Spectral Estimates: From finite to general state spaces

-revised version-

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Abstract

The aim of this paper is to derive lower bounds on the spectral gap for generators of Sub-Markov chains and Sub-Markov jump processes on general state spaces. To this end sets of discrete paths are used to estimate constants appearing in several functional inequalities. We discuss our bounds by considering some examples among them are the circle and the torus. Finally, we examine the relation between the logarithmic-Sobolev constant and isoperimetry to some extend.

Key words: Path Reconstruction Property; Spectral Gap; Log-Sobolev Constant

1 Introduction and Framework

The purpose of this article is to develop methods to control the rate of convergence for reversible, homogenous discrete-time Sub-Markov chains and time-continuous Sub-Markov jump processess $(X_t)_t$, $t \in \mathbb{N}$ or $t \in \mathbb{R}^+$, on a Polish state space $(S, \mathcal{S}, \mu)$.
We consider transition kernels $K : S \times \mathcal{S} \to [0,1]$ of the form

$$(1) \qquad K(x,dy) = k(x,y)\mu(dy), \quad \text{such that} \int K(x,dy) \le 1.$$

We furthermore define the continuous time semigroup associated to K by $H_t = \exp(-t(I-K))$. Its kernel is denoted by $H_t(x,dy)$ which is the distribution at time $t > 0$ of the continuous process started at x. We assume that there exists a stationary measure π for the Sub-Markov chain with density $h > 0$ with respect to the measure μ, i.e. for $A \in \mathcal{S}, \pi(A) = \int 1_A(x)h(x)\,d\mu(x)$. Firstly, we consider Dirichlet forms. For bounded measurable functions f, g we take the closure of

$$(2) \qquad \mathcal{E}(f,g) = \int_S \int_S (f(x) - f(y)(g(x) - g(y))K(x,dy)d\pi(x),$$

on $L^2(S,\mu)$ with respect to the norm $\mathcal{E}(\cdot,\cdot) + \|\cdot\|_2$ and denote it also by $\mathcal{E}$ and its domain by $D(\mathcal{E})$.
The objective of this paper is to deduce quantitative bounds on the rate of convergence of H_t to π in total variation norm using the χ^2-norm. For $p, q \in (0,\infty]$ and an operator $H : L^p \to L^q$ let $\|H\|_{p\to q} := \sup_{\|f\|_p \le 1} \|Hf\|_q$ denote the operator norm. We take $t_1, t_2 \ge 0$ such that $t = t_1 + t_2$ and consider

$$(3) \qquad \|H_t - \pi\|_{2\to\infty} \le \|H_{t_1}\|_{2\to\infty}\|H_{t_2} - \pi\|_{2\to 2}.$$

It is well known that $\|\cdot\|_{2\to 2}$ can be successfully bounded in terms of the *spectral gap*

$$(4) \qquad \lambda_1 := \inf_{f\in D(\mathcal{E})} \left\{ \frac{\mathcal{E}(f,f)}{Var_\pi[f]}; Var_\pi[f] \neq 0 \right\},$$

where

$$Var_\pi[f] = \int_S \left(f(x) - \int_S f(x)\,d\pi(x) \right)^2 d\pi(x)$$

denotes the variance. Thus, if we take $t_1 = 0$ and $t_2 = t$ in (3) we can estimate the rate of convergence by estimating the spectral gap λ_1.
It has been proven useful to consider other values for t_1 and it has turned out that the quantity which becomes important in this case is

$$(5) \qquad \alpha := \inf_{f\in D(\mathcal{E})} \left\{ \frac{\mathcal{E}(f,f)}{\mathcal{L}_\pi[f]}; 0 \neq \mathcal{L}_\pi[f] < +\infty \right\},$$

where

$$(6) \qquad \mathcal{L}_\pi[f] = \int_S f(x)^2 \ln\left(\frac{f(x)^2}{\|f\|_2^2} \right) d\pi(x).$$

The constant α is called *logarithmic-Sobolev constant.*
In order to obtain quantitative bounds on the rate of convergence our goal is to state lower bounds on λ_1 and α.
To this end we use path-based techniques analogous to those for dealing with Markov chains on finite state spaces, e.g. see [DS 91], [S 92], [JS 88], [JS 89] or [SC 97].
In analogy to the finite case we introduce path sets adapted to the Sub-Markov chain under consideration.
Let

$$(7) \qquad \begin{aligned} N : S\times S &\to \mathbb{N}, \\ (x,y) &\mapsto N(x,y) \end{aligned}$$

and consider the map γ,

$$\begin{array}{rcl}\gamma : \{(x,y,n) : x,y \in S, n \leq N(x,y)\} & \rightarrow & S, \\ (x,y,n) & \mapsto & \gamma_{xy}(n).\end{array} \tag{8}$$

For fixed $x, y \in S$ a *discrete path* which connects x and y is the map

$$\begin{array}{rcl}\gamma_{xy} : \{0,1,...,N(x,y)\} & \rightarrow & S, \\ n & \mapsto & \gamma_{xy}(n),\end{array} \tag{9}$$

such that $\gamma_{xy}(0) = x$ and $\gamma_{xy}(N(x,y)) = y$.

Definition 1.1 *A path γ_{xy} connecting x and y is called admissible with respect to $(X_t)_t$ if the map γ_{xy} is injective and adapted, i.e.*

$$k(\gamma_{xy}(n-1), \gamma_{xy}(n)) > 0 \textit{ for } n = 1, ..., N(x,y). \tag{10}$$

A path set $\Gamma = \{\gamma_{xy}\}$ is called admissible if each path γ_{xy} is admissible.

Furthermore, we define the *adapted edge set.*

Definition 1.2 *The adapted edge set $\mathcal{A}^\Gamma$ corresponding to $(X_t)_t$ is*

$$\mathcal{A}^\Gamma := \tag{11}$$
$$\{e = (v,w) : v,w \in S,\ \textit{it exists s.t. } \gamma_{xy} \in \Gamma,\ v = \gamma_{xy}(n-1),\ w = \gamma_{xy}(n)\}.$$

Given an edge $e = (v,w) \in \mathcal{A}^\Gamma$, such that $v = \gamma_{xy}(n-1)$ and $w = \gamma_{xy}(n)$ for some $\gamma_{xy} \in \Gamma$ we say e is of index n.
The boundary ∂A of a set $A \subset S$ is the set

$$\partial A := \{e = (v,w) \in S \times S : v \in A, \quad w \in A^c \textit{ or vice versa}\}. \tag{12}$$

To simplify notation, for and edge $e = (v,w)$ let us introduce

$$Q(v,w) = k(v,w)h(v). \tag{13}$$

Definition 1.3 *Let $\Gamma = \{\gamma_{xy}\}$ be an admissible path set and $p \in \mathbb{R}^+$. For $\gamma_{xy} \in \Gamma$ we set*

$$|||\gamma_{xy}|||_p := \sum_{n=1}^{N(x,y)} Q(\gamma_{xy}(n-1), \gamma_{xy}(n))^{-p}. \tag{14}$$

Remark 1.4 *1) $|||\cdot|||_p$ is well defined, since every edge $e = (v,w)$ of an admissible path satisfies $Q(e) := Q(v,w) > 0$.*
2) If we take $p = 0$ we have $|||\gamma_{xy}|||_0 = \sum_{(v,w)\in\gamma_{xy}} 1 = N(x,y)$. Thus, $|||\cdot|||_0$ is nothing but the length of a path.

The assumption on the state space S we use is that one should be able to find a set of paths $\Gamma = \{\gamma_{xy}\}$ obeying a, as we call it, *path reconstruction property.* The path reconstruction property establishes a relation between the points in S and the edges for admissible paths.
One can think of the p.r.p. in the following geometric terms:
Let $e = (v, w) \in \mathcal{A}$ be an arbitrary edge. Whenever the index of an edge e and the length of the path γ_{xy} containing e are known one is able to determine the starting-point and the ending-point of γ_{xy}.
Once having introduced this property we are in a position to state our lower bounds on the spectral gap λ_1 and the logarithmic-Sobolev constant α.
It is not obvious that such paths exists. But we will put forward a general framework to handle not only convex sets, for which it turns out that such path sets exists, but also star-shaped sets. The extension to star-shaped sets can be used to handle more intricate sets.
The main body of this paper, i.e. Sections 3, 4 and 5, is an application of the p.r.p. to obtain several functional inequalities which can be used to deduce lower bounds on λ_1 and α.
We will see in Section 6 that the performance of our bounds heavily rely on the chosen path set fulfilling the Path Reconstruction Property. Cases where the spectrum of the generator can be computed explicitly, for instance the case taking a circle or a torus, are of special interest because we are able to compare the bounds on λ_1 to the true value.
Finally, we introduce a new inequality, called the Logarithmic Cheeger inequality. This inequality establishes a relation between the logarithmic-Sobolev constant and the isoperimetric constant $I_K^{\ln} := \inf_{A \subset S} -\frac{Q(\partial(A))}{\pi(A)\ln(\pi(A))}$.
We show

$$\frac{\ln(2)(I_K^{\ln})^2}{4\sqrt{2}} \leq \alpha \leq I_K^{\ln}. \tag{15}$$

This can be seen as an analogue of the well known Cheeger inequality for the spectral gap λ_1.
Recently, Yuen [Y 00] considered convergence rates of Markov chains on $\mathbb{R}^n$ assuming regularity conditions similar to the Path Reconstruction Property. The proofs and examples are different from ours.

2 Path Reconstruction Properties

Suppose we are given an admissible collection of paths $\Gamma = \{\gamma_{xy}\}$ in the sense of Definition 1.1. Let $d(\delta n)$ denote the counting measure on $\mathbb{N}$. For the set $S \times S \times \mathbb{N}$ equipped with the measure $\tilde{\nu}_1$ given by $\tilde{\nu}_1 = \mu \otimes \mu \otimes d(\delta n)$ let $E_1 \subset (S \times S \times \mathbb{N})$ denote the set

$$E_1 := \{(x, y, n) : x, y \in S, 1 \leq n \leq N(x, y)\}. \tag{16}$$

Furthermore, denote by $\nu_1 := \tilde{\nu}_1|_{E_1}$ the restriction of $\tilde{\nu}_1$ to E_1. Let us now introduce a map G which maps each point of E_1 to an indexed edge. To this end, take a path $\gamma_{xy} \in \Gamma$ connecting x and y and define G by

$$G : E_1 \rightarrow S \times S \times \mathbb{N} \times \mathbb{N} \tag{17}$$

$$(x, y, n) \mapsto \left(\underbrace{\gamma_{xy}(n-1), \gamma_{xy}(n)}_{\in \mathcal{A}^\Gamma}, \underbrace{n}_{index}, \underbrace{N(x,y)}_{length} \right). \tag{18}$$

We call $G(E_1) =: E_2$ the indexed edge set and introduce a measure ν_2 on E_2 as the restriction of $\tilde{\nu_2} = \mu \otimes \mu \otimes d(\delta n) \otimes d(\delta N)$ to E_2.

(D) Suppose the map G is a measurable injection. Thus, $G : E_1 \rightarrow E_2$ is a bijection. Suppose E_2 is a measurable subset of $S \times S \times \mathbb{N} \times \mathbb{N}$. Furthermore, suppose there exists a density h_1 of the image measure $G(\nu_1)$ with respect to ν_2 and G^{-1} is measurable.

Definition 2.1 *(Path-Reconstruction-Property, p.r.p.)*
An admissible collection of paths $\Gamma = \{\gamma_{xy}\}$ obeys the path reconstruction property if it has property (D).

Let us explain how to apply the p.r.p. in the sequel. For an integrable function $f : E_1 \rightarrow \mathbb{R}$ consider the integral

$$\int_S \int_S \sum_{n=1}^{N(x,y)} f(x, y, n) \, d\mu(x) d\mu(y).$$

Using the measure space E_1 we can write this integral as

$$(*) := \int_{E_1} f \, d\mu d\mu d\delta.$$

Now, we can use the p.r.p. and obtain that (*) is equal to

$$\int_{E_2} f \circ G^{-1} \, dG(\nu_1) = \int_{E_2} f \circ G^{-1} h_1 \, d\mu d\mu d\delta d\delta \int_S \int_S \sum_{\mathbb{N}\times\mathbb{N}} f \circ G^{-1} 1_{E_2} h_1 \, d\mu d\mu.$$

Because h_1 is only definied on E_2 we have to multiply the integrand with 1_{E_2}.

2.0.1 Example

Let S be a convex subset of $\mathbb{R}^d$, $1 \leq d \in \mathbb{N}$. Define admissible path sets:
Fix $N \in \mathbb{N}$ and take $N(x, y) = N$. Define

$$\Gamma^1 = \{\gamma^1_{xy}\}, \gamma^1_{x,y}(n) = \frac{(N-n)x + ny}{N}, n \in \{0, .., N\}. \tag{19}$$

In the example the paths are chosen to be the discrete straight line segments connecting two points $x, y \in S$. A path contains N edges. Let e be an edge of index k. To recover the starting-point we have to choose the left vertex of the edge appearing before $k-1$ steps. To recover the ending-point we take the right vertex of the edge appearing after $N-k$ steps.
Formally, given an edge $e = (v, w)$ of index k we can solve the following equations (20) and (21) to recover x and y from v and w:

$$v = \frac{(N-k+1)x + (k-1)y}{N} \tag{20}$$

$$w = \frac{(N-k)x + ky}{N}. \tag{21}$$

For fixed $l, L \in \mathbb{N}$ the map $G(\cdot, \cdot, l, L)$ is a continuous linear map. The density h_1 is the determinant of the corresponding Jacobian of the linear transformation. Since there can only be the fixed length N we therefore obtain $h_1 = N^d$.
We will use this path set to study several examples.

2.0.2 Extended Path Reconstruction Property

There are simple sets which do not obey the p.r.p. for simple path systems. For instance star- or barbell-shaped sets are not convex and therefore not any two points can be joined by a straight line. In the sequel we want to extentd the p.r.p. to cover a large class of sets including the latter examples. To this end we consider a set S fulfilling

1. S has a finite covering $S = \bigcup_{i=1}^N S_i$ such that each set S_i, $i \in \{1, 2, ..., N\}$ obeys the p.r.p. in the sense of Definition 2.1.

2. For each $i \in \{1, 2, ..., N\}$ there exists $j \in \{1, 2, ..., N\}$, $i \neq j$, such that $S_i \cap S_j \neq \emptyset$.

Let us consider adapted edge sets to the process $(X_t)_t$ on each S_i. For an admissible path set Γ^i we take its adapted edge set $\mathcal{A}_i$. Because of condition 2. from above, we obtain an adapted edge set $\mathcal{A}^\Gamma$ if we set $\mathcal{A}^\Gamma := \bigcup_{i=1}^N \mathcal{A}_i$. The path set obeys an extended p.r.p. in the following sense:

Definition 2.2 *(Extended path reconstruction property)*
Let S satisfy conditions 1. and 2. from above. An admissible collection of paths $\Gamma = \{\gamma_{xy}\}$ obeys the extended path reconstruction property if for all γ_{xy} there exists a decomposition $\gamma_{xy} = \gamma_1 + ... + \gamma_N$ such that γ_i is a path of a path set Γ^i obeying the p.r.p.

We denote by

$$E_1^i := \{(s_{i-1}, s_i, n) : s_{i-1}, s_i \in S_i, 1 \leq n \leq N(s_{i-1}, s_i)\} \tag{22}$$

and

$$\begin{array}{llrcl}(23) & & G_i : E_1^i & \to & S_i \times S_i \times \mathbb{N} \times \mathbb{N} \\ (24) & & (v,w,n) & \mapsto & (\gamma_{xy}(n-1), \gamma_{xy}(n), n, N(s_{i-1}, s_i))\end{array}$$

the corresponding sets and maps to those in Section 1. The p.r.p. on S_i guarantees the existence of a density h_1^i and with $E_2^i = G(E_1^i)$ the change of measure formula becomes

$$\int_{E_1^i} f\, d\nu_1^i = \int_{E_2^i} f \circ G_i^{-1} h_1^i\, d\nu_2^i. \tag{25}$$

Let us illustrate the definition of the extended p.r.p. To this end we consider a set S fulfilling conditions 1. and 2.
For all $i \in \{1, 2, ..., N\}$ the set S_i is convex and therefore it obeys the p.r.p. in the sense of Definition 2.1. For an admissible collection of paths and the corresponding adapted edge set we have $\bigcup_{i=1}^N \mathcal{A}_i \subset \mathcal{A}^\Gamma$. We take for $s_0 = x, s_1 \in S_1, ..., s_{d-1} \in S_d, s_d = y$, denoting $A(l) := \sum_{k=1}^l N_k$ and consider the path set

$$\Gamma^3 := \{\gamma_{xy}^3\} \text{ with} \tag{26}$$

$$\gamma_{xy}^3 = \begin{cases} \frac{(A(1)-n)x+ns_1}{N_1} & , \quad 0 \le n \le A(1), \\ \frac{(A(2)-n)s_1+(n-A(1))s_2}{N_2} & , \quad A(1) \le n \le A(2), \\ ... & \\ \frac{(A(d)-n)s_{d-1}+(n-(A(d-1))y}{N_d} & , \quad A(d-1) \le n \le A(d). \end{cases}$$

Due to the Example in Section 1 each path component, as it is a discrete straight line segment, obeys the p.r.p. in the sense of Definition 2.1. That means we are able to compute the density h_1 for each path component constituting the discrete polygonal path.

3 Poincaré Inequalities

The first classical inequality under consideration is the Poincaré inequality. For $C > 0$ and all $f \in D(\mathcal{E})$

$$Var_\mu[f] \le C\mathcal{E}(f, f). \tag{27}$$

Using the variational characterization of the spectral gap λ_1, (4), the constant C is a lower bound on λ_1, i.e. $\lambda_1 \ge 1/C$.
For $i = 1, 2$ let

$$\begin{array}{rcl} pr_i : E_1 & \to & S, \\ (x_1, x_2, n) & \mapsto & x_i \end{array}$$

denote the projection on the i-th coordinate.
To state our results we have to introduce another projection map on E_1. Let

$$\begin{array}{rcl} pr_{(1,2)} : E_1 & \rightarrow & S \times S, \\ (x, y, n) & \mapsto & (x, y) \end{array}$$

denote the projection on the first two coordinates.
In the sequel we will abbreviate

$$\begin{array}{lll} pr_{(1,2)}(G^{-1}(v, w, m, n)) & \text{by} & pr_{(1,2)}G^{-1}, \\ pr_1(G^{-1}(v, w, m, n)) & \text{by} & pr_1 G^{-1}, \\ pr_2(G^{-1}(v, w, m, n)) & \text{by} & pr_2 G^{-1}. \end{array}$$

With these notational simplifications we can state our first theorem.

Theorem 3.1 *Let $(X_t)_t$ be a reversible Sub-Markov chain in discrete time or a time continuous Sub-Markov jump process on a Polish space $(S, \mathcal{S}, \mu)$. Assume the transition kernel of the chain is given by (1) and there exists an admissible path set Γ obeying the p.r.p.*
Then, $(\mathcal{E}, D(\mathcal{E}))$ satisfies a Poincaré inequality with constant

$$\kappa^{\Gamma}(p) = \tag{28}$$

$$\underset{(v,w) \in \mathcal{A}^{\Gamma}}{\text{esssup}} \left\{ \sum_{m,n \in \mathbb{N}} \| |\gamma_{pr_{1,2}G^{-1}}| \|_p h(pr_1 G^{-1}) h(pr_2 G^{-1}) h_1(v, w, m, n) Q(v, w)^{p-1} 1_{E_2} \right\}.$$

Proof:
For arbitrary points $x, y \in S$ take the path γ_{xy} connecting x and y. For a bounded measurable function f using the triangle inequality, we obtain

$$|f(x) - f(y)| \leq \sum_{n=1}^{N(x,y)} |f(\gamma_{xy}(n-1)) - f(\gamma_{xy}(n))|.$$

Since,

$$2Var_{\pi}[f] = \int_S \int_S |f(x) - f(y)|^2 \, d\pi(x) d\pi(y)$$

we obtain the inequality

$$2Var_{\pi}[f] \leq \int_S \int_S \left(\sum_{n=1}^{N(x,y)} |f(\gamma_{xy}(n-1)) - f(\gamma_{xy}(n))| \right)^2 d\pi(x) d\pi(y).$$

Applying the Cauchy-Schwarz inequality for the counting measure to

$$\left(\sum_{n=1}^{N(x,y)} |f(\gamma_{xy}(n-1)) - f(\gamma_{xy}(n))| \frac{Q(\gamma_{xy}(n-1), \gamma_{xy}(n))^{p/2}}{Q(\gamma_{xy}(n-1), \gamma_{xy}(n))^{p/2}} \right)^2,$$

we obtain that $2Var_\pi[f]$ is bounded by

$$\int_S \int_S \left(\sum_{n=1}^{N(x,y)} |f(\gamma_{xy}(n-1)) - f(\gamma_{xy}(n))|^2 Q(\gamma_{xy}(n-1), \gamma_{xy}(n))^p \right) \\ \times \underbrace{\left(\sum_{[n=1}^{N(x,y)} Q(\gamma_{xy}(n-1), \gamma_{xy}(n))^{-p} \right)}_{=|||\gamma_{xy}|||_p} d\pi(x) d\pi(y).$$

With denoting by F the measurable function

$$F(x,y,n) := |f(\gamma_{xy}(n-1)) - f(\gamma_{xy}(n))|^2 Q(\gamma_{xy}(n-1), \gamma_{xy}(n))^p |||\gamma_{xy}|||_p h(y) h(x),$$

we can apply the p.r.p. and obtain

$$\begin{aligned} \int_S \int_S \sum_{n=1}^{N(x,y)} F(x,y,n)\, d\mu(y) d\mu(x) &= \int_{E_2} F \circ G^{-1} h_1 \, d\mu d\mu d\delta d\delta \\ &= \int_S \int_S \sum_{m,n \in \mathbb{N}} F \circ G^{-1} 1_{E_2} h_1 \, d\mu d\mu =: (*). \end{aligned}$$

For the last equalitiy we used the Fubini theorem. The factor 1_{E_2} has to be added because the density h_1 is only defined on E_2.
The last term (*) is equal to:

$$\begin{aligned} (*) \quad = \quad & \int_S \int_S \sum_{m,n \in \mathbb{N}} |||\gamma_{pr_{1,2}G^{-1}}|||_p h(pr_1 G^{-1}) h(pr_2 G^{-1}) \\ & \times Q(v,w)^{p-1} h_1(v,w,m,n) 1_{E_2} \times Q(v,w) |f(v) - f(w)|^2 d\mu(v) d\mu(w). \end{aligned}$$

And this can now be estimated by

$$\operatorname*{esssup}_{(v,w) \in \mathcal{A}^\Gamma} \left\{ \sum_{m,n \in \mathbb{N}} |||\gamma_{pr_{1,2}G^{-1}}|||_p h(pr_1 G^{-1}) h(pr_2 G^{-1}) h_1(v,w,m,n) Q(v,w)^{p-1} \right\} \\ \times \int_S \int_S Q(v,w) |f(v) - f(w)|^2 \, d\mu(v) d\mu(w).$$

Therefore, we have shown

$$Var_\pi[f] \leq \kappa^\Gamma \mathcal{E}(f,f).$$

This proves the Theorem. □
We apply the bound from Theorem 3.1 to various examples in Section 6. In our setting, i.e. general state spaces, it is not obvious that a spectral gap must exist. As we already mentioned in the introduction all our results can

also be seen as existence results for a spectral gap.

Simplification of the constant
The constant in $\kappa^\Gamma(p)$, (28), seems to be very complicated. It can be simplified by using a somewhat informal notation. Furthermore, applied to the path sets introduced in (20) and (21) the density h_1 is a constant and (28) becomes more simple.
For a given edge $e = (v, w)$ there is a $1:1$ correspondance of the index of e and the length of the path in which e appears. Hence, to keep notation short we suppress the dependence of v, w, m and n in the arguments of $h(\cdot)$ and $\|\cdot\||_p$ and write xy instead of $pr_{1,2}G^{-1}(v, w, m, n)$, x instead of $pr_1G^{-1}(v, w, m, n)$ and y instead of $pr_2G^{-1}(v, w, m, n)$. With these notational simplification the constant becomes

$$\kappa^\Gamma(p) = \operatorname*{esssup}_{(v,\,w)\,\in\,\mathcal{A}^\Gamma} \left\{ \sum_{m,n} \||\gamma_{xy}\||_p h(x)h(y)h_1(v, w, m, n)Q(v, w)^{p-1} 1_{E_2} \right\}.$$

We will use this simplification in the sequel. In the sequel we derive a local Poincaré inequality. We denote by $B_x(r)$ the ball with radius r centered at $x \in S$ with volume $\mathrm{Vol}(B_x(r)) := \int 1_{B_x(r)}(y)d\pi(y)$. We further define

$$f_r(x) := \frac{1}{\mathrm{Vol}(B_x(r))} \int 1_{B_x(r)}(y)f(y)\, d\pi(y).$$

Definition 3.2 *Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form on $L^2(S, \mu)$ and denote by $diam(S)$ the diameter of S. $(\mathcal{E}, D(\mathcal{E}))$ satisfies a local Poincaré inequality if for $r \in [0, diam(S)]$, there exists a constant C_r which only depends on r such that for all $f \in D(\mathcal{E})$*

$$\|f - f_r\|_2^2 \leq C_r\mathcal{E}(f, f). \tag{29}$$

Remark 3.3 *For bounded S we can deduce a Poincaré inequality if we only consider $r = diam(S)$.*

We further introduce the measure space $E_{1,r} \subset E_1$ by

$$E_{1,r} := \{(x, y, n) : x \in S, y \in B_x(r), 1 \leq n \leq N(x, y)\}$$

and define $E_{2,r} := G(E_{1,r})$. Having chosen a path set obeying the p.r.p. one can derive local Poincaré inequalities by the following theorem.

Theorem 3.4 *Let $(X_t)_t$ be a reversible Sub-Markov chain in discrete time or a time continuous Sub-Markov jump process on a bounded Polish space $(S, \mathcal{S}, \mu)$. Assume the transition kernel of the chain is given by (1) and there*

exists an admissible path sets Γ obeying the p.r.p.
Then, $(\mathcal{E}, D(\mathcal{E}))$ satisfies a local Poincaré inequality with constant

$$(30) \qquad \eta^{\Gamma}(p,r) :=$$

$$\operatorname*{esssup}_{(v,\,w)\,\in\,\mathcal{A}^{\Gamma}} \left\{ \frac{1}{Q(v,w)^{1-p}} \sum_{\substack{m,n\in\mathbb{N}\\ d(x,y)\le r}} |||\gamma_{xy}|||_p \frac{h(x)h(y)}{Vol(B_x(r))} h_1(v,w,m,n) 1_{E_{2,r}} \right\}.$$

Proof:
We abbreviate $\mathrm{Vol}(B_x(r))$ by $V_x(r)$. Jensen's inequality implies

$$|f(x) - f_r(x)|^2 \le \frac{1}{V_x(r)} \int_{B_x(r)} |f(x) - f(y)|^2 \, d\pi(y).$$

Consider the integrand on the right hand side. For each $y \in B_x(r)$ take the path γ_{xy} connecting x and y. As in the latter proof we obtain using the triangle inequality and the Cauchy-Schwarz inequality

$$\begin{aligned} (**) \quad &= |f(x) - f(y)|^2 \\ &\le \sum_{n=1}^{N(x,y)} |f(\gamma_{xy}(n-1)) - f(\gamma_{xy}(n))|^2 Q(\gamma_{xy}(n-1), \gamma_{xy}(n))^p |||\gamma_{xy}|||_p. \end{aligned}$$

With

$$F_r(x,y,n) := |f(\gamma_{xy}(n-1)) - f(\gamma_{xy}(n))|^2 Q(\gamma_{xy}(n-1), \gamma_{xy}(n))^p |||\gamma_{xy}|||_p \frac{h(x)h(y)}{V_x(r)}$$

and integrating (*) over S with respect to π yields

$$2\|f - f_r\|_2^2 \le \int_S \int_{B_x(r)} \sum_{n=1}^{N(x,y)} F_r(x,y,n) \, d\mu(y) d\mu(x).$$

Applying the p.r.p. and the simplification of the constant we find

$$\begin{aligned} &\int_S \int_{B_x(r)} \sum_{n=1}^{N(x,y)} F_r(x,y,n) \, d\mu(y) d\mu(x) = \int_{E_{2,r}} F_r \circ G^{-1} h_1 \, d\mu d\mu \\ &\le \operatorname*{esssup}_{(v,\,w)\,\in\,E_2(r)} \left\{ Q(v,w)^{p-1} \sum_{\substack{m,n\in\mathbb{N}\\ d(x,y)\le r}} |||\gamma_{xy}|||_p \frac{h(x)h(y)}{\mathrm{Vol}(B_x(r))} h_1(v,w,m,n) 1_{E_{2,r}} \right\} \\ &\quad \times \int_{S\times S} Q(v,w)(f(v) - f(w))^2 \, d\mu(v) d\mu(w) \\ &= 2\eta^{\Gamma}(p,r)\mathcal{E}(f,f) \end{aligned}$$

This proves the Theorem. $\square$

In the finite setting it has been proven useful to consider general measures on path sets. We refer to [S 92] for further explanations and examples.
With no further difficulties we are able to prove similar results in the setting for general state spaces. In Definition 3.5 we introduce the notions of a weight and use it to prove a more sophisticated version of a Poincaré inequality in Theorem 3.6. All the results in the next two sections can be generalized to weights but we do not apply these bounds to any concrete example.

Definition 3.5 *A weight function ω is a positive function*

$$\begin{aligned} \omega : \mathcal{A} &\to (0,\infty) & (31)\\ e &\mapsto \omega(e). & (32)\end{aligned}$$

The ω-length of a path $\gamma_{xy} \in \Gamma$ is given by

$$|\gamma_{xy}|_\omega := \sum_{n=1}^{N(x,y)} \omega((\gamma_{xy}(n-1), \gamma_{xy}(n)))^{-1}. \tag{33}$$

With the simplification of the constant we can show the following Theorem.

Theorem 3.6 *Let $(X_t)_t$ be a reversible Sub-Markov chain in discrete time or a time continuous jump Sub-Markov process on a bounded Polish space $(S, \mathcal{S}, \mu)$. Assume the transition kernel of the chain is given by (1) and there exists an admissible path sets Γ obeying the p.r.p.*
Then, a weighted Poincaré inequality with constant

$$\kappa^{\Gamma,\omega} = \operatorname*{esssup}_{(v,\,w)\,\in\,\mathcal{A}^\Gamma} \left\{ \frac{\omega((v,w))}{Q(v,w)} \sum_{m,n\in\mathbb{N}} |\gamma^\Gamma_{xy}|_\omega h(x)h(y)h_1(v,w,m,n)1_{E_2} \right\}$$

(34)
holds true.

Proof:
For $x, y \in S$ take the path γ_{xy} connecting x and y and observe that by the Cauchy-Schwarz inequality

$$\begin{aligned} &|f(y) - f(x)|^2 \\ &\leq \left(\sum_{n=1}^{N(x,y)} \frac{1}{\omega(\gamma_{xy}(n-1), \gamma_{xy}(n))} \right) \\ &\quad \times \left(\sum_{n=1}^{N(x,y)} |f(\gamma_{xy}(n-1)) - f(\gamma_{xy}(n))|^2 \omega(\gamma_{xy}(n-1), \gamma_{xy}(n-1)) \right) \\ &= |\gamma_{xy}|_\omega \sum_{n=1}^{N(x,y)} |f(\gamma_{xy}(n-1)) - f(\gamma_{xy}(n))|^2 \omega(\gamma_{xy}(n-1), \gamma_{xy}(n)) \end{aligned}$$

Integration over $S \times S$ with respect to $\pi \otimes \pi$ gives

$$2\|f - \mathbb{E}_\pi[f]\|_2^2 \le$$
$$\int_S \int_S |\gamma_{xy}|_\omega \sum_{n=1}^{N(x,y)} |f(\gamma_{xy}(n-1)) - f(\gamma_{xy}(n))|^2 \omega(\gamma_{xy}(n-1), \gamma_{xy}(n))\, d\pi(x) d\pi(y).$$

Then continuing as in the proof of Theorem 2.3.1. proves the theorem. □

4 Isoperimetric Inequalities

In the previous sections we derived Poincaré inequalities of various types. As mentioned in the introduction, in [LS 88], Lawler and Sokal proved a version of Cheeger's inequality. For the transition kernel in (1) it reads

Lemma 4.1 *([LS 88]) The spectral gap of a Sub-Markov chain with transition kernel given in (1) and the isoperimetric constant* $I_K := \min_{A \subset S} \frac{Q(\partial(A))}{\pi(A)\pi(A^c)}$ *are related by*

$$\frac{I_K^2}{8} \le \lambda_1(K) \le I_K. \tag{35}$$

In the sequel we use paths to state a lower bound on I_K and therefore on λ_1. To prove the following theorem we need a variational characterization of the isoperimetric constant I_K, i.e.

$$I_K = \inf_{\substack{f \in L^2(S,\pi) \\ f \text{ non constant}}} \left\{ \frac{\int\int |f(x) - f(y)|\, K(x, dy) d\pi(x)}{\int \left|f(x) - \int f(x)\, d\pi(x)\right|\, d\pi(x)} \right\}. \tag{36}$$

Theorem 4.2 *Let $(X_t)_t$ be a reversible Sub-Markov chain in discrete time or a time continuous jump Sub-Markov process on a measure space $(S, \mathcal{S}, \mu)$. Assume the transition kernel of the chain is given by (1) and there exists an admissible path set Γ obeying the p.r.p.*
Then, for

$$B^\Gamma := \operatorname*{esssup}_{(v,\, w)\, \in\, \mathcal{A}^\Gamma} \left\{ \sum_{m,n \in \mathbb{N}} Q(v, w)^{-1} h(x) h(y) h_1(v, w, m, n) 1_{E_2} \right\}, \tag{37}$$

we have

$$I_K \ge \frac{1}{B^\Gamma}. \tag{38}$$

Proof:
By the usual argument involving the triangle inequality we have

$$\int_S \int_S |f(x) - f(y)|\, d\pi(x) d\pi(y) \le$$
$$\int_S \int_S \sum_{n=1}^{N(x,y)} |f(\gamma_{xy}(n-1)) - f(\gamma_{xy}(n))|\, d\pi(x) d\pi(y).$$

Denote by LS and RS the left hand side and respectively right hand side of the previous inequality.
The LS is an upper bound on $\int_S |f(x) - \mathbb{E}_\pi[f]|\, d\pi(x)$, whereas using the p.r.p. and the simplification of the constant in the usual manner

$$\begin{aligned} RS &= \int_{S\times S} \sum_{m,n\in\mathbb{N}} \frac{Q(v,w)}{Q(v,w)} |f(v) - f(w)| h(x)h(y) \\ &\qquad \times h_1(v,w,m,n) 1_{E_2}\, d\mu(v) d\mu(w) \\ &\leq \operatorname*{esssup}_{(v,\,w)\,\in\,\mathcal{A}^\Gamma} \left\{ \sum_{m,n\in\mathbb{N}} \frac{h(x)h(y)}{Q(v,w)} h_1(v,w,m,n) 1_{E_2} \right\} \\ &\qquad \times \int_S \int_S |f(v) - f(w)| Q(v,w)\, d\mu(v) d\mu(w). \end{aligned}$$

This proves Theorem 5.2. □

Remark 4.3 *Not even for Markov chains on finite state spaces it is clear if lower bounds on the spectral gap obtained using Theorem 3.1 perform always better than those applying Theorem 5.2. Recently, Fulman and Wilmer in [FW 99] answered this question for several examples in the finite case in the affirmative.*
In the examples in Section 6 we also observe this fact.

5 Comparison

Next, we consider a measure space $(S, \mathcal{S}, \mu)$ and two reversible Sub-Markov chains $(X_t)_t, (Y_t)_t$ in discrete time or time continuous Sub-Markov jump processes having transition kernels $K(x,dy)$ and $\tilde{K}(x,dy)$. The corresponding stationary probability measures are denoted by π and $\tilde{\pi}$, having densities h and $\tilde{h}$ with respect to μ.

PATH SETS:
We have to change the definition of admissibility to be able to compare two Sub-Markov chains. This is necessary because the domain of the corresponding transition kernels can be different.

Definition 5.1 *We say an admissible collection of paths $\Gamma = \{\gamma_{xy}\}$ is an c-admissible collection of paths (comparison admissible) if the conditions of Definition 1.1 are fulfilled for the transition kernels $\tilde{K}$ and K.*

The following theorem can be used to compare the spectral gap and the logarithmic-Sobolev constant for different Sub-Markov chains.

Theorem 5.2 *If there exist constants $a, A \in \mathbb{R}^+$ such that* *(i)* $\tilde{\mathcal{E}}(f,f) \leq A\mathcal{E}(f,f)$, *for all* $f \in L^2(X,\pi)$ *and*

(ii) $a\pi \leq \tilde{\pi}$,

then, we have

$$\tilde{\lambda}_1 \geq \frac{A}{a}\lambda_1 \text{ and } \tilde{\alpha} \geq \frac{A}{a}\alpha \tag{39}$$

Proof:

We use the fact that the variance $Var_\pi[f]$ is given by

$$Var_\pi[f] = \inf_c \left\{ \int_S |f(x) - c|^2 \, d\pi(x) \right\}.$$

Therefore,

$$Var_\pi[f] \leq \frac{1}{a} Var_{\tilde{\pi}}[f].$$

Finally, this leads to

$$\frac{\mathcal{E}(f,f)}{Var_\pi[f]} \geq \frac{a}{A} \frac{\tilde{\mathcal{E}}(f,f)}{Var_{\tilde{\pi}}[f]}.$$

By the variational characterization of the spectral gap λ_1 the first assertion of the theorem follows.

To prove the second assertion we use the variational characterization of the logarithmic-Sobolev functional, i.e.

$$\mathcal{L}_\pi[f] = \inf_c \left\{ \int_S \left(f(x)^2 \log(f(x)^2) - f(x)^2 \log(c) - f(x)^2 + c \right) \, d\pi(x) \right\}.$$

Consequently,

$$\mathcal{L}_\pi[f] \leq \frac{1}{a} \mathcal{L}_{\tilde{\pi}}[f].$$

Similar to the case for the spectral gap, we find

$$\frac{\mathcal{E}(f,f)}{\mathcal{L}_\pi[f]} \geq \frac{a}{A} \frac{\tilde{\mathcal{E}}(f,f)}{\mathcal{L}_{\tilde{\pi}}[f]}.$$

Therefore, the theorem is proved. □

Remark 5.3 *1) The state space may be infinite dimensional.*
2) If $\tilde{\pi} = \pi$ *we can take* $a = 1$ *and we can restrict ourselves to compare the Dirichlet forms.*
3) If the state space is finite we recover the results obtained by Diaconis and Saloff-Coste. They also obtain lower bounds, but in this case the number of elements of the state space comes in.

In applications the necessity for comparing Sub-Markov chains on different state spaces occurs. This is possible with respect to the following remark.

Remark 5.4 *Let K and $\tilde{K}$ be transition kernels on $(S, \mathcal{S}, \mu)$ and $(\tilde{S}, \tilde{\mathcal{S}}, \tilde{\mu})$ respectively. Assume there exists a linear transformation T, such that*

$$\begin{array}{rcl} T: L^2(S, \pi) & \to & L^2(\tilde{\mathcal{S}}, \tilde{\pi}) \\ f & \mapsto & \tilde{f}. \end{array}$$

Then, if there are $A, a_1, a_2, B_1, B_2 \in \mathbb{R}^+$, such that for all $f \in L^2(S, \pi)$

$$\begin{array}{rcl} \tilde{\mathcal{E}}(\tilde{f}, \tilde{f}) & \leq & A\mathcal{E}(f, f), \\ a_1 Var_\pi[f] & \leq & Var_{\tilde{\pi}}[\tilde{f}] + B_1\mathcal{E}(f, f), \\ a_2\mathcal{L}_\pi[f] & \leq & \mathcal{L}_{\tilde{\pi}}[\tilde{f}] + B_2\mathcal{E}(f, f), \end{array}$$

the following inequalities hold true

$$\frac{a_1\lambda_1(\tilde{K})}{A + B_1\lambda_1(\tilde{K})} \leq \lambda_1(K) \text{ and } \frac{a_2\alpha(\tilde{K})}{A + B_2\alpha(\tilde{K})} \leq \alpha(K). \tag{40}$$

The statements of Remark 5.4 follow immediately using the linear tranformation T and the variational characterizations of λ_1, α, $\tilde{\lambda_1}$ and $\tilde{\alpha}$. As in the finite setting these considerations, the simplification of the constant and Theorem 5.2 lead to

Theorem 5.5 *Let $(S, \mathcal{S}, \mu)$ be a measure space and $(X_t)_t, (Y_t)_t$ be two reversible discrete-time Sub-Markov chains or time-continuous Sub-Markov jump processes having transition kernels $K(x, dy)$ and $\tilde{K}(x, dy)$. The corresponding stationary probability measures are denoted by π and $\tilde{\pi}$, having densities h and $\tilde{h}$ w.r.t μ. Assume there exists c-admissible path set in the sense of Definition 5.1 on S obeying the p.r.p. In the above setting we have for $f \in D(\mathcal{E})$*

$$(f, \tilde{H}f)_{\tilde{\pi}} \leq A^\Gamma(p)(f, Hf)_\pi \tag{41}$$

with constant

$$A^\Gamma(p) := \underset{(v,w) \in \mathcal{A}^\Gamma}{esssup} \left\{ \sum_{m,n\in\mathbb{N}} \|\gamma_{xy}\|_p \tilde{d}(x)\tilde{k}(x,y)h_1(v,w,m,n)Q(v,w)^{p-1}1_{E_2} \right\}.$$

__**Proof:**__

The corresponding Dirichlet forms are $\mathcal{E}(f, f) = 2(f, Hf)_\pi$ and $\tilde{\mathcal{E}}(f, f) = 2(f, \tilde{H}f)_{\tilde{\pi}}$. Applying the Cauchy-Schwarz inequality and the p.r.p., we obtain

$$\begin{aligned} &\tilde{\mathcal{E}}(f, f) = 2(f, \tilde{H}f)_{\tilde{\pi}} \\ &\leq \quad \int_S \int_S \left(\sum_{n=1}^{N(x,y)} \left(\frac{Q(\gamma_{xy}(n-1), \gamma_{xy}(n))}{Q(\gamma_{xy}(n-1), \gamma_{xy}(n))} \right)^{\frac{p}{2}} |f(\gamma_{xy}(n-1)) - f(\gamma_{xy}(n))| \right)^2 \end{aligned}$$

$$
\begin{aligned}
& \qquad \times \tilde{h}(x)\tilde{k}(x,y)\, d\mu(x)d\mu(y) \\
\leq \quad & \int_S \int_S |||\gamma_{xy}|||_p \tilde{h}(x)\tilde{k}(x,y) \\
& \times \sum_{n=1}^{N(x,y)} Q(\gamma_{xy}(n-1), \gamma_{xy}(n))^p) |f(\gamma_{xy}(n-1)) - f(\gamma_{xy}(n))|^2 \, d\mu(x)d\mu(y) \\
\overset{p.r.p.}{=} \quad & \int_S \int_S \sum_{m,n\in\mathbb{N}} |||\gamma_{xy}|||_p \tilde{h}(x) Q(v,w)^p |f(v) - f(w)|^2 \\
& \qquad \times h_1(v,w,m,n) 1_{E_2}\, d\mu(v)d\mu(w) \\
\leq \quad & A^\Gamma(p) \int_S \int_S Q(v,w)(f(v)-f(w))^2 \, d\mu(v)d\mu(w) \\
= \quad & 2A^\Gamma(p)(f,Hf)_\pi
\end{aligned}
$$

Hence, the result follows. □

In applications given transition kernels can be very complicated. Computations involving such kernels can be very tricky or even impossible. In these situations one may take advantage of a comparison argument. Sometimes it is possible to find lower and/or upper bounds of the associated Dirichlet form in terms of a Dirichlet form corresponding to a simpler kernel. Then, one can apply Theorem 5.2. Whereas Theorem 5.2 can be useful to find a lower and/or upper bound. We apply both Theorems in Section 6 to some examples.

6 Examples

In the sequel we apply our methods to some examples. The first example illustrates the application of the p.r.p. for a star-shaped set. In the case of a circle, we observe in Example 2 that the lower bounds can be sharp. However the last example, a torus, suggests that our bounds may not be of the right order of magnitude.

We consider more examples in [K 00].

6.1 Example: Crosses

We consider the measure space $(S, \mathcal{S}, \mu) = (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), dx)$ where dx as usual denotes the Lebesgue measure. For $l_1, l_2, b_1, b_2 \in \mathbb{R}^+$ consider the sets

$$A := \{(x,y) \in \mathbb{R}^2 : -b_2 - l_2 \leq y \leq b_2 + l_2, -b_1 \leq x \leq b_1\}$$

and

$$B := \{(x,y) \in \mathbb{R}^2 : -b_1 - l_1 \leq x \leq b_1 + l_1, -b_2 \leq y \leq b_2\}.$$

The cross centered at 0 is the set $C_{l_1,l_2,b_1,b_2} := A \cup B$. In the sequel we abbreviate C_{l_1,l_2,b_1,b_2} by C.

Let $\epsilon > 0$, $x \in C$ and take the (x, ϵ)-cube Q_x^ϵ, i.e.

$$Q_x^\epsilon := \left\{ y \in \mathbb{R}^d : |x_i - y_i| \leq \epsilon, i = 1, 2 \right\}.$$

We consider the transition kernel

$$K(x, dy) = \frac{1}{4\epsilon^2} 1_{Q_x^\epsilon \cap C}(y) dy.$$

With respect to Lebesgue measure, dx, the stationary measure π has density

$$h(x) = \frac{1}{2(l_1 b_2 + l_2 b_1) + b_1 b_2} 1_C(x) = \frac{1}{Vol(C)} 1_C(x).$$

We observe that the length L of the longest path component connecting a point $x \in C$ and 0 can at most be $d_1 \vee d_2$. Choose $d_1 = \sqrt{(l_1 + b_1)^2 + b_2^2}$ and $d_2 = \sqrt{(l_2 + b_2)^2 + b_1^2}$. Without loss of generality we may assume $d_1 \geq d_2$ and hence $L \leq \sqrt{2} d_1$.
Now, we consider the path set Γ^3 introduced in (26). Since we have for the convex sets A and B from above $C = A \cup B$ and $A \cap B \neq \emptyset$, the path set Γ^3 obeys therefore the extended p.r.p.
The constants can be computed by considering the components of a path $\gamma_{xy}^3 = \gamma_{x0}^1 + \gamma_{0y}^1$ where γ_{x0}^1 and $\gamma_{0,y}^1$ are the discrete straight line segments connecting x, 0, and 0, y respectively. We have

$$\begin{aligned}
|||\gamma_{x0}^1|||_p &= \sum_{n=1}^{N_1} Q(\gamma_{x0}^1(n-1), \gamma_{x0}^1(n))^{-p} = N_1 \left(\frac{1}{4\epsilon^2 \mathrm{Vol}(C)} \right)^{-p}, \\
|||\gamma_{0y}^1|||_p &= \sum_{n=1}^{N_2} Q(\gamma_{0y}^1(n-1), \gamma_{0y}^1(n))^{-p} = N_2 \left(\frac{1}{4\epsilon^2 \mathrm{Vol}(C)} \right)^{-p}.
\end{aligned}$$

Therefore, we can compute κ^3 and obtain

$$\kappa^{\Gamma^3}(p) \leq \frac{4\epsilon^2}{\mathrm{Vol}(C)} (N_1^4 + N_2^4).$$

For the lower bound on the isoperimetric constant I_K we have to estimate the constant B^{Γ^3} and with the same considerations we find

$$B^{\Gamma^3} \leq (N_1^3 + N_2^3) \frac{4\epsilon^2}{\mathrm{Vol}(C)}.$$

Since admissibility is assumed, we need $N_1, N_2 \geq d_1/\epsilon$. Thus,

$$\lambda_1 \geq \frac{Vol(C)}{4\epsilon^2 (N_1^4 + N_2^4)} \approx \frac{\mathrm{Vol}(C)\epsilon^2}{8 d_1^4}$$

and respectively

$$\lambda_1 \geq \left(\frac{\mathrm{Vol}(C)^2}{4\epsilon^2(N_1^3+N_2^3)} \right)^2 /8 \approx \frac{\mathrm{Vol}(C)\epsilon^4}{256 d_1^6}.$$

In this case the spectrum cannot be computed explicitely because C has a boundary and one has to observe that the kernel K is not homogenous in the space variable, i.e. it depends explicitely on the starting point. We have not found any results concerning the last example in the literature.
To study the effect of an increase in the dimension on the lower bounds on the spectral gap, we consider $d-$dimensional crosses.
To this end consider the $d-$dim cross $C^d_{r_1,\ldots,r_d,b_1,\ldots,b_d}$, centered at 0. We find by similar computations as above

$$\begin{aligned} \kappa^{\Gamma^3} &= (N_1^{d+2}+N_2^{d+2}) \frac{2^d \epsilon^d}{\mathrm{Vol}(C)} \approx 2^{d+1} \frac{d_1^{d+2}}{\epsilon^2 \mathrm{Vol}(C^d)}, \\ B^{\Gamma^3} &= (N_1^{d+1}+N_2^{d+1}) \frac{2^d \epsilon^d}{\mathrm{Vol}(C^d)} \approx 2^{d+1} \frac{d_1^{d+1}}{\epsilon \mathrm{Vol}(C^d)}. \end{aligned}$$

We observe that increasing the dimension by 1 yields a factor $2d_1$ in our upper bounds on the corresponding constants.
The length D of the longest line connecting a point $x \in C^d_{l_1,\ldots,l_d,b_1,\ldots,b_d}$ and 0 used to state upper bounds on the constants κ^{Γ^4} and B^{Γ^4} can be obtained by computing the length of the diagonals connecting a corner point and 0. Let the diagonals be denoted by $d_1, d_2, ..., d_d$ and if we take $D = \max_{i=1,\ldots,d} d_i$ this yields

$$\kappa^{\Gamma^4} \leq \frac{2^{d+1}\epsilon^d}{\mathrm{Vol}(C^d)} \sum_{n=1}^{\lceil \eta D \rceil} n^{d+2} \text{ and } B^{\Gamma^4} \leq \frac{2^{d+1}\epsilon^d}{\mathrm{Vol}(C^d)} \sum_{n=1}^{\lceil \eta D \rceil} n^{d+1}.$$

With regards to the latter inequalities, the effect of increasing the dimension can directly be seen. It increases the upper bounds on the corresponding constants and therefore decreases the corresponding lower bounds on the spectral gap.

6.2 The Circle

Firstly, we take the circle $C(0,1)$ with radius 1 centered at 0.
We consider the transition kernel

$$K(x,dy) = \frac{1}{2\epsilon} 1_{I_x^\epsilon}(y), \quad 0 < \epsilon < \pi, \quad I_x^\epsilon := \{z \in C(0,1) : d(x,z) < \epsilon\}.$$

Here dy denotes the Lebesgue measure on the circle, i.e. $dy(C(0,1)) = 2\pi$. Then, we have as the stationary measure $\pi(dx) = \frac{1}{2\pi} dx$.

We take as a path set Γ^C the shortest geodesics connecting two points $x, y \in C(0,1)$. These are nothing but the straight line segments connecting two points using local coordinates.
Take a path $\gamma_{xy} \in \Gamma^C$. We have

$$|||\gamma^1_{xy}|||_p = \sum_{n=1}^{N} Q(\gamma^1_{xy}(n-1), \gamma^1_{xy}(n))^{-p} = \left(4\epsilon\pi \sum_{j=1}^{N} 1_{\{\frac{\pi}{N}<\epsilon\}}\right)^p = \left(4\epsilon\pi N 1_{\{\frac{\pi}{N}\leq\epsilon\}}\right)^p.$$

Hence,

$$\kappa^{\Gamma^C} = \left\{ \operatorname*{esssup}_{(v,\,w)\,\in\,\mathcal{A}^\Gamma} \sum_{m,n\in\mathbb{N}} |||\gamma^1_{xy}|||_p h(x)h(y)h_1(v,w,m,n)1_{E_2}Q(v,w)^{p-1} \right\} \leq N^3\frac{\epsilon}{\pi}.$$

Since admissibility is assumed, i.e. $\frac{\pi}{N} \leq \epsilon$ we find

$$\lambda_1 \geq \frac{\pi}{\epsilon}\frac{1}{N} \approx \frac{\epsilon^2}{\pi^2}.$$

We can explicitely calculate the eigenvalues corresponding to the operator under consideration. We take as a basis of orthonormal functions for the circle the system $\left(1, \left(\frac{\cos(kx)}{\sqrt{2\pi}}\right)_{k\in\mathbb{N}}, \left(\frac{\sin(kx)}{\sqrt{2\pi}}\right)_{k\in\mathbb{N}^+}\right)$. It follows that

$$\begin{aligned}
\int_C k(x,y)\frac{\cos(ky)}{\sqrt{2\pi}}dy &= \frac{1}{\sqrt{2\pi}}\frac{1}{2\epsilon}\int_0^{2\pi} 1_{U_\epsilon(x)}(y)\cos(ky)dy \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} \cos(ky)dy \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{2\epsilon}\frac{1}{k}\sin(ky)|_{x-\epsilon}^{x+\epsilon} \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{2k\epsilon}\left(\sin(k(x-\epsilon)) - \sin(k(x+\epsilon))\right) \\
&= \frac{\sin(k\epsilon)}{k\epsilon}\frac{\cos(kx)}{\sqrt{2\pi}}.
\end{aligned}$$

The same computation using the functions 1 and $\frac{\sin(kx)}{\sqrt{2\pi}}$ shows that the eigenvalues of L are given by

$$0, \left(1 - \frac{\sin(k\epsilon)}{k\epsilon}\right)_{k\in\mathbb{N}^+}.$$

This compares well with our estimate because $\lambda_1 \approx \frac{\epsilon^2}{6}$.
Now, take the circle $C(0,r)$ with radius $r \in \mathbb{R}^+$ centerd a 0. We consider the transition kernel

$$K(x,dy) = \frac{1}{2\epsilon}1_{I_x^\epsilon}(y), \quad 0 < \epsilon < \pi, \quad I_x^\epsilon := \{z \in C(0,r) : d(x,z) < \epsilon\}.$$

Similar computations as in the case $r = 1$ yield

$$\lambda_1(C(0,r)) \geq \left(\frac{\epsilon}{\pi r}\right)^2.$$

They are off by a factor $\approx 1/3$.

6.3 The Torus

Let $\epsilon, r_1, r_2 \in \mathbb{R}^+$ and assume $r_1 \geq r_2$ and $\pi r_2/2 \geq \epsilon$. The torus T is the two dimensional submanifold obtained by rotating a circle of radius r_2 around a line lying in the same plane as the circle and having distance r_1 from the circle's midpoint.
We have $\mathrm{Vol}(T) = 4\pi^2 r_1 r_2$. If we take $k(x,y) = \frac{1}{4\epsilon^2} 1_{U_\epsilon(x)(y)}$, then the stationary measure is $d\pi(x) = \frac{1}{Vol(T)} dx$. We choose as a path set Γ^T the shortest geodesics γ_{xy} connecting two points $x, y \in T, x \neq y$. A path γ_{xy} corresponds to the image of the shortest straight line connecting $\varphi(x)$ and $\varphi(y) + \begin{pmatrix} kr_1 \\ lr_2 \end{pmatrix}$, $k, l \in \mathbb{N}$ in local coordinates.
If $L(\gamma)$ denotes the length of a path γ, we set $L = \sup_{\gamma \in \Gamma^T} L(\gamma)$. For an upper bound on L consider the midpoint x_0 of a rectangle $[2\pi k r_1, 2\pi(k+1)r_1) \times [2\pi l r_2, 2\pi(l+1)r_2)$ and one of the four corner points. Owing to the assumption $r_1 \geq r_2$ we obtain

$$L = \sqrt{(\pi r_1)^2 + (\pi r_2)^2} \leq \sqrt{2}\pi r_1.$$

Now, we are able to state our estimates on the spectral gap λ_1.
Firstly, we calculate $|||\cdot|||_p$ for a path γ^1_{xy} connecting $x, y \in T$ and find

$$\begin{aligned} |||\gamma^1_{xy}|||_p &= \sum_{n=1}^{N} Q(\gamma^1_{xy}(n-1), \gamma^1_{xy}(n))^{-p} = N(16\epsilon^2\pi^2 r_1 r_2)^p \\ |||\gamma^2_{xy}|||_p &= N(x,y)(16\epsilon^2\pi^2 r_1 r_2)^p. \end{aligned}$$

For the constant $\kappa^{\Gamma^{T,1}}$ we obtain

$$\kappa^{\Gamma^{T,1}} = N^2 \sum_{m=1}^{N} \frac{N}{16\pi^2 r_1^2 r_2^2} 16\epsilon^2\pi^2 r_1 r_2 = N^4 \frac{\epsilon^2}{\pi^2 r_1 r_2}.$$

Here we used the fact that all paths have length N and in each path there are edges of index 1 up to N.
For the constant $B^{\Gamma^{T,1}}$, appearing in the estimate of I_K,we get

$$B^{\Gamma^{T,1}} = N^3 \frac{\epsilon^2}{\pi^2 r_1 r_2}.$$

Choose N such that $\frac{\pi\sqrt{r_1^2+r_2^2}}{\epsilon} \leq N$. Now, taking $N_0 \geq \left\lceil \frac{\sqrt{2}\pi r_1}{\epsilon} \right\rceil$ would suffice to ensure admissibility. This leads to

$$\lambda_1 \geq \frac{\pi^2 r_1 r_2}{\epsilon^2 N_0^4} \approx \frac{\epsilon^2 r_2}{4\pi^2 r_1^3} \text{ and } \lambda_1 \geq \frac{\pi^2 r_1 r_2}{\epsilon^2 N_0^3} \approx \frac{\epsilon^2 r_2^2}{64\pi^2 r_1^4}, \text{ respectively.}$$

To compute the spectrum of L explicitely we take an orthonormal system of functions on T: $\frac{1}{2\pi}$, $\left(\frac{\cos(xk/r_1)\cos(yl/r_2)}{2\pi}\right)_{k,l\in\mathbb{N}}$, $\left(\frac{\cos(xk/r_1)\sin(yl/r_2)}{2\pi}\right)_{k\in\mathbb{N},l\in\mathbb{N}^+}$, $\left(\frac{\sin(xk/r_1)\cos(yl/r_2)}{2\pi}\right)_{k\in\mathbb{N}^+,l\in\mathbb{N}}$, $\left(\frac{\sin(xk/r_1)\sin(yl/r_1)}{2\pi}\right)_{k,l\in\mathbb{N}^+}$. Proceeding as in the previous example, we get the following spectrum: 1, $\left(1-\frac{r_1\sin(\epsilon k/r_1)}{k\epsilon}\right)_{k\in\mathbb{N}^+}$, $\left(\frac{r_2\sin(\epsilon k/r_2)}{k\epsilon}\right)_{k\in\mathbb{N}^+}$, $\left(1-r_1\frac{\sin(\epsilon k/r_1)r_2\sin(\epsilon l/r_2)}{kl\epsilon^2}\right)_{k,l\in\mathbb{N}^+}$.
Hence, $\lambda_1 \approx \frac{\epsilon^2}{6r_1^2}$. Applying our methods the lower bounds are off by a factor $\approx 1/12 \cdot r_2/r_1$. The factor r_2/r_1 is the consequence of the estimate $\sqrt{r_1^2+r_2^2} \leq \sqrt{2}r_1$. For r_2 being close to r_1 the estimate is of the right order. One can think of tori with a 'small hole'. Whereas if r_2 is much smaller than r_1 the bound becomes worse. This is the case for tori having a very 'big hole'.

6.4 Thetasums

We consider the circle $C(0,1)$ again but now we take the transition kernel

$$K(x,dy) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{|2\pi k+|x-y||^2}{2}\right) dy.$$

Let us proceed by using the path set Γ^1 and computing the weights $|||\cdot|||_p$ for a path $\gamma_{xy}^1 \in \Gamma^1$.

$$\begin{aligned}
|||\gamma_{xy}^1|||_p &= \sum_{n=1}^{N} Q(\gamma_{xy}^1(n-1),\gamma_{xy}^1(n))^{-p} = \sum_{n=1}^{N} \frac{(2\pi)^{3p/2}}{\left(\sum_{k\in\mathbb{Z}}\exp\left(-\frac{|2\pi k+|x-y||^2}{2}\right)\right)^p} \\
&\leq N\frac{(2\pi)^{3p/2}}{\left(\sum_{k\in\mathbb{Z}}\exp\left(-\frac{(2\pi(k+1/N))^2}{2}\right)\right)^p}.
\end{aligned}$$

We take $p=1$. The constant $\kappa^1(1)$ can be estimated as follows:

$$\kappa^1(1) \leq \frac{N^3}{\sum_{k\in\mathbb{Z}}\exp\left(-\frac{(k+1/N)^2 4\pi^2}{2}\right)\sqrt{2\pi}}.$$

The estimate for $\kappa^1(p)$ leads to

$$\lambda_1 \geq \frac{\sum_{k\in\mathbb{Z}}\exp\left(-\frac{(2\pi(k+1/N))^2}{2}\right)\sqrt{2\pi}}{N^3}.$$

If we take the summand $\exp(-(2\pi(k+1/N))^2/2)\sqrt{2\pi}/N^3$ it is maximized for $N = 2\pi/3(\pi k + \sqrt{3})$. Since the summands are monotone decreasing we maximize with respect to the summand corresponding to $k = 0$. This yields $N = 4$ and we find

$$\lambda_1 \geq \frac{\sum_{k\in\mathbb{Z}} \exp\left(-\frac{(2\pi(k+1/4)^2}{2}\right)\sqrt{2\pi}}{4^3} \approx 0.023.$$

As in the two preceeding examples we can calculate the spectrum explicitely. We consider the orthonormal system $(\frac{\exp(i2\pi lx)}{\sqrt{2\pi}})_{l\in\mathbb{Z}}$.

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\sum_{k\in\mathbb{Z}}\exp\left(-\frac{|2\pi k+|x-y||^2}{2}\right)\exp(i2\pi ly)dy \\
&= \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\int_0^{2\pi}\exp\left(-\frac{|2\pi k+|x-y||^2}{2}\right)\exp(i2\pi ly)dy \\
&= \frac{\exp(i2\pi lx)}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\int_{2\pi k-x}^{2\pi(k+1)-x}\exp\left(-\frac{z^2}{2}\right)\exp(i2\pi(-l)z)dz \\
&= \frac{\exp(i2\pi lx)}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left(-\frac{z^2}{2}\right)\exp(i2\pi(-l)z)dz \\
&= \exp\left(-\frac{l^2}{2}\right)\frac{\exp(i2\pi lx)}{\sqrt{2\pi}}.
\end{aligned}$$

Therefore, the spectrum are the numbers $\left(1-\exp\left(-\frac{l^2}{2}\right)\right)_{l\in\mathbb{Z}}$. All eigenvalues, except the case $l = 0$, have multiplicity 2. Especially, we find

$$\lambda_1 = 1 - \exp(-1/2) \approx 0.394.$$

Thus, our bound is far from being of the right order of magnitude!
The latter can be generalized to circles of radius r and tori. For instance we find for the spectrum in the case of a circle with radius r:

$$\left(1-\exp\left(-\frac{l^2}{2r^2}\right)\right)_{l\in\mathbb{Z}}.$$

7 Log-Sobolev - Isoperimetry

As mentioned in the introduction there is a close relation between the analytic inequalities and isoperimetric notions. The well known Cheeger inequality for the spectral gap determines an interval I, namely $I = [I_K^2/4, I_K]$, such that $\lambda_1 \in I$. in the sequel we want to deduce an analogues inequality which we call *Logarithmic-Cheeger inequality.*
As a consequence of the proof we find lower bounds on the logarithmic-Sobolev constant in terms of $I_K^{\ln}$ and λ_1 which we call mixed isoperimetric

bounds. They are summarized in Remark 7.3.
To proceed we observe that Theorem 2.3.3 of [R 85] also holds true in our setting and we have for the discrete gradient $[\nabla_K f](x,y) = f(x) - f(y)$,

Theorem 7.1 *Let $c \geq 0$. Then the following are equivalent*
(i) $c\mathcal{L}_\pi[f] \leq \int_M |\nabla_K f|\, d\pi(x)$, for all $f \in \mathcal{B}(S)$
(ii) $c\max\{\mathcal{L}_\pi[1_A], \mathcal{L}_\pi[-1_A]\} \leq Q(\partial A)$ for $A \subset S$, $\pi(A) \leq \frac{1}{2}$.

The proof is essentially the same as in [R 85], Theorem 2.3.3 and therefore omitted.
Let us shortly motivate how one is led to study I_K^{ln}.
Setting $0\ln(0) := 0$, for finding the best possible constant $c > 0$ in Theorem 7.1, (i), for subsets $A \subset S$ we only need to consider the indicator functions 1_A. We have

$$-c\pi(A)\ln(\pi(A)) = c\mathcal{L}_\pi[1_A] \leq \mathbb{E}_\pi[\int_S |1_A(y) - 1_A(x)| K(x,dy)] = Q(\partial A).$$

Thus, we are led to consider the isoperimetric constant

$$I_K^{ln} = \inf_{A \subset S} -\frac{Q(\partial A)}{\pi(A)\ln(\pi(A))}. \tag{42}$$

Theorem 7.2 *Let $(X_t)_t$ be a reversible Sub-Markov chain in discrete time or a time continuous jump Sub-Markov process on a measure space $(S, \mathcal{S}, \mu)$. Assume the transition kernel of the chain is given by (1).*
For the logarithmic-Sobolev constant we have

$$\frac{\ln(2)}{4\sqrt{2}}(I_K^{\ln})^2 \leq \alpha \leq I_K^{\ln}. \tag{43}$$

Proof:
Take a function $f \in L^2(K,\mu)$, s.t. $\mathcal{L}_\pi[f] < \infty$, applying the triangle inequality we have

$$\begin{aligned}
&I_K^{\ln}\mathcal{L}_\pi[f] \\
&\leq \mathbb{E}_\pi[\int_S |f^2(y) - f^2(x)| K(x,dy)] \\
&\leq \mathbb{E}_\pi[\int_S |f(y) - f(x)|^2 K(x,dy)] + 2\mathbb{E}_\pi[|f(x)| \int_S |f(y) - f(x)| K(x,dy)].
\end{aligned}$$

Applying the Cauchy-Schwarz inequality as well as Jensen's inequality yields

$$\begin{aligned}
&I_K^{\ln}\mathcal{L}_\pi[(f - \mathbb{E}_\pi[f])] \\
&\leq \mathbb{E}_\pi[\int_S |f(y) - \mathbb{E}_\pi[f] - f(x) + \mathbb{E}_\pi[f]|^2 K(x,dy)] \\
&\quad + 2\mathbb{E}_\pi[|f - \mathbb{E}_\pi[f]| \int_S |f(y) - \mathbb{E}_\pi[f] - f(x) + \mathbb{E}_\pi[f]| K(x,dy)]
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathcal{E}(f,f) + 2\sqrt{Var[f]}\sqrt{\mathbb{E}_\pi[\int_S |f(y)-f(x)|K(x,dy)]^2} \\
&\leq \mathcal{E}(f,f) + \frac{2}{\sqrt{\lambda_1}}\mathbb{E}_\pi[\int_S |f(y)-f(x)|^2 K(x,dy)] \\
&= \left(1+\frac{2}{\sqrt{\lambda_1}}\right)\mathcal{E}(f,f)
\end{aligned}
$$

Using an inequality due to Rothaus, [R 85], Lemma 9, i.e.

$$\mathcal{L}_\pi[f] \leq \mathcal{L}_\pi[f-\mathbb{E}_\pi[f]] + 2Var[f],$$

and because the chain is reversible we have

$$\mathbb{E}_\pi[\int_S |f^2(y)-f^2(x)|K(x,dy)] \leq 2\mathbb{E}_\pi[f(x)\int_S |f(y)-f(x)|K(x,dy)].$$

Hence, we obtain

$$2\alpha \geq \frac{\lambda_1 I_K^{\ln}}{\sqrt{\lambda_1}+I_K^{\ln}}. \tag{44}$$

Using $x\ln(2) \leq -x\ln(x)$, $x \in (0,1/2)$, it follows that $\lambda_1 \geq \frac{\ln(2)}{8}(I_K^{\ln})^2$. Togehter with the previous considerations this leads to

$$\alpha \geq \frac{\ln(2)}{4\sqrt{2}}(I_K^{\ln})^2.$$

The upper bound follows immediately by the definiton of $I_K^{\ln}$ and using Theorem 2.8.1. This proves the Theorem. □

For general Markov chains we define the spectral gap by

$$\lambda_1 := \min{}^+ \{|\lambda| : \lambda \text{ eigenvalue of } L\},$$

where $\min^+ = \min(A\backslash\{0\})$ denotes the minimum only over the positive elements for the set A. This conicides with our definition for λ_1 for reversible chains.

Remark 7.3 *From the latter proof we find the mixed isoperimetric bound*

$$\alpha \geq \begin{cases} \frac{\lambda_1 I_K^{\ln}}{2\left(\sqrt{\lambda_1}+I_K^{\ln}\right)+\lambda_1} & : \ K \textit{ is non-reversible}, \\ \frac{\lambda_1 I_K^{\ln}}{2(\sqrt{\lambda_1}+I_K^{\ln})} & : \ K \textit{ is reversible}. \end{cases} \tag{45}$$

Proof:

It remains to show the bound for the non reversible case. As in the proof of the preceding theorem we use Lemma 9 of [R 85] and obtain

$$\alpha \geq \frac{\lambda_1 I_K^{\ln}}{2\left(\sqrt{\lambda_1}+I_K^{\ln}\right)+\lambda_1}. \tag{46}$$

This complements the proof of the statements in Remark 7.3. □

Remark 7.4 *Since the right hand side is a nondecreasing function in λ_1, every lower bound on λ_1 leads to a lower bound on α. Therefore, we can use our path-based lower bounds in the reversible case to estimate the logarithmic-Sobolev constant α.*

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References

[DS 91] *Diaconis, P., Stroock, D.*, (1991), Geometric Bounds for Eigenvalues of Markov Chains. Ann. Appl. Prob., 1, 36-61.

[DS-C 93] *Diaconis P., Saloff-Coste L.*, (1993), Comparison Theorems for reversible Markov chains. Ann. Appl. Prob. 3, 696-730.

[FW 99] *Fulman J., Wilmer E.*, (1999), Comparing eigenvalue bounds for Markov Chains: When does Poincaré beat Cheeger?, Ann. Appl. Probab. 9, no.1, 1-13.

[JS 88] *Jerrum, M., Sinclair, A.*, (1988), Approximating the permanent. SIAM J. Comp., 18, 1149-1178.

[JS 89] *Jerrum, M., Sinclair, A.*, (1989), Approximate counting, uniform generation and rapidly mixing Markov chains. Inf. Comp., 82, 93-113.

[K 00] *Kienitz, J.*, (2000), PhD thesis. University of Bielefeld. In preparation.

[R 85] *Rothaus, O.*, (1985), Analytic Inequalities, Isoperimetric Inequalities and Logarithmic Sobolev Inequalities. J. Funct. Anal., 64, 296-313.

[SC 97] *Saloff-Coste, L.*, (1997), Lectures on finite Markov-chains. École d'Été de Saint Flour 1994. Lecture Notes Math., 1665. Springer, Berlin, 301-413.

[S 92] *Sinclair A.*, (1992), Improved bounds for mixing rates of Markov chains and multicommodity flow, Comb. Prob. Comp. 1, 351-370.

[LS 88] *Lawler, G. F., Sokal, A. D.*, (1988), Bounds on the L^2 spectrum for Markov chains and Markov processes: a generalisation. Trans. Amer. Math. Soc. 309, 557-580.

[Y 00] *Yuen, W. K.*, (2000), Applications of geometric bounds to the convergence rate of Markov chains on $\mathbb{R}^n$. Stoch. Proc. App. 87, 1-23.