Endomorphisms of $\mathfrak{B}_n$, $\mathcal{P}\mathfrak{B}_n$ and $\mathfrak{C}_n$

Volodymyr Mazorchuk

Kyiv Taras Shevchenko University

2000 Mathematics Subject Classification: 20M17, 20M15, 20M10

Abstract

We describe all endomorphisms of the (finite) Brauer semigroup, its partial analogue and the semigroup of all partitions of a $2n$-element set.

1 Introduction

Being one of the most classical objects in the group theory, the full finite symmetric group S_n does not have a uniquely defined analogue in the theory of semigroups. As some of the most natural candidates for such a position, one can take, for example, the *full inverse symmetric semigroup* $\mathcal{IS}_n$, the *full transformation semigroup* $\mathcal{T}_n$ or the *semigroup of all partial transformations* $\mathcal{PT}_n$. The endomorphisms of all these semigroups have been recently described by Schein and Teclezghi in [ST1, ST2, ST3], where it was pointed out that the study of endomorphisms of the classical semigroups seems to be a surprising gap in the general theory of semigroups.

On the other hand, there is a natural generalization of S_n arising in the representation theory. This one is the so-called *Brauer semigroup* $\mathfrak{B}_n$, defined as follows: the elements of $\mathfrak{B}_n$ are all possible decompositions of the set $N_n = \{1, 2, \ldots, n, 1', 2', \ldots, n'\}$ into two-element subsets. The elements of $\mathfrak{B}_n$ can be realized as certain *chips* ([K]) with legs $\{1, 2, \ldots, n\}$ on the one side and $\{1', 2', \ldots, n'\}$ on the other. Then the multiplication is defined by putting two chips together (i.e. identifying the dashed legs of the first chip with regular legs of the second one) and throwing away "dead circles", which may appear in the middle. These dead circles play an important role in the representation theory, cause they define a one-parameter deformation of the semigroup algebra (these deformations are the classical Brauer algebras). However, on the level of finite semigroups they just can be forgotten.

Being a very nice combinatorial object, it does not seem that $\mathfrak{B}_n$ has attracted a lot of attention. Some basic properties of this semigroup were studied in [M], where it was also proposed to generalize $\mathfrak{B}_n$ to the semigroup $\mathcal{P}\mathfrak{B}_n$ of the so-called *partial chips*. The objects of $\mathcal{P}\mathfrak{B}_n$ are all possible decompositions of N_n into two and one-element subsets and the multiplication is defined in the same way as in $\mathfrak{B}_n$. Parallel to this purely combinatorial paper, another generalization of $\mathfrak{B}_n$ appeared in the context of cellular algebras in [Xi]. Although the objects studied there are really algebras and not semigroups, it is easy to derive the corresponding semigroup structure generalizing $\mathfrak{B}_n$. We will call it the semigroup $\mathfrak{C}_n$ of all partitions of N_n, where by a *partition* we mean arbitrary decomposition of N_n into subsets. Again the multiplication is defined in the same way as in $\mathfrak{B}_n$. From the definition it follows immediately that we have the following embeddings: $S_n \subset \mathfrak{B}_n \subset \mathcal{P}\mathfrak{B}_n \subset \mathfrak{C}_n$. It is also clear that $\mathcal{P}\mathfrak{B}_n \supset \mathcal{IS}_n$.

In [M] it was proved that all automorphisms of both $\mathfrak{B}_n$ and $\mathcal{P}\mathfrak{B}_n$ are inner, which means that they are induced by invertible elements. In particular, this implies $\mathrm{Aut}(\mathfrak{B}_n) = \mathrm{Aut}(\mathcal{P}\mathfrak{B}_n) = S_n$. The aim of this paper is to study the endomorphisms of $\mathfrak{B}_n$ and $\mathcal{P}\mathfrak{B}_n$ as good as endo- and automorphisms of $\mathfrak{C}_n$. This seems to be a natural continuation to what is done in [ST1, ST2, ST3, M]. With each semigroup we will deal with separately, naturally dividing the rest of the paper into three sections. For the sake of completeness and also because of the poor availability of [M] we will repeat the description of $\mathrm{Aut}(\mathfrak{B}_n)$ and $\mathrm{Aut}(\mathcal{P}\mathfrak{B}_n)$ at the appropriate places.

2 Semigroup $\mathfrak{B}_n$ and its endomorphisms

In this Section we will study the endomorphisms of the semigroup $\mathfrak{B}_n$ defined above. First we will need some technical lemmas about the structure of $\mathfrak{B}_n$. Set $M_n = \{1, 2, \dots, n\}$. For any $X \subset M_n$ we also set $X' = \{x' | x \in X\}$. For $i, j \in M_n$ we denote by $\pi_{i,j}$ the element of $\mathfrak{B}_n$, which corresponds to the following decomposition of N_n: $\{i, j\} \cup \{i', j'\} \bigcup\limits_{t \notin \{i,j\}} \{t, t'\}$.

The elements $\pi_{i,j}$will be called *atoms*. Clearly, $\pi_{i,j}$ is an idempotent in $\mathfrak{B}_n$. In what follows we will write $\{k, l\} \in \pi$ for $\pi \in \mathfrak{B}_n$, $\{k, l\} \in N_n$ if $\{k, l\}$ belongs to the decomposition of N_n corresponding to π. For $i, j \in M_n$ we will also denote by (i, j) the corresponding transposition in S_n (and will also use an analogous notation for cycles). A subset, $X \subset M_n$, will be called invariant with respect to an element, $\pi \in \mathfrak{B}_n$, provided for any $\{i, j\} \subset N_n$ from $\{i, j\} \in \pi$ it follows $\{i, j\} \subset X \cup X'$ or $\{i, j\} \cap (X \cup X') = \varnothing$. It follows immediately that if X is invariant with respect to π then $M_n \setminus X$ is also invariant with respect to π. Let $\pi \in \mathfrak{B}_n$ and $X \subset M_n$ be invariant with respect to π. Let $\pi|_X$ be the element in $\mathfrak{B}_n$ defined as follows: $\{i, i'\} \in \pi|_X$ if $i \notin X$ and $\{i, j\} \in \pi|_X$ if $\{i, j\} \in \pi$ and $\{i, j\} \subset X \cup X'$. Then for any $\pi \in \mathfrak{B}_n$ and any X invariant with respect to π one has decompositions $\pi = \pi|_X \pi|_{M_n \setminus X} = \pi|_{M_n \setminus X} \pi|_X$. If X is invariant with respect to π, one can also consider π_X, as an element of the smaller semigroup $\mathfrak{B}_{|X|}$.

Lemma 2.1. *$\mathfrak{B}_n$ is generated by S_n and any of $\pi_{i,j}$.*

Proof. Denote by B the subsemigroup of $\mathfrak{B}_n$, generated by S_n and a fixed $\pi_{i,j}$. As all $\pi_{i,j}$ are conjugated under the S_n action, B does not depend on the choice of $\pi_{i,j}$. We will prove that any $\pi \in \mathfrak{B}_n$ belongs to B by downward induction in the number of those i from M_n such that $\{i, i'\} \in \pi$. If this number is n then π is the unit element, hence belongs to $S_n \subset B$. Otherwise, conjugating it with some element from S_n we can assume that $\{i, i'\} \in \pi$ exactly for all $i = 1, 2, \dots, k$, $k < n$. As we can also assume that $\pi \notin S_n$, there exist $k < x \neq y, u \neq v \leq n$ such that $\{x, y\} \in \pi$ and $\{u', v'\} \in \pi$. But then we can write $(k+1, x)(k+2, y)\pi(k+1, u)(k+2, v) = \pi_{k+1,k+2}\hat{\pi}$, where $\{i, i'\} \in \hat{\pi}$ for all $i = 1, 2, \dots, k+2$ and the statement follows by induction. $\square$

Lemma 2.2. *Any non-invertible idempotent of $\mathfrak{B}_n$ decomposes into a product of atoms. Moreover, the atoms can be described as the only non-invertible idempotents, which can not be decomposed into a non-trivial product of other non-invertible idempotents.*

Proof. Let π be an idempotent in $\mathfrak{B}_n$. As it is not invertible, there exists a minimal invariant with respect to π set $X \subset M_n$ which contains more than one element. Clearly both $\pi|_X$ and $\pi|_{M_n \setminus X}$ are idempotents. Thus, decomposing, if necessary, π into a product of commuting idempotents and going to a smaller n, we can assume that $X = M_n$. For $n = 2$ the statement is clear and for $n = 3$ it follows from the observation that, up to an S_3-conjugation, the only idempotent with minimal invariant set M_3 is $\pi_{1,2}\pi_{2,3}$, the last being given as a product of atoms. Now we want to use induction. For this we have to consider two cases, n odd or even, separately.

Let n be odd. Then we necessarily have $\{i, j'\} \in \pi$ for some i and j. From $\pi\pi = \pi$ it follow that there is a sequence of elements $j = j_0, j_1, \dots, j_k = i$ such that $\{j'_{2t}, j'_{2t+1}\} \in \pi$ and $\{j_{2t-1}, j_{2t}\} \in \pi$. In particular, $\{j_0, j_1, \dots, j_k\}$ is invariant with respect to π and hence must coincide with M_n by our assumptions. Now, up to an S_n-conjugation, we can assume $j_i = i+1$ and write $\pi = \pi_{1,2}\hat{\pi}$, where $\hat{\pi}$ is defined as follows: $\{1, 1'\} \in \hat{\pi}$, $\{2, n\} \in \hat{\pi}$ and all other subsets coincide with those from π. We get that the restriction of $\hat{\pi}$ on $\{2, 3, \dots, n\}$ is an idempotent for smaller n, which gives us the inductive step.

Finally, let n be even. Assuming that there is $\{i, j'\} \in \pi$ and using the arguments above, we will get a non-trivial subset in M_n, invariant under π, which contradicts our assumptions. Hence π contains only subsets $\{i, j\}$ or $\{i', j'\}$ for $i, j \in M_n$. Again, from the minimality of the invariant set M_n, one deduces that, up to an S_n conjugation, π contains $\{1, n\}$, $\{2t, 2t+1\}$ and $\{(2t-1)', (2t)'\}$. Decomposing $\pi = \hat{\pi}\pi_{1,2}$, where $\{1, 1'\} \in \hat{\pi}$, $\{n, 2'\} \in \hat{\pi}$ and all other are taken from π, we get the necessary inductive step by the same arguments as above.

The statement about the indecomposability of atoms is now obvious. $\square$

For $\pi \in \mathfrak{B}_n$ set $C(\pi) = \{x \in S_n | x\pi = \pi x\}$. For $X \subset M_n$ let $S(X)$ denote the full symmetric group on X, which is a natural subgroup of S_n.

Lemma 2.3. *1. $C(\pi_{i,j}) = S(\{i, j\}) \oplus S(M_n \setminus \{i, j\})$ for any $i, j \in M_n$.*

2. If $n \neq 4$ and π is an idempotent in $\mathfrak{B}_n \setminus S_n$ such that $C(\pi) \supset S(\{i, j\}) \oplus S(M_n \setminus \{i, j\})$ then $\pi = \pi_{i,j}$.

3. If π is an idempotent in $\mathfrak{B}_4 \setminus S_4$ such that $C(\pi) \supset S(\{i, j\}) \oplus S(M_4 \setminus \{i, j\})$ and $\{k, l\} = M_4 \setminus \{i, j\}$ then there are precisely three possibilities: $\pi = \pi_{i,j}$, $\pi = \pi_{k,l}$ or $\pi = \pi_{i,j}\pi_{k,l}$.

Proof. To prove the first statement we note that $S(\{i,j\}) \oplus S(M_n \setminus \{i,j\}) \subset C(\pi_{i,j})$ is obvious. For the opposite inclusion it is enough to consider a cycle, $c = (i_1, i_2, \dots, i_k) \in S_n$, such that $\{i,j\} \cap \{i_1, \dots, i_k\} \neq \varnothing$ and $(M_n \setminus \{i,j\}) \cap \{i_1, \dots, i_k\} \neq \varnothing$. Considering $\pi_{i,j}c$ and $c\pi_{i,j}$ we get $\{i,j\} \in \pi_{i,j}c$ and $\{i,j\} \notin c\pi_{i,j}$, a contradiction.

Let us prove the second statement. From $\pi\pi = \pi$ and $(i,j)\pi(i,j) = \pi$ we immediately get either $\{i,j\} \in \pi$ and $\{i',j'\} \in \pi$ or $\{i,i'\} \in \pi$ and $\{j,j'\} \in \pi$. In particular, for $n = 3$ this means $\{k,k'\} \in \pi$ for $\{k\} = M_3 \setminus \{i,j\}$ and we are done. For $n > 4$, using the conjugation with any (k,l), $\{k,l\} \subset M_n \setminus \{i,j\}$ we see that it is enough to show that $\{k,k'\} \in \pi$ for some $k \in M_n \setminus \{i,j\}$. First we observe that $\{k,l\} \subset M_n$ and $\{k,l\} \in \pi$ implies $\{k,l\} = \{i,j\}$. Indeed, the arguments above imply that for $\{k,l\} \neq \{i,j\}$ we necessarily have $\{k,l\} \subset M_n \setminus \{i,j\}$. Picking any $t \in M_n \setminus (\{i,j,k,l\})$ we get $(k,t)\pi(k,t) \neq \pi$, a contradiction. In particular, this means that $\pi|_{M_n \setminus \{i,j\}}$ is an idempotent from $S(M_n \setminus \{i,j\})$, thus is the identity. We get $\pi = \pi_{i,j}$ as it is not invertible by our assumptions.

The last statement now easily follows from the proof of the second one. □

Now we are ready to describe all automorphisms of $\mathfrak{B}_n$ ([M, Theorem 5]).

Theorem 2.1. *All automorphisms of* $\mathfrak{B}_n$ *are inner, i.e. have the form* $x \mapsto g^{-1}xg$ *for some* $g \in S_n$. *In particular* $\mathrm{Aut}(\mathfrak{B}_n) \simeq S_n$.

Proof. Let φ be an automorphisms of $\mathfrak{B}_n$. Then $\psi = \varphi|_{S_n}$ is an automorphism of S_n. As it is well-known (see, for example [KM, R]), ψ is an inner automorphism unless $n = 6$. In the last case it is possible that ψ is not inner and, if so, it maps any transposition of S_n into a product of three commuting transpositions ([R, Corollary 7.6]).

We remark that, from the description of atoms, given in Lemma 2.2, we get that φ sends any atom to an (possibly other) atom.

As a first step we will show that ψ will be always an inner automorphism, even in the case $n = 6$. Indeed, suppose not. Then $\varphi(\pi_{1,2}) = \pi_{i,j}$ for some i, j, hence $\varphi((1,2)\pi_{1,2}) = \varphi(\pi_{1,2}) = \pi_{i,j}$. From the other hand, $\varphi((1,2)\pi_{1,2}) = \varphi((1,2))\pi_{i,j} \neq \pi_{i,j}$ as $\varphi((1,2))$ is a product of three pairwise commuting transpositions. The obtained contradiction completes the first step. So, the automorphism ψ is always inner, say, coincide with $\psi_g : x \mapsto g^{-1}xg$ for some $g \in S_n$. We will denote the corresponding inner automorphism of $\mathfrak{B}_n$ by the same symbol.

For the second step we remark that, composing, if necessary, φ with $\psi_{g^{-1}}$, we can assume that ψ is the identity automorphism. To complete the proof we need only to show that under such an assumption φ is also the identity automorphism.

Finally, let us show that $\psi = \mathrm{id}$ implies $\varphi = \mathrm{id}$. It follows from Lemma 2.1 that it is enough to show that $\varphi(\pi_{1,2}) = \pi_{1,2}$. As we have seen before, $\varphi(\pi_{1,2}) = \pi_{i,j}$. From $(1,2)\pi_{1,2} = \pi_{1,2}$ we also get $(1,2)\pi_{i,j} = \pi_{i,j}$, which implies $\{i,j\} = \{1,2\}$. This completes the proof. □

Now we can move to the study of endomorphisms of $\mathfrak{B}_n$. We will do it in two steps. As the first one, we will describe those endomorphisms of $\mathfrak{B}_n$, which stabilize S_n pointwise.

Lemma 2.4. *Let $n \neq 4$ and φ be an endomorphism of $\mathfrak{B}_n$ such that $\varphi(x) = x$ for any $x \in S_n$. Then $\varphi = \mathrm{id}$.*

Proof. It follows from Lemma 2.1, that φ is completely determined by the value $y = \varphi(\pi_{1,2})$. Clearly, y is an idempotent. As φ is an endomorphism and $\varphi(x) = x$ for any $x \in S_n$, we get $C(y) \supset C(\pi_{1,2}) = S(\{1,2\}) \oplus S(M_n \setminus \{1,2\})$. Using Lemma 2.3, we get $y = \pi_{1,2}$ as required. $\square$

In the case $n = 4$ the statement of Lemma 2.4 is no longer true (compare with the case $n = 4$ in [ST1, ST2]). Define the map $\mathfrak{p} : \mathfrak{B}_4 \to \mathfrak{B}_4$ as follows: $\mathfrak{p}(x) = x$, $x \in S_4$; for $x \in \mathfrak{B}_4 \setminus S_4$, such that $\{i,j\} \in x$, $\{k',l'\} \in x$ the element $\pi(x)$ consists of $\{i,j\}$, $M_4 \setminus \{i,j\}$, $\{k',l'\}$ and $M_4' \setminus \{k',l'\}$. One can realize this as "cutting" the through-going legs in all non-invertible elements from $\mathfrak{B}_n$ (this is clearly a well-defined operation only for $n = 4$). This realization also makes it obvious that $\mathfrak{p}$ is an endomorphisms of $\mathfrak{B}_4$.

Lemma 2.5. *Let φ be an endomorphism of $\mathfrak{B}_4$ such that $\varphi(x) = x$ for any $x \in S_n$. Then $\varphi = \mathrm{id}$ or $\varphi = \mathfrak{p}$.*

Proof. As in the proof of Lemma 2.4, φ is uniquely determined by $y = \varphi(x)$. From Lemma 2.3 we get three possibilities for y: $\pi_{1,2}$, $\pi_{3,4}$ and $\pi_{1,2}\pi_{3,4}$. The case $y = \pi_{3,4}$ is not possible because of the same contradiction as in Theorem 2.1: $\pi_{3,4} = \varphi(\pi_{1,2}) = \varphi((1,2)\pi_{1,2}) = (1,2)\pi_{3,4} \neq \pi_{3,4}$. The are only two cases left, so both of them really give us endomorphisms cause we know already two different endomorphisms of $\mathfrak{B}_4$, stabilizing S_n pointwise. These are id and $\mathfrak{p}$. This completes the proof. $\square$

Now we are ready to study all the endomorphisms of $\mathfrak{B}_n$. We will start with the following fact:

Lemma 2.6. *All maximal subgroups in $\mathfrak{B}_n$ are isomorphic to S_m, with $m \leq n$. Moreover, if $m = n$ then this maximal subgroup is the group of all invertible elements in $\mathfrak{B}_n$.*

Proof. Let π be an idempotent in $\mathfrak{B}_n$ and $M_n = \cup_i X_i$ be a decomposition of M_n into a disjoint union of minimal subsets, invariant with respect to π. Then $\pi = \prod_i \pi_i$, $\pi_i = \pi|_{X_i}$, all π_i are idempotents and $\pi_i\pi_j = \pi_j\pi_i$ for all i,j. If x now is an element from the maximal subgroup of $\mathfrak{B}_n$ with the identity element π, then from $\pi x = x\pi = x$ we get $\{i,j\} \in x$ if and only if $\{i,j\} \in \pi$ and $\{i',j'\} \in x$ if and only if $\{i',j'\} \in \pi$. This implies that the corresponding maximal subgroup is isomorphic to S_m, where m is the number of those X_i such that $|X_i|$ is odd. The second statement is now trivial. $\square$

Now we can observe the following: let φ be an endomorphism of $\mathfrak{B}_n$. Then $\psi = \varphi|_{S_n}$ is a homomorphism from S_n to a maximal subgroup of $\mathfrak{B}_n$. There are three possibilities: $\mathrm{Ker}(\psi)$ is trivial, $\mathrm{Ker}(\psi) = A_n$ or $\mathrm{Ker}(\psi) = S_n$. In the first case the image of S_n under π contains $n!$ elements and from Lemma 2.6 we get $\psi(S_n) = S_n$, in other words, ψ is an automorphisms of S_n. Again, we will have to deal with the case $n = 6$ separately.

Lemma 2.7. *Any non-inner automorphism of S_6 can not be continued to an endomorphism of $\mathfrak{B}_6$.*

Proof. Let φ be an endomorphism of $\mathfrak{B}_6$ such that $\psi = \varphi|_{S_6}$ is a non-inner automorphism of S_6. Set $y = \varphi(\pi_{1,2})$. Then from $(1,2)\pi_{1,2} = \pi_{1,2} = \pi_{1,2}(1,2)$ we get that for some $\{i,j,k,l,u,v\} = M_6$ holds $(i,j)(k,l)(u,v)y = y = y(i,j)(k,l)(u,v)$. From this it follows easily that $y = \pi_{i,j}\pi_{k,l}\pi_{u,v}$. In particular, this implies that the image of any element $x \in \mathfrak{B}_6 \setminus S_6$ is an idempotent, which does not contain sets of the form $\{i,j'\}$ for any $i,j \in M_n$. Further, as the idempotents $\pi_{1,2}$ and $\pi_{s,t}$, $\{1,2\} \cap \{s,t\} = \varnothing$ commute, we get that their images commute also and hence $\varphi(\pi_{s,t}) = y$. From this we derive $\pi_{s,t} = y$ for any $\{s,t\}$. Now we will get a contradiction if we choose s,t such that $\psi((s,t)) = (i,k)(l,u)(v,j)$ (such s,t clearly exist by the property of ψ to interchange all transpositions with the all possible products of three commuting transpositions). □

From Lemma 2.7 it now follows that in the case of trivial $\mathrm{Ker}(\psi)$, ψ is an inner automorphism of S_n. Hence, applying the corresponding inner automorphism ψ^{-1} of $\mathfrak{B}_n$ we can assume $\psi = \mathrm{id}$ and the description of all possible φ will now follow from Lemmas 2.4 and 2.5.

Assume that $\mathrm{Ker}(\psi) = S_n$, which means that $\varphi(x) = y = y^2$ for any $x \in S_n$. As usual, ψ is completely determined by $\varphi(\pi_{1,2}) = z = z^2$, which should satisfy $zy = yz = z$ (from $e\pi_{1,2} = \pi_{1,2}e = \pi_{1,2}$). In this case $\varphi(x) = z$ for any $x \in \mathfrak{B}_n \setminus S_n$. From the other hand, Each pair of idempotents $y,z \in \mathfrak{B}_n$, satisfying $zy = yz = z$ determines an endomorphism of $\mathfrak{B}_n$ if we put $\varphi(x) = y$, $x \in S_n$ and $\varphi(x) = z$, $z \in \mathfrak{B}_n \setminus S_n$.

The only case that left is $\mathrm{Ker}(\psi) = A_n$. In this case the image of S_n contains two elements: an idempotent, $y = \varphi(e)$, and an element, z, which belongs to the maximal subgroup of $\mathfrak{B}_n$, corresponding to y and has the order two in this group (i.e. $yz = zy = z$, $z^2 = y$). To determine φ completely, we now have to study $\varphi(\pi_{1,2}) = u = u^2$. We again have $uy = yu = u$ from $e\pi_{1,2} = \pi_{1,2}e = \pi_{1,2}$ and $zu = uz = u$ from $(1,2)\pi_{1,2} = \pi_{1,2}(1,2) = \pi_{1,2}$. Conversely, any triple u,y,z of elements from $\mathfrak{B}_n$, such that $u^2 = u = uy = yu$, $y^2 = y$, $zu = uz = z$ and z is an element of order 2 in the maximal subgroup corresponding to y, determines an endomorphism of $\mathfrak{B}_n$. Altogether, we can combine all the results of this Section into the following two statements.

Theorem 2.2. *1. Choose $g \in S_n$ and define $\varphi_g(t) = g^{-1}tg$, $t \in \mathfrak{B}_n$.*

2. *Choose $x = x^2$ in $\mathfrak{B}_n$ and define $\psi_x(t) = x$, $t \in \mathfrak{B}_n$.*

3. *Choose $y = y^2$ and $z^2 = z = zy = yz$ in $\mathfrak{B}_n$ and define $\psi_{y,z}(t) = y$, $t \in S_n$ and $\psi_{y,z}(t) = z$, $t \in \mathfrak{B}_n \setminus S_n$.*

4. *Choose $y^2 = y = z^2$, $zy = yz = z$, $u^2 = u = yu = uy = uz = zu$ and define $\psi_{y,z,u}(t) = y$, $t \in A_n$; $\psi_{y,z,u}(t) = z$, $t \in S_n \setminus A_n$ and $\psi_{y,z,u}(t) = u$, $t \in \mathfrak{B}_n \setminus S_n$.*

The maps φ_g, ψ_x, $\psi_{y,z}$ and $\psi_{y,z,u}$ are endomorphisms of $\mathfrak{B}_n$, moreover, if $n \neq 4$ then any endomorphism of $\mathfrak{B}_n$ coincides with one of the listed above.

Theorem 2.3. *Additionally to the endomorphisms listed in Theorem 2.2, the semigroup $\mathfrak{B}_4$ has 24 endomorphisms of the form $\mathfrak{p} \circ \varphi_g$ (see Lemma 2.5).*

Remark 1. *Using the notation of Theorem 2.2 and Theorem 2.3 one easily computes the composition of any two endomorphisms thus determining completely the semigroup* $\mathrm{End}(\mathfrak{B}_n)$.

Remark 2. *We will not give any formula for the number of endomorphisms of type* ψ_x, $\psi_{y,z}$ *or* $\psi_{y,z,u}$ *cause even in the first case the known result is so complicated that hardly can be viewed as a satisfactory answer.*

3 Semigroup $\mathcal{P}\mathfrak{B}_n$ and its endomorphisms

In this Section we are going to describe all endomorphisms of the semigroup $\mathcal{P}\mathfrak{B}_n$.

Remark 3. *We want to emphasize one feature, which separates* $\mathcal{P}\mathfrak{B}_n$ *from* $\mathfrak{B}_n$ *and* $\mathfrak{C}_n$. *This is the corresponding deformation of the semigroup algebra. First we recall the construction of the Brauer algebras. Let* $\mathfrak{k}$ *be a field and* $v \in \mathfrak{k}$. *Consider a vectorspace with the basis* $\mathfrak{B}_n$ *and define on it an associative multiplication,* $*$, *setting on the basis elements* $(\pi) * (\tau) = v^{l(\pi,\tau)}(\pi\tau)$, *where* $l(\pi,\tau)$ *is the number of dead circles appeared during the multiplication procedure. The same definition works also for both* $\mathcal{P}\mathfrak{B}_n$ *and* $\mathfrak{C}_n$, *which allows one to consider the corresponding analogue of Brauer algebras (which are called* partition algebras *in the case of* $\mathfrak{C}_n$, *see [Xi]). It happens that in the* $\mathcal{P}\mathfrak{B}_n$ *case one can define a two-parameter analogue of the Brauer algebra as follows: fixing a field* $\mathfrak{k}$ *and two elements* $v, u \in \mathfrak{k}$ *we consider the same vectorspace as above and define the multiplication by* $(\pi) * (\tau) = v^{l(\pi,\tau)}u^{m(\pi,\tau)}(\pi\tau)$, *where* $l(\pi,\tau)$ *is the number of dead circles which do not contain any 1-element subset of the decomposition (of* π *and* τ*) and* $m(\pi,\tau)$ *is the number of dead circles left. The usual deformation then can be obtained taking* $v = u$. *So far it is not known if it is possible to generalize this construction on* $\mathfrak{C}_n$.

We keep the notation from Section 4, in particular, we fix a natural embedding of $\mathfrak{B}_n$ into $\mathcal{P}\mathfrak{B}_n$. For $i \in M_n$ denote by τ_i the element of $\mathcal{P}\mathfrak{B}_n$, which corresponds to the following decomposition of N_n: $\{i\} \cup \{i'\} \bigcup_{t\neq i}\{t,t'\}$. The elements τ_i will be called *partial atoms*. Clearly, τ_i are idempotents in $\mathcal{P}\mathfrak{B}_n$, moreover, S_n together with all τ_i generate in $\mathcal{P}\mathfrak{B}_n$ a standard copy of $\mathcal{IS}_n$. By the *rank*, $\mathrm{Rank}(\pi)$, of the element $\pi \in \mathcal{P}\mathfrak{B}_n$ we will mean the number of subsets of the form $\{i, j'\}$ contained in π. For example, all partial atoms are elements of rank $n-1$ and all atoms are elements of rank $n-2$.

We start again with description of the automorphisms of $\mathcal{P}\mathfrak{B}_n$ ([M, Theorem 6]), however, in order not to repeat the arguments used in Section 2, here we present a different approach.

Lemma 3.1. *Let* π *be an atom and* τ *be a partial atom. Then* $\mathcal{P}\mathfrak{B}_n$ *is generated by* S_n, π *and* τ.

Proof. The proof is analogous to that of Lemma 2.1. We use the downward induction in rank of π and the remark that it is enough to prove that $\mathcal{P}\mathfrak{B}_n$ is generated by the standard copies of $\mathcal{IS}_n$ and $\mathfrak{B}_n$ sitting inside it. If the rank of π is n, then $\pi \in S_n$. Otherwise, multiplying with permutations on both sides, we can reach an element, say σ, containing one of the following subdecompositions: $\{1,2\} \cup \{1',2'\}$ or $\{1,2\} \cup \{1'\} \cup \{2'\}$ or $\{1\} \cup \{2\} \cup \{1',2'\}$ or $\{1\} \cup \{2\} \cup \{1'\} \cup \{2'\}$. In all cases $\sigma|_{\{1,2\}}$ easily decomposes into a product of atoms and partial atoms giving us the induction step. $\square$

Lemma 3.2. *For any* $\pi, \tau \in \mathcal{P}\mathfrak{B}_n$ *the inequality* $\mathrm{Rank}(\pi\tau) \leq \min\{\mathrm{Rank}(\pi), \mathrm{Rank}(\tau)\}$ *holds.*

Proof. Obvious. $\square$

Denote by $\leq$ the natural partial pre-order of divisability on $\mathcal{P}\mathfrak{B}_n$: $\pi \leq \tau$ if and only if there exist $\sigma_1, \sigma_2 \in \mathcal{P}\mathfrak{B}_n$ such that $\pi = \sigma_1\tau\sigma_2$.

Lemma 3.3. *Let* $i < j \in M_n$ *and* $\pi, \tau \in \mathcal{P}\mathfrak{B}_n$ *are such that* $\mathrm{Rank}(\pi) = i$ *and* $\mathrm{Rank}(\tau) = j$. *Then* $\pi < \tau$ *with respect to the pre-order above.*

Proof. That $\tau \not\leq \pi$ follows from Lemma 3.2. To prove that $\pi \leq \tau$ we remark that any element from S_n is invertible, so we can change both π and τ multiplying them with permutations on both sides. Thus we reduce the question to the case, when π and τ are idempotents such that $\{i, j'\} \in \pi$ implies $i = j$ and $\{i, i'\} \in \tau$ (the last is possible because of the inequality $\mathrm{Rank}(\tau) > \mathrm{Rank}(\pi)$). But in the last case $\pi = \pi\tau\pi$ and the statement follows. $\square$

Corollary 3.1. *Any automorphisms of* $\mathcal{P}\mathfrak{B}_n$ *preserves the rank of any element.*

Proof. Using the finiteness of $\mathcal{P}\mathfrak{B}_n$ this follows by the trivial induction in $\mathrm{Rank}(\pi)$. $\square$

Theorem 3.1. *Any automorphism of* $\mathcal{P}\mathfrak{B}_n$ *is inner.*

Proof. Let φ be an automorphism of $\mathcal{P}\mathfrak{B}_n$. Then $\varphi(S_n) = S_n$. Moreover, as the only idempotents of rank $n-1$ are partial atoms, it follows from Corollary 3.1, that φ maps a partial atom to a (possibly different) partial atom. This means, in particular, that $\varphi(\mathcal{IS}_n) \subset \mathcal{IS}_n$, hence φ induces an automorphism of $\mathcal{IS}_n$, which are well-known to be inner. Hence, $\psi = \varphi|_{\mathcal{IS}_n}$ is inner. Using the same argument as in the proof of Theorem 2.1, we can assume ψ to be the identity. Now any atom, in particular $\pi_{1,2}$, should be sent to an idempotent, say x, of rank 2. If x is an element of $\mathfrak{B}_n$ then φ preserves $\mathfrak{B}_n$ and the restriction of φ on $\mathfrak{B}_n$ must be an automorphism. This forces x to be an atom by Theorem 2.2 and moreover, to coincide with its preimage. If $x \notin \mathfrak{B}_n$ then x is an idempotent of rank 2 and hence is either a product of two partial atoms or a product of an atom with a non-commuting partial atom. But for any $i \in M_n$ we have $\pi_{1,2}\tau_i \neq \pi_{1,2} \neq \tau_i\pi_{1,2}$, which implies $x\tau_i \neq x \neq \tau_i x$. However, the last is not true if x contains $\{i\}$ or $\{i'\}$. Hence, x must be an atom, which completes our proof. $\square$

As it was done for $\mathfrak{B}_n$ we first study those endomorphisms of $\mathcal{P}\mathfrak{B}_n$, which stabilize S_n pointwise. Let φ be an endomorphism of $\mathcal{P}\mathfrak{B}_n$ such that $\varphi(x) = x$ for any $x \in S_n$. By

Lemma 3.1, φ is determined by a pair, (u, v), of idempotents in $\mathcal{PB}_n$, where $u = \varphi(\tau_1)$ and $v = \varphi(\pi_{1,2})$. We note that $C(\tau_1) = S(\{2, 3, \dots, n\})$ and $C(\pi_{1,2}) = S(\{1, 2\}) \oplus S(M_n \setminus \{1, 2\})$. From this it follows that $C(u) \supset S(\{2, 3, \dots, n\})$ and $C(v) \supset S(\{1, 2\}) \oplus S(M_n \setminus \{1, 2\})$. We start with determining all possible candidates for u and v. For $X \subset M_n$ we set

$$0(X) = \prod_{i \in X} \tau_i,$$

and note that $0(M_n)$ will not be a zero element in $\mathcal{PB}_n$ ($\mathcal{PB}_n$ does not have any zero element at all).

Lemma 3.4. *1. Let $u^2 = u \in \mathcal{PB}_n$ such that $C(u) \supset S(\{2, 3, \dots, n\})$. Then one of the following holds:*

- *$u = \tau_1$ or $u = e$ or $u = 0(M_n)$ or $u = 0(M_n \setminus \{1\})$.*
- *$n = 2$ and u is arbitrary idempotent.*
- *$n = 3$ and $u = \pi_{3,4}$ or $u = \tau_1\pi_{3,4}$.*

2. Let $v^2 = v \in \mathcal{PB}_n$ such that $C(v) \supset S(\{1, 2\}) \oplus S(M_n \setminus \{1, 2\})$ then one of the following holds:

- *$v = e$ or $v = \pi_{1,2}$ or $v = \tau_1\tau_2$ or $v = \pi_{1,2}0(M_n \setminus \{1, 2\})$ or $v = 0(M_n)$ or $v = \pi_{1,2}\tau_1\tau_2$ or $v = \tau_1\tau_2\pi_{1,2}$ or $v = \pi_{1,2}0(M_n)$ or $v = 0(M_n \setminus \{1, 2\})$ or $v = 0(M_n)\pi_{1,2}$.*
- *$n = 4$ and $v = ab = ba$, where $a \in \{e, \pi_{1,2}, \tau_1\tau_2, \pi_{1,2}\tau_1\tau_2, \tau_1\tau_2\pi_{1,2}\}$ and $b \in \{e, \pi_{3,4}, \tau_3\tau_4, \pi_{3,4}\tau_3\tau_4, \tau_3\tau_4\pi_{3,4}\}$.*

Proof. Let $n > 3$. Then from $(i, j)u(i, j) = u$ for any $i, j > 1$ it follows that either $\{i, i'\} \in u$ for any $i > 1$ or $\{i\}, \{i'\} \in u$ for any $i > 1$. For $n = 2, 3$ the first statement can now be completed by direct verification. Analogously one proves the second statement in the case $n \neq 4$. In particular, one obtains 5 different possibilities for $n = 2$, from which the case $n = 4$ can be easily constructed, if one remarks that $\{1, 2\}$ should be invariant with respect to v. □

Lemma 3.4 says that we have to consider the cases $n = 2, 3, 4$ separately from the general one. We postpone this and first will consider the case $n > 4$. Recall that, in the corresponding case for $\mathfrak{B}_n$, we got only the identity map in the previous Section. As above, we assume that φ is an endomorphism of $\mathcal{PB}_n$, stabilizing S_n pointwise, and $\varphi(\tau_1) = u$, $\varphi(\pi_{1,2}) = v$. Lemma 3.4 now gives us 4 candidates for u and 10 candidates for v.

Lemma 3.5. *Let $n > 4$.*

1. *$v \neq e, 0(M_n \setminus \{1, 2\})$.*
2. *$u \neq e$.*
3. *If $v = \pi_{1,2}$ then $u = \tau_1$.*

4. If $v = 0(M_n)$ then $u \neq \tau_1$.

5. $v \neq \tau_1\tau_2$.

6. $v \neq \pi_{1,2}\tau_1\tau_2$, $v \neq \tau_1\tau_2\pi_{1,2}$.

7. $v \neq \pi_{1,2}0(M_n \setminus \{1,2\})$.

8. $v \neq \pi_{1,2}0(M_n)$, $v \neq 0(M_n)\pi_{1,2}$.

Proof. Applying φ to $(1,2)\pi_{1,2} = \pi_{1,2}$ we get $(1,2)v = v$, which is not true for $v = e, 0(M_n \setminus \{1,2\})$. This proves the first statement.

From $(1,2)\tau_1(1,2) = \tau_2$ we get $(1,2)e(1,2) = e = \varphi(\tau_2)$. In particular $\varphi(\tau_1\tau_2) = ee = e$. But $(1,2)\tau_1\tau_2 = \tau_1\tau_2$ and we get a contradiction as above, proving the second statement.

The third one follows from the following observation: $\pi_{1,2} = \pi_{1,2}\tau_1\pi_{1,2}$. Now if we apply φ, we get $\pi_{1,2} = \pi_{1,2}u\pi_{1,2}$, which is impossible for our candidates for u left unless $u = \tau_1$. Now one proves the forth statement by analogous arguments.

To prove the fifth one, we observe that if $v = \tau_1\tau_2$ then the image of the idempotent $\pi_{1,2}(1,3)$ will be $\tau_2\tau_1(1,3)$, which is not an idempotent.

Using the "mirror symmetry" arguments, the sixth statement need only to be proved for one element, say $v \neq \pi_{1,2}\tau_1\tau_2$. In this case, as in the previous paragraph, the image of the idempotent $\pi_{1,2}(1,3)$ will be $\pi_{1,2}\tau_1\tau_2(1,3)$, which is not an idempotent.

If $v = \pi_{1,2}0(M_n \setminus \{1,2\})$ then

$$z = \varphi(\pi_{3,4}) = \varphi((1,3)(2,4)\pi_{1,2}(1,3)(2,4)) = (1,3)(2,4)v(1,3)(2,4) = \pi_{3,4}0(M_n \setminus \{3,4\}).$$

But $\pi_{1,2}$ and $\pi_{3,4}$ commute, whereas v and z do not. This proves the seventh statement and the last one can be done by the same arguments. $\square$

Lemma 3.6. *Let $n > 4$. Then there exist precisely 3 endomorphisms of $\mathcal{P}\mathfrak{B}_n$, stabilizing S_n pointwise.*

Proof. After Lemma 3.5 we know that this number can not exceed 3. The first endomorphism is the identity, which corresponds to $u = \tau_1$, $v = \pi_{1,2}$. The second straightforward endomorphism is to send all non-invertible elements to $0(M_n)$. As $0(M_n)$ commutes with all elements from S_n, this will really be an endomorphism (it corresponds to $u = v = 0(M_n)$). There is only one non-trivial case left, namely $v = 0(M_n)$, $u = 0(M_n \setminus \{1\})$. Let us prove that this defines an endomorphism of $\mathcal{P}\mathfrak{B}_n$. Indeed, this endomorphism can be defined as follows: any element of rank $< n-1$ goes to $O(M_n)$ and $x\tau_1 y$ goes to $x0(M_n \setminus \{1\})y$ for any $x, y \in S_n$. It is straightforward that this map will be an endomorphism. $\square$

Lemma 3.7. *$\mathcal{P}\mathfrak{B}_2$ has 24 endomorphisms, which stabilize S_2 pointwise. Namely, τ_1 can be mapped to τ_1, τ_2, $\tau_1\tau_2$, $\pi_{1,2}$, $\pi_{1,2}0(M_2)$, $0(M_2)\pi_{1,2}$ and independently $\pi_{1,2}$ can be mapped to $\tau_1\tau_2$, $\pi_{1,2}$, $\pi_{1,2}0(M_2)$, $0(M_2)\pi_{1,2}$.*

Proof. As usual, we will denote by u the image of τ_1 and by v the image of $\pi_{1,2}$. First of all we remark that from the proof of Lemma 3.5 it follows that neither u nor v can be equal to e. From the same proof it follows also that $v \neq \tau_1$ and $v \neq \tau_2$. To complete the proof one has now to show that all other choices for u and v are actually possible. This is a straightforward verification. □

Lemma 3.8. *The endomorphisms of $\mathcal{P}\mathfrak{B}_3$ stabilizing S_3 pointwise coincide with those for $\mathcal{P}\mathfrak{B}_n$, $n > 4$ (Lemma 3.6).*

Proof. Clearly, all 3 endomorphisms, described in Lemma 3.6 still exist in the case $n = 3$. Now we have to go through the proof of Lemma 3.5 to find what can be transfered to the case $n = 3$ directly, and what needs a separate consideration. We also have two additional choices for u, namely $\pi_{2,3}$ and $\tau_1\pi_{2,3}$. Clearly they are impossible, cause they force that $\pi_{1,3}$, resp. $\tau_2\pi_{1,3}$ are images for τ_2 and we obtain that the images of commuting elements τ_1 and τ_2 do not commute. That $v \neq e, 0(\{3\})$ and $v \neq e$ follows directly from the proof of Lemma 3.5. The proof for $v = \pi_{1,2}$, $v = \tau_1\tau_2$, $v = \pi_{1,2}\tau_1\tau_2$, $v = \tau_1\tau_2\pi_{1,2}$ and $v = 0(M_n)$ in Lemma 3.5 is also valid in the case $n = 3$. To complete the proof we have to show that all other choices for v ($v = \pi_{1,2}\tau_3$, $v = \pi_{1,2}0(M_3)$ and $v = 0(M_3)\pi_{1,2}$) are not possible either. As in these cases v does not contain subsets of the form $\{i, j'\}$ and $\tau_1\tau_2 = \tau_1\tau_2\pi_{1,2}\tau_1\tau_2$, the element $\varphi(\tau_1\tau_2)$ also should not contain such subset, which forces $u = 0(M_3)$ or $u = \tau_2\tau_3$. Now consider the element $\tau_3\pi_{1,2} = \pi_{1,2}\tau_3$. Applying the endomorphism, we get that $(1,3)u(1,3)$ and v should commute, which is not true. Indeed, v necessarily contains either $\{i, j\}$ or $\{i', j'\}$ and $(1,3)u(1,3)$ is an element of rank ≤ 1 and belongs to $\mathcal{IS}_3$. This implies $(1,3)u(1,3)v = v(1,3)u(1,3)$ must also belong to $\mathcal{IS}_3$ and have the rank at most 1, which contradicts to the fact that v contains either $\{i, j\}$ or $\{i', j'\}$. □

Lemma 3.9. *The endomorphisms of $\mathcal{P}\mathfrak{B}_4$ stabilizing S_4 pointwise coincide with those for $\mathcal{P}\mathfrak{B}_n$, $n > 4$ (Lemma 3.6).*

Proof. We set $v = ab = ba$ for a, b as in Lemma 3.4. All the arguments from the proof of Lemma 3.5 remain valid and we have only 3 choices for u ($0(M_4)$, τ_1 and $0(\{2,3,4\})$) and 12 choices for v ($a \neq e$, $b \neq e, 0(\{3,4\})$) left. We use the fact that $\tau_1\tau_2$ commutes with $\pi_{3,4}$, which implies that $(1,2)(3,4)v(1,2)(3,4)$ commutes with $u(1,2)u(1,2) \in \mathcal{IS}_4$. In particular, $u(1,2)u(1,2)$ always contains $\{1\}$, $\{2\}$, $\{1'\}$ and $\{2'\}$. As $(1,2)(3,4)v(1,2)(3,4)$ always contains either $\{1,2\}$ or $\{1',2'\}$, we get a contradiction. This shows that all other cases are not possible, completing the proof. □

Lemma 3.10. *Let φ be an endomorphism of $\mathcal{P}\mathfrak{B}_6$ such that $\psi = \varphi|_{S_6}$ is a non-inner automorphism of S_6. Then $\varphi(\mathcal{P}\mathfrak{B}_6 \setminus S_6) = 0(M_6)$, the latter really defining an endomorphism of $\mathcal{P}\mathfrak{B}_6$.*

Proof. The second statement is clear and we have to prove only the first one. It is enough to prove that both $u = \varphi(\tau_1)$ and $v = \varphi(\pi_{1,2})$ equal $0(M_6)$. We start with v and use the same arguments as in the proof of Lemma 2.7 obtaining that v does not contain any subset of the form $\{i, j'\}$, v is stable under left and right multiplication with $(i, j)(k, l)(u, v)$ and v equals $\varphi(\pi_{s,t})$ for any s, t. Choosing s, t such that $\psi((s, t)) = (i, k)(l, u)(v, j)$ we get that v is stable under the multiplication with $(i, k)(l, u)(v, j)$ and hence v can not contain any 2-element subset. This implies $v = 0(M_6)$.

Now we proceed with u. The same arguments as above applied to $\tau_1\tau_j$ give us $\varphi(\tau_1\tau_j) = \varphi(\tau_1(1, j)\tau_1(1, j)) = u(k, l)(s, t)(u, v)u(k, l)(s, t)(u, v) = 0(M_n)$. This implies

$$u(k, l)(s, t)(u, v)u = 0(M_n),$$

which means, in particular, $u \in \mathcal{IS}_n$, say $u = 0(X)$, $\varnothing \neq X \subset M_6$. Now use the fact that τ_1 commutes with all (i, j), $i, j \neq 1$, which forces u to commute with all $\varphi((i, j))$. If $\varphi((i, j)) = (s, t)(u, v)(k, l)$, this means that if $X \cap \{s, t\} \neq \varnothing$ (resp. $\{u, v\}$, $\{k, l\}$) then $X \supset \{s, t\}$ (resp. $\{u, v\}$, $\{k, l\}$). As $X \neq \varnothing$, changing $i, j \neq 1$ we get that $X = M_n$ and hence $u = 0(M_n)$. $\square$

Remark 4. *As part of the proof of Lemma 3.9, we got the following: if φ is an endomorphism of $\mathcal{IS}_6$ such that $\psi = \varphi|_{S_6}$ is a non-inner automorphism of $\mathcal{IS}_6$ then $\varphi(\mathcal{IS}_6 \setminus S_6) = 0(M_6)$. This result seems to be lacking in [ST1].*

We have already finished the most difficult part and now can turn to the description of endomorphisms of $\mathcal{P}\mathfrak{B}_n$, which annihilate some part of S_n.

Lemma 3.11. *All maximal subgroups in $\mathcal{P}\mathfrak{B}_n$ are isomorphic to S_m, with $m \leq n$. Moreover, if $m = n$ then this maximal subgroup is the group of all invertible elements in $\mathcal{P}\mathfrak{B}_n$.*

Proof. Analogous to that of Lemma 2.6. $\square$

Let φ be an endomorphism of $\mathcal{P}\mathfrak{B}_n$ and $\psi = \varphi|_{S_n}$ be the corresponding restriction, which will be a homomorphism from S_n to a maximal subgroup of $\mathcal{P}\mathfrak{B}_n$. Assume $\mathrm{Ker}(\psi) = A_n$. Then the image of S_n consists of an idempotent, x, of $\mathcal{P}\mathfrak{B}_n$ and an element, y, of order two in the corresponding maximal subgroup. The case $n = 2$ in this situation is trivial, so we can assume $n > 2$. Set $u = \varphi(\tau_1)$ and $v = \varphi(\pi_{1,2})$. Certainly, $ux = xu = u$ and $vx = xv = v$. As $n > 3$ we can find an even permutation t sending 1 to 3. Then $t\tau_1t^{-1} = \tau_3$, which commutes with $\pi_{1,2}$ and $\varphi(\tau_3) = \varphi(t\tau_1t^{-1}) = xux = u$. Hence $uv = vu$. It is also clear that $yv = vy = v$. Clearly, there also exists a transposition, commuting with τ_1, hence $yu = uy$. Further, $\varphi(\tau_2) = u$ and thus $\varphi(\tau_1\tau_2) = u$, hence $uy = yu = u$ (here the condition $n > 2$ is again important). Now use $\pi_{1,2}\tau_1\tau_2\pi_{1,2} = \pi_{1,2}$. This implies $vuv = v$. Analogously one gets $uvu = u$. Altogether we get $uv = vu = u = v$. Conversely, if the choice of elements $x, y, u = v$ satisfies $x = x^2 = y^2$, $xy = yx = y$, $xu = ux = u = u^2$, $yu = uy = u$, it obviously defines an endomorphism of $\mathcal{P}\mathfrak{B}_n$.

Finally, assume that $\varphi(S_n) = x = x^2$. Then again $ux = xu = u = u^2$ and $vx = xv = v = v^2$. In particular, $\varphi(\tau_i) = u$ for all i. By the same arguments as above for $n > 2$ we get $v = uv = vu = u$ and the endomorphism is defined by x, u such that $x = x^2$ and $xu = ux = u = u^2$. In the case $n = 2$ we can take any u, v satisfying $uvu = u$ and $vuv = v$ obtaining additional endomorphisms with the image x, u, v, uv, vu. Now we can combine the results of this Section into the following statements.

Theorem 3.2. *1. Choose $g \in S_n$ and define $\varphi_g(t) = g^{-1}tg$, $t \in \mathcal{P}\mathfrak{B}_n$.*

2. Choose $g \in S_n$ and define $\psi_g^{(1)}(t) = g^{-1}tg$, $t \in S_n$ and $\psi_g^{(1)}(t) = 0(M_n)$, $t \in \mathcal{P}\mathfrak{B}_n \setminus S_n$.

3. Choose $g \in S_n$ and define $\psi_g^{(2)}(t) = g^{-1}tg$, $t \in S_n$; $\psi_g^{(2)}(h\tau_i) = g^{-1}h0(M_n \setminus \{i\})g$, $h \in S_n$ and $\psi_g^{(2)}(t) = 0(M_n)$, $t \in \mathcal{P}\mathfrak{B}_n \setminus (S_n \cup \{h\tau_1, h\tau_2, \dots, h\tau_n | h \in S_n\})$.

4. Choose $x, y, u \in \mathcal{P}\mathfrak{B}_n$ such that $x = x^2 = y^2$, $xy = yx = y$, $xu = ux = u = u^2$, $yu = uy = u$ and define $\psi_{x,y,u}(t) = x$, $t \in A_n$; $\psi_{x,y,u}(t) = y$, $t \in S_n \setminus A_n$; $\psi_{x,y,u}(t) = u$, $t \in \mathcal{P}\mathfrak{B}_n \setminus S_n$.

5. Choose $x, u \in \mathcal{P}\mathfrak{B}_n$ such that $x = x^2$ and $xu = ux = u = u^2$ and define $\psi_{x,u}(t) = x$, $t \in S_n$; $\psi_{x,u}(t) = u$, $t \in \mathcal{P}\mathfrak{B}_n \setminus S_n$.

Then the maps φ_g, $\psi_g^{(1)}$, $\psi_g^{(2)}$, $\psi_{x,y,u}$ and $\psi_{x,u}$ are endomorphisms of $\mathcal{P}\mathfrak{B}_n$. Moreover, if $n \neq 2, 6$ then any endomorphism of $\mathcal{P}\mathfrak{B}_n$ coincides with one of the listed above.

Theorem 3.3. *Additionally to the endomorphisms listed in Theorem 3.2, the semigroup $\mathcal{P}\mathfrak{B}_6$ has endomorphisms φ_ψ, which are indexed by non-inner automorphisms of S_6. If ψ is a non-inner automorphism of S_6, then $\varphi_\psi(t) = \psi(t)$, $t \in S_6$ and $\varphi_\psi(t) = 0(M_6)$, $t \in \mathcal{P}\mathfrak{B}_n \setminus S_6$.*

Theorem 3.4. *Additionally to the endomorphisms listed in Theorem 3.2, the semigroup $\mathcal{P}\mathfrak{B}_2$ has endomorphisms stabilizing S_2, which send τ_1 to τ_1, τ_2, $\tau_1\tau_2$, $\pi_{1,2}$, $\pi_{1,2}0(M_2)$, $0(M_2)\pi_{1,2}$ and independently $\pi_{1,2}$ to $\tau_1\tau_2$, $\pi_{1,2}$, $\pi_{1,2}0(M_2)$, $0(M_2)\pi_{1,2}$; and endomorphisms defined by a triple x, u, v of idempotents satisfying $xu = ux = u$, $xv = vx = v$, $uvu = u$ and $vuv = v$, these endomorphisms send S_2 to x, $\pi_{1,2}$ to u and τ_1 to v.*

Remark 5. *We note that some endomorphisms of $\mathcal{P}\mathfrak{B}_2$ listed in Theorem 3.4 can coincide with endomorphisms listed in Theorem 3.2. It is easy to separate different ones, but this will lead to a much worse form of presentation of these endomorphisms. This is the reason why we decided to keep the formulation of Theorem 3.4 as above.*

4 Semigroup $\mathfrak{C}_n$ and its endomorphisms

In this last Section we will deal with the most complicated case – the semigroup $\mathfrak{C}_n$. We keep all the notation from the previous sections and, as it was done above, also start with automorphisms. For $i, j \in M_n$ we set $\xi_{i,j}$ (resp. $\zeta_{i,j}$) to be the idempotent defined as follows: it contains $\{t, t'\}$ for all $t \neq i, j$ and it also contains $\{i, j, j'\}$ and $\{i'\}$ (resp. $\{j, i', j'\}$ and $\{i\}$). For $X \subset M_n$ we set $\alpha(X)$ (resp. $\beta(X)$) to be the element containing $\{i, i'\}$, $i \notin X$ and $X \cup X'$ (resp. X and X'). One can easily check that $\xi_{i,j}\zeta_{i,j} = \alpha(\{i, j\})$; $\zeta_{i,j}\xi_{i,j} = \tau_i$ and $\xi_{i,j}\zeta_{j,i} = \pi_{i,j} = \beta(\{i, j\})$. We also remark that, as in the case of $\mathcal{P}\mathfrak{B}_n$, the semigroup $\mathfrak{C}_1$ coincides with $\mathcal{P}\mathfrak{B}_1$ coincides with $\mathcal{IS}_1$, so we can assume $n > 1$. We will say that elements $a, b \in N_n$ are *connected* with respect to (or by) $t \in \mathfrak{C}_n$ if t contains a subset, $X \subset N_n$, containing both a and b.

Lemma 4.1. *Fix $i, j \in M_n$. Then $\mathfrak{C}_n$ is generated by S_n and $\xi_{i,j}$, $\zeta_{i,j}$.*

Proof. The statement is trivial for $n = 2$ and we will reduce the general case to this one. We use the downward induction in the number k of the subsets of the form $\{i, i'\}$ contained in $t \in \mathfrak{C}_n$. For $k = n$ everything is trivial. Now we prove the induction step. Multiplying t, if necessary, with permutations on both sides, we can assume that the following are satisfied: if $i \in M_n$ (resp. $i' \in M_n'$) is connected with $j \in M_n$ (resp. $j' \in M_n'$), $j > i$ then i is connected with all $t \in \{i+1, \dots, j\}$ (resp. $t' \in \{(i+1)', \dots, j'\}$); if $i < j \in M_n$, i connected with some element $l' \in M_n'$ and j is connected with some element $s' \in M_n'$ then l and s can be chosen such that $l \leq s$. Consider the element $\hat{t} \in \mathfrak{C}_n$ defined as follows: $\hat{t}$ contains $\{1, 1'\}$ and $\hat{t}|_{M_n \setminus \{1\}} = t|_{M_n \setminus \{1\}}$. Now from trivial $n = 2$ calculation it follows that $t = g\hat{t}h$, where $\{1, 2\}$ is invariant with respect to g, h and $g|_{M_n \setminus \{1,2\}} = h|_{M_n \setminus \{1,2\}}$ equals the identity. The induction now completes the proof. □

From Lemma 4.1 it follows that in order to determine an endomorphism, (or an automorphism), φ, of $\mathfrak{C}_n$ one has to determine the image of S_n (which should be S_n for any automorphism) and two elements $u = \varphi(\xi_{1,2})$ and $v = \varphi(\zeta_{1,2})$. At the moment we assume that φ is an automorphisms. Let $t \in \mathfrak{C}_n$. By the *rank* of t (denoted $\mathrm{Rank}(t)$) we will mean the number of subsets in t intersecting both M_n and M_n'. It is clear that the rank of the product of two elements is less or equal to the minimum of the ranks of these elements. We also note that $rank(\xi_{i,j}) = \mathrm{Rank}(\zeta_{i,j}) = n - 1$ and if $\mathrm{Rank}(t) = n - 1$ then t contains at least $n - 3$ subsets of the form $\{i, i'\}$.

Lemma 4.2. *Any non-inner automorphisms ψ of S_6 can not be continued to an automorphism of $\mathfrak{C}_n$.*

Proof. ψ sends $(1, 2)$ to some $(i, j)(k, l)(t, s)$. Using $(1, 2)\xi_{1,2} = \xi_{1,2}$ and $\zeta_{1,2}(1, 2) = \zeta_{1,2}$ we get $(i, j)(k, l)(t, s)u = u$ and $v(i, j)(k, l)(t, s) = v$. But such u and v can not contain any subset of the form $\{i, i'\}$, which means that their ranks are strictly less than $n - 1 = 5$. This implies that the elements of rank 5 can not belong to the image of φ, hence φ can not be an automorphism. □

By Lemma 4.2, the restriction $\psi = \varphi|_{S_n}$ is an inner automorphism of S_n, hence, composing it with an inner autoimorphism of $\mathfrak{C}_n$ we can assume that ψ is the identity map.

Theorem 4.1. *Any automorphism of $\mathfrak{C}_n$ is inner.*

Proof. Using the notation above we have to prove that $\psi = id$ implies $\varphi = id$. For this it is sufficient to prove that $u = \xi_{1,2}$ and $v = \zeta_{1,2}$. Let us determine the candidates for u and v. First we note that $C(\xi_{1,2}) = C(\zeta_{1,2}) = S(M_n \setminus \{1, 2\})$. As ψ is an identity, we get $C(u) = C(v) = S(M_n \setminus \{1, 2\})$. Moreover, for any $\sigma \in S(M_n \setminus \{1, 2\})$ holds $\sigma\xi_{1,2} \neq \xi_{1,2}$ and $\sigma\zeta_{1,2} \neq \zeta_{1,2}$, thus $\sigma u \neq u$ and $\sigma v \neq v$. From this we deduce that both u and v contain $\{i, i'\}$ for $i \in M_n \setminus \{1, 2\}$. From $(1, 2)\xi_{1,2} = \xi_{1,2}$ it follows $(1, 2)u = u$, which means that either 1 and 2 are connected by u or u contains $\{1\}$ and $\{2\}$. In the last case we get $u(1, 2) = (1, 2)$, which is impossible as $\xi_{1,2}(1, 2) \neq \xi_{1,2}$. Hence 1 and 2 are connected by u. Dually (with respect to the mirror symmetry), $1'$ and $2'$ are connected by v. From $\xi_{1,2}(1,2) \neq \xi_{1,2}$ it also follows that 1 is connected with exactly one of $1'$ and $2'$. This means that u is either $\xi_{1,2}$ or $\xi_{1,2}(1,2)$. Analogously v is either $\zeta_{1,2}$ or $(1,2)\zeta_{1,2}$.

Let $u = \xi_{1,2}(1,2)$ and $v = (1,2)\zeta_{1,2}$. Then

$$\varphi(\tau_1) = \varphi(\zeta_{1,2}\xi_{1,2}) = vu = (1,2)\zeta_{1,2}\xi_{1,2}(1,2) = \tau_2.$$

If $n > 2$ the last is impossible because of $C(\tau_2) \neq C(\tau_1)$. If $n = 2$, in the case $\varphi(\mathcal{IS}_2) \subset \mathcal{IS}_2$ we can always assume $\varphi(\tau_1) = \tau_1$ applying the unique non-trivial inner automorphism. A contradiction.

If only one of u or v does not coincide with its preimage, then, calculating the image of the idempotent τ_1 we will obtain a non-idempotent element, which is impossible. This shows that $u = \xi_{1,2}$ and $v = \zeta_{1,2}$ completing the proof. $\square$

Now we can turn to the proper endomorphisms of $\mathfrak{C}_n$. We start with the case of a non-inner automorphism of S_6.

Lemma 4.3. *Let φ be an endomorphisms of $\mathfrak{C}_6$ such that $\psi = \varphi|_{S_6}$ is an outer automorphism of S_6. Then either $\varphi(\mathfrak{C}_6 \setminus S_6) = 0(M_6)$ or $\varphi(\mathfrak{C}_6 \setminus S_6) = \alpha(M_6)$ or $\varphi(\mathfrak{C}_6 \setminus S_6) = \beta(M_6)$ or $\varphi(\mathfrak{C}_6 \setminus S_6) = \alpha(M_6)0(M_6)$ or $\varphi(\mathfrak{C}_6 \setminus S_6) = 0(M_6)\alpha(M_6)$. Moreover, all five possibilities listed above define endomorphisms of $\mathfrak{C}_6$.*

Proof. The second statement is clear and we will prove only the first one, for which we use the property of ψ to send any transposition to a product of three commuting. From $(1,2)\xi_{1,2} = \xi_{1,2}$, $\zeta_{1,2}(1,2) = \zeta_{1,2}$, $(i,j)\xi_{1,2}(i,j) = \xi_{1,2}$ and $(i,j)\zeta_{1,2}(i,j) = \zeta_{1,2}$, $i,j > 2$ we derive that u either contains a subset containing M_6 or contains all $\{i\}$, $i \in M_6$ and v either contains a subset containing M_6' or contains all $\{i'\}$, $i \in M_6$. Then, using

$$\xi_{1,2}((1,2)(3,4)\zeta_{1,2}(1,2)(3,4)) = ((1,2)(3,4)\zeta_{1,2}(1,2)(3,4))\xi_{1,2}$$

we get that u commutes with gvg for some $g \in S_6$ of order two. This immediately implies that if u contains a subsets containing M_6 (resp. contains a subsets containing M_6', resp. contains all $\{i\}$, $i \in M_n$, resp. contains all $\{i'\}$, $i \in M_n$) then v contains a subsets containing M_6 (resp. contains a subsets containing M_6', resp. contains all $\{i\}$, $i \in M_n$, resp. contains all $\{i'\}$, $i \in M_n$). The last observation forces that both u and v can be chosen only among $0(M_6)$, $\alpha(M_6)$, $\beta(M_6)$, $\alpha(M_6)0(M_6)$ or $0(M_6)\alpha(M_6)$. Moreover, as their left and right legs should be connected in the same way and using $uvu = u$, $vuv = v$, we also derive that $u = v$, completing the proof. $\square$

Now we have to describe all endomorphisms φ of $\mathfrak{C}_n$, whose restriction ψ to S_n is the identity map. We first consider the case $n = 2$.

Lemma 4.4. *$\mathfrak{C}_2$ has precisely 25 endomorphisms stabilizing S_2, namely 9 endomorphisms sending $\xi_{1,2}$ to $\xi_{1,2}$, $\xi_{1,2}(1,2)$, $\alpha(M_2)$ and $\zeta_{1,2}$ independently to $\zeta_{1,2}$, $(1,2)\zeta_{1,2}$, $\alpha(M_2)$ and* 16 *endomorphisms sending $\xi_{1,2}$ and $\zeta_{1,2}$ independently to $\tau_1\tau_2$, $\pi_{1,2}\tau_1\tau_2$, $\tau_1\tau_2\pi_{1,2}$, $\pi_{1,2} = \beta(M_2)$ and $\zeta_{1,2}$ (two inner automorphisms are also counted here).*

Proof. Use $(1,2)\xi_{1,2}$ and $\zeta_{1,2}(1,2) = \zeta_{1,2}$ to get $(1,2)u = u$ and $v(1,2) = v$. As $\xi_{1,2}$ and $\zeta_{1,2}$ are inverse to each other, we get $uvu = u$ and $vuv = v$, which implies that if one of u or v does not contain any subset intersecting both M_n and M'_n, then the other one does not contain any such subset either. Thus we get that u can be chosen among $\xi_{1,2}$, $\xi_{1,2}(1,2)$, $\alpha(M_2)$ and v independently among $\zeta_{1,2}$, $(1,2)\zeta_{1,2}$, $\alpha(M_2)$ or u and v can be chosen among $\tau_1\tau_2$, $\pi_{1,2}\tau_1\tau_2$, $\tau_1\tau_2\pi_{1,2}$, $\pi_{1,2} = \beta(M_2)$ and $\zeta_{1,2}$. It is a trivial calculation that all the choices do really define endomorphisms of $\mathfrak{C}_2$. $\square$

Now we consider the case $n > 3$.

Lemma 4.5. *For $n > 3$ the semigroup $\mathfrak{C}_n$ has precisely 6 endomorphisms, stabilizing S_n pointwise. They are: the identity and 5 endomorphisms sending $\mathfrak{C}_n \setminus S_n$ to $0(M_n)$, $\alpha(M_n)$, $\beta(M_n)$, $\alpha(M_n)0(M_n)$, $0(M_n)\alpha(M_n)$.*

Proof. We divide the proof into two parts. In the first one we will show that there are only six possibilities for u and v: $\xi_{1,2}$ (resp. $\zeta_{1,2}$) and $0(M_n)$, $\alpha(M_n)$, $\beta(M_n)$, $\alpha(M_n)0(M_n)$, $0(M_n)\alpha(M_n)$. In the second part we will show that $u \neq \xi_{1,2}$ implies $v = u$ and $v \neq \zeta_{1,2}$ implies $u = v$. These two facts together clearly will prove our lemma. Let us prove the statement for u (the case of v can be treated analogously). We will use the following trivial identities in $\mathfrak{C}_n$:

1. $(1,2)\xi_{1,2} = \xi_{1,2}$ implying $(1,2)u = u$;
2. $(i,j)\xi_{1,2} = \xi_{1,2}(i,j)$, $i,j > 2$ implying $(i,j)u = u(i,j)$, $i,j > 2$;
3. $\xi_{1,2}\xi_{1,3} = \xi_{1,3}\xi_{1,2} = \xi_{1,2}$ implying $u(2,3)u(2,3) = (2,3)u(2,3)u = u$;
4. $\xi_{1,2}\xi_{i,j} = \xi_{i,j}\xi_{1,2}$, $i,j > 2$ implying $(1,i)(1,j)u(1,i)(1,j)u = u(1,i)(1,j)u(1,i)(1,j)$, $i,j > 2$.

From the first one we get that either u contains $\{1\}$ and $\{2\}$ or it contains a subset containing $\{1,2\}$. From the second one we get that there are only following possibilities for $u|_{M_n\setminus\{1,2\}}$: the identity, $0(M_n \setminus \{1,2\})$, $\alpha(M_n \setminus \{1,2\})$, $\beta(M_n \setminus \{1,2\})$, $\alpha(M_n \setminus \{1,2\})0(M_n \setminus \{1,2\})$, $0(M_n \setminus \{1,2\})\alpha(M_n \setminus \{1,2\})$. The third one forbids, in particular, all cases, where $u|_{M_n\setminus\{1,2\}}$ is the identity (and automatically $\{1,2\}$ is invariant under u because of the second condition) except $u = \xi_{1,2}$ and, together with the forth one, implies that either u contains all $\{i\}$ (resp. $\{i'\}$) or a subset containing M_n (resp. M'_n). This gives us 5 more possibilities for u: $0(M_n)$, $\alpha(M_n)$, $\beta(M_n)$, $\alpha(M_n)0(M_n)$, $0(M_n)\alpha(M_n)$.

Now we will prove the second part. As we have already seen, there are only six possibilities for u and v: $\xi_{1,2}$ (resp. $\zeta_{1,2}$) and $0(M_n)$, $\alpha(M_n)$, $\beta(M_n)$, $\alpha(M_n)0(M_n)$, $0(M_n)\alpha(M_n)$. Among these elements there is only one pair of non-equal commuting ones, namely $\alpha(M_n)$ and $\beta(M_n)$. The case when one of u, v equals $\alpha(M_n)$ and other one equals $\beta(M_n)$ is impossible because of $uvu = u$ and $vuv = v$. Let $u \neq \xi_{1,2}$ (the case $v \neq \zeta_{1,2}$ is analogous), then $(1,3)(2,4)u(1,3)(2,4) = u = \varphi(\xi_{3,4})$ and the last one commutes with $\zeta_{1,2}$. Hence $uv = vu$ and we get $v = u$ completing the proof. $\square$

Now, there is only one case left, namely $n = 3$.

Lemma 4.6. *The semigroup $\mathfrak{C}_3$ has 6 endomorphisms stabilizing S_3 pointwise defined as in Lemma 4.5.*

Proof. Although the statement of the Lemma is the same as of Lemma 4.5, the proof will be a little bit different cause there are no analogue of the forth identity. Instead we will use the fact that $(1,3)vu(1,3)$ commutes with both u and v coming from the fact that $(1,3)\zeta_{1,2}\xi_{1,2}(1,3)$ commutes with both $\xi_{1,2}$ and $\zeta_{1,2}$ in $\mathfrak{C}_3$. Following the proof of Lemma 4.5 we will get three additional possibilities for u: $\alpha(M_3)\tau_{1,2}$, $\alpha(M_3)\pi_{2,3}$ and $\alpha(M_3)\pi_{2,3}\tau_{1,2}$ and three dual possibilities for v. As none of them commute with $0(M_3)$, $\alpha(M_3)$, $\beta(M_3)$, $\alpha(M_3)0(M_3)$, $0(M_3)\alpha(M_3)$, if u is one of the last five, we get $v = u$ by the same arguments as in Lemma 4.5. Hence, we can suppose that u is one from $\alpha_{M_3}\tau_{1,2}$, $\alpha_{M_3}\pi_{2,3}$ and $\alpha_{M_3}\pi_{2,3}\tau_{1,2}$ and v is one from $\zeta_{1,2}$, $\tau_{1,2}\alpha_{M_3}$, $\pi_{2,3}\alpha_{M_3}$ and $\pi_{2,3}\tau_{1,2}\alpha_{M_3}$. But in all these cases $(1,3)vu(1,3)$ can not commute with u, cause all elements from M_3 are connected by u and $(1,3)vu(1,3)$ necessarily contains a proper subset of M_n. This completes the proof. □

We are almost done, and the only cases left are those, where the image of S_n is smaller that S_n. For this we will need the following fact.

Lemma 4.7. *Let $\pi \in \mathfrak{C}_n$ be an idempotent. Then the maximal subgroup of $\mathfrak{C}_n$, corresponding to π, contains only elements of rank* $\mathrm{Rank}(\pi)$*. In particular, it is a subgroup of* $S_{\mathrm{Rank}(\pi)}$.

Proof. The first statement follows from the fact that rank can only decrease in the product. The second statement is an immediate corollary of the first one. □

Thus, either $\psi = \varphi|_{S_n}$ is an automorphism of S_n or its image contains one or two elements. First we assume that $\mathrm{Ker}(\psi) = A_n$. Then the image of S_n consists of an idempotent, x, and an element, y, of order two in the maximal subgroup of $\mathfrak{C}_n$, corresponding to x. Again the case $n = 2$ is trivial. Let $u^2 = u = \varphi(\xi_{1,2})$ and $v^2 = v = \varphi(\zeta_{1,2})$. We have $ux = xu = u$, $vx = xv = v$, $yu = u$ and $vy = v$. Assume $n > 3$. Then there is an even permutation, say g, such that $g^{-1}\zeta_{1,2}g$ commutes with $\xi_{1,2}$, hence $uv = vu$. This means $v = vuv = vu = uvu = u$. Thus, our endomorphism sends $\mathfrak{C}_n \setminus S_n$ to u. From the other hand, if we choose x, y, u such that $x^2 = x$, $y^2 = x$, $xy = yx = y$, $u^2 = u = ux = xu = yu = uy$, we will get an endomorphism of $\mathfrak{C}_n$ sending A_n to x, $S_n \setminus A_n$ to y and $\mathfrak{C}_n \setminus S_n$ to u. For $n = 3$ we use that τ_3 commutes with $\xi_{1,2}$ and $\zeta_{1,2}$. We have $\tau_3 = (1,2,3)\zeta_{1,2}\xi_{1,2}(1,3,2)$ implying $\tau_3 = vu$. Hence $(vu)u = u(vu)$ and $(vu)v = v(vu)$. Again we have $v = vuv = v(uv) = (uv)v = uvv = uv = uuv = u(uv) = (uv)u = uvu = u$ and thus the result coincides with that in case $n > 3$.

Finally, assume $\varphi(S_n) = x = x^2$. Then $ux = xu = u = u^2$ and $vx = xv = v = v^2$. By the same arguments as above we get $u = v$ in the case $n > 2$ obtaining $\varphi(\mathfrak{C}_n \setminus S_n) = u$ and the endomorphisms is defined by a pair of idempotents x and u such that $ux = xu = u$. In the case $n = 2$ we can take any u, v satisfying $uvu = u$ and $vuv = v$ obtaining additional endomorphisms with the image x, u, v, uv, vu. Combining all the results above, we can formulate the following statements:

Theorem 4.2. *1. Choose $g \in S_n$ and define $\varphi_g(t) = g^{-1}tg$, $t \in \mathfrak{C}_n$.*

2. *Choose $g \in S_n$ and define $\psi_g^{(1)}(t) = g^{-1}tg$, $t \in S_n$ and $\psi_g^{(1)}(t) = 0(M_n)$, $t \in \mathfrak{C}_n \setminus S_n$.*

3. *Choose $g \in S_n$ and define $\psi_g^{(2)}(t) = g^{-1}tg$, $t \in S_n$ and $\psi_g^{(2)}(t) = \alpha(M_n)$, $t \in \mathfrak{C}_n \setminus S_n$.*

4. *Choose $g \in S_n$ and define $\psi_g^{(3)}(t) = g^{-1}tg$, $t \in S_n$ and $\psi_g^{(3)}(t) = \beta(M_n)$, $t \in \mathfrak{C}_n \setminus S_n$.*

5. *Choose $g \in S_n$ and define $\psi_g^{(4)}(t) = g^{-1}tg$, $t \in S_n$ and $\psi_g^{(4)}(t) = \beta(M_n)0(M_n)$, $t \in \mathfrak{C}_n \setminus S_n$.*

6. *Choose $g \in S_n$ and define $\psi_g^{(5)}(t) = g^{-1}tg$, $t \in S_n$ and $\psi_g^{(5)}(t) = 0(M_n)\beta(M_n)$, $t \in \mathfrak{C}_n \setminus S_n$.*

7. *Choose $x, y, u \in \mathfrak{C}_n$ such that $x = x^2 = y^2$, $xy = yx = y$, $xu = ux = u = u^2$, $yu = uy = u$ and define $\psi_{x,y,u}(t) = x$, $t \in A_n$; $\psi_{x,y,u}(t) = y$, $t \in S_n \setminus A_n$; $\psi_{x,y,u}(t) = u$, $t \in \mathfrak{C}_n \setminus S_n$.*

8. *Choose $x, u \in \mathfrak{C}_n$ such that $x = x^2$ and $xu = ux = u = u^2$ and define $\psi_{x,u}(t) = x$, $t \in S_n$; $\psi_{x,u}(t) = u$, $t \in \mathfrak{C}_n \setminus S_n$.*

Then the maps φ_g, $\psi_g^{(i)}$, $i = 1, 2, 3, 4, 5$, $\psi_{x,y,u}$ and $\psi_{x,u}$ are endomorphisms of $\mathfrak{C}_n$. Moreover, if $n \neq 2, 6$ then any endomorphism of $\mathfrak{C}_n$ coincides with one of the listed above.

Theorem 4.3. *Additionally to the endomorphisms listed in Theorem 4.2, the semigroup $\mathfrak{C}_6$ has endomorphisms $\varphi_\psi^{(i)}$, $i = 1, 2, 3, 4, 5$, which are indexed by non-inner automorphisms of S_6. If ψ is a non-inner automorphism of S_6, then $\varphi_\psi^{(i)}(t) = \psi(t)$, $t \in S_6$ and $\varphi_\psi^{(1)}(t) = 0(M_6)$, $\varphi_\psi^{(2)}(t) = \alpha(M_6)$, $\varphi_\psi^{(3)}(t) = \beta(M_6)$, $\varphi_\psi^{(4)}(t) = \beta(M_6)0(M_6)$, $\varphi_\psi^{(5)}(t) = 0(M_6)\beta(M_6)$, $t \in \mathfrak{C}_n \setminus S_6$.*

Theorem 4.4. *Additionally to the endomorphisms listed in Theorem 4.2, the semigroup $\mathcal{P}\mathfrak{B}_2$ has endomorphisms stabilizing S_2, which send $\xi_{1,2}$ to $\xi_{1,2}$, $\xi_{1,2}(1,2)$, $\alpha(M_2)$ and $\zeta_{1,2}$ independently to $\zeta_{1,2}$, $(1,2)\zeta_{1,2}$, $\alpha(M_2)$; which send $\xi_{1,2}$ and $\zeta_{1,2}$ independently to $\tau_1\tau_2$, $\pi_{1,2}\tau_1\tau_2$, $\tau_1\tau_2\pi_{1,2}$, $\pi_{1,2} = \beta(M_2)$ and $\zeta_{1,2}$; and endomorphisms defined by a triple x, u, v of idempotents satisfying $xu = ux = u$, $xv = vx = v$, $uvu = u$ and $vuv = v$, these endomorphisms send S_2 to x, $\pi_{1,2}$ to u and τ_1 to v.*

5 Acknowledgments

The paper was written during the authors visit to SFB-343, Bielefeld University, whose hospitality and financial support are gratefully acknowledged.

References

[KM] *M.I.Kargapolov, Yu.I.Merzlyakov*, Osnovy teorii grupp. (Russian) [Fundamentals of group theory] Fourth edition. Fizmatlit "Nauka", Moscow, 1996.

[K] *S.V.Kerov*, Realizations of representations of the Brauer semigroup. (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 164 (1987), Differentsialnaya Geom. Gruppy Li i Mekh. IX, 188-193, 199.

[M] *V.Mazorchuk*, On the structure of Brauer semigroup and its partial analogue. Problems in Algebra 13 (1998), Gomel: University Press, 29-45.

[R] *J.J.Rotman*, An introduction to the theory of groups. Fourth edition. Graduate Texts in Mathematics, 148. Springer-Verlag, New York, 1995.

[ST1] *B.M.Schein and B.Teclezghi*, Endomorphisms of finite symmetric inverse semigroups. J. Algebra 198 (1997), 300-310.

[ST2] *B.M.Schein and B.Teclezghi*, Endomorphisms of finite full transformation semigroups. Proc. AMS 126 (1998), 2579-2587.

[ST3] *B.M.Schein and B.Teclezghi*, Endomorphisms of symmetric semigroups of functions on a finite set. Comm. Algebra 26 (1998), 3921-3938.

[Xi] *Changchang Xi*, Partition algebras are cellular. Compositio Math. 119 (1999), 99-109.

Volodymyr Mazorchuk, Algebra, Department of Mechanics and Mathematics, Kyiv Taras Shevchenko University, 64, Volodymyrska st., 01033, Kyiv, Ukraine,
e-mail: `mazor@mechmat.univ.kiev.ua`, `mazor@mathematik.uni-bielefeld.de`