu)$, we have $|\nabla_\Lambda W_\Lambda| \in L^2(\mu_\Lambda)$, where $W_\Lambda = \sum_{i\in\Lambda} W_i(x_i)$. Hence $\mathrm{d}\mu_\Lambda = p_\Lambda \mathrm{d}\eta_\Lambda$ with $\sqrt{p_\Lambda} \in H^{2,1}(\eta_\Lambda)$. Indeed, $p_\Lambda = f_\Lambda \exp(-W_\Lambda)$, where both factors have modifications which are absolutely continuous along almost all coordinate lines in any fixed local chart. Clearly, $\sqrt{p_\Lambda}$ has the same property. In addition, the mapping $\nabla_\Lambda\sqrt{p_\Lambda}$ evaluated by means of such modifications coincides η_Λ-a.e. with $\frac{1}{2}(\nabla_\Lambda f_\Lambda/\sqrt{f_\Lambda} - \nabla_\Lambda W_\Lambda\sqrt{f_\Lambda})\exp(-W_\Lambda/2)$, which is η_Λ-square integrable, since $|\nabla_\Lambda f_\Lambda|^2/f_\Lambda$ and $|\nabla_\Lambda W_\Lambda|^2 f_\Lambda$ are in $L^1(\lambda_\Lambda)$. By Lemma 2.1, $\nabla_\Lambda W_\Lambda$ and $\nabla_\Lambda f_\Lambda/f_\Lambda = L^2(\mu)$-$\lim_{\varepsilon\to 0}\nabla_\Lambda \log(f_\Lambda + \varepsilon)$ belong to $\Gamma(\mu_\Lambda)$. On the other hand, the vector field

$$Z_\Lambda^\mu - \frac{\nabla_\Lambda f_\Lambda}{f_\Lambda} = \nabla_\Lambda W_\Lambda + Y_\Lambda^\mu - \frac{\nabla_\Lambda f_\Lambda}{f_\Lambda}$$

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is orthogonal to $\Gamma(\mu_\Lambda)$ in the space $L^2(\mu_\Lambda, \mathrm{Vec}(M_\Lambda))$. Therefore, we obtain the orthogonal decomposition

$$Y_\Lambda^\mu = \left(\frac{\nabla_\Lambda f_\Lambda}{f_\Lambda} - \nabla_\Lambda W_\Lambda\right) + \left(Z_\Lambda^\mu - \frac{\nabla_\Lambda f_\Lambda}{f_\Lambda}\right),$$

whence

$$\int\limits_{M^\Lambda} \left|\nabla_\Lambda W_\Lambda - \frac{\nabla_\Lambda f_\Lambda}{f_\Lambda}\right|^2 \mathrm{d}\mu_\Lambda \leq \int\limits_{M^\Lambda} \left|Y_\Lambda^\mu\right|^2 \mathrm{d}\mu_\Lambda.$$

Therefore, one has

$$\begin{aligned}\int\limits_{M^\Lambda} \left|\nabla_\Lambda \sqrt{p_\Lambda}\right|^2 \mathrm{d}\eta_\Lambda &= \frac{1}{4}\int\limits_{M^\Lambda} \left|\nabla_\Lambda W_\Lambda - \frac{\nabla_\Lambda f_\Lambda}{f_\Lambda}\right|^2 \mathrm{d}\mu_\Lambda \\ &\leqslant \frac{1}{4}\int\limits_{M^\Lambda} \left|Y_\Lambda^\mu\right|^2 \mathrm{d}\mu_\Lambda \leqslant \frac{1}{4}\int |Y|^2\,\mathrm{d}\mu.\end{aligned} \tag{4.6}$$

Let $\Lambda_n = \{1, \ldots, n\}$ and let $\sigma_n = \mathcal{B}(M^{\Lambda_n}) \times M^{\Lambda_n^c}$. By (4.4) and (4.6), one has

$$\begin{aligned}\int\limits_{M^S} p_{\Lambda_n} \log p_{\Lambda_n} \mathrm{d}\eta &= \int\limits_{M^{\Lambda_n}} p_{\Lambda_n} \log p_{\Lambda_n} \mathrm{d}\eta_{\Lambda_n} \\ &\leqslant \alpha \int \left|\nabla_{\Lambda_n} \sqrt{p_{\Lambda_n}}\right|^2 \mathrm{d}\eta_{\Lambda_n} \leqslant \frac{\alpha}{4}\int |Y|^2\,\mathrm{d}\mu.\end{aligned}$$

This means that $\{p_{\Lambda_n}\}_{n\geqslant 1}$ is uniformly integrable with respect to η. Furthermore, it is readily seen that $(p_{\Lambda_n})_{n\geqslant 1}$ is a σ_n-martingale under η. Then, $p_{\Lambda_n} \to p$ in $L^1(\eta)$ for some $p \in L^1(\eta)$. We have $\sqrt{p} \in H^{2,1}(\eta)$ by (4.6). On the other hand, we have for $f \in \mathcal{FC}_0^\infty$ and large enough n

$$\int f\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu_{\Lambda_n} = \int f p_{\Lambda_n}\,\mathrm{d}\eta_{\Lambda_n} = \int f p_{\Lambda_n}\,\mathrm{d}\eta = \lim_{k\to\infty}\int f p_{\Lambda_k}\,\mathrm{d}\eta = \int f p\,\mathrm{d}\eta,$$

hence $\mathrm{d}\mu = p\,\mathrm{d}\eta$. □

Additional results about regularity in infinite dimensions are given in Section 7 devoted to the so called finite range case.

5. Existence results in infinite dimensions

We keep the notation introduced in Section 4, in particular, given vector fields Z_i, $i \in S$, on M^S such that $Z_i(x) \in T_{x_i}M^i$, and we consider the elliptic operators L_i and (heuristically) $L_Z = \sum_i L_i$.

Recall that $o_i \in M^i$ are fixed points.

Let $A = (a_{i,j})_{i,j\in S}$ be an infinite symmetric matrix with $a_{i,j} \geqslant 0$. Given a collection of positive numbers $q = (q_i)_{i\in S}$ such that $\sum_{i\in S} q_i < \infty$, we denote by $l^1(q)$ the space L^1 with respect to the discrete measure on S that assigns q_i to i. Given a collection $\xi = (\xi_i)_{i\in S}$ of nonnegative numbers, we write $A\xi \leqslant \lambda\xi$ for some $\lambda \geqslant 0$ if $\sum_{j\in S} a_{i,j}\xi_j \leqslant \lambda\xi_i$ for all $i \in S$.

We shall assume that S is a union of an increasing sequence of finite sets Λ_n, $n \in \mathbb{N}$, which is always possible, since S is countable; a typical example is $S = \mathbb{Z}^d$, $\Lambda_n = \{(z_1, \ldots, z_d)\colon |z_i| \leqslant n\}$.

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Given a family $(C_i)_{i\in S}$ of real numbers, we write $C_i \to +\infty$ if $\lim_{n\to\infty} \inf_{i\in S\setminus\Lambda_n} C_i = +\infty$. The restriction of a function $f : M^S \to \mathbb{R}$ to M^{Λ_n} is defined by the equality $f_{\Lambda_n|o}(x_{\Lambda_n}) := f(x_{\Lambda_n} \times o_{\Lambda_n^c})$ and is denoted by $f_{\Lambda_n|o} : M^{\Lambda_n} \to \mathbb{R}$. Correspondingly, we set

$$Z_{\Lambda_n|o}(x_{\Lambda_n}) := Z(x_{\Lambda_n} \times o_{\Lambda_n^c}), \qquad L_{\Lambda_n|o} f := \Delta_{\Lambda_n} f_{\Lambda_n|o} + Z_{\Lambda_n} f_{\Lambda_n|o}$$

for $f : M^S \to \mathbb{R}$ such that $f_{\Lambda_n|o} \in C^2(M^{\Lambda_n})$.

Let us introduce a class of test functions that will be employed below. Given nonnegative functions $G_i \in C(M^i)$, we set

$$\Psi(x) := \sum_{i\in S} q_i G_i(x).$$

Suppose that the set $\Omega := \{\Psi < \infty\}$ is equipped with some completely regular topology τ such that the natural embedding $(\Omega, \tau) \to M^S$ is continuous.

By $\mathcal{K}_\Psi$ we denote the class of all bounded functions $f : M^S \to \mathbb{R}$ with the following properties:

(1) f is zero outside of one of the sets $S_r := \{\Psi \leqslant r\}$,

(2) f has partial derivatives of all orders whose restrictions to M^{Λ_n} are continuous, and such that the restrictions of f and of the $L_i f$'s to Ω are τ-continuous,

(3) all the functions $L_i f$ are bounded and the series $\sum_{i\in S} L_i f$ converges uniformly on S_r.

Note that $f_{\Lambda_n|o} \in C_0^\infty(M^{\Lambda_n})$ for all $f \in \mathcal{K}_\Psi$.

Throughout, we use the following convention: every measure μ on M^{Λ_n} is considered as a measure on M^S (i.e., as $\mu \otimes \delta_{o_{\Lambda_n^c}}$).

The existence results in this section are based on the following simple lemma.

LEMMA 5.1. – *Suppose that for all $n \in \mathbb{N}$ the restrictions of the fields Z_i to $M^{\Lambda_n} \times \{o_{\Lambda_n^c}\}$ are locally integrable in power bigger than* $\dim M^{\Lambda_n}$ *(e.g., are Borel and locally bounded). Let $\|A\|_{l^1(q)} \le \lambda$ and $\sum_{j\in S} a_{i,j} \leqslant \lambda$ for all $i \in S$. Assume that for each $i \in S$, there exist nonnegative compact functions $V_i \in C^2(M^i)$ and $G_i \in C(M^i)$ such that $G_i(o_i) = 0$ and, for some $c, \delta > 0$, one has for all $x \in M^S$ and $n \in \mathbb{N}$*

$$\begin{aligned}(5.1)\qquad &\Delta_i V_i(x_i) + \bigl\langle \nabla_i V_i(x_i), Z_i(x_{\Lambda_n} \times o_{\Lambda_n^c})\bigr\rangle \\ &\quad\leqslant c - (\lambda+\delta) G_i(x_i) + \sum_{j\in\Lambda_n} a_{i,j} G_j(x_j), \quad i \in \Lambda_n.\end{aligned}$$

Then there exist measures $\mu_n \in \mathcal{P}(M^{\Lambda_n})$ such that $L^*_{\Lambda_n|o}\mu_n = 0$ with respect to $C_0^\infty(M^{\Lambda_n})$ and

$$(5.2)\qquad \int G_i \, d\mu_n \leqslant \frac{c}{\delta}, \quad \forall n \in \mathbb{N},\ i \in S.$$

In particular, $\int \Psi \, d\mu_n \leqslant \frac{c}{\delta} \sum_{i\in S} q_i$ and the sequence $\{\mu_n\}$ is relatively weakly compact. Moreover, the same is true if $V_i = F_i \circ \varrho_i$, where $F_i \in C^2[0,\infty)$ is such that $\lim_{r\to\infty} F_i(r) = +\infty$ and (5.1) *holds for all $x = (x_i)_{i\in S} \in M^S$ with $x_i \notin \mathrm{cut}(o_i)$, $\forall i \in S$.*

Proof. – We may assume that $S = \mathbb{N}$, $\Lambda_n = \{1, \ldots, n\}$, and $\sum_{i=1}^\infty q_i = 1$. Let $o := (o_1, o_2, \ldots)$ and

$$\Psi_n(x_{\Lambda_n}) := \sum_{i\leqslant n} q_i G_i(x_i), \qquad \Phi_n(x_{\Lambda_n}) := \sum_{i\leqslant n} q_i V_i(x_i).$$

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By (5.1) and the estimate $\|A\|_{l^1(q)} \leqslant \lambda$, which means that

$$\sum_{i,j} q_i a_{i,j}|z_j| \leqslant \lambda \sum_j q_j|z_j|,$$

one has (recall that $G_j(o_j) = 0$)

$$L_{\Lambda_n|o}\Phi_n(x_{\Lambda_n}) \leqslant c - \delta\Psi_n(x_{\Lambda_n}).$$

Therefore, by Theorem 1.1, there exists $\mu_n \in \mathcal{P}(M^{\Lambda_n})$ such that

$$L^*_{\Lambda_n|o}\mu_n = 0$$

with respect to $C_0^\infty(M^{\Lambda_n})$ and, by the last statement of Proposition 3.2 one has

$$\int \Psi_n \,\mathrm{d}\mu_n \leqslant \frac{c}{\delta}.$$

In the case when V_i is replaced by $F_i \circ \varrho_i$ for $F_i \in C^2[0,\infty)$ and $\lim_{r\to\infty} F(r) = \infty$, we modify the proof of Theorem 3.1 for the product manifold M^{Λ_n}. For $i \in \Lambda_n$, let $B_{l,i} = B(o_i, l)$ be the closed geodesic ball in M^i with center o_i and radius l. Let $B_l = \prod_{i\in\Lambda_n} B_{l,i}$. Let $h_{i,\varepsilon}$ be chosen for F_i as h_ε in the proof of Theorem 3.1 has been chosen for F. We obtain

$$L_{\Lambda_n|o}\left(\sum_{i\in\Lambda_n} q_i h_{i,\varepsilon} \circ F_i \circ \varrho_i\right) \leqslant c - \delta \sum_{i\in\Lambda_n} q_i h'_{i,\varepsilon}(F_i \circ \varrho_i) G_i(x_i). \tag{5.3}$$

Let $\mu^l \in \mathcal{P}(B_l)$ with density $p^l \in H^{\alpha,1}(B_l; \lambda_{M^{\Lambda_n}})$ for some $\alpha > \dim M^{\Lambda_n}$ be such that (3.2) holds with $L_{\Lambda_n|o}$ and M^{Λ_n} in place of L_Z and M, respectively. Then (3.4) holds for $\lambda_{M^{\Lambda_n}}$ in place of λ_M. Finally, let $\{D^i_m\}$ be taken as in the proof of Theorem 3.1 for $B^o_{l,i}$ and $\mathrm{cut}(o_i)$. Let $D_m := \prod_{i\in\Lambda_n} D^i_m$. Then it is easy to obtain an analogue of (3.5) for the present situation, which in turn leads to the existence of $\mu_n \in \mathcal{P}(M^{\Lambda_n})$ such that $L^*_{\Lambda_n|o}\mu_n = 0$ and $\int \Psi_n \,\mathrm{d}\mu_n \leqslant c/\delta$.

Let us regard μ_n as a probability on M^S. Then $\int \Psi \,\mathrm{d}\mu_n \leqslant c/\delta$, which yields that the sequence $\{\mu_n\}$ is uniformly tight. Let us show (5.2). Let n be fixed and let $\xi_i = \int G_i \,\mathrm{d}\mu_n$. Then $\xi = (\xi_i) \in l^1(q)$. It follows by the above reasoning that

$$\xi \leqslant (\lambda+\delta)^{-1}c + (\lambda+\delta)^{-1}A\xi,$$

where the sequence $(c, c, \ldots,) \in l^1(q)$ is denoted by c. Iterating this inequality and using the estimate $\|A\|_{l^1(q)} \leqslant \lambda$, we obtain

$$\xi \leqslant \frac{c}{\lambda+\delta}\sum_{n=0}^\infty \left(\frac{A}{\lambda+\delta}\right)^n 1 \leqslant \frac{c}{\lambda+\delta}\sum_{n=0}^\infty \left(\frac{\lambda}{\lambda+\delta}\right)^n 1 = \frac{c}{\delta}1,$$

since $\sum_{j\in S} a_{i,j} \leqslant \lambda$, so that $A(1) \leqslant \lambda$. □

PROPOSITION 5.2. – *Let A, q, V_i, G_i, and Z_i be as in Lemma* 5.1 *such that* (5.1) *holds. Suppose that the sets* $\{\sum_{i\in S} C_i q_i G_i \leqslant r\}$ *are τ-compact whenever $C_i \in \mathbb{R}_+$ such that $C_i \to +\infty$*

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and that the restrictions of Z_i's to these sets are τ-continuous. Then there exists $\mu \in \mathcal{P}(M^S)$ such that

$$\int G_i \,\mathrm{d}\mu \leqslant \frac{c}{\delta} \tag{5.4}$$

and $L_Z^*\mu = 0$ with respect to the class $\mathcal{K}_\Psi$. Moreover, the same is true if $V_i = F_i \circ \varrho_i$, where $F_i \in C^2[0,\infty)$ is such that $\lim_{r\to\infty} F_i(r) = +\infty$ *and* (5.1) *holds for all $x = (x_i)_{i\in S} \in M^S$ with $x_i \notin \mathrm{cut}(o_i)$, $\forall i \in S$.*

Proof. – Let us keep the same notation as in the proof of Lemma 5.1. The sequence $\{\mu_n\}$ constructed in that lemma is uniformly tight, hence there exists a measure $\mu \in \mathcal{P}(M^S)$ which is the weak limit of some subsequence $\{\mu_{n_j}\}_{j\in\mathbb{N}}$. It follows from (5.2) that there exist positive numbers $(C_i)_{i\in S}$ with $C_i \to +\infty$ such that

$$\sup_n \int \sum_{i\in S} C_i q_i G_i \,\mathrm{d}\mu_n < \infty,$$

and the same is true for μ in place of μ_n. Hence by assumption the sequence $\{\mu_n\}$ is also uniformly tight on Ω with respect to the topology τ, hence $\mu_{n_j} \to \mu$ weakly on (Ω, τ) as $j \to \infty$. By the definition of $L_{\Lambda_n|o}$, we obtain for every $f \in \mathcal{K}_\Psi$ that

$$\int \sum_{i=1}^n L_i f \,\mathrm{d}\mu_n = \int L_{\Lambda_n|o} f_{\Lambda_n|o} \,\mathrm{d}\mu_n = 0.$$

Let K be the support of f on Ω. By definition, the bounded τ-continuous functions $\sum_{i=1}^n L_i f$ converge to Lf uniformly on K, whence we obtain the desired conclusion. □

PROPOSITION 5.3. – *Consider the situation of Proposition* 5.2. *In addition, suppose that for every $i \in S$, there exist $\gamma_i \in C[0,\infty)$ and $\psi_i \in C(M)$ such that* $\lim_{r\to\infty} \gamma_i(r)/r = 0$ *and*

$$\bigl|Z_i(x)\bigr| \leqslant \psi_i(x_i) + \gamma_i\Bigl(\sum_{j\in S} q_j G_j(x_j)\Bigr), \quad x \in M^S,\ i \in S. \tag{5.5}$$

If there exists $h \in C^2[0,\infty)$ such that $h' \geqslant 0$, $h'' \leqslant 0$, $h(\infty) = \infty$, and $\|\nabla_i h(V_i)\|_\infty < \infty$, $i \in S$, then there exists $\mu \in \mathcal{P}(M^S)$ such that (5.4) *holds and $L_Z^*\mu = 0$ with respect to the class $\mathcal{FC}_0^\infty$. If, in addition,* (5.5) *holds for $\psi_i = \gamma_i(G_i)$, then $|Z_i| \in L^1(\mu)$ and $\int L_Z f \,\mathrm{d}\mu = 0$ for every $f \in \mathcal{FC}_b^\infty$.*

Finally, under the assumptions of Corollary 2.3 *for each M^i in case $\psi_i = \gamma_i(G_i)$ but without the assumption on the above function h, the above results hold for $V_i = F_i \circ \varrho_i$ with $F_i \in C^2[0,\infty)$ provided that* $\lim_{r\to\infty} F_i(r) = \infty$ *and* (5.1) *holds for all $x = (x_i)_{i\in S} \in M^S$ with $x_i \notin \mathrm{cut}(o_i)$, $\forall i \in S$.*

Proof. – Without loss of generality, we assume that the sequence $\{\mu_n\}$ in Lemma 5.1 converges weakly to μ. We only have to verify the equality $\int L_Z f \,\mathrm{d}\mu = 0$ for every $f \in \mathcal{FC}_0^\infty$, where μ is the measure constructed in Lemma 5.1. We see that $\sum_{i\leqslant n} q_i h(V_i)$ is a function satisfying the assumption of Remark 2.4 for M^{Λ_n} and $L_{\Lambda_n|o}$. Suppose that $f(x) = f_0(x_\Lambda)$, where $f_0 \in C_0^\infty(M^\Lambda)$, and let n be so large that $\Lambda \subset \Lambda_n$. Then by Remark 2.4

$$\int L_Z f \,\mathrm{d}\mu_n = \int \sum_{i\in\Lambda} L_i f \,\mathrm{d}\mu_n = \int \sum_{i\in\Lambda_n} L_i f \,\mathrm{d}\mu_n = 0.$$

Clearly, by the weak convergence, we have

$$\int \sum_{i\in\Lambda} \Delta_i f \,\mathrm{d}\mu_n \to \int \sum_{i\in\Lambda} \Delta_i f \,\mathrm{d}\mu.$$

In addition, for every fixed $i \in \Lambda$, one has

$$\int_{\Omega} \langle Z_i, \nabla_i f\rangle \,\mathrm{d}\mu_n \to \int_{\Omega} \langle Z_i, \nabla_i f\rangle \,\mathrm{d}\mu. \tag{5.6}$$

Indeed, the function $g = \langle Z_i, \nabla_i f\rangle$ is τ-continuous and μ-integrable. This follows by (5.5), since $\psi_i(x_i)$ is bounded on the support of f_0 and the function $\gamma_i(\sum_{j\in S} q_j G_j)$ is μ-integrable by (5.4). We obtain from (5.5) that

$$\lim_{R\to+\infty} \sup_n \int_{|g|>R} |g| \,\mathrm{d}\mu_n = 0.$$

This together with the weak convergence of $\{\mu_n\}$ yields (5.6). The second assertion is proved by a similar argument. The last assertion can be proved in the same way by using Corollary 2.3 instead of Remark 2.4. □

Let us consider a typical example of a topology τ that can be used in Proposition 5.2.

Example 5.4. – Let $G_i(x) := \varrho_i(x_i)^p$, where $p \geqslant 1$. Let τ be the topology on the set $\{\Psi < \infty\}$ generated by the metric

$$d_p(x, y) := \left(\sum_{i\in S} q_i \varrho_i(x_i, y_i)^p\right)^{1/p}.$$

Then the sets $K_r := \{\sum_{i\in S} C_i q_i G_i \leqslant r\}$ are τ-compact if $C_i \in \mathbb{R}_+$ and $C_i \to \infty$.

Proof. – Let $x^{(j)} = (x_i^{(j)})_{i\in S}$ be a sequence of points in the set K_r. It is readily seen that K_r is compact in the product topology of M^S, hence there exist a subsequence $\{y^{(k)}\} \subset \{x^{(j)}\}$ and a point $x = (x_i) \in K_r$ such that $y_i^{(k)} \to x_i$ as $k \to \infty$ for every fixed $i \in S$. Given $\varepsilon > 0$, we can pick n_0 such that $C_i \geqslant \varepsilon^{-1}$ if $i \notin \Lambda_{n_0}$. Noting that

$$\sum_{i\notin\Lambda_{n_0}} q_i G_i(z) \leqslant \varepsilon \sum_{i\in S} C_i q_i G_i(z) \leqslant r\varepsilon$$

for all $z \in K_r$, we obtain that $d_p(y^{(k)}, x) < 2(r\varepsilon)^{1/p} + \varepsilon$ for all k sufficiently large, i.e., x is a cluster point of $\{x^{(j)}\}$ with respect to τ. □

THEOREM 5.5. – *Assume that* $\sup_{i\in S} \dim M^i < \infty$ *and that each* M^i *satisfies the assumptions of Corollary* 2.3 *with Ricci curvature bounded below by a constant independent of* i. *Let*

$$Z_i(x) = Y_i(x_i) + \widehat{Y}_i(x),$$

where the mappings $\widehat{Y}_i$ *are continuous on* M^S, *the* Y_i*'s are continuous on* M^i *and*

$$\langle Y_i, \nabla_i \varrho_i\rangle \leqslant c' - (\lambda+\delta)\varrho_i^p, \qquad \bigl|\widehat{Y}_i(x)\bigr| \leqslant c' + \sum_j a_{i,j}\varrho_j(x_j)^p$$

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for some c', $\delta > 0$, $p \geqslant 1$ *and* $(a_{i,j})$ *given as in Lemma* 5.1. *If* (5.4) *holds for* $G_i = \varrho_i^p$, *then there exists* μ *such that* $L_Z^*\mu = 0$ *with respect to* $\mathcal{F}\mathcal{C}_0^\infty$. *Moreover, one has* $\sup_{i\in S} \int \varrho_i^p \,\mathrm{d}\mu < \infty$.

Proof. – By the Laplacian comparison theorem, our assumptions imply $\Delta_i \varrho_i \leqslant c_0(1 + \varrho_i^{-1})$ outside $\mathrm{cut}(o_i) \cup \{o_i\}$ for some $c_0 > 0$ and all $i \in S$. Then (5.1) holds for some $c > c'$, all $x = (x_i) \in M^S$ with $x_i \notin \mathrm{cut}(o_i)$, $G_i = \varrho_i^p$, and $V_i = F \circ \varrho_i$ for some $F \in C^\infty[0,\infty)$ such that $0 \notin \operatorname{supp} F$ and $F(r) = r$ for $r \geqslant 1, \forall i \in S$. For τ we take the topology generated by the metric d_p discussed in Theorem 5.4. Therefore, the assertion follows from Proposition 5.3. □

Finally, let us observe that the condition $\|A\|_{l^1(q)} \leqslant \lambda$ is satisfied if $Aq \leqslant \lambda q$ in the above sense. A simple sufficient condition for the estimate $Aq \leqslant \lambda q$ is this: $a_{i,j} = a(i-j)$, where $S = \mathbb{Z}^d$, $0 \leqslant a(i) \leqslant c_1 q_i^2$, $a(i) = a(-i)$, $\sum_i q_i \leqslant c_2$, $q_{i-j} q_j \leqslant c_3 q_i$, $\lambda = c_1 c_2 c_3$. For example, if $d = 1$, it suffices that $q_i = |i|^{-r}$, $a(i) \leqslant |i|^{-2r}$, $r > 1$. In particular, as we shall see in the next section, Example 5.5 yields the existence of Gibbs measures for many models with the finite radius of interaction.

6. Applications to Gibbs measures

In this section, we discuss an important special case of the elliptic equation $L_Z^*\mu = 0$ where the Z_i's are logarithmic derivatives of μ along the x_i's; then every term in (4.3) vanishes separately which simplifies certain technical issues. Note that if in this case $Z_i \in L^2(\mu)$, then the operator L_Z is symmetric on $L^2(\mu)$. We shall now introduce a suitable concept of differentiability of measures (and a local version of logarithmic derivative). Let div_i denote the divergence of vector fields on M^i.

DEFINITION 6.1. – *Let* μ *be a Radon measure on* M^S, *let* $i \in S$ *be fixed, and let* $\mathcal{K}$ *be a certain class of functions differentiable along* x_i *and separating the measures on* M^S. *We say that* μ *has the logarithmic derivative* β_i *along* x_i *with respect to* $\mathcal{K}$, *if* β_i *is a* μ*-measurable vector field such that* $\beta_i(x) \in T_{x_i} M^i$ *and, for every* $\psi \in \mathcal{K}$ *and every* $v \in \mathrm{Vec}_0^\infty(M^i)$, *the set of all compactly supported* C^∞*-vector fields on* M^i, *one has* $\langle \nabla_i \psi, v\rangle$, $\psi \,\mathrm{div}_i v + \psi \langle v, \beta_i\rangle \in L^1(\mu)$ *and*

$$\int \langle \nabla_i \psi, v\rangle \,\mathrm{d}\mu = -\int \psi\left(\mathrm{div}_i\, v + \langle v, \beta_i\rangle\right) \mathrm{d}\mu. \tag{6.1}$$

The logarithmic derivative β_i of μ will also be denoted by β_i^μ.

We shall see that under broad assumptions, a measure μ with the logarithmic derivatives Z_i along x_i satisfies the elliptic equation $L_Z^*\mu = 0$. This follows from the formal integration by parts on every term in (4.3), but requires a justification.

Let v be a fixed smooth compactly supported vector field on a Riemannian manifold M and let U_t^v, $t \in \mathbb{R}^1$, be the corresponding flow, i.e., $U_t^v(x)$ solves the ordinary differential equation $x'(t) = v(x(t))$, $x(0) = x$.

The following lemma is a straightforward modification of a result in [49] proved in the linear case for globally integrable logarithmic derivatives (cf. also [6] and [7] for the manifold case). Although the reasoning is essentially the same as in [49], we include a proof for completeness, since some additional technicalities arise. This lemma shows how Gibbs measures fit into the above framework of elliptic equations.

LEMMA 6.2. – *Let* $X = M \times Y$, *where* M *is a finite-dimensional Riemannian manifold and* $(Y, \mathcal{F})$ *is a measurable space, let* μ *be a measure on* $\mathcal{B} = \mathcal{B}(M) \otimes \mathcal{F}$ *with the regular conditional measures* μ^y *on* $M \times \{y\}$, *and let* ν *be the projection of* $|\mu|$ *to* Y. *Suppose that* $\mathcal{K}$ *is a class of bounded* $\mathcal{B}$*-measurable functions that satisfies the following conditions*:

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(i) *for every* $\psi \in \mathcal{K}$ *and* $y \in Y$, *the function* $x \mapsto \psi(x, y)$ *is continuously differentiable and* $\nabla_x \psi$ *is bounded;*
(ii) $(x, y) \mapsto \psi(U_t^v(x), y) \in \mathcal{K}$ *and* $\varphi \circ \psi \in \mathcal{K}$ *whenever* $\varphi \in C_0^\infty(\mathbb{R}^1)$, $\varphi(0) = 0$, $\psi \in \mathcal{K}$, $t \in \mathbb{R}^1$, $v \in \mathrm{Vec}_0^\infty(M)$, *and* $\psi_1\psi_2 \in \mathcal{K}$ *if* $\psi_1, \psi_2 \in \mathcal{K}$*;*
(iii) *the class* $\mathcal{K}$ *separates the measures on* $\mathcal{B}$.

Let $(x, y) \mapsto \beta(x, y) \in T_x M$ *be a* μ*-measurable mapping (i.e.,* $\langle \beta, v\rangle$ *is* μ*-measurable for all smooth vector fields* v *on* M*) such that* $\psi|\beta| \in L^1(\mu)$ *for every* $\psi \in \mathcal{K}$ *and*

$$\int \langle \nabla_x \psi, v\rangle \,\mathrm{d}\mu = -\int \psi \,\mathrm{div}\, v \,\mathrm{d}\mu - \int \psi \langle \beta, v\rangle \,\mathrm{d}\mu \tag{6.2}$$

for every $\psi \in \mathcal{K}$ *and every* $v \in \mathrm{Vec}_0^\infty(M)$. *Then, for* ν*-a.e.* y, μ^y *admits a density* f^y *on the fiber* $M \times \{y\}$ *such that*

$$f^y \in W_{\mathrm{loc}}^{1,1}(\lambda_M) \quad \text{and} \quad \beta(x, y) = \nabla_x f^y(x)/f^y(x). \tag{6.3}$$

Proof. – We can find an increasing sequence of measurable sets $A_j \subset X$ such that $\bigcup_j A_j$ has full measure and there exist functions $\varphi_j \in \mathcal{K}$ with $\varphi_j > 0$ on A_j. Indeed, let $\mathcal{K}_0 = \{\psi \in \mathcal{K}\colon 0 \leqslant \psi \leqslant 1\}$. By [27, Theorem IV.11.6], there is a sequence $\varphi_j \in \mathcal{K}_0$ such that, for every $\psi \in \mathcal{K}_0$, one has $\psi \leqslant \sup_j \varphi_j$ μ-a.e. Then the union of the sets $A_j = \{\varphi_j > 0\}$ has full measure. Indeed, if $\sup_j \varphi_j = 0$ on a positive measure set A, then for every $\psi \in \mathcal{K}_0$, one has $\psi = 0$ μ-a.e. on A, hence the same is true for every $\varphi \in \mathcal{K}$, which easily follows by taking compositions with smooth compactly supported functions vanishing at the origin. Thus, the measure $\mu|_A$ and the zero measure are not separated by $\mathcal{K}$, which is a contradiction. Moreover, we may assume that $\varphi_j = 1$ on A_j. Indeed, every φ_j can be replaced by the sequence of functions $\zeta_k \circ \varphi_j$, where $\zeta_k \in C_0^\infty(\mathbb{R}^1)$, $0 \leqslant \zeta_k \leqslant 1$, $\zeta_k(t) = 0$ if $t \leqslant 0$, $\zeta_k(t) = 1$ if $k^{-1} \leqslant t \leqslant k$, and $\zeta_k(t) = 0$ if $t \geqslant k+1$. Let us consider the measure

$$\mu_j = \varphi_j \,\mu.$$

Letting $\beta_j = \beta + \nabla_x \varphi_j/\varphi_j$, and using that $\psi_1\psi_2 \in \mathcal{K}$ for all $\psi_1, \psi_2 \in \mathcal{K}$, we obtain from (6.2) that

$$\int \langle \nabla_x \psi, v\rangle \,\mathrm{d}\mu_j = -\int \psi \,\mathrm{div}\, v \,\mathrm{d}\mu_j - \int \psi \langle \beta_j, v\rangle \,\mathrm{d}\mu_j$$

for every $\psi \in \mathcal{K}$ and every smooth compactly supported vector field v on M. In addition, $|\beta_j| \in L^1(\mu_j)$. Let v be a fixed smooth compactly supported vector field on M and let U_t^v, $t \in \mathbb{R}^1$, be the corresponding flow. Then we have

$$\begin{aligned} &\int_X \bigl[\psi\bigl(U_t^v(x), y\bigr) - \psi(x, y)\bigr] \,\mathrm{d}\mu_j \\ &\quad = -\int_0^t \int_X \psi\bigl(U_s^v(x), y\bigr)\bigl(\mathrm{div}\, v(x) + \bigl\langle \beta_j(x, y), v(x)\bigr\rangle\bigr) \,\mathrm{d}\mu_j \,\mathrm{d}s \end{aligned} \tag{6.4}$$

for all $\psi \in \mathcal{K}$, which is proved as follows. Both sides of (6.4) are continuously differentiable in t and vanish at $t = 0$. We observe that for every $\varphi \in \mathcal{K}$, one has

$$\frac{\partial}{\partial \tau} \varphi\bigl(U_\tau^v(x), y\bigr)\big|_{\tau=0} = \bigl\langle \nabla_x \varphi(x, y), v(x)\bigr\rangle.$$

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Therefore,

$$\frac{\partial}{\partial t}\psi\big(U_t^v(x),y\big)=\frac{\partial}{\partial\tau}\psi\big(U_t^v\big(U_\tau^v(x)\big),y\big)\big|_{\tau=0}=\big\langle\nabla_x\big[\psi\big(U_t^v(x),y\big)\big],v(x)\big\rangle.$$

Hence the derivatives of the left and right sides of (6.4) are given by

$$\int_X\big\langle\nabla_x\big[\psi\big(U_t^v(x),y\big)\big],v(x)\big\rangle\,\mathrm{d}\mu_j$$

and

$$-\int_X\psi\big(U_t^v(x),y\big)\big(\operatorname{div}v(x)+\big\langle\beta_j(x,y),v(x)\big\rangle\big)\,\mathrm{d}\mu_j,$$

respectively, hence are equal. The left-hand side of (6.4) equals the integral of ψ with respect to the measure $(\mu_j)_t-\mu_j$, where $(\mu_j)_t$ is the image of μ_j under the shift $(x,y)\mapsto(U_t^v(x),y)$. The right-hand side of (6.4) is the integral of ψ against the measure

$$\sigma_j^t:=\int_0^t\big(\big[\operatorname{div}v+\langle\beta_j,v\rangle\big]\mu_j\big)_s\,\mathrm{d}s.$$

Hence, by our assumption on $\mathcal{K}$, we have

$$(\mu_j)_t-\mu_j=\sigma_j^t.$$

This implies that (6.4) holds for all bounded $\mathcal{B}$-measurable functions ψ, which enables us to reduce the claim to the case $M=\mathbb{R}^d$ (however, with a Riemannian structure possibly different from the standard one). Indeed, (6.4) is true, in particular, for all functions of the form $\psi(x,y)=f(x)\psi_0(x,y)$, where $f\in C_0^\infty(M)$ has support in a local chart U and $\psi_0\in\mathcal{K}$. Differentiating (6.4) at $t=0$, we arrive at the equality

$$\int\langle\nabla_x\psi_0,v\rangle\,f\,\mathrm{d}\mu_j=-\int\langle\nabla_x f,v\rangle\,\psi_0\,\mathrm{d}\mu_j-\int\psi_0\big(\operatorname{div}v+\langle\beta_j,v\rangle\big)f\,\mathrm{d}\mu_j$$

for every $v\in\mathrm{Vec}_0^\infty(M)$. This shows that the measure $f\,\mu_j$ satisfies the same condition as μ_j with $\beta_j+\nabla_x f/f$ in place of β_j. Therefore, it suffices to consider the case where μ_j has support in U. Moreover, by considering vector fields v that are constant on the support of f, we may assume that (6.4) is true for the constant fields e_i, $i=1,\dots,d$, where $\{e_i\}$ is a standard basis in $\mathbb{R}^d$, and all t from a fixed interval. Let us set

$$\widehat{\beta}_j(x)=\big(\langle\beta_j(x),e_1\rangle+\operatorname{div}e_1,\dots,\big\langle\beta_j(x),e_d\big\rangle+\operatorname{div}e_d\big),$$

where div and $\langle\cdot,\cdot\rangle$ correspond to the Riemannian structure of M (so that the divergences of the constant fields e_i may be nonzero). By the assumption that μ_j has support in U, we have $|\widehat{\beta}_j|\in L^1(\mu_j)$. We shall denote by $(\cdot,\cdot)$ the standard inner product in $\mathbb{R}^d$. Then, for every vector v from the unit ball U_d in $\mathbb{R}^d$ and every $t\in[0,1]$, we obtain the relation

$$(\mu_j)_{tv}-\mu_j=\int_0^t\big(\big(\widehat{\beta}_j,v\big)\,\mu_j\big)_{sv}\,\mathrm{d}s. \tag{6.5}$$

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We set

$$\mu_j^y = \varphi_j \mu^y, \quad \text{i.e.,} \quad \mu_j(B) = \int_Y \mu_j^y(B)\,\nu(\mathrm{d}y).$$

Now (6.5) yields the absolute continuity of the measures μ_j^y for ν-a.e. y. Indeed, let p be a probability density on $\mathbb{R}^d$ with support in U_d, $p_\varepsilon(t) = \varepsilon^{-d} p(t/\varepsilon)$, $\gamma_\varepsilon = p_\varepsilon\,\mathrm{d}x$, $\varepsilon \in (0,1)$, and let

$$\pi_\varepsilon(B) = \int_Y \mu_j^y * \gamma_\varepsilon(B)\,\nu(\mathrm{d}y).$$

Then for every bounded Borel function g, one has

$$\begin{aligned}\int_X g(x,y)\,\mathrm{d}\pi_\varepsilon &= \int_Y \int_{\mathbb{R}^d\times\{y\}} \int_{\mathbb{R}^d} g(x+\varepsilon z, y)\,p(z)\,\mathrm{d}z\,\mu_j^y(\mathrm{d}x)\,\nu(\mathrm{d}y)\\ &= \int_{\mathbb{R}^d}\int_X g(x+\varepsilon z, y)\,p(z)\,\mathrm{d}\mu_j\,\mathrm{d}z. \qquad (6.6)\end{aligned}$$

It follows from (6.5) and (6.6) that

$$\begin{aligned}\left|\int_X g\,\mathrm{d}\mu_j - \int_X g\,\mathrm{d}\pi_\varepsilon\right| &= \left|\int_{U_d}\int_X g\left[\mathrm{d}(\mu_j) - \mathrm{d}(\mu_j)_{\varepsilon z}\right] p(z)\,\mathrm{d}z\right|\\ &= \left|\int_{U_d}\int_0^\varepsilon\int_X g(x+sz,y)\big(\widehat{\beta}_j, z\big)\,\mathrm{d}\mu_j\,\mathrm{d}s\,p(z)\,\mathrm{d}z\right| \leqslant \varepsilon \sup|g|\,\big\|\,\big|\widehat{\beta}_j\big|\,\mu_j\big\|,\end{aligned}$$

since $|(\widehat{\beta}_j, z)| \leqslant |\widehat{\beta}_j|$ on the support of p. Therefore,

$$\|\mu_j - \pi_\varepsilon\| \leqslant 2\varepsilon \big\|\widehat{\beta}_j\big\|_{L^1(\mu_j,\mathbb{R}^d)}.$$

Clearly, every measure π_ε with $\varepsilon > 0$ has absolutely continuous conditional measures on $\mathbb{R}^d \times \{y\}$. Hence, for ν-a.e. y, the conditional measure μ_j^y admits a density $q_j^y(x)$ with respect to Lebesgue measure. Thus, we obtain from (6.5) that there exists a measurable set Y_0 of full ν-measure such that for every $i = 1, \ldots, d$, every rational t, and every $y \in Y_0$, one has for a.e. x

$$q_j^y(x+te_i) - q_j^y(x) = \int_0^t \big(\big(\widehat{\beta}_j, e_i\big)\,q_j^y\big)(x+se_i)\,\mathrm{d}s.$$

Therefore, for every $y \in Y_0$, one has $q_j^y \in W^{1,1}_{\mathrm{loc}}$ with $D_x q_j^y(x)/q_j^y(x) = \widehat{\beta}_j(x,y)$, where D_x stands for the standard gradient on $\mathbb{R}^d$. The Riemannian volume λ on M is given by a smooth positive density q with respect to Lebesgue measure on the coordinate neighborhood U we deal with. Hence $q_j^y = q f_j^y$. Therefore,

$$\nabla_x f_j^y(x)/f_j^y(x) = \beta_j(x,y). \qquad (6.7)$$

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Indeed, $\partial_{e_i} q_j^y = f_j^y \partial_{e_i} q + q \partial_{e_i} f_j^y$, so that

$$\frac{\partial_{e_i} q}{q} + \frac{\partial_{e_i} f_j^y}{f_j^y} = \langle \beta_j, e_i \rangle + \operatorname{div} e_i.$$

Now (6.7) follows from the identity $\partial_{e_i} q / q = \operatorname{div} e_i$, which is readily verified by the integration by parts formula:

$$\begin{aligned}\int \psi \operatorname{div} e_i \,\mathrm{d}\lambda &= -\int \langle \nabla\psi, e_i\rangle \,\mathrm{d}\lambda = -\int \partial_{e_i}\psi \, q \,\mathrm{d}x \\ &= \int \psi \, \partial_{e_i} q \,\mathrm{d}x = \int \psi \, \frac{\partial_{e_i} q}{q} \,\mathrm{d}\lambda, \quad \forall \psi \in C_0^\infty(U).\end{aligned}$$

Recall that $\varphi_j \mu^y = \mu_j^y$ for ν-a.e. y, i.e., $\varphi_j(x,y) f^y(x) = f_j^y(x)$ for a.e. x. In addition, $\nabla_x \varphi_j = 0$ μ-a.e. on A_j, since the derivative of any differentiable function F on $\mathbb{R}^d$ vanishes almost everywhere on the set $\{F = 1\}$. Now the claim follows, since the union of the A_j's has full μ-measure (hence $(\mathbb{R}^d \times \{y\}) \cap (\bigcup_j A_j)$ has full μ^y-measure for ν-a.e. y). □

COROLLARY 6.3. – *Suppose that the hypotheses of Lemma* 6.2 *are fulfilled. Then equality* (6.2) *is valid for every function* $\psi \in L^1(\mu)$ *such that* $\psi(\,\cdot\,, y) \in W^{1,1}_{\mathrm{loc}}(\lambda_M)$ *for* ν*-a.e.* y *and* $|\nabla_x \psi|,\ \psi|\beta| \in L^1(\mu)$.

Proof. – For ν-a.e. y, we have by the integration by parts formula

$$\int_M \langle \nabla_x \psi, v\rangle \,\mathrm{d}\mu^y = -\int_M \psi \big[\operatorname{div} v + \langle \beta, v\rangle\big] \,\mathrm{d}\mu^y.$$

Integrating in y, we arrive at (6.2). □

Remark 6.4. – (i) It is clear from the above proof that the separation assumption (iii) on $\mathcal{K}$ can be weakened; e.g., it would be enough to replace it by the following condition:

(iii)′ there exists a measurable set $\Omega \subset X$ of full measure with respect to all shifts $(\mu)_t$ generated by the fields v as above such that $\mathcal{K}$ separates the measures on the set Ω.

In particular, it is the case when Ω has full μ-measure and is mapped by the transformations U_t^v into itself.

(ii) In turn, the collection of fields v involved in the formulation may be considerably reduced. For example, it suffices to have (6.2) for countably many fields $v_k \in \mathrm{Vec}_0^\infty(M)$ such that for every point $m \in M$ one can find a local chart $\mathcal{O}_m$ containing m and fields $v_{m_1}, \ldots, v_{m_d}$ that are constant and linearly independent on $\mathcal{O}_m$. Obviously, in the case $M = \mathbb{R}^d$, it suffices to have (6.2) for d linearly independent constant vectors v. Next we observe that the requirement $\varphi \circ \psi \in \mathcal{K}$ for all $\psi \in \mathcal{K}$ and $\varphi \in C_0^\infty(\mathbb{R}^1)$ with $\varphi(0) = 0$ in condition (ii) can be replaced by the following assumption:

there exist functions $\psi_j \in \mathcal{K}$ such that the sets $\{\psi_j = 1\}$ cover $M \times Y$ up to a μ-measure zero set. Finally, if $\mathcal{K}$ is a linear space and is stable under compositions with C_0^∞-functions vanishing at 0, then it is stable under multiplication, i.e., $\mathcal{K}$ is an algebra.

Remark 6.5. – (i) If μ is a Gibbs measure on M^S specified by conditional densities $f^\Lambda \in W^{1,1}_{\mathrm{loc}}$ on the fibers $M^\Lambda \times \{y\}$, $y \in M^{S\setminus\Lambda}$ such that $\nabla_\Lambda f^\Lambda / f^\Lambda$ is locally λ_{M^Λ}-integrable, then the previous lemma yields an "integration by parts characterization" of μ, i.e., every probability measure μ' on M^S that satisfies the analog of (6.2) with every $\beta_\Lambda = \nabla_\Lambda f^\Lambda / f^\Lambda$ in place of β

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has the same conditional measures as μ. This is obvious from the fact that a probability measure on M is uniquely determined by its logarithmic gradient if it is locally λ_M-integrable (cf. [15]).

(ii) It is worth noting that one can always find a suitable class $\mathcal{K}$ that satisfies conditions (i)–(iii) from the above lemma and (6.2). We shall deal with the version of μ^y such that $\varrho^y \in W^{1,1}_{\mathrm{loc}}(\lambda_M)$ for every $y \in Y$. We take an increasing sequence of compact sets K_j that cover M. Such sets can be chosen with the property that, for every fixed t, v, and i, the set $U_t^v(K_i)$ is contained in one of the K_j's, which is obviously possible. Then we consider the sets $\Omega_{j,r} \in Y$, $j, r \in \mathbb{N}$, such that

$$\int_{K_j} \left|\nabla_x f^y(x)\right| \lambda_M(\mathrm{d}x) \leqslant r, \quad \forall y \in \Omega_{j,r}.$$

Then the sets $\Omega_{j,r}$ cover Y. Now we take for $\mathcal{K}$ the class of all functions of the form $f(x,y) = \theta(\psi_1(x)\varphi_1(y), \ldots, \psi_n(x)\varphi_n(y))$, where $\theta \in C_0^\infty(\mathbb{R}^n)$, $\theta(0) = 0$, $\psi_i \in C_0^\infty(M)$ has support in one of the sets K_j, and φ_i is a bounded measurable function on Y with support in one of the sets $\Omega_{j,r}$. It is readily seen that $\mathcal{K}$ satisfies conditions (i)–(iii) in the lemma. Equality (6.2) is true, since $f(x,y) = 0$ if $y \notin \Omega_{j,r}$ and $\|f(\,\cdot\,,y)|\beta^y|\,\|_{L^1(\mu^y)} \leqslant r\sup|\theta|$ if $y \in \Omega_{j,r}$. Taking into account Remark 6.4(i), one could use the smaller class formed by the products $\psi_i(x)\varphi_i(y)$ as above, since it satisfies the aforementioned modification of conditions (i) and (ii).

COROLLARY 6.6. – *Suppose that $\mathcal{K}$ is a certain class of bounded $\mathcal{B}$-measurable functions on M^S that satisfy the hypotheses of Lemma* 6.2 *with respect to every representation $M^S = M^i \times M^{i^c}$ and, in addition, that the second derivatives $\nabla^2_{x_i}\psi$, $\psi \in \mathcal{K}$, are bounded. Let μ be a Radon measure on M^S having the logarithmic derivatives Z_i along the x_i's with respect to $\mathcal{K}$ such that $\langle Z_i, \nabla_i \psi\rangle \in L^1(\mu)$ for all $\psi \in \mathcal{K}$ and $i \in S$. Then $L_Z^*\mu = 0$ with respect to $\mathcal{K}$ in the sense of Definition* 4.1.

Proof. – It suffices to show that

$$\int \Delta_i \psi \,\mathrm{d}\mu = -\int \langle Z_i, \nabla_i \psi\rangle \,\mathrm{d}\mu, \quad \forall \psi \in \mathcal{K},\ i \in S. \tag{6.8}$$

For a fixed $i \in S$, in accordance with Lemma 6.2, we have by the integration by parts formula

$$\int_{M^i} \Delta_i \psi\big(x_i, x_i^c\big)\, \mu\big(\mathrm{d}x^i | x_i^c\big) = -\int_{M^i} \langle Z_i, \nabla_i \psi\rangle\, \mu\big(\mathrm{d}x^i | x_i^c\big).$$

Integrating this relation in x_i^c, we arrive at (6.8). □

Example 6.7. – Let $q_n > 0$ be such that $\sum_{n\in S} q_n < \infty$ and let $\mu \in \mathcal{P}(M^S)$ be such that $\Psi = \sum_{n \in S} q_n \varrho_n^p < \infty$ μ-a.e. for some $p \geqslant 2$. Suppose that the regular conditional probabilities $\mu(\,\cdot\,|x_{i^c})$ on $M^i \times \{x_{i^c}\}$, $x_{i^c} \in M^{i^c}$, have continuously differentiable densities $x_i \mapsto \exp V_i(x_i, x_{i^c})$ such that the mappings $x \mapsto Z_i(x) = \nabla_i V_i(x_i, x_{i^c})$ are continuous on M^S. If $\varrho_i^p \in C^2(M^i)$, then $L_Z^*\mu = 0$ with respect to the class $\mathcal{K}$ of all functions φ with supports in the sets $\{\Psi \leqslant r\}$ and bounded derivatives $\nabla_i \varphi$, $\nabla_i^2 \varphi$. In addition, $\mathcal{K}$ separates the Borel measures on the set $\{\Psi < \infty\}$.

Proof. – We only have to show that $\mathcal{K}$ separates the Borel measures on $\{\Psi < \infty\}$. This is obvious, since $\mathcal{K}$ contains all functions of the form $f\theta(\Psi)$, $f \in \mathcal{F}C_0^\infty$, $\theta \in C_0^\infty(\mathbb{R}^1)$. Note that we could employ the class $\mathcal{K}_0$ of functions

$$\varphi(x) = \zeta\left(f_1(x_{\Lambda_n})\theta_1\big(\Psi(x)\big), \ldots, f_n(x_{\Lambda_n})\theta_n\big(\Psi(x)\big)\right),$$

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where $\theta_i \in C_0^\infty(\mathbb{R}^1)$, $f_i \in C_0^\infty(M^{\Lambda_n})$, $\zeta \in C_0^\infty(\mathbb{R}^n)$, $\zeta(0) = 0$. We remark that $\mathcal{K}_0$ satisfies the hypotheses of Lemma 6.2, so that this class can be used to derive the existence of conditional densities. □

PROPOSITION 6.8. – *Suppose that in the situation of Proposition* 5.3, *for all finite sets $\Lambda \subset S$, there exists continuously differentiable functions U_Λ on M^Λ such that*

$$Z_\Lambda\bigl(x_\Lambda \times o^{\Lambda^c}\bigr) = \nabla_\Lambda U_\Lambda(x_\Lambda).$$

Then there exists $\mu \in \mathcal{P}(M^S)$ such that L with domain $\operatorname{span}\mathcal{FC}_0^\infty$ *is symmetric on $L^2(\mu)$, and for every $f \in \mathcal{FC}_0^\infty$ and every $v \in \mathrm{Vec}_b^1(M^i)$, one has*

$$\int \langle \nabla_i f, v\rangle \,\mathrm{d}\mu = -\int f\langle v, Z_i\rangle \,\mathrm{d}\mu - \int f \operatorname{div}_i v \,\mathrm{d}\mu. \tag{6.9}$$

Proof. – The claim follows by the proof of Proposition 5.3 applied to the probability measures $\mu_n = c_n \exp U_{\Lambda_n}$ (see Remark 3.5(i)). □

Remark 6.9. – Clearly, if Z_i is continuous in the product topology of M^S, then it is τ-continuous. It is easily seen that the continuity assumption on Z_i cannot be completely dropped. However, in the situation of Proposition 5.3 or Proposition 6.8, in place of continuity of Z_i, it is sufficient to have continuous fields $\widehat{Z}_i$ such that $|Z_i - \widehat{Z}_i| \leqslant \varepsilon_i \Psi$ with $\varepsilon_i \to 0$. This is readily seen from the proof.

Example 6.10. – Suppose that the fields Z_i are continuous on M^S and that, for all finite sets $\Lambda \subset S$, there exist continuously differentiable functions U_Λ on M^Λ such that

$$Z_\Lambda\bigl(x_\Lambda \times o^{\Lambda^c}\bigr) = \nabla_\Lambda U_\Lambda(x_\Lambda).$$

Let $V_i(x_i) = G_i(x_i) = \varrho_i(x_i)^p$, $p > 1$, and let q and A be as in Lemma 5.1. Assume that (5.1) holds outside of $\mathrm{cut}(o_i)$. Then there exists $\mu \in \mathcal{P}(M^S)$ such that the regular conditional probabilities $\mu(\,\cdot\,|x_i^c)$ are given by continuously differentiable densities $f(\,\cdot\,|x_i^c)$ with $\nabla_i f(x_i|x_i^c)/f(x_i|x_i^c) = Z_i(x)$. In particular, if $\Delta_i \varrho_i \leqslant c_0(1 + \varrho_i^{-1})$, then it suffices to have the estimate

$$\varrho_i^{p-1}\langle \nabla_i \varrho_i, Z_i\rangle \leqslant \widehat{c} - \widehat{\lambda}\varrho_i^p + p^{-1}\sum_{j\in S} a_{i,j}\varrho_j^p,$$

where $\widehat{c} = c - 2c_0 p - p(p-1)$, $\widehat{\lambda} = p^{-1}(\lambda + \varepsilon) + 2c_0 p + p(p-1)$.

Proof. – As observed above, the functions $\exp U_\Lambda$ are integrable on M^Λ with respect to the Riemannian volumes. Let μ_n be the probability measure on M^{Λ_n} with the density $c_n \exp U_{\Lambda_n}$. This measure will be regarded as a measure on $M^{\Lambda_n} \times \{o\}$. By Lemma 5.1, the sequence of probability measures μ_n on $M^{\Lambda_n} \times \{o\}$ has a weak limit point μ that is concentrated on the set $\Omega = \{\Psi < \infty\}$. Moreover, the sequence $\{\mu_n\}$ is uniformly tight with respect to the metric

$$d_p(x, y) = \biggl(\sum_i q_i \varrho_i(x_i, y_i)^p\biggr)^{1/p},$$

hence we may assume that $\mu_n \to \mu$ weakly on (Ω, d_p). Let us take for $\mathcal{K}$ the collection of all bounded functions ψ on Ω such that: (i) ψ is continuous with respect the metric d_p and has d_p-bounded support, (ii) the functions $x_i \mapsto \psi(\dots, x_i, \dots)$ on M^i are continuously differentiable in

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x_i with bounded gradients. We observe that $\mathcal{K}$ is a linear space and contains all functions of the form $\varphi\theta(\Psi)$, where $\varphi \in \mathcal{FC}_0^\infty$, $\theta \in C_0^\infty(\mathbb{R}^1)$. Hence $\mathcal{K}$ separates measures on Ω. In addition, $\psi(f) \in \mathcal{K}$ whenever $\psi \in C_0^\infty(\mathbb{R}^1)$, $\psi(0) = 0$, $f \in \mathcal{K}$. Finally, if $v \in \mathrm{Vec}_0^\infty(M^i)$ and $f \in \mathcal{K}$, then the function $x \mapsto f\bigl(x_1, \ldots, U_t^v(x_i), x_{i+1}, \ldots\bigr)$ belongs to $\mathcal{K}$. Now, in order to apply Lemma 6.2, it remains to verify that, for every $\psi \in \mathcal{K}$ and every smooth compactly supported vector field v on M^i, one has

$$\int \langle \nabla_i \psi, v\rangle \, \mathrm{d}\mu = -\int \psi \operatorname{div} v \, \mathrm{d}\mu - \int \psi \langle Z_i, v\rangle \, \mathrm{d}\mu.$$

This follows from the corresponding relations for the μ_n's by the weak convergence of $\{\mu_n\}$ to μ on (Ω, d_p), since the functions $\langle \nabla_i \psi, v\rangle$, $\psi \operatorname{div} v$, $\psi\langle Z_i, v\rangle$ are d_p-continuous and bounded (note that Z_i is bounded on d_p-bounded sets by their compactness in M^S). □

We observe that the above example enables one to construct measures with given conditional distributions on $M^i \times \{y^i\}$, $y^i = (y_j)_{j\neq i} \in M^{i^c}$, provided that these distributions have continuously differentiable densities $x_i \mapsto c_i \exp U_i(x_i, y^i)$, where the fields $Z_i = \nabla_i U_i$ satisfy the corresponding assumptions. For example, if the M^i's have Ricci curvatures bounded below and the o_i's are poles, then it suffices that $U_i(x_i, y^i) = -\varrho_i(x_i)^p + w_i(x_i, y^i)$, where $|\nabla_i w_i(x_i, y^i)| \leqslant c + \sum_j a_{i,j}\varrho_j(y_j)^{p-1}$, where we set $y_i = x_i$.

Remark 6.11. – The idea of constructing measures on an infinite-dimensional space X with a given logarithmic gradient β as invariant measures of a diffusion process with drift $\beta/2$ goes back to S. Albeverio and R. Høegh-Krohn [2], who introduced the concept of vector logarithmic gradient. Lyapunov functions technique has been applied for this purpose in [37,38] and in a more general setting in [15]. Concerning applications of Lyapunov functions in the case where there exists a diffusion process with generator L, see [25,39,40], and the references therein. The approach initiated in [37,38] has been recently developed in [3] and [4] in order to cover a broad class of Gibbs measures. The above results in the flat case yield extensions of the analogous results from [3] and [4]. For further extensions in the linear case, see [17].

In the next section we shall consider Gibbs measures in the finite range case.

7. Finite range vector fields

Let $S = \mathbb{Z}^m$ and let $M^S = \prod_{i\in S} M^i$, where the M^i's are Riemannian manifolds which satisfy the hypotheses of Corollary 2.3, hence as shown in the proof of Corollary 2.3 they also satisfy the conditions of Theorem 2.2. Suppose that we are given a family $Z = (Z_i)_{i\in S}$ of Borel vector fields Z_i on M^S such that $Z_i(x) \in T_{x_i}M^i$, $i \in S$. We shall say that Z is of finite range R if, for every $i \in S$, Z_i depends only on the coordinates x_j with $j \in i + \Lambda_1$, where $\Lambda_k = \{s = (s_1, \ldots, s_m) \in \mathbb{Z}^m\colon |s_j| \leqslant kR\}$.

Given a measure ν on a manifold M, we define the divergence of a ν-measurable vector field Z on M with $|Z| \in L^1_{\mathrm{loc}}(\nu)$ as a function $\operatorname{div}^\nu Z \in L^1_{\mathrm{loc}}(\nu)$ such that

$$\int_M \langle Z, \nabla f\rangle \, \mathrm{d}\nu = -\int_M f \operatorname{div}^\nu Z \, \mathrm{d}\nu, \quad \forall f \in C_0^\infty(M), \tag{7.1}$$

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if such a function exists. It is easily seen that if $\nu = p\,\mathrm{d}\lambda_M$, where $p \in W^{1,1}_{\mathrm{loc}}(\lambda_M)$ and Z is continuously differentiable, then

$$\operatorname{div}^{\nu} Z = \operatorname{div} Z + \left\langle Z, \frac{\nabla p}{p} \right\rangle,$$

where $\operatorname{div} Z$ is the usual divergence with respect to the Riemannian volume (i.e., the trace of the derivative). In this case, we also have $\operatorname{div}^{\lambda_M} Z = \operatorname{div} Z$. We shall denote $\operatorname{div}^{\lambda_M} Z$ by $\operatorname{div} Z$ even if the latter exists only in the sense of (7.1).

By analogy, one can define a divergence of a vector field on an infinite-dimensional manifold with a measure. For example, if $\lambda_{M^i}(M^i) = 1$ and λ^S is the corresponding product-measure on M^S, then we shall say that a λ^S-measurable vector field Z_i with $Z_i(x) \in T_{M^i}(x_i)$ and $|Z_i| \in L^1(\lambda^S)$ has a divergence $\operatorname{div} Z_i$ (which can be denoted also by $\operatorname{div}_i Z_i$) with respect to λ^S if $\operatorname{div} Z_i \in L^1(\lambda^S)$ is a function on M^S such that

$$\int_{M^S} \langle \nabla_i \varphi, Z_i \rangle \,\mathrm{d}\lambda^S = -\int_{M^S} \varphi \operatorname{div} Z_i \,\mathrm{d}\lambda^S, \quad \forall \varphi \in \mathcal{FC}_0^\infty.$$

The divergence of a vector field $Z = (Z_i)_{i \in S}$ can be defined as $\sum_{i \in S} \operatorname{div} Z_i$ provided that the divergences $\operatorname{div} Z_i$ exist and the series converges to a function from $L^1(\mu)$ in a suitable sense, e.g., in $L^1(\mu)$ or with respect to the duality with the linear span of $\mathcal{FC}_0^\infty$. We shall only use the divergence of components Z_i.

LEMMA 7.1. – (i) *Let $\mu \in \mathcal{P}(M^S)$ and $L_Z^* \mu = 0$ with respect to $\mathcal{FC}_0^\infty$, where $Z = (Z_i)_{i \in S}$ is of finite range R and $|Z_i| \in L^2(\mu)$. Let $k \in \mathbb{N}$ and suppose that $\nu_k \in \mathcal{P}(M^{\Lambda_k})$ is such that $\nu_k = \exp(W_k)\,\lambda_{\Lambda_k}$ and $\mu_{\Lambda_k} = f_k\,\nu_k$, where $W_k \in W^{1,1}_{\mathrm{loc}}(\lambda_{\Lambda_k})$, $\sqrt{f_k} \in H^{2,1}(\nu_k)$. Assume that* (4.5) *holds for $H^{2,1}(\nu_k)$. Let $\beta_i^{\nu_k} = \nabla_i W_k \in L^2_{\mathrm{loc}}(\nu_k) \cap L^2(\mu_{\Lambda_k})$, $i \in \Lambda_k$. Then*

$$\begin{aligned}\int_{M^{\Lambda_k}} \frac{|\nabla f_k|^2}{f_k}\,\mathrm{d}\nu_k &= \sum_{i \in \Lambda_{k-1}} \int_{M^{\Lambda_k}} \big\langle Z_i - \beta_i^{\nu_k}, \nabla_i f_k \big\rangle \,\mathrm{d}\nu_k \\ &\quad + \sum_{i \in \Lambda_k \setminus \Lambda_{k-1}} \int_{M^S} \left\langle Z_i - \beta_i^{\nu_k}, \frac{\nabla_i f_k}{f_k} \right\rangle \mathrm{d}\mu.\end{aligned} \tag{7.2}$$

(ii) *Assume, in addition, that there exists $\nu_{k+1} \in \mathcal{P}(M^{\Lambda_{k+1}})$ such that ν_k is the projection of ν_{k+1} onto M^{Λ_k}, $\mu_{\Lambda_{k+1}} = f_{k+1}\,\nu_{k+1}$, $\sqrt{f_{k+1}} \in H^{2,1}(\nu_{k+1})$, $\beta_i^{\nu_{k+1}}$ exists and $\beta_i^{\nu_{k+1}} \in L^2_{\mathrm{loc}}(\nu_{k+1}) \cap L^2(\mu_{\Lambda_{k+1}})$ for every $i \in \Lambda_k$. Assume* (4.5) *also holds for $H^{2,1}(\nu_{k+1})$. Then*

$$\begin{aligned}\sum_{i \in \Lambda_{k-1}} \int_{M^{\Lambda_k}} \frac{|\nabla_i f_k|^2}{f_k}\,\mathrm{d}\nu_k &= \sum_{i \in \Lambda_{k-1}} \int_{M^{\Lambda_k}} \big\langle Z_i - \beta_i^{\nu_k}, \nabla_i f_k \big\rangle \,\mathrm{d}\nu_k \\ &\quad - \sum_{i \in \Lambda_k \setminus \Lambda_{k-1}} \int_{M^{\Lambda_{k+1}}} \left\langle \frac{\nabla_i f_k}{f_k}, \frac{\nabla_i f_{k+1}}{f_{k+1}} + \beta_i^{\nu_{k+1}} - Z_i \right\rangle \mathrm{d}\mu_{\Lambda_{k+1}}.\end{aligned} \tag{7.3}$$

Proof. – (i) First note that since (4.5) holds we have

$$\int \frac{|\nabla_i f_k|^2}{f_k^2}\,\mathrm{d}\mu_k = 4 \int \frac{|\nabla_i \sqrt{f_k}|^2}{f_k}\,\mathrm{d}\mu_k = 4 \int \big|\nabla_i \sqrt{f_k}\big|^2 \,\mathrm{d}\nu_k < \infty$$

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and since $|\beta_i^{\nu_k}|, |Z_i| \in L^2(\mu)$, both integrals on the right-hand side of (7.2) exist. Let $\varphi \in C_0^\infty(M^{\Lambda_k})$. Then

$$\int\limits_{M^{\Lambda_k}} \Delta_{\Lambda_k}\varphi\, f_k\,\mathrm{d}\nu_k + \sum_{i\in\Lambda_k}\int\limits_{M^S}\langle Z_i, \nabla_i\varphi\rangle\,\mathrm{d}\mu = 0.$$

Approximating f_k by $(n \wedge \sqrt{f_k})^2 \in H^{2,1}(\nu_k), n \in \mathbb{N}$, allows to integrate by parts, so using that Z_i, $i \in \Lambda_{k-1}$, depends only on x_{Λ_k}, we obtain again using (4.5)

$$\begin{aligned}\int\limits_{M^{\Lambda_k}} \langle\nabla\varphi, \nabla f_k\rangle\,\mathrm{d}\nu_k &= -\sum_{i\in\Lambda_k}\int\limits_{M^{\Lambda_k}}\left\langle\beta_i^{\nu_k}, \nabla_i\varphi\right\rangle f_k\,\mathrm{d}\nu_k + \sum_{i\in\Lambda_k}\int\limits_{M^S}\left\langle Z_i, \nabla_i\varphi\right\rangle\mathrm{d}\mu\\ &= \sum_{i\in\Lambda_{k-1}}\int\limits_{M^{\Lambda_k}}\left\langle Z_i - \beta_i^{\nu_k}, \nabla_i\varphi\right\rangle f_k\,\mathrm{d}\nu_k + \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}}\int\limits_{M^S}\left\langle Z_i - \beta_i^{\nu_k}, \nabla_i\varphi\right\rangle\mathrm{d}\mu. \end{aligned}\tag{7.4}$$

The desired equality follows if we put $\nabla_i\varphi = \nabla_i f_k/f_k$, but this requires some justification. We observe that

$$\int\limits_{M^S}\langle Z_i, \nabla_i\varphi\rangle\,\mathrm{d}\mu = \int\limits_{M^{\Lambda_k}}\langle E_i, \nabla_i\varphi\rangle f_k\,\mathrm{d}\nu_k,$$

where E_i is the conditional expectation of Z_i with respect to μ and the σ-field generated by x_j, $j \in \Lambda_k$. Thus, $|E_i| \in L^2(\mu_{\Lambda_k})$. Since $|\beta_i^{\nu_k}|, |Z_i|, |\nabla f_k/f_k| \in L^2(\mu_{\Lambda_k})$, it suffices to show that there exists a sequence of functions $\varphi_i \in C_0^\infty(M^{\Lambda_k})$ such that $|\nabla\varphi_i - \nabla f_k/f_k| \to 0$ in $L^2(\mu_{\Lambda_k})$, i.e., $\nabla f_k/f_k \in \Gamma(\mu_{\Lambda_k})$. We have $\mu_{\Lambda_k} = p\lambda_{\Lambda_k}$, where $p = f_k \exp W_k$ and $\sqrt{p} \in H^{2,1}(\lambda_{\Lambda_k})$. Hence

$$\frac{\nabla f_k}{f_k} = \frac{\nabla p}{p} - \nabla W_k.$$

It remains to note that $\nabla W_k \in \Gamma(\mu_{\Lambda_k})$ by Lemma 2.1 and $\nabla p/p \in \Gamma(\mu_k)$ by Theorem 2.2 (which applies by our assumptions stated at the beginning of this section).

Equality (7.3) is proved in a similar manner taking into account that for every $i \in \Lambda_k\setminus\Lambda_{k-1}$, one has

$$\begin{aligned}\int\limits_{M^S}\left[\Delta_i\varphi + \langle Z_i, \nabla_i\varphi\rangle\right]\mathrm{d}\mu &= \int\limits_{M^{\Lambda_{k+1}}}\left[\Delta_i\varphi + \langle Z_i, \nabla_i\varphi\rangle\right] f_{k+1}\,\mathrm{d}\nu_{k+1}\\ &= \int\limits_{M^{\Lambda_{k+1}}}\left[-\langle\nabla_i\varphi, \nabla_i f_{k+1}\rangle + \left\langle\nabla_i\varphi, Z_i - \beta_i^{\nu_{k+1}}\right\rangle f_{k+1}\right]\mathrm{d}\nu_{k+1}.\end{aligned}$$

With the above justification, one can replace $\nabla_i\varphi$ by $\nabla_i f_k/f_k$. □

LEMMA 7.2.– (i) *Let $\mu_{k+1} \in \mathcal{P}(M^{\Lambda_{k+1}})$ have the logarithmic derivative $\beta_i^{\mu_{k+1}}$ along x_i for some $i \in \Lambda_k$ and let $|\beta_i^{\mu_{k+1}}| \in L^2(\mu_{k+1})$. Let μ_k be the projection of μ_{k+1} to M^{Λ_k}. Then μ_k has the logarithmic derivative $\beta_i^{\mu_k}$ along x_i and*

$$\int\limits_{M^{\Lambda_k}}\left|\beta_i^{\mu_k}\right|^2\mathrm{d}\mu_k \leqslant \int\limits_{M^{\Lambda_{k+1}}}\left|\beta_i^{\mu_{k+1}}\right|^2\mathrm{d}\mu_{k+1}.\tag{7.5}$$

(ii) *Let, in addition, $\nu_{k+1} \in \mathcal{P}(M^{\Lambda_{k+1}})$ be such that $\mu_{k+1} = f_{k+1}\,\nu_{k+1}$ and $\mu_k = f_k\,\nu_k$, where $\sqrt{f_k} \in H^{2,1}(\nu_k)$, $\sqrt{f_{k+1}} \in H^{2,1}(\nu_{k+1})$. Assume that both $H^{2,1}(\nu_k)$ and $H^{2,1}(\nu_{k+1})$ satisfy* (4.5).

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Then, for every $i \in \Lambda_k$ such that $\beta_i^{\nu_{k+1}} \in L^2_{\mathrm{loc}}(\nu_k)$ exists and depends only on the variables x_j, $j \in \Lambda_k$, one has

$$\int_{M^{\Lambda_k}} \frac{|\nabla_i f_k|^2}{f_k}\,\mathrm{d}\nu_k \leqslant \int_{M^{\Lambda_{k+1}}} \frac{|\nabla_i f_{k+1}|^2}{f_{k+1}}\,\mathrm{d}\nu_{k+1}. \tag{7.6}$$

Proof. – It is easily verified that the conditional expectation of $\beta_i^{\mu_{k+1}}$ with respect to the measure μ_{k+1} and the σ-field σ_k generated by the variables x_j, $j \in \Lambda_k$, serves as $\beta_i^{\mu_k}$. Hence we obtain (7.5). In order to prove (7.6), let us note that as shown in the proof of Lemma 7.1 both integrals exist and that $\beta_i^{\nu_k} = \beta_i^{\nu_{k+1}}$ since $\beta_i^{\nu_{k+1}}$ only depends on the variables x_j, $j \in \Lambda_k$. The left-hand side in (7.6) is equal to the square of the norm of $|\nabla_i f_k/f_k|$ in $L^2(\mu_k)$, hence coincides with the supremum of

$$\left(\int_{M^{\Lambda_k}} \left\langle \frac{\nabla_i f_k}{f_k}, v\right\rangle \mathrm{d}\mu_k\right)^2$$

over all $v \in \mathrm{Vec}_0^\infty(M^{\Lambda_k})$ such that $v(x) \in T_{x_i}M^i$ and $\|\,|v|\,\|_{L^2(\mu_k)} \leqslant 1$. Given such a field, we have

$$\mathrm{div}^{\nu_{k+1}} v = \mathrm{div}\, v + \langle v, \beta_i^{\nu_{k+1}}\rangle = \mathrm{div}\, v + \langle v, \beta_i^{\nu_k}\rangle = \mathrm{div}^{\nu_k} v.$$

Therefore, by (7.1) and the hypotheses that $\beta_i^{\nu_{k+1}} \in L^2_{\mathrm{loc}}(\nu_k)$, $\sqrt{f_k} \in H^{2,1}(\nu_k)$ and $\sqrt{f_{k+1}} \in H^{2,1}(\nu_{k+1})$, one obtains, by approximating f_k by $(n \wedge \sqrt{f_k})^2 \in H^{2,1}(\nu_k)$, $n \in \mathbb{N}$, and the same for f_{k+1}, that

$$\begin{aligned}
\int_{M^{\Lambda_k}} \langle \nabla_i f_k, v\rangle\,\mathrm{d}\nu_k &= -\int_{M^{\Lambda_k}} f_k \mathrm{div}^{\nu_k} v\,\mathrm{d}\nu_k \\
&= -\int_{M^{\Lambda_{k+1}}} f_{k+1}\mathrm{div}^{\nu_k} v\,\mathrm{d}\nu_{k+1} = \int_{M^{\Lambda_{k+1}}} \langle \nabla_i f_{k+1}, v\rangle\,\mathrm{d}\nu_{k+1} \\
&\leqslant \left(\int_{M^{\Lambda_{k+1}}} \frac{|\nabla_i f_{k+1}|^2}{f_{k+1}^2}\,\mathrm{d}\mu_{k+1}\right)^{1/2} \left(\int_{M^{\Lambda_{k+1}}} |v|^2\,\mathrm{d}\mu_{k+1}\right)^{1/2} \\
&\leqslant \left(\int_{M^{\Lambda_{k+1}}} \frac{|\nabla_i f_{k+1}|^2}{f_{k+1}^2}\,\mathrm{d}\mu_{k+1}\right)^{1/2}.
\end{aligned}$$

This completes the proof. □

We assume in the next theorem that $M^i = M = \mathbb{T}^1$ is a circle of unit length, λ is (normalized) Lebesgue measure on M. It is well-known that one has the following log-Sobolev inequality

$$\int_{M^{\Lambda_k}} u^2 \log u\,\mathrm{d}\lambda_k \leqslant \int_{M^{\Lambda_k}} |\nabla u|^2\,\mathrm{d}\lambda_k, \quad u \in H^{2,1}(\lambda_k),\ \|u\|_{L^2(\lambda_k)} = 1. \tag{7.7}$$

Next, we shall employ the following lemma analogous to Ramirez's inequality in [42, Lemma 5].

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LEMMA 7.3. – *Let $M=\mathbb{T}^1$ and let $u\in H^{2,1}(\lambda)$ be nonnegative and continuous. If $\psi\in L^2(\lambda)$ and $\sqrt{u}\in H^{2,1}(\lambda)$, then*

$$\int_M \psi u\,\mathrm{d}\lambda \leqslant \int_M \psi^2 u\,\mathrm{d}\lambda + \frac{1}{4}\int_M \frac{|\nabla u|^2}{u}\,\mathrm{d}\lambda + (\min u)\int_M \psi\,\mathrm{d}\lambda. \tag{7.8}$$

Proof. – Let $c=\min u$. Then

$$\begin{aligned}\int_M \psi u\,\mathrm{d}\lambda &= \int_M \psi(u-c)\,\mathrm{d}\lambda + c\int_M \psi\,\mathrm{d}\lambda\\ &\leqslant \left(\int_M \psi^2\big|\sqrt{u}+\sqrt{c}\big|^2\,\mathrm{d}\lambda \int_M \big|\sqrt{u}-\sqrt{c}\big|^2\,\mathrm{d}\lambda\right)^{1/2} + c\int_M \psi\,\mathrm{d}\lambda\\ &\leqslant \int_M \psi^2 u\,\mathrm{d}\lambda + \max\big|\sqrt{u}-\sqrt{c}\big|^2 + c\int_M \psi\,\mathrm{d}\lambda.\end{aligned}$$

It remains to note that $|\sqrt{u}-\sqrt{c}|$ is majorized by the integral of $|\nabla\sqrt{u}|$ and apply the Cauchy inequality. □

THEOREM 7.4. – *Let λ^S be the product measure on M^S, where $M=\mathbb{T}^1$. Suppose that $\mu\in\mathcal{P}(M^S)$ satisfies the equation $L_Z^*\mu=0$ with respect to $\mathcal{FC}_0^\infty$, where Z is of finite range R and $\sup_i|Z_i|\leqslant\kappa<\infty$. Assume also that* $\operatorname{div} Z_i\in L^\infty(\lambda^S)$, *$i\in S$, exist and*

$$\eta := \sum_{i\in S}\|\operatorname{div} Z_i\|_\infty < \infty, \quad \textit{where } \|\operatorname{div} Z_i\|_\infty := \sup_{M^S}|\operatorname{div} Z_i|.$$

Then $\mu=f^2\,\mathrm{d}\lambda^S$ with $f\in H^{2,1}(\lambda^S)$ and $\int_{M^S}|\nabla f|^2\,\mathrm{d}\lambda^S\leqslant\eta/4$.

Proof. – We set $\mu_k:=\mu_{\Lambda_k}$ and $\lambda_k:=\lambda^{\Lambda_k}$ for simplicity. We know by Theorem 4.2 that $\mu_k=f_k\,\lambda_k$ and $f_k\in H^{2,1}(\lambda_k)$ has a continuous strictly positive version. Moreover,

$$\int_{M^{\Lambda_k}} \frac{|\nabla f_k|^2}{f_k^2}\,\mathrm{d}\mu_k \leqslant \int_{M^{\Lambda_k}} \big|Z^\mu_{\Lambda_k}\big|^2\,\mathrm{d}\mu_k \leqslant \kappa^2\operatorname{card}\Lambda_k \leqslant \kappa^2 R^m(2k+1)^m. \tag{7.9}$$

For fixed $x_{\Lambda_{k+1}\setminus\{i\}}$ and $i\in\Lambda_k\setminus\Lambda_{k-1}$, we shall apply (7.8) to the functions

$$u(x_i)=f_{k+1}(x_{\Lambda_{k+1}\setminus\{i\}}\times x_i), \qquad \psi(x_i)=\left\langle Z_i,\frac{\nabla_i f_k}{f_k}\right\rangle(x_{\Lambda_{k+1}\setminus\{i\}}\times x_i).$$

We have

$$\int_M \psi(x_i)\,\lambda(\mathrm{d}x_i) = -\int_M \operatorname{div} Z_i \log f_k\,\lambda(\mathrm{d}x_i) \leqslant \|\operatorname{div} Z_i\|_\infty \int_M |\log f_k|\,\lambda(\mathrm{d}x_i).$$

Since $f_k=\int_{M^{\Lambda_{k+1}\setminus\Lambda_k}} f_{k+1}\,\mathrm{d}\lambda^{\Lambda_{k+1}\setminus\Lambda_k}$, we have

$$\int_{M^{\Lambda_{k+1}\setminus\Lambda_k}} \min_{x_i} f_{k+1}\,\mathrm{d}\lambda^{\Lambda_{k+1}\setminus\Lambda_k}\int_M |\log f_k|\,\lambda(\mathrm{d}x_i)$$

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$$\leqslant \int\limits_M \Bigl(\min_{x_i} f_k\Bigr) |\log f_k|\, \lambda(\mathrm{d}x_i) \leqslant 1 + \int\limits_M (f_k \log f_k)\, \lambda(\mathrm{d}x_i).$$

By (7.8), (7.7) we obtain

$$\begin{aligned} &\int\limits_{M^{\Lambda_{k+1}}} \left\langle Z_i, \frac{\nabla_i f_k}{f_k}\right\rangle f_{k+1}\, \mathrm{d}\lambda_{k+1} \\ &\leqslant \kappa^2 \int\limits_{M^{\Lambda_{k+1}}} \frac{|\nabla_i f_k|^2}{f_k^2} f_{k+1}\, \mathrm{d}\lambda_{k+1} \\ &\quad + \frac{1}{4} \int\limits_{M^{\Lambda_{k+1}}} \frac{|\nabla_i f_{k+1}|^2}{f_{k+1}}\, \mathrm{d}\lambda_{k+1} + \|\mathrm{div}\, Z_i\|_\infty \Biggl(1 + \frac{1}{2} \int\limits_{M^{\Lambda_k}} \frac{|\nabla f_k|^2}{f_k}\, \mathrm{d}\lambda_k\Biggr). \end{aligned} \tag{7.10}$$

By Lemma 7.2, for any $i \in \Lambda_k$, one has

$$\int\limits_{M^{\Lambda_{k+1}}} \frac{|\nabla_i f_k|^2}{f_k^2} f_{k+1}\, \mathrm{d}\lambda_{k+1} = \int\limits_{M^{\Lambda_k}} \frac{|\nabla_i f_k|^2}{f_k^2} f_k\, \mathrm{d}\lambda_k \leqslant \int\limits_{M^{\Lambda_{k+1}}} \frac{|\nabla_i f_{k+1}|^2}{f_{k+1}^2} f_{k+1}\, \mathrm{d}\lambda_{k+1}. \tag{7.11}$$

Let $\varepsilon_k := \sum_{i \notin \Lambda_{k-1}} \|\mathrm{div}\, Z_i\|_\infty$, which goes to 0 as $k \to \infty$. Then, by (7.10) and (7.11) we obtain

$$\begin{aligned} &\sum_{i \in \Lambda_k \setminus \Lambda_{k-1}} \int\limits_{M^{\Lambda_{k+1}}} \left\langle Z_i, \frac{\nabla_i f_k}{f_k}\right\rangle f_{k+1}\, \mathrm{d}\lambda_{k+1} \\ &\leqslant \left(\kappa^2 + \frac{1}{4}\right) \sum_{i \in \Lambda_k \setminus \Lambda_{k-1}} \int\limits_{M^{\Lambda_{k+1}}} \frac{|\nabla_i f_{k+1}|^2}{f_{k+1}}\, \mathrm{d}\lambda_{k+1} + \varepsilon_k + \frac{\varepsilon_k}{2} \int\limits_{M^{\Lambda_k}} \frac{|\nabla f_k|^2}{f_k}\, \mathrm{d}\lambda_k. \end{aligned}$$

Combining this with (7.2) and realizing that $\beta_i^{\nu_k} = 0$ (since $\nu_k = \lambda_k$) and that

$$\sum_{i \in \Lambda_{k-1}} \int\limits_{M_k^{\Lambda}} \langle Z_i, \nabla_i f_k\rangle\, \mathrm{d}\lambda_k = \sum_{i \in \Lambda_{k-1}} \int\limits_{M^{\Lambda_k}} (\mathrm{div}\, Z_i) f_k\, \mathrm{d}\lambda_k \leqslant \eta \int\limits_{M^{\Lambda_k}} f_k\, \mathrm{d}\lambda_k = \eta,$$

we obtain

$$\int\limits_{M^{\Lambda_k}} \frac{|\nabla f_k|^2}{f_k}\, \mathrm{d}\lambda_k \leqslant \frac{\eta + \varepsilon_k}{1 - \varepsilon_k} + \frac{\kappa^2 + 1/4}{1 - \varepsilon_k} \sum_{i \in \Lambda_k \setminus \Lambda_{k-1}} \int\limits_{M^{\Lambda_{k+1}}} \frac{|\nabla_i f_{k+1}|^2}{f_{k+1}}\, \mathrm{d}\lambda_{k+1} \tag{7.12}$$

for all k such that $\varepsilon_k < 1$. Therefore, letting $\Lambda_0 := \emptyset$ and

$$T_k := \sum_{i \in \Lambda_k \setminus \Lambda_{k-1}} \int\limits_{M^{\Lambda_{k+1}}} \frac{|\nabla_i f_{k+1}|^2}{f_{k+1}}\, \mathrm{d}\lambda_{k+1}, \quad k \in \mathbb{N}, \tag{7.13}$$

we obtain by (7.11) and (7.12) that

$$\sum_{j=1}^{k-1} T_j \leqslant \sum_{j=1}^{k-1} \sum_{i \in \Lambda_j \setminus \Lambda_{j-1}} \int\limits_{M^{\Lambda_k}} \frac{|\nabla_i f_k|^2}{f_k}\, \mathrm{d}\lambda_k \leqslant \frac{\eta + \varepsilon_k}{1 - \varepsilon_k} + \frac{\kappa^2 + 1/4}{1 - \varepsilon_k} T_k$$

for all k with $\varepsilon_k < 1$. Let us show that $\sum_{j=1}^{k} T_j \leqslant \eta$ for all k. Otherwise, we have $\sum_{j=1}^{k_0} T_j \geqslant (1+\varepsilon)(\eta+\varepsilon_{k_0})/(1-\varepsilon_{k_0})$ for some $k_0 \geqslant 1$ and $\varepsilon > 0$. Then there exists $c > 0$ such that

$$\sum_{j=1}^{k-1} T_j \leqslant \frac{T_k}{c}, \qquad \forall k > k_0.$$

This implies

$$\sum_{j=1}^{k} T_j \geqslant (1+c)\sum_{j=1}^{k-1} T_j \geqslant \cdots \geqslant (1+c)^{k-k_0}\sum_{j=1}^{k_0} T_j$$

for all $k > k_0$, which is impossible by (7.9). Thus, we have $T_k \to 0$ as $k \to \infty$. Noting that $\int_{M^{\Lambda_k}} \frac{|\nabla f_k|^2}{f_k}\,\mathrm{d}\lambda_k$ is nondecreasing in k according to (7.11), we obtain form (7.12) that

$$\sup_k \int\limits_{M^S} \frac{|\nabla f_k|^2}{f_k}\,\mathrm{d}\lambda^S = \lim_{k\to\infty} \int\limits_{M^S} \frac{|\nabla f_k|^2}{f_k}\,\mathrm{d}\lambda^S \leqslant \eta < \infty.$$

As in the proof of Theorem 4.3, this yields that the sequence $\{\sqrt{f_k}\}$ converges weakly in $H^{2,1}(\lambda^S)$ to some $f \in H^{2,1}(\lambda^S)$ and that $\mu = f^2\lambda^S$. In addition, $\int_{M^S} |\nabla f|^2\,\mathrm{d}\lambda^S \leqslant \eta/4$. □

Remark 7.5. – The proof of Theorem 7.4 enables us to generalize [42, Theorem 4]. Namely, suppose that Z is of finite range with $\operatorname{div} Z_i = 0$ for all $i \in S$. If μ is a probability measure on M^S such that $\sup_i \|Z_i\|_\infty < \infty$ and $L_Z^*\mu = 0$ with respect to the class $\mathcal{FC}_0^\infty$, then $\mu = \lambda^S$ (simply note that in this case $\eta = 0$). Unlike [42, Theorem 4], no smoothness of Z_i is required.

Now we shall consider a more general situation when M^i are complete Riemannian manifolds and Lebesgue product measure is replaced by some probability measure ν on M^S. In the rest of this section we shall refer to the following assumptions.

(A1) For every i, M^i is a complete Riemannian manifold of finite dimension satisfying the hypotheses of Corollary 2.3.

(A2) The projection ν_k of ν to M^{Λ_k}, where Λ_k is the same as above, satisfies condition (4.5), has a density $\exp(W_k)$ with respect to the Riemannian volume such that $W_k \in W^{1,1}_{\mathrm{loc}}(\lambda_{\Lambda_k})$ and $|\nabla_i W_k| \in L^2_{\mathrm{loc}}(\nu_k)$. Set $\beta^{\nu_k} = (\beta_i^{\nu_k})_{i\in\Lambda_k}$, $\beta_i^{\nu_k} = \nabla_i W_k$, where we fix some Borel versions.

(A3) $H^{2,1}(\nu)$ is well-defined (i.e., the linear span of $\mathcal{FC}_0^\infty$ with norm $\|\cdot\|_{H^{2,1}(\nu)}$ is closable on $L^2(\nu)$) and the logarithmic Sobolev inequality (4.4) holds for ν.

THEOREM 7.6. – *Let $\nu \in \mathcal{P}(M^S)$ be such that* (A1), (A2), (A3) *are fulfilled. Let $\mu \in \mathcal{P}(M^S)$ satisfy $L_Z^*\mu = 0$ with respect to $\mathcal{FC}_0^\infty$, where $Z = (Z_i)_{i\in S}$ is of finite range R, $|Z_i| \in L^2(\mu)$, $|\beta_i^{\nu_k}| \in L^2(\mu)$, $i \in \Lambda_k$, and let*

$$\kappa^2 := \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int\limits_{M^S} \left|Z_i - \beta_i^{\nu_k}\right|^2 \mathrm{d}\mu < \infty. \tag{7.14}$$

Suppose that, for all $i \in \Lambda_{k-1}$, one has $|Z_i| \in L^2_{\mathrm{loc}}(\nu_k)$ and that $\operatorname{div}^{\nu_k}(\beta_i^{\nu_k} - Z_i)$ exists and

$$J := \sup_k \int\limits_{M^S} \left|\sum_{i\in\Lambda_{k-1}} \operatorname{div}^{\nu_k}\left(\beta_i^{\nu_k} - Z_i\right)\right| \mathrm{d}\mu < \infty. \tag{7.15}$$

Then $\mu = f^2\,\nu$, *where* $f \in H^{2,1}(\nu)$.

Proof. – Let $\mu_k := \mu_{\Lambda_k}$ and $f_k := \mathrm{d}\mu_k/\mathrm{d}\nu_k$. By the same reasoning as in Theorem 4.3, we have that f_k exists and that $\sqrt{f_k} \in H^{2,1}(\nu_k)$. Let $V_i := Z_i - \beta_i^{\nu_k}$, $i \in \Lambda_{k-1}$. We shall show that

$$\sum_{i\in\Lambda_{k-1}} \int\limits_{M^{\Lambda_k}} \langle V_i, \nabla_i f_k\rangle\,\mathrm{d}\nu_k = -\int\limits_{M^{\Lambda_k}} \sum_{i\in\Lambda_{k-1}} \mathrm{div}^{\nu_k} V_i\; f_k\,\mathrm{d}\nu_k \leqslant J. \tag{7.16}$$

By (7.15), it is enough to justify the integration by parts in the equality on the left in (7.16). We shall do this for any nonnegative f_k such that $\sqrt{f_k} \in H^{2,1}(\nu_k)$ and $|V_i|^2 f_k$ and $\sum_{i\in\Lambda_{k-1}} \mathrm{div}^{\nu_k} V_i\; f_k$ are in $L^1(\nu_k)$ (here we do not use that f_k is related to μ_k). Then it suffices to prove (7.16) for bounded f_k passing to the functions $\min(f_k^{1/2}, n)^2$ and letting $n \to \infty$ (note that $|V_i|\sqrt{f_k}$, $|\nabla f_k|/\sqrt{f_k} \in L^2(\nu_k)$ due to our assumptions). Moreover, we may assume that f_k has compact support by passing to functions $\zeta_j f_k$, where $\zeta_j \in C_0^\infty(M^{\Lambda_k})$ are nonnegative uniformly bounded functions with uniformly bounded gradients and $\zeta_j = 1$ on $B(o, j)$. There exists a sequence $\{\psi_j\} \subset C_0^\infty(M^{\Lambda_k})$ of nonnegative uniformly bounded functions with supports in a compact set K such that $\psi_j \to \sqrt{f_k}$ in $H^{2,1}(\nu_k)$ and $\psi_j \to \sqrt{f_k}$ ν_k-a.e. Since the desired integration by parts formula holds for ψ_j^2 in place of f_k, it remains to note that $|\sqrt{f_k}V_i - \psi_j V_i| + |2\nabla\psi_j - \nabla f_k\big/\sqrt{f_k}| \to 0$ in $L^2(\nu_k)$ due to our assumptions $|\beta_i^{\nu_k}|, |Z_i| \in L^2_{\mathrm{loc}}(\nu_k)$, so that $|V_i| \in L^2_{\mathrm{loc}}(\nu_k)$. Thus, (7.16) is established.

By Lemma 7.1 we obtain

$$\begin{aligned}
&\int\limits_{M^{\Lambda_k}} \frac{|\nabla f_k|^2}{f_k^2}\, f_k\,\mathrm{d}\nu_k \\
&\quad= \sum_{i\in\Lambda_{k-1}} \int\limits_{M^{\Lambda_k}} \big\langle Z_i - \beta_i^{\nu_k}, \nabla_i f_k\big\rangle \mathrm{d}\nu_k + \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int\limits_{M^{\Lambda_{k+1}}} \left\langle Z_i - \beta_i^{\nu_k}, \frac{\nabla_i f_k}{f_k}\right\rangle f_{k+1}\,\mathrm{d}\nu_{k+1} \\
&\quad\leqslant J + \frac{1}{2} \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int\limits_{M^{\Lambda_{k+1}}} \left[\big|Z_i - \beta_i^{\nu_k}\big|^2 + \frac{|\nabla_i f_k|^2}{f_k^2}\right] f_{k+1}\,\mathrm{d}\nu_{k+1} \\
&\quad\leqslant J + \frac{1}{2}\kappa^2 + \frac{1}{2} \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int\limits_{M^{\Lambda_k}} \frac{|\nabla_i f_k|^2}{f_k}\,\mathrm{d}\nu_k.
\end{aligned} \tag{7.17}$$

Hence $\||\nabla f_k|/\sqrt{f_k}\|_{L^2(\nu_k)} \le 2J + \kappa^2$. By the same reasoning as in Theorem 4.3 we obtain that the sequence $\sqrt{f_k}$ converges weakly in $H^{2,1}(\nu)$ to some function f. The logarithmic Sobolev inequality yields that $\mu = f^2\,\nu$ (cf. the end of the proof of Theorem 4.3). □

We observe that in the case of the one-dimensional lattice ($S = \mathbb{Z}$) condition (7.14) is equivalent to the condition $\sup_i \int_{M^{\mathbb{Z}}} |Z_i - \beta_i^{\nu_k}|^2\,\mathrm{d}\mu < \infty$. We also note that, as one can notice from (7.17), it is enough to replace Z_i in (7.14) by the conditional expectation $\mathbb{E}^{\mu}_{\Lambda_k} Z_i$ of Z_i with respect to μ and the σ-field generated by x_{Λ_k}. Clearly, (7.15) is fulfilled if $\beta_i^\nu = Z_i$, since then $Z_i = \beta_i^{\nu_k}$ for all $i \in \Lambda_{k-1}$.

COROLLARY 7.7. – *Assume that in the situation of Theorem* 7.6 *one has* $Z_i = \beta_i^\nu = \beta_i^\mu$ *for some Borel versions. Then* $\mu = \nu$.

Proof. – By Theorem 7.6, $\mu = f^2\nu$, where $f \in H^{2,1}(\nu)$. It is readily seen that $\beta_i^\mu = 2\nabla_i f/f + \beta_i^\nu$. Hence $\nabla_i f = 0$ ν-a.e. since it holds μ-a.e. but also holds ν-a.e. on the set $\{f = 0\}$. Then by

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the logarithmic Sobolev inequality we have $\int f^2 \log f \,\mathrm{d}\nu \leqslant 0$, which yields that $f = 1$ (see, e.g., [10, Lemma 1.7.7]). □

We shall now see that, even without the assumption about the log-Sobolev inequality every solution of the elliptic equation with a symmetric solution is also symmetric provided it satisfies condition (7.18) below. This means, in particular, that every invariant measure satisfying (7.18) is Gibbsian provided that there is a Gibbsian invariant measure. This result extends well-known results by Holley and Stroock [35] and Fritz [28,29]. Some ideas of the above cited papers are used below.

THEOREM 7.8. – *Let $\nu \in \mathcal{P}(M^S)$ be such that* (A1) *and* (A2) *are fulfilled and that it has logarithmic derivatives β_i^ν with respect to $\mathcal{FC}_0^\infty$ along x_i for all $i \in S$. Set $Z_i := \beta_i^\nu$, where we fix some Borel version. Assume that $Z := (Z_i)_{i\in S}$ is of finite range R. Let $\mu \in \mathcal{P}(M^S)$ be such that $L_Z^*\mu = 0$ with respect to $\mathcal{FC}_0^\infty$, where $|Z_i| \in L^2(\mu)$. Letting $\mathbb{E}_{\Lambda_k}^\mu Z_i$ be the conditional expectation of Z_i with respect to the σ-field generated by x_{Λ_k} and the measure μ, assume that for some $c > 0$*

$$\int_{M^S} \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \left|\mathbb{E}_{\Lambda_k}^\mu Z_i - \beta_i^{\nu_k}\right|^2 \mathrm{d}\mu \leqslant ck, \quad \forall k \in \mathbb{N}, \tag{7.18}$$

where $\beta_i^{\nu_k}$ are fixed Borel versions. Then β_i^μ exists and coincides with Z_i for every $i \in S$. In particular, μ is Gibbsian.

Proof. – Let $\mu_k := \mu_{\Lambda_k}$ and $f_k := \mathrm{d}\mu_k/\mathrm{d}\nu_k$ where as above $\nu_k := \nu_{\Lambda_k}$. By the same reasoning as in the proof of Theorem 4.3, we obtain that f_k exists and that $\sqrt{f_k} \in H^{2,1}(\nu_k)$. Due to the equality $Z_i = \beta_i^\nu$ and the finite range assumption, we have $\beta_i^{\nu_k} = Z_i$ whenever $i \in \Lambda_{k-1}$. Therefore, we obtain by (7.3):

$$\begin{aligned} &\sum_{i\in\Lambda_{k-1}} \int_{M^{\Lambda_k}} \frac{|\nabla_i f_k|^2}{f_k}\,\mathrm{d}\nu_k \\ &\quad = - \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^{\Lambda_{k+1}}} \left\langle \frac{\nabla_i f_k}{f_k}, \frac{\nabla_i f_{k+1}}{f_{k+1}} \right\rangle \mathrm{d}\mu_{k+1} \\ &\quad \leqslant \left(\sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^{\Lambda_k}} \frac{|\nabla_i f_k|^2}{f_k^2}\,\mathrm{d}\mu_k \right)^{1/2} \left(\sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^{\Lambda_{k+1}}} \frac{|\nabla_i f_{k+1}|^2}{f_{k+1}^2}\,\mathrm{d}\mu_{k+1} \right)^{1/2}. \end{aligned} \tag{7.19}$$

We observe that the first factor on the right in (7.19) is majorized by $\sqrt{ck}$. Indeed, by (7.2), (7.18) and the equality $\beta_i^{\nu_k} = Z_i$ for every $i \in \Lambda_{k-1}$, we have

$$\begin{aligned} \int_{M^{\Lambda_k}} \frac{|\nabla f_k|^2}{f_k^2}\,\mathrm{d}\mu_k &= \sum_{i\in\Lambda_k\setminus\Lambda_{k-1}} \int_{M^S} \left\langle \mathbb{E}_{\Lambda_k}^\mu Z_i - \beta_i^{\nu_k}, \frac{\nabla_i f_k}{f_k} \right\rangle \mathrm{d}\mu \\ &\leqslant \sqrt{ck} \left(\int_{M^{\Lambda_k}} \frac{|\nabla f_k|^2}{f_k^2}\,\mathrm{d}\mu_k \right)^{1/2}. \end{aligned}$$

This implies $\||\nabla f_k|/f_k\|_{L^2(\mu_k)} \leqslant \sqrt{ck}$. Let T_k be defined by (7.13) with ν_{k+1} in place of λ_{k+1}. Then we obtain by (7.19) and (7.6) that

$$\sum_{j=1}^{k-1} T_j \leqslant \sqrt{ckT_k}.$$

This yields that $T_k = 0$ for all k. Indeed, let g be the function on $[1, +\infty)$ which equals T_j on $[j, j+1)$. By the above inequality and the estimate $T_k \leqslant \sqrt{2ckT_k}$, which follows from the estimate $T_k \leqslant \||\nabla f_{k+1}|/f_{k+1}\|^2_{L^2(\mu_{k+1})} \leqslant c(k+1) \leqslant 2ck$, we obtain that the function $G(t) = \int_1^t g(s)\,\mathrm{d}s$ satisfies the inequality $\sqrt{c_1 t G'(t)} \geqslant G(t)$ for some $c_1 > c$. It remains to note that any positive solution of the inequality $G^2(t) \leqslant c_1 t G'(t)$ explodes in finite time which leads to a contradiction. Hence (7.19) implies $\nabla_i \sqrt{f_k} = 0$ ν_k-a.e. for all $i \in \Lambda_{k-1}$. Since (4.5) holds for $H^{2,1}(\nu_k)$, this implies that f_k only depends on $x_{\Lambda_k \setminus \Lambda_{k-1}}$. Therefore $\beta_i^{\mu_k} = \beta_i^{\nu_k}$ μ_k-a.e. for all $i \in \Lambda_{k-1}$. Since $Z_i = \beta_i^{\nu_k}$ ν_k-a.e. for all $i \in \Lambda_{k-1}$, it follows directly from Definition 6.1 that β_i^{μ} exists and $\beta_i^{\mu} = Z$. $\square$

We observe that condition (7.18) is fulfilled if

$$Z_i(x) = Z_i^{(1)}(x_{\Lambda_k}) + Z_i^{(2)}(x), \quad i \in \Lambda_k, \tag{7.20}$$

where

$$\sup_{i \in \Lambda_k \setminus \Lambda_{k-1}} \big|Z_i^{(2)}(x)\big| \leqslant c' k^{2-m}, \quad \forall k \geqslant 1,$$

for some $c' > 0$. Indeed, in this case, we have

$$\mathbb{E}^{\nu}_{\Lambda_k} Z_i = Z_i^{(1)} + \mathbb{E}^{\nu}_{\Lambda_k} Z_i^{(2)} = \mathbb{E}^{\mu}_{\Lambda_k} Z_i + \mathbb{E}^{\nu}_{\Lambda_k} Z_i^{(2)} - \mathbb{E}^{\mu}_{\Lambda_k} Z_i^{(2)}, \quad \mu_k\text{-a.e.}$$

We observe that if μ satisfies $L_Z^*\mu = 0$ and $|Z_i| \in L^2(\mu)$ for $i \in S$, then μ_k is absolutely continuous with respect to ν_k.

For example, (7.18) is fulfilled if $m \leqslant 2$ and one has (7.20) with $\sup_i |Z_i^{(2)}| < \infty$. Clearly, this is the case if $m \leqslant 2$ and the Z_i's are uniformly bounded. Thus, in the case of the two-dimensional lattice, the above theorem gives broad sufficient conditions for the reversibility of every stationary measure of the stochastic system associated with a Gibbs measure. A detailed discussion of such applications and of the relation to [28,29] and [35] will be addressed in a forthcoming paper. Certain a priori estimates which can be used for the verification of (7.18) are discussed in the next section.

It should be also noted that the technical condition (4.5) is ensured by the existence of continuous strictly positive densities of the measures ν_k, which, in turn, follows, by [14, Proposition 2.18], from the following condition: $\exp(\varepsilon_i |Z_i|) \in L^1(\nu)$ for some $\varepsilon_i > 0$.

Finally, let us note that analogous results are valid for more general elliptic equations which involve non-constant diffusion coefficients a_i which depend on x_j with $j \in \Lambda_i$.

8. Estimates of solutions in infinite dimensions

In this section, we establish some a priori estimates for arbitrary probability measures solving the elliptic equations considered in the previous sections. In particular, we show the integrability of Ψ with respect to every probability measure satisfying the corresponding elliptic equation and not only with respect to the above constructed solution μ (the interest in such estimates is due to possible non-uniqueness of solutions in the class of all probability measures). The basic idea behind such a priori estimates is simple and well known: given a positive number c and a positive function Ψ such that $L_Z\Phi \leqslant c - \Psi$ for a suitable function Φ, one obtains the estimate $\int \Psi\,\mathrm{d}\mu \leqslant c$ provided that $L_Z^*\mu = 0$. The subsequent results give a justification of this formal procedure. In this section, we fix a collection $Z = (Z_i)_{i\in S}$ of Borel maps on M^S such that $Z_i(x) \in T_{x_i}M^i$.

LEMMA 8.1. – *Let $\mu \in \mathcal{P}(M^S)$ satisfy equation $L_Z^*\mu = 0$ with respect to some class $\mathcal{K}$ (cf.* Definition 4.1). *Suppose that V is a nonnegative μ-a.e. finite Borel function on M^S such that $\nabla_i V$, $\Delta_i V$ exist for all $i \in S$ and such that $\varphi \circ V \in \mathcal{K}$ for every $\varphi \in C_0^\infty(\mathbb{R}^1)$. Let Θ be a nonnegative Borel function on M^S that is μ-integrable on the sets $\{V \leqslant c\}$, $c \in [0,\infty)$ (e.g., let $\Theta = \chi \circ V$, where χ is a nonnegative locally bounded Borel function on $\mathbb{R}^1$). Assume, in addition, that $L_Z V \leqslant C - \Theta$ μ-a.e. in the following sense: there exist μ-measurable functions λ_i such that the series $\sum_{i\in S}\lambda_i$ converges in $L^1(\mu)$ on the sets $\{V \leqslant c\}$, $c \in [0,\infty)$, and one has*

$$L_i V \leqslant \lambda_i \quad \mu\text{-a.e.} \quad \text{and} \quad \sum_{i\in S}\lambda_i(x) \leqslant C - \Theta(x) \quad \mu\text{-a.e.}$$

Then

$$\int_{M^S} \Theta\,\mathrm{d}\mu \leqslant C. \tag{8.1}$$

Proof. – Certainly, (8.1) follows trivially by integrating the estimate $L_Z V \leqslant C - \Theta$ and making use of the equality $\int L_Z V\,\mathrm{d}\mu = 0$. However, due to the above interpretation of both relations, some justification is needed. By our hypothesis, we have (4.3) with $\psi = \varphi \circ V$ for every $\varphi \in C_0^\infty(\mathbb{R}^1)$. Then the same is true for every $\varphi \in C^\infty(\mathbb{R}^1)$ such that $\varphi = \text{const}$ outside some interval, since $\varphi - \text{const} \in C_0^\infty(\mathbb{R}^1)$ and (4.3) is trivially true for $\psi = \text{const}$. Now let us fix an even function $\zeta \in C^\infty(\mathbb{R}^1)$ such that $\zeta(t) = t$ if $|t| \leqslant 1$, $\zeta(t) = 2$ if $t \geqslant 3$, $0 \leqslant \zeta'(t) \leqslant 1$, and $\zeta''(t) \leqslant 0$ if $t \geqslant 0$. Set $\zeta_j(t) = j\zeta(t/j)$ if $t \geqslant 0$ and $\zeta_j(t) = \zeta_j(-t)$ if $t \leqslant 0$. Clearly, $0 \leqslant \zeta_j'(t) \leqslant 1$ and $\zeta_j''(t) \leqslant 0$ if $t \geqslant 0$. In addition, $\zeta_j(t) = t$ if $t \in [0, j]$ and $\zeta_j(t) = 2$ if $t \geqslant 2j$. Hence, (4.3) is satisfied for $\psi = \zeta_j \circ V$. We observe that

$$L_i(\zeta_j \circ V) = \zeta_j' \circ V\, L_i V + \zeta_j'' \circ V\, |\nabla_i V|^2 \leqslant \zeta_j' \circ V\, L_i V \leqslant (\zeta_j' \circ V)\lambda_i.$$

By (4.3), the convergence of the series $\sum_{i\in S}(\zeta_j' \circ V)\lambda_i$ in $L^1(\mu)$, and the hypothesis $\sum_{i\in S}\lambda_i \leqslant C - \Theta$, we arrive at the estimate

$$\int_{M^S} (\zeta_j' \circ V)\Theta\,\mathrm{d}\mu \leqslant C \int_{M^S} \zeta_j' \circ V\,\mathrm{d}\mu \leqslant C,$$

whence the desired estimate follows by Fatou's lemma, since one has $\zeta_j' \circ V \geqslant 0$ and $\lim_{j\to\infty} \zeta_j' \circ V \to 1$ μ-a.e. □

Remark 8.2. – Suppose that the functions λ_i in the above lemma can be written as $\lambda_i = u_i - w_i$, where u_i and w_i are nonnegative functions μ-integrable on the sets $\{V \leqslant r\}$. Then the convergence of the series $\sum_{i\in S}\lambda_i$ in $L^1(\mu)$ on the sets $\{V \leqslant r\}$ is equivalent to the integrability of the series $\sum_{i\in S} u_i$ on the sets $\{V \leqslant r\}$. Indeed, let ζ_r be the function introduced in the proof

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of Lemma 8.1. Then, as we have seen,

$$L_i(\zeta_r \circ V) \leqslant (\zeta_r' \circ V)\lambda_i = (\zeta_r' \circ V)u_i - (\zeta_r' \circ V)w_i.$$

Since the sum of the integrals of $L_i(\zeta_r \circ V)$ is zero, it follows that

$$\sum_{i\in S} \int_{V\leqslant r} w_i \,\mathrm{d}\mu \leqslant \sum_{i\in S} \int_{M^S} (\zeta_r' \circ V)w_i \,\mathrm{d}\mu \leqslant \sum_{i\in S} \int_{M^S} (\zeta_r' \circ V)u_i \,\mathrm{d}\mu \leqslant \sum_{i\in S} \int_{V\leqslant 3r} u_i \,\mathrm{d}\mu < \infty,$$

since $0 \leqslant \zeta_r' \leqslant 1$, $\zeta_r' \circ V = 1$ on $\{V \leqslant r\}$, and $\zeta_r' \circ V = 0$ outside $\{V \leqslant 3r\}$.

THEOREM 8.3. – *Suppose that in the situation of Proposition* 5.2, $\mu \in \mathcal{P}(M^S)$ *satisfies* $L_Z^*\mu = 0$ *with respect to the class* $\mathcal{K}$ *such that* $\varphi \circ V \in \mathcal{K}$ *for every* $\varphi \in C_0^\infty(\mathbb{R}^1)$, *where* $V = \sum_{i\in S} q_i V_i$ *is finite* μ*-a.e. Assume also that* $G_i \leqslant V_i$. *Then*

$$\int \sum_{i\in S} q_i G_i \,\mathrm{d}\mu \leqslant \frac{c}{\delta} \sum_{j\in S} q_j. \tag{8.2}$$

Moreover,

$$\int G_i \,\mathrm{d}\mu \leqslant \frac{c}{\delta}, \quad \forall i \in S. \tag{8.3}$$

Proof. – We may assume that $\sum_i q_i = 1$. Let

$$\lambda_i = c - (\lambda + \delta)G_i + \sum_j a_{i,j} G_j.$$

Clearly, $L_i V \leqslant q_i\lambda_i$ and $\sum_i q_i\lambda_i \leqslant c - \delta\Psi$. It is readily seen that the series $q_i c + q_i \sum_j a_{i,j} G_j$ converges in $L^1(\mu)$ on every set $M_r = \{V \leqslant r\}$. Indeed, for every $x \in M_r$, one has $\{G_j(x)\} \in l^1(q)$ due to the estimate $G_i \leqslant V_i$, hence $\sum_i (q_i \sum_j a_{i,j} G_j(x)) \leqslant \lambda \sum_j q_j G_j(x) \leqslant \lambda r$. Estimate (8.3) follows in the same manner as in the proof of Lemma 5.1. □

In the Gibbsian case, a priori estimates follow trivially from the finite-dimensional case.

PROPOSITION 8.4. – *Let* $\mu \in \mathcal{P}(M^S)$ *be such that the conditional probabilities* $\mu(\,\cdot\,|x_{i^c})$ *on* $M^i \times \{x_{i^c}\}$, $x_{i^c} \in M^{i^c}$, *have continuously differentiable densities* $x_i \mapsto \exp V_i(x_i, x_{i^c})$ *such that the mappings* $x \mapsto Z_i(x) = \nabla_i V_i(x_i, x_{i^c})$ *are continuous on* M^S. *Let* $\|A\|_{l^1(q)} \leqslant \lambda$ *and* $\sum_{j\in S} a_{i,j} \leqslant \lambda$ *for all* $i \in S$. *Assume that, for each* $i \in S$, *there exist nonnegative functions* $G_i \in C(M^n)$ *and nonnegative compact functions* $V_i \in C^2(M^i)$ *such that, for some* $c, \delta > 0$, *one has*

$$\Delta_i V_i(x_i) + \bigl\langle \nabla_i V_i(x_i), Z_i(x)\bigr\rangle \leqslant c - (\lambda + \delta)G_i(x_i) + \sum_{j\in S} a_{i,j} G_j(x_j).$$

Suppose that $\Phi = \sum_{i\in S} q_i V_i < \infty$ μ*-a.e. and that* $G_i \leqslant V_i$. *Then*

$$\int G_i \,\mathrm{d}\mu \leqslant \frac{c}{\delta}, \quad \forall i \in S.$$

Moreover, the same is true if $V_i = F_i \circ \varrho_i$, *where* $F_i \in C^2[0,\infty)$ *is such that* $\lim_{r\to\infty} F_i(r) = \infty$ *and* (5.1) *holds for all* $x = (x_i)_{i\in S}$ *with* $x_i \notin \mathrm{cut}(o_i)$, $\forall i \in S$.

Proof. – Let ζ_j be the same as in the proof of Lemma 8.1. Let ν_i be the projection of μ to M^{i^c}. We know that, for ν_i-a.e. x_{i^c}, the regular conditional probability $\mu(\cdot\,|x_{i^c})$ on $M^i \times \{x_{i^c}\}$, $x_{i^c} \in M^{i^c}$, has the logarithmic gradient $x_i \mapsto Z_i(x_i, x_{i^c})$, hence satisfies the elliptic equation $L_i^*\mu(\cdot\,|x_{i^c}) = 0$ with respect to $C_0^\infty(M^i)$. Note that

$$L_i(\zeta_j \circ \Phi) = q_i\zeta_j' \circ \Phi\, L_i V_i + \zeta_j'' \circ \Phi |\nabla_i V_i|^2 \leqslant q_i \zeta_j' \circ \Phi \Bigl(c - (\lambda+\delta)G_i + \sum_{j\in S} a_{i,j}G_j\Bigr).$$

According to Proposition 3.2, we obtain

$$\int_{M^i} \Bigl(q_i c\zeta_j' \circ \Phi - q_i(\lambda+\delta)G_i\zeta_j' \circ \Phi + q_i\zeta_j' \circ \Phi \sum_{j\in S} a_{i,j}G_j\Bigr)\,\mu(\mathrm{d}x_i|x_{i^c}) \geqslant 0.$$

Integrating this inequality with respect to ν_i and summing over $i \in S$, we arrive at the estimate

$$\begin{aligned}\int_{M^S} (\lambda+\delta)\zeta_j' \circ \Phi \sum_{i\in S} q_i G_i\,\mathrm{d}\mu &\leqslant c\sum_{i\in S} q_i + \int_{M^S} \zeta_j' \circ \Phi \sum_{i,j} q_i a_{i,j} G_j\,\mathrm{d}\mu\\ &\leqslant c\sum_{i\in S} q_i + \int_{M^S} \lambda\zeta_j' \circ \Phi \sum_{i\in S} q_i G_i\,\mathrm{d}\mu,\end{aligned}$$

since $\{G_i(x_i)\} \in l^1(q)$ for μ-a.e. x by the estimate $G_i \leqslant V_i$ and the assumption that $\Phi < \infty$ μ-a.e. Therefore,

$$\delta \int_{M^S} \zeta_j' \circ \Phi \sum_{i\in S} q_i G_i\,\mathrm{d}\mu \leqslant c\sum_{i\in S} q_i.$$

Letting $j \to \infty$ and noting that $\zeta_j' \circ \Phi \to 1$ μ-a.e. and $0 \leqslant \zeta_j' \leqslant 2$, we obtain the estimate

$$\int_{M^S} \sum_{i\in S} q_i G_i\,\mathrm{d}\mu \leqslant \frac{c}{\delta}\sum_{i\in S} q_i.$$

Now the desired estimate follows in the same manner as in Lemma 5.1. □

Example 8.5. – The assertion of Proposition 8.4 is valid if μ is as in the proposition with $V_i = G_i = \varrho_i^p$, where $p \geqslant 2$, and

$$\varrho_i^{p-1}\langle \nabla_i \varrho_i, Z_i\rangle \leqslant p^{-1}c - \varrho_i^{p-1}\Delta_i\varrho_i - (p-1)\varrho_i^{p-2} - p^{-1}(\lambda+\delta)\varrho_i^p + p^{-1}\sum_{j\in S} a_{i,j}\varrho_j^p$$

holds for all $x = (x_i)_{i\in S} \in M^S$ with $x_i \notin \mathrm{cut}(o_i)$, $\forall i \in S$. In particular, if $\Delta_i\varrho_i \leqslant c_0(1+\varrho_i^{-1})$ outside $\mathrm{cut}(o_i) \cup \{o_i\}$, for some $c_0 > 0$ and all $i \in S$, then it suffices to have the estimate

$$\varrho_i^{p-1}\langle \nabla_i \varrho_i, Z_i\rangle \leqslant \widehat{c} - \widehat{\lambda}\varrho_i^p + p^{-1}\sum_{j\in S} a_{i,j}\varrho_j^p,$$

where $\widehat{c} = c - 2c_0p - p(p-1)$, $\widehat{\lambda} = p^{-1}(\lambda+\delta) + 2c_0p + p(p-1)$.

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Acknowledgment

This work has been supported in part by the Russian Foundation of Fundamental Research project 00-15-99267, the INTAS Grant 94-378, the INTAS-RFBR Grant 95-0099, the DFG Grant 436 RUS 113/343/0(R), the SFB-343, the EU-TMR-Project ERB-FMRX-CT96-0075, the NSFC (19631060), Fok Ying-Tung Educational Foundation, and the Alexander von Humboldt Foundation. We thank P. Malliavin for useful discussions.

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