A CHARACTERIZATION OF HEREDITARY CATEGORIES WITH TILTING OBJECT

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Introduction

Let k be an algebraically closed field and $\mathcal{H}$ a connected abelian $k-$category. We assume that $\mathcal{H}$ is hereditary, that is the Yoneda $\mathrm{Ext}^2(-,-)$ vanishes, and we assume that $\mathcal{H}$ has finite dimensional homomorphism and extension spaces. In addition $\mathcal{H}$ has a tilting object, that is some object T with $\mathrm{Ext}^1_{\mathcal{H}}(T,T) = 0$ such that $\mathrm{Hom}_{\mathcal{H}}(T,X) = 0 = \mathrm{Ext}^1_{\mathcal{H}}(T,X)$ implies $X = 0$. This concept was introduced in [HRS] to obtain a common treatment of both the class of tilted algebras (compare [HRi]) and the class of canonical algebras (compare [R2] or [LP]). This common treatment lead to the definition of a quasitilted algebra. A quasitilted algebra is the endomorphism algebra $\mathrm{End}_{\mathcal{H}}T$ of a tilting object $T \in \mathcal{H}$. In [HRS] quasitilted algebras are characterized by the following homological property. This class coincides with the class of finite dimensional $k-$algebras of global dimension at most 2 whose finitely generated indecomposable modules have either projective or injective dimension at most 1.

It was shown in [HRe1] that any hereditary category which is derived equivalent to a hereditary category with tilting object automatically has a tilting object. So one is interested in a description of hereditary categories with tilting object up to derived equivalence. Possible approaches to this problem include a thorough investigation of quasitilted algebras via the homological characterization or a detailled inspection of the essential features of the main examples. We will follow here the second approach.

There are two main known types of such categories $\mathcal{H}$; those derived equivalent to $\mathrm{mod}\, H$ for some finite dimensional hereditary $k-$algebra H and those derived equivalent to some category $\mathrm{coh}\,\mathbb{X}$ of coherent sheaves on a weighted projective line $\mathbb{X}$, in the sense of [GL]. Because of the simple description of the corresponding bounded derived category $D^b(\mathcal{H})$, it is possible to give a description of those in the same derived equivalence class (see for example [H1] and also [LS]). Note that the tilted algebras are those coming from $\mathrm{mod}\, H$ using an arbitrary tilting object, in this case called tilting module, and the canonical algebras are those coming from $\mathrm{coh}\,\mathbb{X}$ using a special type of tilting object. The first type of hereditary categories is characterized by the existence of some indecomposable directing object C [HRe1]. Recall that C is directing if it does not lie on a cycle of nonzero nonisomorphisms between indecomposable objects.

The aim of this paper is to show that these two types are the only possible hereditary categories containing a tilting object, and thereby proving a conjecture stated for example in [Re].

The result has quite a number of consequences. We only mention that it follows that a connected quasitilted algebra Λ (or even a connected piecewise hereditary algebra) satisfies that either $H^1(\Lambda) = 0$ or $H^2(\Lambda) = 0$, where we have denoted by $H^i(\Lambda)$ the $i-$th Hochschild cohomology space of Λ. Clearly, $H^0(\Lambda) \simeq k$ and

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$H^i(\Lambda) = 0$ for all $i \geq 3$. For details we refer to [H2] and [Re]. Moreover we have that the characterization of tame quasitilted algebras in [Sk] is an easy consequence.

This conjecture was previously shown under additional assumptions. In joint work with Reiten (see [H1]) it is proved that a hereditary category with tilting object which contains nonzero projective objects is equivalent to $\operatorname{mod} H$ for some finite dimensional hereditary $k-$algebra H. In [HRS] it was shown that representation-finite quasitilted are in fact tilted algebras, so the conjecture holds for this representation type. It was shown in [Sk] that tame quasitilted algebras can only arise as endomorphism algebras of tilting objects in hereditary categories from the list above. In [L] it was shown that any noetherian hereditary category with tilting object which does not contain a nonzero projective object is a category $\operatorname{coh} \mathbb{X}$ of coherent sheaves on a weighted projective line $\mathbb{X}$. This was generalized in [HRe2] in the following way. Any hereditary category with tilting object which contains a simple object is derived equivalent to a category of coherent sheaves or derived equivalent to $\operatorname{mod} H$ for some finite dimensional hereditary $k-$algebra H.

We point out that the main result of this article implies that any hereditary category with tilting object is derived equivalent to a noetherian hereditary category with tilting object.

Note that any tilting object T in $\mathcal{H}$ induces a torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ on $\mathcal{H}$, where $\mathcal{T}(T) = \operatorname{Fac} T$, the factors of finite direct sums of copies of T. We say that $X \in \mathcal{H}$ is torsionisable, if $X \in \mathcal{T}(T)$ for some tilting object $T \in \mathcal{H}$. We also say that $E \in \mathcal{H}$ is exceptional, if $\operatorname{Ext}^1_{\mathcal{H}}(E, E) = 0$.

The basic strategy of the proof of the main result is as follows. Assume that $\mathcal{H}$ is not equivalent to some $\operatorname{mod} H$ where H is a finite dimensional hereditary $k-$algebra. When $\mathcal{H}$ has some simple object, then we know that $\mathcal{H}$ is derived equivalent to some category $\operatorname{coh} \mathbb{X}$ [HRe2]. Note that the converse does not hold. So we may assume that $\mathcal{H}$ does not contain a simple object. But then it was shown in [HRe2] that for an indecomposable torsionisable exceptional object $E \in \mathcal{H}$ the right perpendicular category

$$E^{\perp} = \{X \in \mathcal{H} \,|\, \operatorname{Hom}_{\mathcal{H}}(E, X) = \operatorname{Ext}^1_{\mathcal{H}}(E, X) = 0\}$$

is equivalent to $\operatorname{mod} H$ for some finite dimensional hereditary $k-$algebra H. We will recall some more features on perpendicular categories in section 1. It is easy to see that one may assume that H is wild [HRe3]. In this case we will construct a directing object in $\mathcal{H}$, and the result will follow from [HRe1]. Actually we will assume to the contrary that there are no directing objects and will come up with a tilting object and an infinite chain of proper epimorphisms over the corresponding quasitilted algebra. We should point out that some arguments were inspired from the presentation of the results in [Ke]. These more technical results will be presented in section 2. The main result will be dealt with in the final section.

The category $\mathcal{H}$ is known to have Auslander-Reiten sequences (almost split sequences), and (when $\mathcal{H}$ is not equivalent to some $\operatorname{mod} H$ for a finite dimensional hereditary $k-$algebra H) there is an equivalence $\tau : \mathcal{H} \to \mathcal{H}$ with the property that if $0 \to A \to B \to C \to 0$ is an almost split sequence, then $\tau C \simeq A$. It is also known that the Grothendieck group $K_0(\mathcal{H})$ is free abelian of finite rank [HRS]. For more basic facts on hereditary categories with tilting object we refer to [H1], [HRS] and [HRe2].

We denote the composition of morphisms $f : X \to Y$ and $g : Y \to Z$ in a given category $\mathcal{K}$ by fg. When dealing with finite dimensional $k-$algebras we will consider finite dimensional left modules. For unexplained representation-theoretic terminology we refer to [R1] or [ARS].

1. Perpendicular categories

In this section let $\mathcal{H}$ be a connected hereditary abelian $k-$category with tilting object T. Denote by $\mathcal{H}_0$ the full subcategory of $\mathcal{H}$ whose objects are those of finite length, and by $\mathcal{H}_\infty$ the full subcategory of $\mathcal{H}$ where each indecompoable summand of the objects has infinite length. Let $\mathcal{H}_{\text{dir}}$ be the full subcategory of $\mathcal{H}$ where each indecomposable summand of the objects is directing. Recall that an indecomposable object $X \in \mathcal{H}$ is said to be directing, if X does not lie on a cycle of nonzero nonisomorphisms in $\mathcal{H}$ between indecomposable objects.

We will frequently use the concept of minimal right (resp. left) approximations (compare [AS]). Given two objects $X, Y \in \mathcal{H}$, then we have a map $f : X^t \to Y$ (resp. $g : X \to Y^t$) with the property that the induced map $\text{Hom}_{\mathcal{H}}(X, f) : \text{Hom}_{\mathcal{H}}(X, X^t) \to \text{Hom}_{\mathcal{H}}(X, Y)$ is surjective (resp. $\text{Hom}_{\mathcal{H}}(g, Y) : \text{Hom}_{\mathcal{H}}(Y^t, Y) \to \text{Hom}_{\mathcal{H}}(X, Y)$) is surjective. Note that $t = \dim_k \text{Hom}_{\mathcal{H}}(X, Y)$ and that as components of f (resp. g) one uses a $k-$basis of the vector space $\text{Hom}_{\mathcal{H}}(X, Y)$. We call f the minimal right $\text{add}\, X-$approximation of Y, where $\text{add}\, X$ is the full subcategory of $\mathcal{H}$ formed by finite direct sums of direct summands of X. We call g the minimal left $\text{add}\, Y-$approximation of X. In case that $\text{End}\, X = k$ we also have that the induced map $\text{Hom}_{\mathcal{H}}(X, f)$ is an isomorphism.

If $T \in \mathcal{H}$ is a tilting object we have the associated torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ where

$$\mathcal{T}(T) = \text{Fac}\, T = \{X \in \mathcal{H} \mid \text{Ext}^1_{\mathcal{H}}(T, X) = 0\}$$

and

$$\mathcal{F}(T) = \{Y \in \mathcal{H} \mid \text{Hom}_{\mathcal{H}}(T, Y) = 0\}.$$

Recall that for an indecomposable exceptional object $E \in \mathcal{H}$ there are defined the right and left perpendicular category $E^\perp$ and ${}^\perp E$ by

$$E^\perp = \{X \in \mathcal{H} \mid \text{Hom}_{\mathcal{H}}(E, X) = 0 = \text{Ext}^1_{\mathcal{H}}(E, X)\}$$

$${}^\perp E = \{X \in \mathcal{H} \mid \text{Hom}_{\mathcal{H}}(X, E) = 0 = \text{Ext}^1_{\mathcal{H}}(X, E)\}$$

If $\mathcal{H}$ is not equivalent to $\text{mod}\, H$ for a finite dimensional hereditary $k-$algebra H, we know that the Auslander-Reiten translation $\tau = \tau_{\mathcal{H}}$ is an equivalence on $\mathcal{H}$. In this case it easy to see that τ induces an equivalence from ${}^\perp E$ to $E^\perp$.

If E is indecomposable we have the Auslander-Reiten sequence

$$0 \to \tau E \to M \to E \to 0.$$

The following collection of known results is essential for the subsequent development. The first two assertions are contained in [HRe2] as 4.11 and 2.10. The third follows from [LM] and [R1], while the last is contained as 3.4 in [HS].

Theorem 1.1. *If $E \in \mathcal{H}_\infty$ is an indecomposable torsionisable exceptional object, and*

$$0 \to \tau E \to M \to E \to 0$$

is the Auslander-Reiten sequence, then the following holds.

(i) $E^\perp = \text{mod}\, H$ *for some finite dimensional hereditary $k-$algebra H with $M \in E^\perp$ and $rk\, K_0(H) = rk\, K_0(\mathcal{H}) - 1$.*

(ii) ${}_H H \oplus E$ *is a tilting object in $\mathcal{H}$ and $\text{End}_{\mathcal{H}}({}_H H \oplus E) = H[M]$ (the one-point extension algebra of H by M).*

(iii) *If H is a tame algebra, then $\mathcal{H}$ is derived equivalent to $coh\,\mathbb{X}$ for a weighted projective line $\mathbb{X}$ or to $mod\,H'$ for a finite dimensional hereditary $k-$algebra H'.*

(iv) *If $H[M]$ is not a tilted algebra and M is decomposable, then $M = M_1 \oplus M_2$ with M_1, M_2 indecomposable. M_1, M_2 are pairwise orthogonal and exactly one of the indecomposable summands is a direct summand of H.*

We point out that in the situation of 1.1 (iv) it was shown in [HS] that one component of $E^{\perp}$ is representation-finite of type $\mathbb{A}_r$ for some $r \in \mathbb{N}$. Later we will assume that the other component is not of tame representation type, i.e. that the underlying graph is not a Euclidean diagram.

The following two easy lemmas will be useful in the next section. They give for special tilting objects alternative descriptions of the torsion and torsionfree classes.

Lemma 1.2. *Let $E \in \mathcal{H}_\infty$ be an indecomposable torsionisable exceptional object with $E^{\perp} = mod\,H$ for some finite dimensional hereditary $k-$algebra H. If $T = {}_HH \oplus E$, then $\mathcal{T}(T) = \{X \in \mathcal{H} \mid Ext^1_{\mathcal{H}}(E, X) = 0\}$ and $\mathcal{F}(T) = Sub\,\tau E$, the subobjects of finite direct summands of copies of τE. Moreover for each $X \in \mathcal{F}(T)$ there exists an exact sequence $0 \to X \to \tau E^r \to Y \to 0$ with Y an injective $H-$module and some $r \in \mathbb{N}$.*

Proof: Trivially, $\mathcal{T}(T) \subset \{X \in \mathcal{H} \mid \mathrm{Ext}^1_{\mathcal{H}}(E, X) = 0\}$. For the converse inclusion let $X \in \mathcal{H}$ with $\mathrm{Ext}^1_{\mathcal{H}}(E, X) = 0$. If $\mathrm{Hom}_{\mathcal{H}}(E, X) = 0$, then $X \in E^{\perp}$, so $\mathrm{Ext}^1_{\mathcal{H}}({}_HH, X) = 0$ and so $X \in \mathcal{T}(T)$. If $\mathrm{Hom}_{\mathcal{H}}(E, X) \neq 0$ consider the minimal right $\mathrm{add}\,E-$approximation $f : E^t \to X$. Then $\mathrm{cok}\,f \in E^{\perp} \subset \mathcal{T}(T)$ and $\mathrm{im}\,f \in \mathrm{Fac}\,T = \mathcal{T}(T)$, so $X \in \mathcal{T}(T)$, since $\mathcal{T}(T)$ is closed under extensions.

Trivially $\mathrm{Sub}\,\tau E \subset \mathcal{F}(T)$, since $\tau E \in \mathcal{F}(T)$ and $\mathcal{F}(T)$ is closed under subobjects. For the converse inclusion let $X \in \mathcal{F}(T)$, so $0 \neq \mathrm{Ext}^1_{\mathcal{H}}(E, X) \simeq D\mathrm{Hom}_{\mathcal{H}}(X, \tau E)$. Consider the minimal left $\mathrm{add}\,\tau E-$approximation $g : X \to \tau E^r$ of X. Thus we have that $\mathrm{Hom}_{\mathcal{H}}(\ker g, \tau E) = 0$, so $\ker g \in \mathcal{T}(T)$ by the first part. But then $\mathrm{Hom}_{\mathcal{H}}(\ker g, X) = 0$, so $\ker g = 0$, hence $X \in \mathrm{Sub}\,\tau E$.

For the last part of the assertion, consider for $X \in \mathcal{F}(T)$ the minimal left $\mathrm{add}\,\tau E-$approximation $g : X \to \tau E^r$ which we know to be injective. So we have an exact sequence

$$(*) \quad 0 \to X \to \tau E^r \to Y \to 0.$$

Apply $\mathrm{Hom}_{\mathcal{H}}(E, -)$ to this sequence and use that $X, \tau E \in \mathcal{F}(T)$ then yields that the following sequence

$$0 \to \mathrm{Hom}_{\mathcal{H}}(E, Y) \to \mathrm{Ext}^1_{\mathcal{H}}(E, X) \xrightarrow{\ \phi\ } \mathrm{Ext}^1_{\mathcal{H}}(E, \tau E^r) \to \mathrm{Ext}^1_{\mathcal{H}}(E, Y) \to 0$$

is exact. Since $\mathrm{End}\,E \simeq k$, we infer that ϕ is an isomorphism, so $Y \in E^{\perp}$. Let $Z \in E^{\perp}$. Apply $\mathrm{Hom}_{\mathcal{H}}((Z, -)$ to the exact sequence $(*)$. This yields a surjection $\mathrm{Ext}^1_{\mathcal{H}}(Z, \tau E^r) \to \mathrm{Ext}^1_{\mathcal{H}}(Z, Y) \to 0$. Now $\mathrm{Ext}^1_{\mathcal{H}}(Z, \tau E^r) \simeq D\mathrm{Hom}_{\mathcal{H}}(\tau E^r, \tau Z) \simeq D\mathrm{Hom}_{\mathcal{H}}(E^r, Z) = 0$ shows that $\mathrm{Ext}^1_{\mathcal{H}}(Z, Y) = 0$, hence Y is an injective $H-$module.

We will also need the dual of 1.2.

Lemma 1.3. *Let $E \in \mathcal{H}_\infty$ be an indecomposable torsionisable exceptional object with ${}^{\perp}E = mod\,H$ for some finite dimensional hereditary $k-$algebra H. Then $T = E \oplus D(H_H)$ is a tilting object where $D(H_H)$ is the injective cogenerator in $mod\,H$. Moreover $\mathcal{T}(T) = Fac\,E$ and $\mathcal{F}(T) = \{Y \in \mathcal{H} \mid Hom_{\mathcal{H}}(E, Y) = 0\}$.*

Proof: Let $0 \to E \to \tau^- M \to \tau^- E \to 0$ be the Auslander-Reiten sequence starting at E. Then it is straightforward to check that $\tau^- M \in {}^{\perp}E$. But then it follows that $\mathrm{Ext}^1_{\mathcal{H}}(T, T) = 0$, since $\mathrm{Ext}^1_{\mathcal{H}}(\tau^- M, D(H_H)) = 0$, for $D(H_H)$ is injective.

Let $0 \neq X \in \mathcal{H}$ with $\mathrm{Hom}_{\mathcal{H}}(T,X) = \mathrm{Ext}^1_{\mathcal{H}}(T,X) = 0$. Then $\mathrm{Hom}_{\mathcal{H}}(\tau^- X, T) = \mathrm{Ext}^1_{\mathcal{H}}(\tau^- X, T) = 0$ shows that $\tau^- X \in {}^\perp E$ and so $\mathrm{Hom}_{\mathcal{H}}(\tau^- X, D(H_H)) \neq 0$, a contradiction. So T is a tilting object in $\mathcal{H}$.

Trivially $\mathcal{F}(T) \subset \{Y \in \mathcal{H} \,|\, \mathrm{Hom}_{\mathcal{H}}(E,Y) = 0\}$. Conversely, let $Y \in \mathcal{H}$ with $\mathrm{Hom}_{\mathcal{H}}(E,Y) = 0$. So $\mathrm{Ext}^1_{\mathcal{H}}(\tau^- Y, E) = 0$. If $\mathrm{Hom}_{\mathcal{H}}(\tau^- Y, E) = 0$, then $\tau^- Y \in {}^\perp E$. But then $0 = \mathrm{Ext}^1_{\mathcal{H}}(\tau^- Y, D(H_H)) \simeq \mathrm{Hom}_{\mathcal{H}}(D(H_H), Y)$, so $Y \in \mathcal{F}(T)$. If $\mathrm{Hom}_{\mathcal{H}}(\tau^- Y, E) \neq 0$ consider the minimal left $\mathrm{add}\, E-$approximation $f : \tau^- Y \to E^r$ of $\tau^- Y$. Then $\ker f \in {}^\perp E$, so $\mathrm{Ext}^1_{\mathcal{H}}(\ker f, D(H_H)) = 0$. Since $\mathrm{im}\, f$ is cogenerated by E we infer that $\mathrm{Ext}^1_{\mathcal{H}}(\mathrm{im}\, f, D(H_H)) = 0$. Thus also $0 = \mathrm{Ext}^1_{\mathcal{H}}(\tau^- Y, D(H_H)) = \mathrm{Hom}_{\mathcal{H}}(D(H_H), Y)$, hence $Y \in \mathcal{F}(T)$.

Trivially $\mathrm{Fac}\, E \subset \mathcal{T}(T)$. Conversely let $0 \neq X \in \mathcal{T}(T)$. Thus $\mathrm{Hom}_{\mathcal{H}}(\tau^- X, T) = 0$. So $\tau^- X \notin {}^\perp E$, since $D(H_H)$ is an injective cogenerator. Thus we have that $0 \neq \mathrm{Ext}^1_{\mathcal{H}}(\tau^- X, E) \simeq \mathrm{Hom}_{\mathcal{H}}(E, X)$. Consider the minimal left $\mathrm{add}\, E-$approximation $g : E^s \to X$ of X. By the first part of the proof we deduce from $\mathrm{Hom}_{\mathcal{H}}(E, \mathrm{cok}\, f) = 0$ that $\mathrm{cok}\, f \in \mathcal{F}(T)$. Since $X \in \mathcal{T}(T)$ we infer that $\mathrm{cok}\, f = 0$, or equivalently that $X \in \mathrm{Fac}\, E$.

We will also need the following consequence of 1.2.

Corollary 1.4. *Let $E \in \mathcal{H}_\infty$ be an indecomposable torsionisable exceptional object. If $X \in \mathcal{H}$ satisfies $\mathrm{Ext}^1_{\mathcal{H}}(E,X) = 0$, then X is torsionisable.*

2. Special torsion pairs

We keep the notation from the previous section. From now on we will assume that $\mathcal{H}$ is a connected hereditary abelian $k-$category with tilting object satisfying

(a) $\mathcal{H} = \mathcal{H}_\infty$,
(b) $\mathcal{H}_{\mathrm{dir}} = \emptyset$ and
(c) if $E \in \mathcal{H}$ is an indecomposable torsionisable exceptional object with $E^\perp = \mathrm{mod}\, H$ for some finite dimensional hereditary $k-$algebra H, then no connected component of H is tame.

Note that these assumptions in particular imply that τ is an equivalence. Note that under the assumption (b) it follows from [HS] by using the result in [HRe1] that the perpendicular category of any indecomposable torsionisable exceptional object has at most two connected components and that for any tilting object $T \in \mathcal{H}$ we have that $\mathrm{End}_{\mathcal{H}} T$ is quasitilted and not tilted again using [HRe1].

The main aim of this section is to show Corollary 2.11. For this we will need a series of lemmas which will show certain properties of the torsion pair associated with an indecomposable torsionisable exceptional object $E \in \mathcal{H}$. If E is such an object with $E^\perp = \mathrm{mod}\, H$ for some finite dimensional hereditary $k-$algebra H, then we put $T_E = {}_H H \oplus E$ and denote by $(\mathcal{T}_E, \mathcal{F}_E)$ the associated torsion pair. As above we denote by $0 \to \tau E \to M \to E \to 0$ the Auslander-Reiten sequence ending at E. Let $\Lambda = H[M] = \mathrm{End}_{\mathcal{H}} T_E$ be the corresponding quasitilted algebra. We denote by $\mathrm{Hom}_{\mathcal{H}}(T_E, -)$ the functor from $\mathcal{H}$ to $\mathrm{mod}\, \Lambda$. We will use that this functor is an equivalence from $\mathcal{T}_E$ onto a certain subcategory of $\mathrm{mod}\, \Lambda$. For details we refer to [HRS]. If the middle term M is decomposable, we know from 1.1 that $M = M_1 \oplus M_2$ where M_1, M_2 are both indecomposable. We denote the unique direct summand of M which is a direct summand of T_E by M_1.

Lemma 2.1. *Using the above notation we have that the induced map $M_1 \to E$ is mono and that $\mathrm{End}_{\mathcal{H}} M_2 = k$. Also we have that the induced map $M_2 \to E$ is epi. Moreover there exists an integer $r \leq \mathrm{rk}\, K_0(\mathcal{H}) - 3$ and a chain of irreducible monomorphisms*

$$M_1^{(r)} \rightarrowtail M_1^{(r-1)} \rightarrowtail \cdots \rightarrowtail M_1^{(1)} \rightarrowtail E$$

where $M_1^{(r)}$ has an indecomposable middle term in the Auslander-Reiten sequence ending at $M_1^{(r)}$. Moreover E is filtered by $\tau^{-j} M_1^{(r)}$ for $0 \leq j \leq r$.

Proof: Indeed, since M_1 is indecomposable exceptional we have that $\mathrm{End}_{\mathcal{H}} M_1 = k$ and so $\dim_k \mathrm{Hom}_{\mathcal{H}}(M_1, E) = 1$. Since $0 = \mathrm{Ext}^1_{\mathcal{H}}(M_1, T) \simeq D\mathrm{Hom}_{\mathcal{H}}(T, \tau M_1)$, we infer that $\tau M_1 \in \mathcal{F}_E$. So it follows by using 1.2 that the induced map $\tau M_1 \to \tau E$ is mono and so is $M_1 \to E$. Since there exists an irreducible map $f : M_2 \to E$ we have that f is either epi or mono. Since $M_2 \in E^{\perp}$ the map f is epi, since otherwise M_2 would be projective in $E^{\perp}$ in contradiction to 1.1 (iv). Since $H[M]$ is quasitilted we conclude from [HRS] and [HS] that $\mathrm{End}_{\mathcal{H}} M_2 = k$.

Iterating this argument gives for each indecomposable torsionisable exceptional $E \in \mathcal{H}$, with $E^{\perp} = \mathrm{mod}\, H$ for some finite dimensional hereditary $k-$algebra H, a chain of irreducible monomorphisms

$$M_1^{(r)} \rightarrowtail M_1^{(r-1)} \rightarrowtail \cdots \rightarrowtail M_1^{(1)} \rightarrowtail E$$

where $M_1^{(i+1)} \in M_1^{(i)^{\perp}}$ is projective for $1 \leq i \leq r-1$. We prove by induction on i that $M_1^{(i)} \in E^{\perp}$. For $i = 1$ this holds by assumption. Since $M_1^{(i+1)} \rightarrowtail M_1^{(i)}$ is mono and $M_1^{(i)} \in E^{\perp}$ we have that $\mathrm{Hom}_{\mathcal{H}}(E, M_1^{(i+1)}) = 0$. By induction, applied to the pair $M_1^{(i+1)}, M_1^{(1)}$, we have that $M_1^{(i+1)} \in M_1^{(1)^{\perp}}$. Also $\tau E \in M_1^{(1)^{\perp}}$. Moreover τE and $M_1^{(i+1)}$ belong to different connected components of $M_1^{(1)^{\perp}}$, hence $0 = \mathrm{Hom}_{\mathcal{H}}(M_1^{(i+1)}, \tau E) \simeq D\mathrm{Ext}^1_{\mathcal{H}}(E, M_1^{(i+1)})$ shows that $M_1^{(i+1)} \in E^{\perp}$. So the $M_1^{(i)}$ are pairwise nonisomorphic indecomposable $H-$projectives for $1 \leq i \leq r$. Since H is assumed to be wild it follows by using 1.1 (i) that $r \leq \mathrm{rk}\, K_0(\mathcal{H}) - 3$. Choose r maximal then $M_1^{(r)}$ has an indecomposable middle term in the Auslander-Reiten sequence ending at $M_1^{(r)}$.

We will show the last assertion by induction on r. For $r = 0$ there is nothing to show. By induction we have that $M_1^{(1)}$ is filtered by $\tau^{-j} M_1^{(r)}$ for $0 \leq j \leq r-1$. We consider the Auslander-Reiten sequence starting at $M_1^{(1)}$

$$0 \to M_1^{(1)} \xrightarrow{(\pi, \mu)} \tau^{-} M_1^{(2)} \oplus E \xrightarrow{\begin{pmatrix} \mu' \\ \pi' \end{pmatrix}} \tau^{-} M_1^{(1)} \to 0$$

Again by induction we infer that $\mathrm{cok}\, \mu' = \tau^{-r} M_1^{(r)}$. But then also $\mathrm{cok}\, \mu = \tau^{-r} M_1^{(r)}$, hence the assertion.

In analogy to the theory of wild hereditary algebras we will call objects X, whose middle term in the Auslander-Reiten sequence is indecomposable, quasisimple objects.

Dually, given $E \in \mathcal{H}$ indecomposable torsionisable exceptional, with ${}^{\perp}E = \mathrm{mod}\, H$ for some finite dimensional hereditary $k-$algebra H, there exists an integer $s \leq \mathrm{rk}\, K_0(\mathcal{H}) - 3$ and a chain of irreducible epimorphisms

$$E \twoheadrightarrow N_1^{(1)} \twoheadrightarrow N_1^{(2)} \twoheadrightarrow \cdots \twoheadrightarrow N_1^{(s)}$$

with $N_1^{(s)}$ quasisimple. Moreover E is filtered by $\tau^{j} N_1^{(s)}$ for $0 \leq j \leq s$.

For later reference we state the following result which was basically shown in [HRe3].

Lemma 2.2. *Let E be an indecomposable torsionisable exceptional object, then E is not $\tau-$periodic, so $\tau^i E \neq E$ for all $i \neq 0$.*

Proof: If E is $\tau-$periodic, then each object in the connected component of the Auslander-Reiten quiver of $\mathcal{H}$ containing E is $\tau-$periodic (see for example [HPR]). By the previous lemma we thus have a quasisimple torsionisable exceptional F which is $\tau-$periodic. Let M be the indecomposable middle term in the Auslander-Reiten sequence ending at F. Then it was shown in the proof of 1.1 in [HRe3] that M is $\tau_{F^\perp}-$periodic. Thus $F^\perp$ is tame which contradicts the assumption (c).

We will first consider this special type of objects and will then extend these results to arbitrary objects.

Lemma 2.3. *Let E be an indecomposable torsionisable exceptional quasisimple object. Then $Ext^1_{\mathcal{H}}(E,\tau^i E) = 0$ for all $i \neq 1, i \geq 0$ and $Hom_{\mathcal{H}}(\tau^i E, E) = 0$ for all $i \geq 1$. In particular $\tau^i E \in \mathcal{T}_E$ for all $i \neq 1$ and $i \geq 0$.*

Proof: Since E is exceptional we have that $\mathrm{Ext}^1_{\mathcal{H}}(E,E) = 0$. Since E is quasisimple the Auslander-Reiten sequence $0 \to \tau E \to M \to E \to 0$ satisfies that M is indecomposable. Since $\mathrm{End}_{\mathcal{H}} M = k$ and E is exceptional it follows that $\mathrm{Hom}_{\mathcal{H}}(\tau E, E) = 0$. But then $\mathrm{Ext}^1_{\mathcal{H}}(E, \tau^2 E) \simeq D\mathrm{Hom}_{\mathcal{H}}(\tau E, E) = 0$. We will show the assertion by induction on i. So assume that $\mathrm{Hom}_{\mathcal{H}}(\tau^j E, E) = 0$. for all $1 \leq j \leq i$ and $i \geq 1$. Then $\mathrm{Ext}^1_{\mathcal{H}}(E, \tau^{i+1}E) \simeq D\mathrm{Hom}_{\mathcal{H}}(\tau^{i+1}E, \tau E) = D\mathrm{Hom}_{\mathcal{H}}(\tau^i E, E) = 0$. If $\mathrm{Hom}_{\mathcal{H}}(\tau^{i+1}E, E) \neq 0$ then any nonzero map $f : \tau^{i+1}E \to E$ is either mono or epi, since $\mathrm{Ext}^1_{\mathcal{H}}(E, \tau^{i+1}E) = 0$ by [HRi]. First we assume that we have a monomorphism $\tau^{i+1}E \to E$. By 2.2 this map is proper. Then $\mathrm{Hom}_{\mathcal{H}}(E, \tau^{i+1}E) = 0$ and so $\tau^{i+1}E \in E^\perp = \mathrm{mod}\, H$ for some finite dimensional hereditary $k-$algebra H. Moreover it follows from 2.3 in [HRe2] that $\tau^{i+1}E$ is a projective $H-$module. Let $\Lambda = H[M]$ be the one-point extension algebra of H by M. Since Λ is quasitilted and not tilted we infer that M is a regular $H-$module. Let Γ be the connected component of the Auslander-Reiten quiver of Λ containing M. We have a unique (up to isomorphism) non-split exact sequence $0 \to \tau^2 E \to N \to M \to 0$ in $\mathcal{H}$, since $\dim_k \mathrm{Ext}^1_{\mathcal{H}}(M, \tau^2 E) = \dim_k D\mathrm{Hom}_{\mathcal{H}}(\tau^2 E, \tau M) = 1$. Since $\tau^2 E, M \in \mathcal{T}_E$, we have that $N \in \mathcal{T}_E$. Let

$$0 \to \tau M \xrightarrow{(\pi,\mu)} \tau E \oplus X \xrightarrow{\begin{pmatrix}\mu' \\ \pi'\end{pmatrix}} M \to 0$$

be the Auslander-Reiten sequence in $\mathcal{H}$ ending at M. Then $\tau^2 E = \ker \pi = \ker \pi'$ shows that $N \simeq X$. Since $\tau E \in \mathcal{F}_E$ it follows that $0 \to \tau^2 E \to N \to M \to 0$ is a relative Auslander-Reiten sequence in $\mathcal{T}_E$.

Using the tilting functor $\mathrm{Hom}_{\mathcal{H}}(T_E, -)$ we see that $\mathrm{Hom}_{\mathcal{H}}(T_E, \tau^2 E) \in \Gamma$. By the inductive hypothesis we have that $\tau^j E \in \mathcal{T}_E$ for $2 \leq j \leq i+1$. Since $\mathcal{T}_E$ is closed under extensions we also have that $0 \to \tau^{j+1}E \to \tau^j M \to \tau^j E \to 0$ for $2 \leq j \leq i$ are relative Auslander-Reiten sequences in $\mathcal{T}_E$. Using again the tilting functor we see that $\mathrm{Hom}_{\mathcal{H}}(T_E, \tau^j E) \in \Gamma$ for $2 \leq j \leq i+1$. But $\tau^{i+1}E$ is a projective $H-$module. So M and $\mathrm{Hom}_{\mathcal{H}}(T_E, \tau^{i+1}E)$ both lie in Γ. This shows that Γ coincides with the preprojective component of $\mathrm{mod}\, H$, hence M is a preprojective $H-$module, a contradiction.

If $\tau^{i+1}E \to E$ is an epimorphism it follows that $E \in {}^\perp(\tau^{i+1}E)$ is injective, or equivalently that $\tau^{-i-1}E \in {}^\perp E$ is injective. So the dual argument applies by using the tilting object $T' = E \oplus D(H_H)$, where $D(H_H)$ is the injective cogenerator in ${}^\perp E$. The inductive hypothesis then shows by using 1.3 that $\tau^{-j}E \in \mathcal{F}(T')$ for $1 \leq j \leq i$. Also $M \in \mathcal{F}(T')$. As above we may show that $\mathrm{Ext}^1_{\mathcal{H}}(T', M)$ and $\mathrm{Ext}^1_{\mathcal{H}}(T', \tau^{-j}E)$ for $1 \leq j \leq i$ both lie in the same connected component of the Auslander-Reiten quiver of $\Lambda = \mathrm{End}_{\mathcal{H}} T'$. Note that this endomorphism algebra is isomorphic to the one-point coextension algebra $[\tau^- M]H$ of H by $\tau^- M$. Since Λ is not tilted we infer that $\tau^- M$ is a regular $H-$module. Now $\operatorname{Ext}^1_{\mathcal{H}}(T', \tau^{-i}E)$ is an injective $H-$module, so $\operatorname{Ext}^1_{\mathcal{H}}(T', M) = D\operatorname{Hom}_{\mathcal{H}}(\tau^- M, D(H_H)) = \tau^- M$ is preinjective, a contradiction.

Hence $\tau^i E \in \mathcal{T}_E$ for all $i \neq 1$ and $\operatorname{Hom}_{\mathcal{H}}(\tau^i E, E) = 0$ for all $i \geq 1, i \geq 0$ which finishes the proof of the lemma.

Lemma 2.4. *Let E be an indecomposable torsionisable exceptional quasisimple object. Then there exists $m \geq 0$ such that $Hom_{\mathcal{H}}(E, \tau^j E) \neq 0$ for all $j \geq m$.*

Proof: By the previous lemma we have that $\tau^j E \in \mathcal{T}_E$ for all $j \neq 1$. Assume to the contrary that for each $m > 1$ there is some $j \geq m$ with $\operatorname{Hom}_{\mathcal{H}}(E, \tau^j E) = 0$. Since $\tau^j E \in \mathcal{T}_E$ we have that $\tau^j E \in E^{\perp} = \operatorname{mod} H$ for some finite dimensional hereditary $k-$algebra H. If $\operatorname{Hom}_{\mathcal{H}}(E, \tau^{j-1}E) \neq 0$, then any nonzero map $f : E \to \tau^{j-1}E$ is either mono or epi, since $\operatorname{Ext}^1_{\mathcal{H}}(\tau^{j-1}E, E) \simeq D\operatorname{Hom}_{\mathcal{H}}(E, \tau^j E) = 0$ (see [HRi]). If f is epi, then $\tau^{j-1}E \in {}^{\perp}E$ is injective. Since $\tau : {}^{\perp}E \to E^{\perp}$ is an equivalence, we infer that $\tau^j E$ is injective in $E^{\perp}$. If f is mono, then $E \in (\tau^{j-1}E)^{\perp}$ is projective, or equivalently $\tau^{-j+1}E \in E^{\perp}$ is projective. Since E is not $\tau-$periodic by 2.2 and there are only finitely many indecomposable projectives and injectives in $E^{\perp}$ up to isomorphism there exists $m > 1$ such that $\operatorname{Hom}_{\mathcal{H}}(E, \tau^j E) = 0$ for all $j \geq m$. But then $\tau^j E \in E^{\perp}$ for all $j \geq m$. As above let $\Lambda = H[M]$, so we have that $\tau^i_{\Lambda} X = \tau^i_H X$ for all $i \geq 1$ and $X = \operatorname{Hom}_{\mathcal{H}}(T_E, \tau^m E)$. But then $\operatorname{Hom}_H(M, \tau^i_H X) = 0$ for all $i \geq 0$ by 2.5.5 in [R1]. Since H is wild, X is not a regular $H-$module by 10.5 in [Ke]. Since $\tau_H^{-j} M$ is a sincere $H-$module for all j sufficiently large by 10.3 in [Ke] we infer that X is not a preinjective $H-$module. So X is a preprojective $H-$module, but this contradicts the fact that $\tau^i_H X$ is indecomposable for all $i > 0$.

The next two assertions extend the two previous results to not necessarily quasisimple exceptional objects.

Lemma 2.5. *Let E be an indecomposable torsionisable exceptional object. Then there exists $m \geq 0$ such that $Hom_{\mathcal{H}}(E, \tau^j E) \neq 0$ for all $j \geq m$.*

Proof: Let $\pi : E \twoheadrightarrow F$ be the composition of the chain of irreducible epimorphisms such that F is quasisimple. Assume that there are r such irreducible epimorphisms in this chain. By 2.4 there exists $m \geq 0$ such that $\operatorname{Hom}_{\mathcal{H}}(F, \tau^j F) \neq 0$ for all $j \geq m$. Now there is a monomorphism $\mu : \tau^r F \to E$. Let $j \geq m$ and consider for $0 \neq f \in \operatorname{Hom}_{\mathcal{H}}(F, \tau^{j+r}F)$ the composition $g = \pi f \tau^j(\mu) : E \to \tau^j E$. Clearly $g \neq 0$, so $\operatorname{Hom}_{\mathcal{H}}(E, \tau^j E) \neq 0$ for all $j \geq m$.

Lemma 2.6. *Let E be an indecomposable torsionisable exceptional object. Then there exists $m \geq 0$ such that $\tau^j E \in \mathcal{T}_E$ for all $j \geq m$.*

Proof: Let $\pi : E \twoheadrightarrow F$ be as above the composition of the chain of irreducible epimorphisms such that F is quasisimple. Assume that there are r such irreducible epimorphisms in this chain. Let $\tilde{F} = \oplus_{i=0}^{r} \tau^i F$. By 2.3 we have that $\operatorname{Ext}^1_{\mathcal{H}}(\tilde{F}, \tau^j F) = 0$ for all $j > r+1$. But then it follows that also $\operatorname{Ext}^1_{\mathcal{H}}(E, \tau^j F) = 0$ for all $j > r+1$, since E has a filtration by $\tau^r F, \ldots, F$ (compare 2.1). For $j \geq 0$ we have that $\tau^j E$ is filtered by $\tau^{j+r}F, \ldots, \tau^j F$, thus $\operatorname{Ext}^1_{\mathcal{H}}(E, \tau^j E) = 0$ for all $j > r+1$.

We will also need the dual of the last lemma, which we state without proof.

Lemma 2.7. *Let E be an indecomposable torsionisable exceptional object. Then there exists $m \geq 0$ such that $Hom_{\mathcal{H}}(E, \tau^{-j}E) = 0$ for all $j \geq m$. So for $j \geq m$ we have that $\tau^{-j}E$ is torsionfree for the torsion pair induced by $T' = E \oplus I$, where I is an injective cogenerator for ${}^{\perp}E$.*

The last two assertions of the next lemma will be quite essential in the proof of the main theorem.

Lemma 2.8. *Let E be an indecomposable torsionisable exceptional object. Then the following hold.*

(i) *There exists $m \geq 0$ such that $\tau^j E \in Fac\, E$ for all $j \geq m$.*
(ii) *For $X \in \mathcal{T}_E$ there exists m' such that $\tau^j X \in \mathcal{T}_E$ for all $j \geq m'$.*
(iii) *For $X \in \mathcal{H}$ with $Hom_{\mathcal{H}}(E, X) = 0$ there exists an integer $m \geq 0$ such that $Hom_{\mathcal{H}}(E, \tau^{-j} X) = 0$ for all $j \geq m$.*

Proof: Let $E^{\perp} = \operatorname{mod} H$ for some finite dimensional hereditary $k-$algebra H. Let $S_1, \ldots, S_{n-1}$ be the simple $H-$modules. Since $\operatorname{mod} H \in \mathcal{T}_E$ by 1.2 we have that all S_i are torsionisable and clearly exceptional, since H is finite dimensional. By lemma 2.5 there exists m_i such that $\operatorname{Hom}_{\mathcal{H}}(S_i, \tau^j S_i) \neq 0$ for all $j \geq m_i$ and $1 \leq i \leq n-1$. Also we choose by 2.5 and 2.6 an integer r such that $\operatorname{Hom}_{\mathcal{H}}(E, \tau^j E) \neq 0$ and $\tau^j E \in \mathcal{T}_E$ for all $j \geq r$. Let $m = \max(r, m_1, \ldots, m_{n-1})$. Let $j \geq m$ and consider $f : E^t \to \tau^j E$ the minimal right $\operatorname{add} E-$approximation of $\tau^j E$. By the choice of m we infer that $f \neq 0$. Assume that f is not epi and let $Q = \operatorname{cok} f$. Since $\tau^j E \in \mathcal{T}_E$ we have that $Q \in \mathcal{T}_E$ and by construction we have that $\operatorname{Hom}_{\mathcal{H}}(E, Q) = 0$, hence $Q \in E^{\perp}$. Thus Q has a simple factor S_i. So we have an epimorphism $\pi : \tau^j E \to S_i$. Since $j \geq m \geq m_i$ we have that $\operatorname{Hom}_{\mathcal{H}}(S_i, \tau^j S_i) \neq 0$, so $\operatorname{Hom}_{\mathcal{H}}(\tau^j E, \tau^j S_i) \neq 0$, since π is epi. But $\operatorname{Hom}_{\mathcal{H}}(\tau^j E, \tau^j S_i) = \operatorname{Hom}_{\mathcal{H}}(E, S_i) = 0$, since $S_i \in E^{\perp}$, a contradiction, and so f is epi, and therefore $\tau^j E \in \operatorname{Fac} E$ for all $j \geq m$.

The second assertion now follows easily from the first part. Let $X \in \mathcal{T}_E$. So there exists $\tilde{T} \in \operatorname{add} T_E$ and an epimorphism $\pi : \tilde{T} \to X$. By (i) there is $m \geq 0$ such that $\tau^j \tilde{T} \in \operatorname{Fac} T_E$ for all $j \geq m$, just take m as the maximum of the corresponding numbers for the indecomposable direct summands of T_E. Since $\tau^j \pi : \tau^j \tilde{T} \to \tau^j X$ is epi for all $j \geq 0$, we have that $\tau^j X \in \mathcal{T}_E$ for all $j \geq m$.

Finally, let $X \in \mathcal{H}$ with $\operatorname{Hom}_{\mathcal{H}}(E, X) = 0$. By (i) we may choose m such that $\tau^j E \in \operatorname{Fac} E$ for all $j \geq m$. Thus for all $j \geq m$ we have an epimorphism $\pi : \tau^{-j} \tilde{E} \to E$ for some $\tilde{E} \in \operatorname{add} E$. But then $0 = \operatorname{Hom}_{\mathcal{H}}(E, X) = \operatorname{Hom}_{\mathcal{H}}(\tau^{-j} E, \tau^{-j} X)$ shows that $\operatorname{Hom}_{\mathcal{H}}(E, \tau^{-j} X) = 0$.

Lemma 2.9. *Let E be an indecomposable torsionisable exceptional object and let P be an indecoposable projective in $E^{\perp}$ with $Hom_{\mathcal{H}}(P, E) \neq 0$. Then the minimal $add\, P-$approximation $f : P^t \to E$ of E is either mono or epi.*

Proof: Assume that f is not epi, then $\operatorname{im} f$ is a proper subobject of E which lies in $E^{\perp}$ and therefore is projective. If $\ker f \neq 0$, then $0 \to \ker f \to P^t \to \operatorname{im} f \to 0$ is an exact sequence in $E^{\perp}$, hence splits, but this contradicts the minimality of f.

The following lemma and its immediate corollary are crucial for the proof of the main theorem in the next section.

Lemma 2.10. *For each integer $n \geq 2$ there exists E indecomposable torsionisable exceptional such that $Hom_{\mathcal{H}}(E, \tau^n E) = 0$.*

Proof: Let $n \geq 2$ and assume that for all torsionisable exceptional objects E we have that $\operatorname{Hom}_{\mathcal{H}}(E, \tau^n E) \neq 0$. Consider $f : E \to (\tau^n E)^r$ and $g : E^s \to \tau^n E$ be the minimal left (resp.right) approximations. We claim that f is mono and g is epi.

First we show that f is mono. Assume that $\ker f \neq 0$. Consider the induced exact sequence $0 \to \ker f \to E \to \operatorname{im} f \to 0$. By construction we infer that $\operatorname{Hom}_{\mathcal{H}}(\ker f, \tau^n E) = 0$. Thus $\operatorname{Ext}^1_{\mathcal{H}}(\tau^{n-1} E, \ker f) = 0$, shows that $\ker f$ is torsionisable by 1.4. Since $\operatorname{Hom}_{\mathcal{H}}(\ker f, \tau^n E) = 0$ and $\operatorname{im} f \to (\tau^n E)^r$ is mono we have that $\operatorname{Hom}_{\mathcal{H}}(\ker f, \operatorname{im} f) = 0$. Now $\operatorname{Ext}^1_{\mathcal{H}}(\ker f, E) = 0$ shows that $\ker f$ is exceptional by applying $\operatorname{Hom}_{\mathcal{H}}(\ker f, -)$ to the exact sequence from above. Now also

$0 \to \tau^n \ker f \to \tau^n E \to \tau^n \operatorname{im} f \to 0$ is exact, so $\operatorname{Hom}_{\mathcal{H}}(\ker f, \tau^n \ker f) = 0$, since $\operatorname{Hom}_{\mathcal{H}}(\ker f, \tau^n E) = 0$. But this is a contradiction, so $\ker f = 0$, and thus f is mono.

Next we show that g is epi. Assume that $\operatorname{cok} g \neq 0$. Consider the induced exact sequence $0 \to \operatorname{im} g \to \tau^n E \to \operatorname{cok} g \to 0$. By construction we infer that $\operatorname{Hom}_{\mathcal{H}}(E, \operatorname{cok} g) = 0$. Now $\operatorname{Ext}^1_{\mathcal{H}}(\tau^n E, \operatorname{cok} g) = 0$, shows that $\operatorname{cok} g$ is torsionisable by 1.4. Since $\operatorname{Hom}_{\mathcal{H}}(E, \operatorname{cok} g) = 0$ and $E^s \to \operatorname{im} g$ is epi we have that $\operatorname{Hom}_{\mathcal{H}}(\operatorname{im} g, \operatorname{cok} g) = 0$. Now $\operatorname{Ext}^1_{\mathcal{H}}(\tau^n E, \operatorname{cok} g) = 0$ shows that $\operatorname{cok} g$ is exceptional by applying the functor $\operatorname{Hom}_{\mathcal{H}}(\operatorname{cok} g, -)$ to the exact sequence from above. Next we apply $\operatorname{Hom}_{\mathcal{H}}(-, \tau^n \operatorname{cok} g)$ to the exact sequence from above. This shows that $\operatorname{Hom}_{\mathcal{H}}(\operatorname{cok} g, \tau^n \operatorname{cok} g) = 0$, since $0 = \operatorname{Hom}_{\mathcal{H}}(E, \operatorname{cok} g) = \operatorname{Hom}_{\mathcal{H}}(\tau^n E, \tau^n \operatorname{cok} g)$. But this is a contradiction, so $\operatorname{cok} g = 0$, and thus g is epi.

Now we choose E an indecomposable torsionisable exceptional object such that the vector space $\operatorname{Hom}_{\mathcal{H}}(E, \tau^n E)$ is of minimal $k-$dimension. Let $E^{\perp} = \operatorname{mod} H$ for some finite dimensional hereditary $k-$algebra H and let $0 \to \tau E \to M \to E \to 0$ be the Auslander-Reiten sequence. By [H1] we may assume that M is a sincere $H-$module. Let $P(\alpha)$ be a simple $H-$projective and let $I(\omega)$ be a simple injective in ${}^{\perp}E$. Consider $f : P(\alpha)^t \to E$ and $g : E \to I(\omega)^r$ the minimal right (resp. left) approximations. Since M is sincere both maps are nonzero. By 2.9 and its dual we know that f is either mono or epi and that the same holds for g. Since $P(\alpha)$ is simple projective in $E^{\perp}$ we have that $\tau^{-}P(\alpha)$ is simple projective in ${}^{\perp}E$. We may choose α and ω auch that $0 = \operatorname{Hom}_{{}^{\perp}E}(\tau^{-}P(\alpha), I(\omega)) \simeq D\operatorname{Ext}^1_{\mathcal{H}}(I(\omega), P(\alpha))$. If f is epi, then $\operatorname{Ext}^1_{\mathcal{H}}(I(\omega), P(\alpha)) = 0$ implies that g is not mono, hence g is epi. So we have that either f is mono or g is epi.

We will first assume that f is mono. So we have an exact sequence

$$0 \to P(\alpha)^t \to E \to Q \to 0.$$

Clearly $Q \in \mathcal{T}_E$ and $\operatorname{Hom}_{\mathcal{H}}(P(\alpha), Q) = 0$. This implies that Q is exceptional. Since $\operatorname{End} E = k$ and $\operatorname{Ext}^1_{\mathcal{H}}(E, P(\alpha)) = 0$ we infer that $\operatorname{Hom}_{\mathcal{H}}(E, Q) = k$, and so Q is indecomposable. We also consider the exact sequence

$$0 \to \tau^n P(\alpha)^t \to \tau^n E \to \tau^n Q \to 0.$$

Since $P(\alpha) \in \mathcal{T}_E$ and the fact that the minimal right $\operatorname{add} P(\alpha)-$approximation $P(\alpha)^s \to \tau^n P(\alpha)$ is epi by the first part of the proof we infer that $\tau^n P(\alpha) \in \mathcal{T}_E$. Applying $\operatorname{Hom}_{\mathcal{H}}(E, -)$ to the second sequence then yields that

$$0 \to \operatorname{Hom}_{\mathcal{H}}(E, \tau^n P(\alpha)^t) \to \operatorname{Hom}_{\mathcal{H}}(E, \tau^n E) \xrightarrow{\pi} \operatorname{Hom}_{\mathcal{H}}(E, \tau^n Q) \to 0$$

is exact. Applying $\operatorname{Hom}_{\mathcal{H}}(-, \tau^n Q)$ to the first sequence yields that

$$0 \to \operatorname{Hom}_{\mathcal{H}}(Q, \tau^n Q) \to \operatorname{Hom}_{\mathcal{H}}(E, \tau^n Q)$$

is exact. The minimality assumption on E now implies that π is injective, or equivalently that $\operatorname{Hom}_{\mathcal{H}}(E, \tau^n P(\alpha)) = 0$, and so $\tau^n P(\alpha) \in E^{\perp}$. Since $P(\alpha)$ is simple projective in $E^{\perp}$ there is no proper epi from $P(\alpha)^s$ onto $\tau^n P(\alpha)$, so $s = 1$ and $P(\alpha) = \tau^n P(\alpha)$. But this contradicts 2.2.

Next we consider the case that $g : E \to I(\omega)^r$ is epi. So we obtain an exact sequence

$$0 \to K \to E \to I(\omega)^r \to 0.$$

By construction we have that $0 = \operatorname{Hom}_{\mathcal{H}}(K, I(\omega)) = \operatorname{Ext}^1_{\mathcal{H}}(\tau^{-}I(\omega), K)$, hence K is torsionisable. As above we may show that K is also indecomposable and exceptional.

We als have the exact sequence

$$0 \to \tau^{-n}K \to \tau^{-n}E \to \tau^{-n}I(\omega)^r \to 0.$$

Now $\mathrm{Ext}^1_{\mathcal{H}}(I(\omega), E) = 0$. By the first part of the proof we have that $\tau^{-n}I(\omega)$ is cogenerated by $I(\omega)$. Therefore we obtain by applying $\mathrm{Hom}_{\mathcal{H}}(-, E)$ to this exact sequence an exact sequence

$$\mathrm{Hom}_{\mathcal{H}}(\tau^{-n}I(\omega)^r, E) \to \mathrm{Hom}_{\mathcal{H}}(\tau^{-n}E, E) \xrightarrow{\pi} \mathrm{Hom}_{\mathcal{H}}(\tau^{-n}K, E) \to 0.$$

Also we have from the first exact sequence that

$$0 \to \mathrm{Hom}_{\mathcal{H}}(\tau^{-n}K, K) \to \mathrm{Hom}_{\mathcal{H}}(\tau^{-n}K, E)$$

is exact. Thus by the minimality assumption on E we have that π is injective, or equivalently we have that $\tau^{-n}I(\omega)^r \in {}^{\perp}E$. But then as above we conclude that $I(\omega) = \tau^{-n}I(\omega)$, a contradiction to 2.2.

The proof of the next corollary is clearly inspired by some wing arguments in the theory of wild hereditary algebras (compare for example [Ke]).

Corollary 2.11. *There exists an indecomposable torsionisable exceptional object E and some integer $n \geq 2$ such that $\mathrm{Hom}_{\mathcal{H}}(E, \tau^{n+1}E) = 0$ and $\mathrm{Hom}_{\mathcal{H}}(E, \tau^n E) \neq 0$.*

Proof: Let $n > \mathrm{rk}\, K_0(\mathcal{H})$. By 2.10 choose E an indecomposable torsionisable exceptional object with $\mathrm{Hom}_{\mathcal{H}}(E, \tau^{n+1}E) = 0$. Assume that $\mathrm{Hom}_{\mathcal{H}}(E, \tau^j E) = 0$ for all j with $2 \leq j \leq n$.

First we show that we may assume without loss of generality that E is quasisimple. Indeed, let

$$E = E_r \to E_{r-1} \to \cdots \to E_1 \to E_0$$

be the chain of irreducible epimorphisms such that E_0 is quasisimple which exists by the dual of 2.1. For each $1 \leq i \leq r$ we have that $E_0 \in {}^{\perp}E_i$. Since $\tau^i E_0$ is a subobject of E_i we have that $\mathrm{Hom}_{\mathcal{H}}(E_0, \tau^i E_0) = 0$ for $1 \leq i \leq r$. If $r < j \leq n+1$, then $\tau^j E_0$ is a subobject of $\tau^{j-r}E_r$ and the epimorphism $E_r \to E_0$ then shows that $\mathrm{Hom}_{\mathcal{H}}(E_0, \tau^j E_0) = 0$ for all j with $2 \leq j \leq n$.

So we have an indecomposable torsionisable exceptional object E which is quasisimple and satisfies $\mathrm{Hom}_{\mathcal{H}}(E, \tau^j E) = 0$ for all j with $2 \leq j \leq n+1$.

We now show by induction on r that for all $r \leq n$ we have a chain of irreducible epimorphisms

$$E_r \twoheadrightarrow E_{r-1} \twoheadrightarrow \cdots \twoheadrightarrow E_1 \twoheadrightarrow E_0 = E$$

with all E_i exceptional for $1 \leq i \leq r$.

Let E_1 be the indecomposable summand of the middle term of the Auslander-Reiten sequence ending at E_0. We know that E_1 is not a direct summand of T_{E_0}. First we claim that $\mathrm{Hom}_{\mathcal{H}}(E_0, \tau^2 E_0) = 0$ implies that E_1 is exceptional. We consider the exact sequence $0 \to \tau E_0 \to E_1 \to E_0 \to 0$. Indeed, $0 = \mathrm{Hom}_{\mathcal{H}}(E_0, \tau^2 E_0) \simeq D\mathrm{Ext}^1_{\mathcal{H}}(\tau E_0, E_0)$, shows that $\mathrm{Ext}^1_{\mathcal{H}}(\tau E_0, E_1) = 0$ by applying $\mathrm{Hom}_{\mathcal{H}}(\tau E_0, -)$ to the Auslander-Reiten sequence and using that τE_0 is exceptional. Since $E_1 \in E_0^{\perp}$ this shows that E_1 is exceptional by applying the functor $\mathrm{Hom}_{\mathcal{H}}(-, E_1)$ to the above sequence.

Assume inductively that we have constructed a chain of irreducible epimorphisms

$$E_{r-1} \twoheadrightarrow E_{r-2} \twoheadrightarrow \cdots \twoheadrightarrow E_1 \twoheadrightarrow E_0 = E$$

with all E_i exceptional for $1 \leq i \leq r-1$ and exact sequences

$$0 \to \tau E_{j-1} \to E_j \to E_0 \to 0$$

$$0 \to \tau^j E_0 \to E_j \to E_{j-1} \to 0$$

for $1 \le j \le r-1$ and that $\mathrm{Ext}^1_{\mathcal{H}}(\tau^j E_0, E_j) = 0$ for $1 \le j \le r-1$.

Let E_r be the unique indecomposable summand of the middle term of the Auslander sequence ending at E_{r-1} which is not a direct summand of $T_{E_{r-1}}$. By 2.1 we have an epimorphism $E_r \to E_{r-1}$. We claim that $\mathrm{Hom}_{\mathcal{H}}(E_0, \tau^{r+1}E_0) = 0$ implies that E_r is exceptional. Let

$$0 \to \tau E_{r-1} \xrightarrow{(\mu, \pi)} E_r \oplus \tau E_{r-2} \xrightarrow{\begin{pmatrix}\pi' \\ \mu'\end{pmatrix}} E_{r-1} \to 0$$

be the Auslander-Reiten sequence in $\mathcal{H}$ ending at E_{r-1}. Then by induction we have that $E_0 = \mathrm{cok}\,\mu'$ which is isomorphic to $\mathrm{cok}\,\mu$. Thus we have an exact sequence

$$(*) \quad 0 \to \tau E_{r-1} \to E_r \to E_0 \to 0.$$

Since $0 = \mathrm{Hom}_{\mathcal{H}}(E_0, \tau^{r+1}E_0) = \mathrm{Ext}^1_{\mathcal{H}}(\tau^r E_0, E_0)$ and $\mathrm{Ext}^1_{\mathcal{H}}(\tau^{r-1}E_0, E_{r-1}) = 0$ by the induction hypothesis we infer that $\mathrm{Ext}^1_{\mathcal{H}}(\tau^r E_0, E_r) = 0$ by applying the functor $\mathrm{Hom}_{\mathcal{H}}(\tau^r E_0, -)$ to the sequence $(*)$.

By induction we have that $\tau^{r-1}E_0 = \ker \tau^- \pi$. So $\tau^r E_0 = \ker \pi = \ker \pi'$. Thus we also have an exact sequence

$$(**) \quad 0 \to \tau^r E_0 \to E_r \to E_{r-1} \to 0.$$

Applying $\mathrm{Hom}_{\mathcal{H}}(-, E_r)$ to $(**)$ and using that $E_r \in E_{r-1}^{\perp}$ and $\mathrm{Ext}^1_{\mathcal{H}}(\tau^r E_0, E_r) = 0$ then shows that E_r is exceptional.

So we have constructed a chain of irreducible epimorphisms between exceptional objects of length equal to $\mathrm{rk}\, K_0(\mathcal{H})$, which contradicts the dual of 2.1.

Thus we have an indecomposable torsionisable exceptional object E and some integer $n \ge 2$ such that $\mathrm{Hom}_{\mathcal{H}}(E, \tau^{n+1}E) = 0$ and $\mathrm{Hom}_{\mathcal{H}}(E, \tau^n E) \ne 0$.

3. The main result

We keep the basic notation from the previous sections. We are now able to show the characterization theorem of hereditary abelian $k-$categories containing a tilting object.

Theorem 3.1. *Let $\mathcal{H}$ be a connected hereditary abelian $k-$category with tilting object. Then $\mathcal{H}$ is derived equivalent to mod$H$ for some finite dimensional hereditary $k-$algebra H or derived equivalent to coh$\mathbb{X}$ for some weighted projective line $\mathbb{X}$.*

Proof: As pointed out in the introduction we may assume without loss of generality that $\mathcal{H}$ satisfies the extra assumptions stated at the beginning of section 2 by using the results from [HRe1], [HRe2] and [HS]. By 2.11 we have an indecomposable exceptional object E and some integer $n \ge 2$ with $\mathrm{Hom}_{\mathcal{H}}(E, \tau^{n+1}E) = 0$ and $\mathrm{Hom}_{\mathcal{H}}(E, \tau^n E) \ne 0$. As before we see that any nonzero map $f : E \to \tau^n E$ is either mono or epi. We will first deal with the case that f is epi. So we have an exact sequence

$$0 \to K \to E \to \tau^n E \to 0.$$

Now $\tau^n E \in {}^{\perp}E$ is injective, so $\tau^{n+1}E = I(\alpha) \in E^{\perp}$ is injective. We also have an exact sequence

$$(*) \quad 0 \to \tau K \to \tau E \to \tau^{n+1}E = I(\alpha) \to 0.$$

We claim that there exists an indecomposable torsionisable exceptional object F such that the exact sequence $(*)$ is contained in $\mathcal{T}_F$.

If there exists an indecomposable projective $E^{\perp}-$module $P(\beta)$ with the property that $\mathrm{Hom}_{\mathcal{H}}(P(\beta), I(\alpha)) = 0$, then we apply $\mathrm{Hom}_{\mathcal{H}}(P(\beta), -)$ to the sequence $(*)$ and obtain the following exact sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{H}}(P(\beta), \tau K) \to \mathrm{Ext}^1_{\mathcal{H}}(P(\beta), \tau E) \to \mathrm{Ext}^1_{\mathcal{H}}(P(\beta), I(\alpha)) \to 0.$$

Now $0 = \mathrm{Hom}_{\mathcal{H}}(E, P(\beta)) \simeq D\mathrm{Ext}^1_{\mathcal{H}}(P(\beta), \tau E)$ shows that for $F = P(\beta)$ we have that $(*)$ is contained in $\mathcal{T}_F$.

If for all indecomposable projective $E^{\perp}-$modules $P(\beta)$ we have the fact that $\mathrm{Hom}_{\mathcal{H}}(P(\beta), I(\alpha)) \neq 0$, then we infer that $P(\alpha)$ is simple projective, where $P(\alpha)$ is the projective cover of the socle of $I(\alpha)$. But then there exists a simple injective $E^{\perp}-$module $I(\omega)$ with $\mathrm{Hom}_{\mathcal{H}}(I(\omega), I(\alpha)) = 0$. Apply $\mathrm{Hom}_{\mathcal{H}}(I(\omega), -)$ to the sequence $(*)$. So we obtain the following exact sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{H}}(I(\omega), \tau K) \to \mathrm{Ext}^1_{\mathcal{H}}(I(\omega), \tau E) \to \mathrm{Ext}^1_{\mathcal{H}}(I(\omega), I(\alpha)) \to 0.$$

Now $0 = \mathrm{Hom}_{\mathcal{H}}(E, I(\omega)) \simeq D\mathrm{Ext}^1_{\mathcal{H}}(I(\omega), \tau E)$ shows that for $F = I(\omega)$ we have that $(*)$ is contained in $\mathcal{T}_F$.

By 2.8 (ii) there exists an integer $m \geq 1$ such that

$$0 \to \tau^i K \to \tau^i E \to \tau^{n+i} E \to 0.$$

is contained in $\mathcal{T}_F$ for all $i \geq m$. So we obtain an infinite chain of proper epimorphisms

$$\tau^m E \xrightarrow{\ \pi\ } \tau^{m+n} E \xrightarrow{\ \tau^n \pi\ } \tau^{m+2n} E \xrightarrow{\ \tau^{2n}\pi\ } \cdots$$

contained in $\mathcal{T}_F$. Moreover we infer that $\ker \tau^{jn}\pi \in \mathcal{T}_F$ for all $j \geq 0$. Let $F^{\perp} = \mathrm{mod}\, H'$ for some finite dimensional hereditary $k-$algebra H' and let $\Lambda = H'[M']$ be the corresponding quasitilted algebra where M' is the middle term of the Auslander-Reiten sequence ending at F. Then $\mathrm{Hom}_{\mathcal{H}}(T_F, \tau^{jn}\pi)$ is an infinite chain of proper epimorphisms of $\Lambda-$modules, a contradiction.

Next we consider the case that the map $f : E \to \tau^n E$ is mono. Now $P(\alpha) = \tau^{-n} E \in E^{\perp}$ is projective. We also have an exact sequence

$$(**) \quad 0 \to \tau P(\alpha) \to \tau E \to Q \to 0.$$

We claim that there exists an indecomposable torsionisable exceptional object F such that the sequence $(**)$ is torsionfree for the torsion pair associated with the tilting object $T' = F \oplus I$, where I is an injective cogenerator of ${}^{\perp}F$. We denote this torsionfree class by $\mathcal{F}$.

If there exists an indecomposable injective $E^{\perp}-$module $I(\beta)$ with the property that $\mathrm{Hom}_{\mathcal{H}}(P(\alpha), I(\beta)) = 0$, then we apply $\mathrm{Hom}_{\mathcal{H}}(I(\beta), -)$ to the sequence $(**)$ and obtain the following exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{H}}(I(\beta), \tau P(\alpha)) \to \mathrm{Hom}_{\mathcal{H}}(I(\beta), \tau E) \to \mathrm{Hom}_{\mathcal{H}}(I(\beta), Q) \to 0.$$

Now $\mathrm{Hom}_{\mathcal{H}}(I(\beta), \tau E) \simeq D\mathrm{Ext}^1_{\mathcal{H}}(E, I(\beta)) = 0$ shows that we may choose $F = I(\beta)$.

If for all indecomposable injective $E^{\perp}-$modules $I(\beta)$ we have the fact that $\mathrm{Hom}_{\mathcal{H}}(P(\alpha), I(\beta)) \neq 0$, then we infer that $I(\alpha)$ is simple injective, where $I(\alpha)$ is the injective envelope of the top of $P(\alpha)$. But then there exists a simple projective $E^{\perp}-$module $P(\omega)$ with $\mathrm{Hom}_{\mathcal{H}}(P(\alpha), P(\omega)) = 0$. Apply $\mathrm{Hom}_{\mathcal{H}}(P(\omega), -)$ to the sequence $(**)$. So we obtain the following exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{H}}(P(\omega), \tau P(\alpha)) \to \mathrm{Hom}_{\mathcal{H}}(P(\omega), \tau E) \to \mathrm{Hom}_{\mathcal{H}}(P(\omega), Q) \to 0.$$

Now $\mathrm{Hom}_{\mathcal{H}}(P(\omega), \tau E) \simeq D\mathrm{Ext}^1_{\mathcal{H}}(E, P(\omega))$ shows that we may choose $F = P(\omega)$.

By 2.8 (iii) and 1.3 there exists an integer $m \geq 1$ such that

$$0 \to \tau^{-i-n}E \to \tau^{-i}E \to \tau^{-i-1}Q \to 0.$$

is contained in $\mathcal{F}$ for all $i \geq m$. So we obtain an infinite chain of proper monomorphisms

$$\cdots \tau^{-m-3n}E \xrightarrow{\tau^{-2n}\mu} \tau^{-m-2n}E \xrightarrow{\tau^{-n}\mu} \tau^{-m-n}E \xrightarrow{\mu} \tau^{-m}E$$

contained in $\mathcal{F}$. Moreover we infer that $\mathrm{cok}\,\tau^{-jn}\mu \in \mathcal{F}$ for all $j \geq 0$. Let ${}^{\perp}F = \mathrm{mod}\,H''$ for some finite dimensional hereditary $k-$algebra H'' and let $\Lambda' = [M'']H''$ be the corresponding quasitilted algebra where M'' is the middle term of the Auslander-Reiten sequence starting at F. Then $\mathrm{Ext}^1_{\mathcal{H}}(T', \tau^{-jn}\mu)$ is an infinite chain of proper monomorphisms of $\Lambda'-$modules, a contradiction.

This finishes the proof of the theorem.

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