On double Ringel–Hall algebras*

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Contents

Introduction

By Green's theorem [G], the Ringel–Hall algebra of a finite dimensional hereditary algebra Λ together with its torus algebra can be endowed with Hopf algebra structure in an explicit way (see [G] and [X]). It is natural to construct its Drinfeld double $\mathcal{D}(\Lambda)$. An important result of Green and Ringel (see [R2] and [G]) states that the Drinfeld double $\mathcal{C}(\Lambda)$ of the composition subalgebra of a Ringel–Hall algebra provides a complete realization of the Lusztig form U of the quantized enveloping algebra of the corresponding Kac–Moody algebra $\mathfrak{g}$. So a natural question is how to measure the difference and commom sense

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between $\mathcal{D}(\Lambda)$ and U. A remarkable observation in [SV2] shows that the Ringel–Hall algebra is the positive part of the quantized enveloping algebra of a generalized Kac–Moody algebra. Our motivation in the present paper is to understand the main results of Sevenhant and Van den Bergh [SV2]. It seems to us that it is necessary to give an explicit formulation and a complete proof of their main results for the double Ringel–Hall algebras of all finite–dimesional hereditary algebras. Note that the results in [SV2] are considered for the quiver case. However, our formulation is based on a result of Hua [H] which counts the number of indecomposables over arbitrary finite dimensional hereditary algebra over a finite field.

The purpose of the present paper is two–fold. On one hand, we verify that Ringel–Hall algebras belong to the class of quantum groups which provide solutions of quantum Yang–Baxter equation. By decomposing the double Ringel–Hall algebra $\mathcal{D}(\Lambda)$ in terms of its skew–Hopf pairing, we obtain that $\mathcal{D}(\Lambda)$ is a restricted non–degenerate member of some datum in the sense of Green [G]. As a consequence, this phenomenon shows that the Ringel–Hall algebra is independent of the orientation of Λ and the canonical isomorphism (not unique) is consistent with the Green's isomorphism for the composition algebra. Note that in the quiver case this fact is pointed out by Lusztig and proved by Sevenhant and Van den Bergh by using Fourier transformations (see [L1] and [SV1]). Further, we can define in a natural way the highest weight module category $\mathcal{O}$ and integrable modules over $\mathcal{D}(\Lambda)$. The pairing φ between the positive and negative Ringel–Hall algebras provides us an R–matrix Θ^f. The action of Θ^f in $\mathcal{O}$ induces the $\mathcal{D}(\Lambda)$–module isomorphism $M \otimes M' \cong M' \otimes M$ for all modules $M, M' \in \mathcal{O}$. Moreover, the operator Θ^f satisfies a fundamental symmetry relation: quantum Yang–Baxter equation. It is also shown that there exist enough irreducible integrable highest weight modules on which the action of Θ^f provides solutions of quantum Yang–Baxter equation. Moreover, we show the complete irreducibility and the well–known Kac–Weyl character formula for the integrable highest weight modules with strongly dominant highest weights. On the other hand, in the last section, we present the theorem of Sevenhant and Van den Bergh [SV2] and its proof, which claims that the Drinfeld double of a Ringel–Hall algebra is the quantized enveloping algebra of a generalized Kac–Moody algebra. The proof of our main results depends on an important property, which claims that the dimension vectors of new primitive generators for $\mathcal{D}(\Lambda)$ belong to the fundamental set of the imaginary roots of the Kac–Moody algebra. We give its proof in Section 6 by applying the action of Lusztig's symmetries.

1 Preliminaries

1.1 By a vauled graph (Γ, d) we mean a finite set Γ (of vertices) together with non-negative integers d_{ij} for all $i,\, j \in \Gamma$ such that $d_{ii} = 0$ and there exist positive integers $\{\varepsilon_i\}_{i\in\Gamma}$ satisfying

$$d_{ij}\varepsilon_j = d_{ji}\varepsilon_i \quad \text{for all } \ i,\, j \in \Gamma.$$

An orientation Ω of a valued graph (Γ, d) is given by prescribing for each edge $\{i,\, j\}$ of $(\Gamma,\, d)$ an order (indicated by an arrow $i \longrightarrow j$). We call $(\Gamma,\, d,\, \Omega)$, or simply Ω, a valued quiver. For $i \in \Gamma$, we can define a new orentation $\sigma_i\Omega$ of $(\Gamma,\, d)$ by reversing the arrows along all edges containing i.

1.2 Let k be a finite field and $(\Gamma,\, d,\, \Omega)$ a valued quiver. From now on, we shall always assume that $(\Gamma,\, d,\, \Omega)$ is connected and contains no oriented cycles. Further, let $\mathcal{S} = (F_i,\, {}_iM_j)_{i,j\in\Gamma}$ be a reduced k-species of type Ω, that is, for all $i,\, j \in \Gamma$, ${}_iM_j$ is an F_i-F_j-bimodule, where F_i and F_j are finite extensions of k in an algebraic closure of k and $\dim\,({}_iM_j)_{F_j} = d_{ij}$ and $\dim{}_k F_i = \varepsilon_i$.

By definition, a k–representation $(V_i,\, {}_j\varphi_i)$ of $\mathcal{S}$ consists of vector spaces $(V_i)_{F_i}$, $i \in \Gamma$, and of F_j-linear map ${}_j\varphi_i$: $V_i \otimes {}_iM_j \to V_j$ for each arrow $i \to j$. Such a representation is called finite dimensional if $\sum \dim{}_k V_i < \infty$. By rep–$\mathcal{S}$ we then denote the category of finite dimensional representations of $\mathcal{S}$ over k. Note that the category rep–$\mathcal{S}$ is equivalent to the category $\operatorname{mod}\Lambda$ of finite dimensional modules over the tensor algebra Λ of $\mathcal{S}$ which is a finite diemsional hereditary k–algebra. In the following, we shall simply identify representations of $\mathcal{S}$ with Λ–modules. Moreover, each finite dimensional hereditary k–algebra is obtained from a k–species (see [DR]).

1.3 Let $\mathcal{S} = (F_{i,i} M_j)_{i,j\in\Gamma}$ be a k-species with $\varepsilon_i = \dim{}_k F_i$ and $d_{ij} = \dim{}_i M_j{}_{F_j}$. For each representation $V = (V_i,\, {}_j\varphi_i)$ in rep–$\mathcal{S}$, the dimension vector of V is defined to be $\underline{\dim} V = (\dim{}_{F_i} V_i)_{i\in\Gamma} \in \mathbb{Z}^\Gamma$. For $V, W \in \text{rep} - \mathcal{S}$, we define

$$\langle \alpha, \beta \rangle = \sum_{i\in\Gamma} \varepsilon_i a_i b_i - \sum_{i\to j} d_{ij}\varepsilon_j a_i b_j,$$

where $\alpha = \underline{\dim} V = (a_1, \cdots, a_n)$ and $\beta = \underline{\dim} W = (b_1, \cdots, b_n)$. It is well–known that

$$\langle \alpha, \beta \rangle = \dim{}_k \mathrm{Hom}_\Lambda(V, W) - \dim{}_k \mathrm{Ext}\,^1_\Lambda(V, W).$$

Furhter, we set

$$(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle.$$

Then both $\langle -, - \rangle$ and $(-, -)$ are well defined on the Grothendieck group $G_0(\Lambda)$ of rep-$\mathcal{S}$ which can be identified with $\mathbb{Z}^\Gamma$. The bilinear forms $\langle -, - \rangle$

and $(-,-)$ are called Euler form and symmetric Euler form, respectively. In fact, the Grothendieck group with the symmetric Euler form is a Cartan datum in the sense of Lusztig. Moreover, each Cartan datum can be realized in this way (see [R2]).

1.4 Let k be a finite field with q elements, and set $v = \sqrt{q}$. Let Λ be a finite dimensional hereditary algebra over k. From now on, By $\mathcal{P}$ we denote the set of isomorphism classes of finite dimensional Λ–modules, and $I \subset \mathcal{P}$ the set of isomorphism classes of simple Λ-modules. Finally, we set $\mathcal{P}_1 = \mathcal{P}\backslash\{0\}$. For each $\alpha \in \mathcal{P}$, we fix a representative V_α in the isoclass α. By abuse of notation, we write $\alpha = \underline{\dim} V_\alpha$ fro $\alpha \in \mathcal{P}$. The Euler form $\langle , \rangle$ and its symmetrization $(-,-)$ are then defined on $\mathbb{Z}[I]$.

Given $\alpha, \beta, \lambda \in \mathcal{P}$, let $g^\lambda_{\alpha\,\beta}$ be the number of submodules M of V_λ such that $M \cong V_\beta$ and $V_\lambda/M \cong V_\alpha$. More generally, given $\alpha_1, \cdots, \alpha_t, \lambda \in \mathcal{P}$, we let $g^\lambda_{\alpha_1,\cdots,\alpha_t}$ be the number of the filtrations

$$0 = M_t \subseteq M_{t-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = V_\lambda,$$

such that $M_{i-1}/M_i \cong V_{\alpha_i}$ for all $1 \le i \le t$. For each $\lambda \in \mathcal{P}$, we set $a_\lambda = |\mathrm{Aut}\,(V_\alpha)|$, the order of the automorphism group of V_α.

We now recall the definition of the Ringel–Hall algebra of Λ and its double. Let R be a subfield of the real number field $\mathbb{R}$ containing v. Let $= \mathcal{H}^+(\Lambda)$ be an R-vector space with basis $\{\, K_\alpha u^+_\lambda : \alpha \in \mathbb{Z}[I], \lambda \in \mathcal{P}\}$. Then $\mathcal{H}^+(\Lambda)$ becomes a Hopf algebra in the following sense:

(a) Multiplication (Ringel [R1]):

$$\begin{array}{l} u^+_\alpha u^+_\beta = v^{\langle\alpha,\beta\rangle} \sum_{\lambda\in\mathcal{P}} g^\lambda_{\alpha,\beta} u^+_\lambda, \text{ for all } \alpha, \beta \in \mathcal{P}, \\ K_\alpha u^+_\beta = v^{(\alpha,\beta)} u^+_\beta K_\alpha, \text{ for all } \alpha \in \mathbb{Z}[I], \beta \in \mathcal{P}, \\ K_\alpha K_\beta = K_{\alpha+\beta}, \text{ for all } \alpha, \beta \in \mathbb{Z}[I], \end{array}$$

with unit $1 = u^+_0 = K_0$.

(b) Comultiplication (Green [G]):

$$\begin{array}{l} \Delta(u^+_\lambda) = \sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle} \frac{a_\alpha a_\beta}{a_\lambda} g^\lambda_{\alpha,\beta} u^+_\alpha K_\beta \otimes u^+_\beta, \text{ for all } \lambda \in \mathcal{P}, \\ \Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \text{ for all } \alpha \in \mathbb{Z}[I], \end{array}$$

with counit $\epsilon(u^+_\lambda) = 0$, for $\lambda \neq 0$ in $\mathcal{P}$, and $\epsilon(K_\alpha) = 1$.

(c) Antipode (see [X]):

$$\sigma(u^+_\lambda) = \delta_{\lambda 0} + \sum_{m\ge 1} (-1)^m \sum_{\pi,\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\Sigma_{i<j}\langle\lambda_i,\lambda_j\rangle} \times \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g^\lambda_{\lambda_1,\cdots,\lambda_m} g^\pi_{\lambda_1,\cdots,\lambda_m} K_{-\lambda} u^+_\pi,$$

for all $\lambda \in \mathcal{P}$, and $\sigma(K_\alpha) = K_{-\alpha}$ for all $\alpha \in \mathbb{Z}[I]$.

The Hopf algebra $\mathcal{H}^+(\Lambda)$ is called the (extended twisted) Ringel-Hall algebra of Λ. The subspace of $\mathcal{H}^+(\Lambda)$ with basis $\{u_\lambda^+ : \lambda \in \mathcal{P}\}$ is an associative subalgebra (but not closed under comultiplication), we denoted it by $\mathfrak{h}^+(\Lambda)$. We point out that, if V_i $(i \in I)$ is a simple Λ-module, then

$$\Delta(u_i^+) = u_i^+ \otimes 1 + K_i \otimes u_i^+, \text{ and } \sigma(u_i^+) = -K_{-i}u_i^+.$$

It is easy to see that $\mathcal{H}^+(\Lambda)$ and $\mathfrak{h}^+(\Lambda)$ have the canonical $\mathbb{N}[I]$-gradation.

1.5 Dually, one can define a Hopf algebra $\mathcal{H}^-(\Lambda)$ with basis $\{K_\alpha u_\lambda^- : \alpha \in \mathbb{Z}[I], \lambda \in \mathcal{P}\}$ and Hopf structure as follows:

(a) Multiplication:

$$\begin{aligned}
&u_\alpha^- u_\beta^- = v^{\langle \alpha,\beta\rangle} \textstyle\sum_{\lambda\in\mathcal{P}} g_{\alpha,\beta}^\lambda u_\lambda^-, \text{ for all } \alpha,\beta \in \mathcal{P},\\
&K_\alpha u_\beta^- = v^{-(\alpha,\beta)} u_\beta^- K_\alpha, \text{ for all } \alpha \in \mathbb{Z}[I], \beta \in \mathcal{P},\\
&K_\alpha K_\beta = K_{\alpha+\beta}, \text{ for all } \alpha,\beta \in \mathbb{Z}[I],
\end{aligned}$$

with unit $1 = u_0^- = K_0$.

(b) Comultiplication:

$$\begin{aligned}
&\Delta(u_\lambda^-) = \textstyle\sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle} \frac{a_\alpha a_\beta}{a_\lambda} g_{\alpha,\beta}^\lambda u_\beta^- \otimes u_\alpha^- K_{-\beta}, \text{ for all } \lambda \in \mathcal{P},\\
&\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \text{ for all } \alpha \in \mathbb{Z}[I],
\end{aligned}$$

with counit $\epsilon(u_\lambda^-) = 0$, for $\lambda \neq 0$ in $\mathcal{P}$, and $\epsilon(K_\alpha) = 1$.

(c) Antipode and its inverse:

$$\sigma(u_\lambda^-) = \delta_{\lambda 0} + \sum_{m\geq 1}(-1)^m \sum_{\pi,\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g_{\lambda_1,\cdots,\lambda_m}^\lambda g_{\lambda_m,\cdots,\lambda_1}^\pi u_\pi^- K_\lambda$$

for all $\lambda \in \mathcal{P}$, and $\sigma(K_\alpha) = K_{-\alpha}$ for all $\alpha \in \mathbb{Z}[I]$.

$$\begin{aligned}
\sigma^{-1}(u_\lambda^-) = \quad & \delta_{\lambda 0} + \textstyle\sum_{m\geq 1}(-1)^m \sum_{\pi,\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\Sigma_{i<j}\langle\lambda_i,\lambda_j\rangle}\\
& \times \tfrac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g_{\lambda_1,\cdots,\lambda_m}^\lambda g_{\lambda_1,\cdots,\lambda_m}^\pi K_\lambda u_\pi^-
\end{aligned}$$

for all $\lambda \in \mathcal{P}$, and $\sigma^{-1}(K_\alpha) = K_{-\alpha}$, for all $\alpha \in \mathbb{Z}[I]$. In particular, we have

$$\Delta(u_i^-) = u_i^- \otimes K_{-i} + 1 \otimes u_i^-, \text{ and } \sigma(u_i^-) = -u_i^- K_i, \text{ for all } i \in I.$$

By $\mathfrak{h}^-(\Lambda)$ we denote the subalgebra of $\mathcal{H}^-(\Lambda)$ with basis $\{\, u_\lambda^- : \lambda \in \mathcal{P}\}$.

1.6 Following Ringel (see [X]), there exists a bilinear form $\varphi : \mathcal{H}^+(\Lambda) \times \mathcal{H}^-(\Lambda) \to R$ defined by:

$$\varphi(K_\alpha u_\beta^+, K_{\alpha'} u_{\beta'}^-) = v^{-(\alpha,\alpha')-(\beta,\alpha')+(\alpha,\beta')} \frac{|V_\beta|}{a_\beta} \delta_{\beta\beta'}.$$

By [X], the bilinear form φ is a skew Hopf-paring (see the definition in [Jo] or [X]).

It is easy to see that the restriction of φ on $\mathfrak{h}^+(\Lambda)_\alpha \times \mathfrak{h}^-(\Lambda)_\beta$ for any $\alpha, \beta \in \mathbb{N}[I]$ is zero unless $\alpha = \beta$ and the restriction of φ on $\mathfrak{h}^+(\Lambda)_\alpha \times \mathfrak{h}^-(\Lambda)_\alpha$ is non-degenerate for any $\alpha \in \mathbb{N}[I]$. Therefore, we can form the Drinfeld double of $(\mathcal{H}^+(\Lambda), \mathcal{H}^-(\Lambda), \varphi)$. Its ideal generated by $\{ K_\alpha \otimes K_{-\alpha} - 1 : \alpha \in \mathbb{Z}[I]\}$, or equivalently by $\{K_\alpha \otimes 1 - 1 \otimes K_\alpha : \alpha \in \mathbb{Z}[I]\}$, is a Hopf ideal. The corresponding quotient inherits a Hopf structure, which is called the reduced Drinfeld double of the Ringel-Hall algebra of Λ and is denoted by $\mathcal{D}(\Lambda, R)$ (or simply $\mathcal{D}(\Lambda)$, see the construction in [X]). Obviously, we have the triangular decomposition

$$\mathcal{D}(\Lambda) = \mathfrak{h}^-(\Lambda) \otimes \mathcal{T} \otimes \mathfrak{h}^+(\Lambda),$$

where $\mathcal{T}$ denotes the torus generated by $\{K_\alpha : \alpha \in \mathbb{Z}[I]\}$.

The subalgebra of $\mathcal{D}(\Lambda)$ generated by $\{u_i^\pm, K_i : i \in I\}$ is called the double composition algebra of Λ and will be denoted by $\mathcal{C}(\Lambda)$. It is easy to see that $\mathcal{C}(\Lambda)$ is a Hopf subalgebra of $\mathcal{D}(\Lambda)$ and admits a triangular decomposition

$$\mathcal{C}(\Lambda) = \mathfrak{c}^-(\Lambda) \otimes \mathcal{T} \otimes \mathfrak{c}^+(\Lambda),$$

where $\mathfrak{c}^+(\Lambda)$ is the composition algebra of Λ, which is generated by $\{u_i^+ : i \in I\}$ and is still $\mathbb{N}[I]$-graded, and $\mathfrak{c}^-(\Lambda)$ is defined dually. Moreover, we have that, for any $\alpha \in \mathbb{N}[I]$, the restriction $\varphi : \mathfrak{c}^+(\Lambda)_\alpha \times \mathfrak{c}^-(\Lambda)_\alpha \to R$ is non-degenerate (see [HX, Lemma 1.5.2]).

1.7 Let (Γ, d, Ω) be a valued quiver (connected and without oriented cycles), $\mathcal{S} = (F_i, {}_iM_j)_{i,j\in\Gamma}$ a k-species of type Ω, and Λ the tensor algebra of $\mathcal{S}$. Let i be a sink or a source of (Γ, Ω), we define $\sigma_i\mathcal{S}$ to be the k-species obtained from $\mathcal{S}$ by replacing ${}_rM_s$ by its k-dual for $r = i$ or $s = i$; then $\sigma_i\mathcal{S}$ is a reduced k-species of type $\sigma_i\Omega$. By $\sigma_i\Lambda$ we denote the tensor algebra of $\sigma_i\mathcal{S}$.

In case i is a sink or a source, we have Bernstein-Gelfand-Ponomarev reflection functors $\sigma_p^\pm : \operatorname{rep}\mathcal{S} \to \operatorname{rep}\sigma_p\mathcal{S}$ which induces exact equivalences $\operatorname{rep}\mathcal{S}\langle i\rangle \to \operatorname{rep}\sigma_i\mathcal{S}\langle i\rangle$, where $\operatorname{rep}\mathcal{S}\langle i\rangle$ (resp. $\operatorname{rep}\sigma_i\mathcal{S}\langle i\rangle$) denotes the subcategory of $\operatorname{rep}\mathcal{S}$ (resp. $\operatorname{rep}\sigma_i\mathcal{S}$) of all representations which do not have a direct summand isomorphic to the simple V_i (see [BGP] and [DR]).

Let again i be a sink for $\mathcal{S}$. Set

$$\mathfrak{h}^+(\Lambda)\langle i\rangle = \mathfrak{h}^+(\Lambda)|\operatorname{rep}\mathcal{S}\langle i\rangle,$$

i.e. the R-subspace of $\mathfrak{h}^+(\Lambda)$ generated by u_α^+ with $V_\alpha \in \operatorname{rep}\mathcal{S}\langle i\rangle$. It is easy to see that $\mathfrak{h}^+(\Lambda)\langle i\rangle$ is an R-subalgebra of $\mathfrak{h}^+(\Lambda)$, hence of $\mathcal{H}^+(\Lambda)$. Similarly, set

$$\mathfrak{h}^+(\sigma_i\Lambda)\langle i\rangle = \mathfrak{h}^+(\sigma_i\Lambda)|\operatorname{rep}\sigma_i\mathcal{S}\langle i\rangle,$$

the R-subalgebra of $\mathfrak{h}^+(\sigma_i\Lambda)$ generated by u_α^+ with $V_\alpha \in \operatorname{rep}\sigma_i\mathcal{S}\langle i\rangle$.

Then we have

$$\mathfrak{h}^+(\Lambda) = \sum_{s\geq 0}(u_i^+)^s\mathfrak{h}^+(\Lambda)\langle i\rangle \quad \text{and} \quad \mathfrak{h}^+(\sigma_i\Lambda) = \sum_{s\geq 0}\mathfrak{h}^+(\sigma_i\Lambda)\langle i\rangle(u_i^+)^s.$$

Dually, the subalgebras $\mathfrak{h}^-(\Lambda)\langle i\rangle$ and $\mathfrak{h}^-(\sigma_i\Lambda)\langle i\rangle$ can be defined.

For each $i \in I$, there are derivations r_i and r_i' on $\mathfrak{h}(\Lambda)$ (thus also on $\mathfrak{h}^+(\Lambda)$ and $\mathfrak{h}^-(\Lambda)$, see [CX]) such that

$$r_i(1) = r_i'(1) = 0 \ \text{ and } \ r_i(u_j) = \frac{\delta_{ij}}{v^{(i,i)} - 1} = r_i'(u_j)$$

for all $i, j \in I$. Further, we have for all $\lambda_1, \lambda_2 \in \mathcal{P}$

$$r_i(u_{\lambda_1}u_{\lambda_2}) = u_{\lambda_1}r_i(u_{\lambda_2}) + v^{(i,\lambda_2)}r_i(u_{\lambda_1})u_{\lambda_2}$$

and

$$r_i'(u_{\lambda_1}u_{\lambda_2}) = v^{(i,\lambda_1)}u_{\lambda_1}r_i'(u_{\lambda_2}) + r_i'(u_{\lambda_1})u_{\lambda_2}.$$

If i is a sink, then it is easy to see that

$$\begin{aligned} &\mathfrak{h}^+(\Lambda)\langle i\rangle = \{x \in \mathfrak{h}^+(\Lambda) : r_i'(x) = 0\} \\ \text{and} \quad &\mathfrak{h}^+(\sigma_i\Lambda)\langle i\rangle = \{x \in \mathfrak{h}^+(\sigma_i\Lambda) : r_i(x) = 0\}. \end{aligned}$$

1.8 Let i be a sink. We recall from [XY] (see also [SV1]) the definition of an R–algebra isomorphism $T_i^R : \mathcal{D}(\Lambda) \to \mathcal{D}(\sigma_i\Lambda)$ for each $i \in I$ (Here we use notation T_i^R to indicate that this isomorphism is induced by the BGP–relection functor). To keep the notation in [XY], for each $\alpha \in \mathcal{P}$, we set $\langle u_\alpha^\pm\rangle = v^{-\dim V_\alpha + \langle\alpha,\alpha\rangle}u_\alpha^\pm$. Given a $\lambda \in \mathcal{P}$, we can write $V_\lambda = V_{\lambda_0} \oplus tV_i$ such that V_{λ_0} has no direct summand isomorphic to V_i. The $T_i^R : \mathcal{D}(\Lambda) \to \mathcal{D}(\sigma_i\Lambda)$ is defined by (see [XY, Theorem 4.5])

$$\begin{aligned} &T_i^R(\langle u_\lambda^+\rangle) = v^{\langle\lambda,\ ti\rangle}K_{ti}\langle u_i^-\rangle^{(t)}\langle u_{\sigma_i\lambda_0}^+\rangle, \\ &T_i^R(\langle u_\lambda^-\rangle) = v^{\langle\lambda,ti\rangle}K_{-ti}\langle u_i^+\rangle^{(t)}\langle u_{\sigma_i\lambda_0}^-\rangle, \\ &T_i^R(K_\alpha) = K_{s_i(\alpha)}. \end{aligned}$$

Conversely, one can define an R–algebra isomorphism $T_i' : \mathcal{D}(\sigma_i\Lambda) \to \mathcal{D}(\Lambda)$ which is the inverse of T_i.

Moreover, one can easily show that

$$\begin{aligned} &\mathfrak{h}^+(\Lambda)\langle i\rangle = \{x \in \mathfrak{h}^+(\Lambda) : T_i(x) \in \mathcal{H}^+(\sigma_i\Lambda)\} \\ \text{and} \quad &\mathfrak{h}^+(\sigma_i\Lambda)\langle i\rangle = \{x \in \mathfrak{h}^+(\sigma_i\Lambda) : T_i'(x) \in \mathcal{H}^+(\Lambda)\}. \end{aligned}$$

2 The decomposition of a double Ringel–Hall algebra

In [HX] the structure of double Ringel–Hall algebras of tame hereditary algebras has been studied. This section is devoted to generalizing results in [HX] to arbitrary hereditary algebras.

2.1 Let k be a field, (Γ, d, Ω) a valued quiver without oriented cycles, and $\mathcal{S}$ a k–species of type Ω. By Λ we denote the tensor algebra of $\mathcal{S}$, by $\mathcal{D}(\Lambda)$ the double Ringel–Hall algebra of Λ.

For $\alpha, \beta \in \mathbb{N}[I]$, we write $\alpha \leq \beta$ to signify that $\beta - \alpha \in \mathbb{N}[I]$. We now assume that π_1 is the minimal element in $\mathbb{N}[I]$ such that $\mathfrak{c}(\Lambda)_{\pi_1} \neq \mathfrak{h}(\Lambda)_{\pi_1}$. This implies that $\mathfrak{c}(\Lambda)_\alpha = \mathfrak{h}(\Lambda)_\alpha$ for all $\alpha < \pi_1$. Note that π_1 is necessarily the dimension vector of an indecomposable A–module because $\{u_\beta : \beta \in \mathcal{P}$ and V_β is indecomposable$\}$ gives rise to a universal PBW–basis of $\mathfrak{h}(\Lambda)$ (see [GP, Theorem 3.1]).

Further, we define

$$L^+_{\pi_1} = \{x^+ \in \mathfrak{h}^+(\Lambda)_{\pi_1} : \varphi(x^+, \mathfrak{c}^-(\Lambda)_{\pi_1}) = 0\}$$

and

$$L^-_{\pi_1} = \{y^- \in \mathfrak{h}^-(\Lambda)_{\pi_1} : \varphi(\mathfrak{c}^+(\Lambda)_{\pi_1}, y^-) = 0\}.$$

By the non–degeneracy of φ, we obtain

$$\mathfrak{h}^+(\Lambda)_{\pi_1} = \mathfrak{c}^+(\Lambda)_{\pi_1} \oplus L^+_{\pi_1}, \quad \mathfrak{h}^-(\Lambda)_{\pi_1} = \mathfrak{c}^-(\Lambda)_{\pi_1} \oplus L^-_{\pi_1},$$

$$\mathfrak{c}^+(\Lambda)_{\pi_1} = \{x^+ \in \mathfrak{h}^+(\Lambda)_{\pi_1} : \varphi(x^+, L^-_{\pi_1}) = 0\},$$

$$\mathfrak{c}^-(\Lambda)_{\pi_1} = \{y^- \in \mathfrak{h}^-(\Lambda)_{\pi_1} : \varphi(L^+_{\pi_1}, y^-) = 0\}.$$

Finally, we denote by $\mathfrak{d}^\pm_1(\Lambda)$ the subalgebra of $\mathfrak{h}^\pm(\Lambda)$ generated by $\mathfrak{c}^\pm(\Lambda)$ and $L^\pm_{\pi_1}$, respectively. Then both $\mathfrak{d}^-_1(\Lambda)$ and $\mathfrak{d}^+_1(\Lambda)$ are $\mathbb{N}[I]$–graded, and the restriction of φ to $\mathfrak{d}^+_1(\Lambda) \times \mathfrak{d}^-_1(\Lambda)$ is non–degenerated. Set $\mathcal{D}^\pm_1(\Lambda) = \mathfrak{d}^\pm_1(\Lambda) \otimes \mathcal{T}$ and $\mathcal{D}_1(\Lambda) = \mathfrak{d}^-_1(\Lambda) \otimes \mathcal{T} \otimes \mathfrak{d}^+_1(\Lambda)$.

Inductively, for $m > 1$, suppose that $\mathfrak{d}^\pm_{m-1}(\Lambda)$ and $\mathcal{D}_{m-1}(\Lambda)$ have been constructed. We then let π_m be the minimal element in $\mathbb{N}[I]$ such that $\mathfrak{d}^+_{m-1}(\Lambda)_{\pi_m} \neq \mathfrak{h}^+(\Lambda)_{\pi_m}$. This implies that π_m is the dimension vector of an indecomposable A–module.

As above, we define

$$L^+_{\pi_m} = \{x^+ \in \mathfrak{h}^+(\Lambda)_{\pi_m} : \varphi(x^+, \mathfrak{d}^-_{m-1}(\Lambda)_{\pi_m}) = 0\}$$

and

$$L^-_{\pi_m} = \{y^- \in \mathfrak{h}^-(\Lambda)_{\pi_m} : \varphi(\mathfrak{d}^+_{m-1}(\Lambda)_{\pi_m}, y^-) = 0\}.$$

By the non–degeneracy of the restriction of φ to $\mathfrak{d}^+_{m-1}(\Lambda)\times\mathfrak{d}^-_{m-1}(\Lambda)$, we obtain

$$\mathfrak{h}^+(\Lambda)_{\pi_m}=\mathfrak{d}^+_{m-1}(\Lambda)_{\pi_m}\oplus L^+_{\pi_m},\quad \mathfrak{h}^-(\Lambda)_{\pi_m}=\mathfrak{d}^-_{m-1}(\Lambda)_{\pi_m}\oplus L^-_{\pi_m},$$

$$\mathfrak{d}^+_{m-1}(\Lambda)_{\pi_m}=\{x^+\in\mathfrak{h}^+(\Lambda)_{\pi_m}:\varphi(x^+,L^-_{\pi_m})=0\},$$
$$\mathfrak{d}^-_{m-1}(\Lambda)_{\pi_m}=\{y^-\in\mathfrak{h}^-(\Lambda)_{\pi_m}:\varphi(L^+_{\pi_m},y^-)=0\}.$$

We now denote by $\mathfrak{d}^\pm_m(\Lambda)$ the subalgebra of $\mathfrak{h}^\pm(\Lambda)$ generated by $\mathfrak{d}^\pm_{m-1}(\Lambda)$ and $L^\pm_{\pi_m}$, respectively. Then both $\mathfrak{d}^-_m(\Lambda)$ and $\mathfrak{d}^+_m(\Lambda)$ are $\mathbb{N}[I]$–graded, and the restriction of φ to $\mathfrak{d}^+_m(\Lambda)\times\mathfrak{d}^-_m(\Lambda)$ is non–degenerated. Set $\mathcal{D}^\pm_m(\Lambda)=\mathfrak{d}^\pm_m(\Lambda)\otimes\mathcal{T}$ and $\mathcal{D}_m(\Lambda)=\mathfrak{d}^-_m(\Lambda)\otimes\mathcal{T}\otimes\mathfrak{d}^+_m(\Lambda)$.

As a conclusion, we obtain chains of subalgebras of $\mathfrak{h}^\pm(\Lambda)$ and of $\mathcal{D}(\Lambda)$

$$\mathfrak{d}^\pm_0(\Lambda)\subset\mathfrak{d}^\pm_1(\Lambda)\subset\cdots\mathfrak{d}^\pm_m(\Lambda)\subset\cdots\subset\mathfrak{h}^\pm(\Lambda),$$

$$\mathcal{D}_0(\Lambda)\subset\mathcal{D}_1(\Lambda)\subset\cdots\mathcal{D}_m(\Lambda)\subset\cdots\subset\mathcal{D}(\Lambda),$$

where we set $\mathfrak{d}_0(\Lambda)^\pm=\mathfrak{c}^\pm(\Lambda)$ and $\mathcal{D}_0(\Lambda)=\mathcal{C}(\Lambda)$.

2.2 Analogously to Lemmas 2.3.1 and 2.3.2 in [HX], we have the following lemma.

Lemma. *(1) For $m\geq 1$, the elements in $L^+_{\pi_m}$ are primitive, that is, for each $x^+\in L^+_{\pi_m}$, we have $\Delta(x^+)=x^+\otimes 1+K_{\pi_m}\otimes x^+$ and $\sigma(x^+)=-K_{-\pi_m}x^+$. Dually, for each $y^-\in L^-_{\pi_m}$, we have $\Delta(y^-)=y^-\otimes K_{-\pi_m}+1\otimes y^-$ and $\sigma(y^-)=-K_{\pi_m}y^-$.*

(2) For $x^+\in L^+_{\pi_m}$, $y^-\in L^-_{\pi_m}$, we have that

$$x^+y^--y^-x^+=-\varphi(x^+,y^-)(K_{\pi_m}-K_{-\pi_m})$$

2.3 For each $i\in I$, set $E_i(0)=u_i^+$ and $F_i(0)=-v_i^{-1}u_i^-$, $i\in I$, where $v_i=v^{(i,i)/2}$. Then, for $i,j\in I$, it holds

$$E_i(0)F_j(0)-F_j(0)E_i(0)=\frac{K_i-K_i^{-1}}{v_i-v_i^{-1}}\delta_{ij}.$$

For $m\geq 1$, let $\eta_m=\dim_R L^+_{\pi_m}=\dim_R L^+_{\pi_m}$. We then take an R–basis $\{E_p(m):1\leq p\leq\eta_m\}$ of $L^+_{\pi_m}$ and an R–basis $\{F_p(m):1\leq p\leq\eta_m\}$ of $L^-_{\pi_m}$ such that

$$\varphi(E_p(m),F_q(m))=\frac{-1}{v^m-v^{-m}}\delta_{pq}.$$

By Lemma 2.2(2), we have

$$E_p(m)F_q(n)-F_q(n)E_p(m)=\frac{K_{\pi_m}-K_{-\pi_m}}{v^m-v^{-m}}\delta_{pq}\delta_{mn}$$

for all $m, n \geq 1$, $1 \leq p \leq \eta_m$ and $1 \leq q \leq \eta_n$. In particular, we have

$$E_i(0)F_p(m) - F_p(m)E_i(0) = 0$$

and

$$E_p(m)F_i(0) - F_i(0)E_p(m) = 0$$

for $i \in I$, $m \geq 1$, and $1 \leq p \leq \eta_m$.

2.4 The following theorem is essential for the further study of the structure of a double Ringel–Hall algebra and its representation theory. The proof of the theorem will be given in Section 6. Note that the statement of the theorem has been mentioned in [SV2], but no proof was given there.

Theorem. *All π_m, $m \geq 1$, arising in the decomposition of the double Ringel–Hall algebra $\mathcal{D}(\Lambda)$ lie in the fundamental set, that is, $(\pi_m, i) \leq 0$ for all $i \in I$, and each π_m is a dimension vector of some indecomposable representation in rep $\mathcal{S}$.*

3 Uniqueness of skew–Hopf pairings

In [G] it is shown that certain pairings associated with a datum $(I, (\ ,\ ))$ are canonically unique. In this section we consider pairings associated with a datum together with some extra data. The uniqueness of such nondegenerate pairings is proved. As an application we see that the double Ringel–Hall algebras and their subalgebras constructed in the section 2 admit the structure of those pairings. As a result, it is shown that the Ringel–Hall algebras are independent of the orientations.

3.1 Following [G,3.1], by a datum we mean a pair $(I, (\ ,\ ))$ consisting of a set I and a symmetric, bilinear, $\mathbb{Z}$–valued form on $\mathbb{Z}[I]$ (the free Abelian group with I as basis). Such a datum is called a Cartan datum if I is a finite set and the following conditions are satisfied:

(i) $(i, i) \in \{2, 4, 6, \cdots\}$ for each $i \in I$,

(ii) $2(i, j)/(i, i) \in \{0, -1, -2, \cdots\}$ for any $i \neq j$ in I.

Given a datum $(I, (\ ,\ ))$, we choose certain $0 \neq \delta_j \in \mathbb{N}[I]$ for $j \in J$, where J is an index set. Note that δ_j and $\delta_{j'}$ may coincide for $j \neq j'$ in J. We shall always assume that for each fixed $j \in J$, the set $\{j' \in J : \delta_{j'} = \delta_j\}$ is finite. By $\tilde{C}$ we denote the triple $(I, (\ ,\ ), \{\delta_j : j \in J\})$. For convenience, in the following we set $\delta_i = i$ for each $i \in I$,.

3.2 Given a triple $\tilde{C} = (I, (\ ,\ ), \{\delta_j : j \in J\})$ as in 3.1, a skew–Hopf pairing (A^+, A^-, φ) is said to belong to $\tilde{C}$ or to be a member of $\mathcal{L}(\tilde{C})$ if the following conditions are satisfied:

(A^+1) $A^+ = \oplus_{\nu \in \mathbb{N}[I]} A^+_\nu$ is an $\mathbb{N}[I]$–graded, associative R–algebra generated by elements $x_i^+ \in A^+_{\delta_i}$, $i \in I \cup J$ (the disjoint union of I and J), and by $A^+_0 = \mathcal{T}$ such that

$$K_\alpha x_i^+ = v^{(\alpha,\delta_i)} x_i^+ K_\alpha$$

for all $i \in I \cup J$ and all $\alpha \in \mathbb{Z}[I]$.

(A^-1) $A^- = \oplus_{\nu \in \mathbb{N}[I]} A^-_\nu$ is an $\mathbb{N}[I]$–graded, associative R–algebra generated by elements $x_i^- \in A^-_{\delta_i}$, $i \in I \cup J$, and by $A^-_0 = \mathcal{T}$ such that

$$K_\alpha x_i^- = v^{-(\alpha,\delta_i)} x_i^- K_\alpha$$

for all $i \in I \cup J$ and all $\alpha \in \mathbb{Z}[I]$.

(A2)

$$\begin{array}{ll} \Delta_{A^+}(x_i^+) = x_i^+ \otimes 1 + K_{\delta_i} \otimes x_i^+, & \Delta_{A^+}(K_\alpha) = K_\alpha \otimes K_\alpha, \\ \Delta_{A^-}(x_i^-) = x_i^- \otimes K_{-\delta_i} + 1 \otimes x_i^-, & \Delta_{A^-}(K_\alpha) = K_\alpha \otimes K_\alpha \end{array}$$

for all $i \in I \cup J$ and all $\alpha \in \mathbb{Z}[I]$.

(A3)

$$\begin{array}{ll} \varphi(x_i^+, x_i^-) \neq 0 & \text{for all } i \in I \cup J, \\ \varphi(x_i^+, x_j^-) = 0 & \text{for all } i \neq j \in I \cup J, \\ \varphi(K_\alpha, K_\beta) = v^{-(\alpha,\beta)}, & \varphi(x_i^+, K_\beta) = \varphi(K_\alpha, x_i^-) = 0 \end{array}$$

for all $i \in I \cup J$ and $\alpha, \beta \in \mathbb{Z}[I]$.

Such a pairing (A^+, A^-, φ) is called a restricted nondegenerate member of $\mathcal{L}(\tilde{C})$ if its restricted form $\varphi : B^+ \times B^- \to R$ is nondegenerate, where B^+ (resp. B^-) denotes the subalgebra of A^+ (resp. A^-) generated by x_i^+ (resp. x_i^-), $i \in I \cup J$.

3.3 Examples. Let Λ be the hereditary algebra associated with a species $\mathcal{S}$ over a finite field k. Let $(I, (\ ,\ ))$ be the Cartan datum determined by Λ. By 2.1, we have subalgebras $\mathcal{D}_m(\Lambda)$, $m \geq 0$, of $\mathcal{D}(\Lambda)$. For each $m \geq 1$, we set $J_m = \{(t,p) : 1 \leq t \leq m, 1 \leq p \leq \eta_t\}$. We further set $J_0 = \emptyset$ and $J_\infty = \cup_{m\geq 1} J_m$. Finally, we set $\delta_{(m,p)} = \pi_m$ for all $m \geq 1$ and $1 \leq p \leq \eta_m$. Then it is easy to see that for each $m \geq 0$, $(\mathcal{D}^+_m(\Lambda), \mathcal{D}^-_m(\Lambda), \varphi)$ is a restricted nondegenerate skew–Hopf pairing belonging to $\tilde{C}_m := (I, (\ ,\ ), \{\delta_j : j \in J_m\})$ and that $(\mathcal{H}^+(\Lambda), \mathcal{H}^-(\Lambda), \varphi)$ is a restricted nondegenerate skew–Hopf pairing belonging to $\tilde{C}_\infty := (I, (\ ,\ ), \{\delta_j : j \in J_\infty\})$.

As the second example, we use the construction in [L, Chapter 1] to define the free object in $\mathcal{L}(\tilde{C})$ for a given $\tilde{C} = (I, (\ ,\ ), \{\delta_j : j \in J\})$. Let $\tilde{f}^+$ be the free R–algebra generated by Θ_i^+, $i \in I \cup J$ and $\tilde{F}^+$ the R–algebra generated by $\tilde{f}^+$ and $\mathcal{T}$ subject to the relations

$$K_\alpha \Theta_i^+ = v^{(\alpha,\delta_i)} \Theta_i^+ K_\alpha$$

for all $i \in I \cup J$ and $\alpha \in \mathbb{Z}[I]$.

The algebra $\tilde{F}^+$ has a natural $\mathbb{N}[I]$–gradation. Indeed, for any $\nu \in \mathbb{N}[I]$, we denote by $\tilde{F}^+_\nu$ the $\mathcal{T}$–submodule of $\tilde{F}^+$ spanned by all monomials $\Theta_{i_1}^+\Theta_{i_2}^+ \cdots \Theta_{i_s}^+$ such that $\sum_{t=1}^s \delta_{i_t} = \nu$. The Hopf algebra structure of $\tilde{F}^+$ is given by

$$\begin{aligned}
&\Delta(\Theta_i^+) = \Theta_i^+ \otimes 1 + K_{\delta_i} \otimes \Theta_i^+, \quad \Delta(K_\alpha) = K_\alpha \otimes K_\alpha,\\
&\varepsilon(\Theta_i^+) = 0, \quad \varepsilon(K_\alpha) = 0,\\
&\sigma(\Theta_i^+) = -K_{-\delta_i}\Theta_i^+, \quad \sigma(K_\alpha) = K_{-\alpha}
\end{aligned}$$

for all $i \in I \cup J$ and $\alpha \in \mathbb{Z}[I]$. It is easy to verify that $\tilde{F}^+$ is a Hopf algebra.

Dually, we define $\tilde{f}^-$ to be the free R–algebra generated by $\Theta_i^-, i \in I \cup J$ and $\tilde{F}^-$ to be the R–algebra generated by $\tilde{f}^-$ and $\mathcal{T}$ subject to the relations

$$K_\alpha \Theta_i^- = v^{-(\alpha,\delta_i)} \Theta_i^- K_\alpha$$

for all $i \in I \cup J$ and $\alpha \in \mathbb{Z}[I]$.

The algebra $\tilde{F}^-$ has also a natural $\mathbb{N}[I]$–gradation. The Hopf algebra structure of $\tilde{F}^-$ is given by

$$\begin{aligned}
&\Delta(\Theta_i^-) = \Theta_i^- \otimes K_{\delta_i} + 1 \otimes \Theta_i^-, \quad \Delta(K_\alpha) = K_\alpha \otimes K_\alpha,\\
&\varepsilon(\Theta_i^-) = 0, \quad \varepsilon(K_\alpha) = 0,\\
&\sigma(\Theta_i^-) = -\Theta_i^- K_{\delta_i}, \quad \sigma(K_\alpha) = K_{-\alpha}
\end{aligned}$$

for all $i \in I \cup J$ and $\alpha \in \mathbb{Z}[I]$.

Analogusly to [Lusztig, Proposition 1.2.3], there exists a unique pairing $\varphi : \tilde{F}^+ \times \tilde{F}^- \to R$ satifying $\varphi(1,1) = 1$ and

(a) $\varphi(\Theta_i^+, \Theta_j^-) = \delta_{ij}(1 - v^{-(\delta_i,\delta_i)})^{-1}$, $\varphi(K_\alpha, K_\beta) = v^{-(\alpha,\beta)}$ for all $i, j \in I \cup J$ and all $\alpha, \beta \in \mathbb{Z}[I]$,

(b) $\varphi(x, yy') = \varphi(\Delta(x), y \otimes y')$ for all $x \in \tilde{F}^+$, $y, y' \in \tilde{F}^-$,

(c) $\varphi(xx', y) = \varphi(x \otimes x', \Delta^{\mathrm{opp}}(y))$ for all $x, x' \in \tilde{F}^+$, $y \in \tilde{F}^-$.

Moreover, the pairing φ is a skew–Hopf pairing belonging to $\tilde{C}$.

With the pairing $(\tilde{F}^+, \tilde{F}^-, \varphi)$ we now associate a restricted nondegenerate pairing in $\mathcal{L}(\tilde{C})$ as follows.

We define

$$\mathcal{I}_0^+ = \{x \in \tilde{f}^+ : \varphi(x, \tilde{f}^-) = 0\} = \{x \in \tilde{f}^+ : \varphi(x, \tilde{F}^-) = 0\}$$

and set $\mathcal{I}^+ = \mathcal{T}\mathcal{I}_0^+ \cong \mathcal{T} \otimes \mathcal{I}_0^+$. It is easy to show that $\mathcal{I}_0^+$ and $\mathcal{I}^+$ are respectively two–sided ideals of $\tilde{f}^+$ and $\tilde{F}^+$.

Dually, we define

$$\mathcal{I}_0^- = \{y \in \tilde{f}^- : \varphi(\tilde{f}^+, y) = 0\} = \{y \in \tilde{f}^- : \varphi(\tilde{F}^+, y) = 0\}$$

and set $\mathcal{I}^- = \mathcal{T}\mathcal{I}_0^- \cong \mathcal{T} \otimes \mathcal{I}_0^-$. They are respectively two–sided ideals of $\tilde{f}^-$ and $\tilde{F}^-$.

Further, we have

$$\Delta(\mathcal{I}^+) \subseteq \mathcal{I}^+ \otimes \tilde{F}^+ + \tilde{F}^+ \otimes \mathcal{I}^+ \text{ and } \Delta(\mathcal{I}^-) \subseteq \mathcal{I}^- \otimes \tilde{F}^- + \tilde{F}^- \otimes \mathcal{I}^-,$$

that is, $\mathcal{I}^+$ and $\mathcal{I}^-$ are respectively Hopf ideals of $\tilde{F}^+$ and $\tilde{F}^-$.

Finally, we set $f^\pm = \tilde{f}^\pm/\mathcal{I}_o^\pm$ and $F^\pm = \tilde{F}^\pm/\mathcal{I}^\pm$. Since $\mathcal{I}_0^\pm$ and $\mathcal{I}^\pm$ are compatible with the weight decompositions of $\tilde{f}^\pm$ and $\tilde{F}^\pm$, $f^\pm$ and $F^\pm$ have also $\mathbb{N}[I]$–gradation. Moreover, F^+ and F^- admit Hopf algebra structure induced by that of $\tilde{F}^+$ and $\tilde{F}^-$, respectively. Then the pairing $(\tilde{F}^+, \tilde{F}^-, \varphi)$ induces a skew–Hopf pairing (F^+, F^-, ϕ) which is obviously a restricted nondegenerate member in $\mathcal{L}(\tilde{C})$.

3.4 Let $C = (I, (\ ,\ ), \{\delta_j : j \in J\})$ be a triple in 3.1. For each $\mu = \sum_{i \in I \cup J} \mu_i i \in \mathbb{N}[I \cup J]$, we set $w(\mu) = \sum_{i \in I \cup J} \mu_i \delta_i \in \mathbb{N}[I]$. Furhter, we define $I(\mu)$ to be the set of all sequences $a = (a_1, a_2, \cdots, a_p)$ with $a_s \in I \cup J$ which satisfy

$$\mu_i = |\{s : 1 \leq s \leq p, a_s = i\}|$$

for all $i \in I \cup J$. By convention, we define $I(0)$ to consist of a single element $\emptyset$. Finally, we set

$$I(\infty) = \bigcup_{\nu \in \mathbb{N}[I]} \bigcup_{\mu, w(\mu) = \nu} I(\mu).$$

Let (A^+, A^-, φ) be a skew–Hopf pairing in $\mathcal{L}(\tilde{C})$. Then for each $\nu \in \mathbb{N}[I]$, A_ν^+ (resp. A_ν^-) is the $\mathcal{T}$–span of the monomials $x_a^+ = x_{a_1}^+ x_{a_2}^+ \cdots x_{a_p}^+$ (resp. $x_a^- = x_{a_1}^- x_{a_2}^- \cdots x_{a_p}^-$) with $a \in \cup_{w(\mu)=\nu} I(\mu)$. In case $\nu = 0$, we set $x_\emptyset^\pm = 1$.

Lemma. *Let (A^+, A^-, φ) be a skew–Hopf pairing in $\mathcal{L}(\tilde{C})$. Then for $a \in I(\mu), b \in I(\mu'), \alpha, \beta \in \mathbb{N}[I]$, we have*

$$\varphi(K_\alpha x_a^+, K_\beta x_b^-) = \begin{cases} v^{-(\alpha,\beta)-(\nu,\beta)+(\alpha,\nu)} \varphi(x_a^+, x_b^-) & \text{if } \mu = \mu' \\ 0 & \text{if } \mu \neq \mu', \end{cases}$$

where $\nu = w(\mu)$.

Proof. Let $a = (a_1, \cdots, a_p) \in I(\mu)$ and $b = (b_1, \cdots, b_q) \in I(\mu')$. Then $x_a^+ = x_{a_1}^+ \cdots x_{a_p}^+$ and $x_b^- = x_{b_1}^- \cdots x_{b_q}^-$.

If $p=0$ or $q=0$, the equality in the lemma holds obviously. In case $p=1$ and $q\geq 1$, we have

$$\begin{array}{rl} & \varphi(K_\alpha x_a^+, K_\beta x_b^-) = \varphi(\Delta_{A^+}(K_\alpha x_{a_1}^+), K_\beta x_{b_1}^- \cdots x_{b_{q-1}}^- \otimes x_{b_q}^-) \\ = & \varphi(K_\alpha x_{a_1}^+ \otimes K_\alpha + K_{\alpha+\delta_{a_1}} \otimes K_\alpha x_{a_1}^+, K_\beta x_{b_1}^- \cdots x_{b_{q-1}}^- \otimes x_{b_q}^-) \\ = & \varphi(K_{\alpha+\delta_{a_1}}, K_\beta x_{b_1}^- \cdots x_{b_{q-1}}^-)\varphi(K_\alpha x_{a_1}^+, x_{b_q}^-). \end{array}$$

This implies that $\varphi(K_\alpha x_a^+, K_\beta x_b^-)=0$ unless $q=1$ and $a_1=b_1$, i.e. $\mu=\mu'$. Moreover, we have

$$\begin{array}{rl} & \varphi(K_\alpha x_{a_1}^+, K_\beta x_{a_1}^-) = v^{-(\alpha+\nu,\beta)}\varphi(K_\alpha \otimes x_{a_1}^+, \Delta_{A^-}^{\mathrm{opp}}(x_{a_1}^-)) \\ = & v^{-(\alpha,\beta)-(\nu,\beta)}\varphi(K_\alpha \otimes x_{a_1}^+, K_{-\nu}\otimes x_{a_1}^- + x_{a_1}^-\otimes 1) \\ = & v^{-(\alpha,\beta)-(\nu,\beta)+(\alpha,\nu)}\varphi(x_{a_1}^+, x_{a_1}^-) \end{array}$$

Now let $p>1$ and $q\geq 1$. We then have

$$\begin{array}{rl} & \varphi(K_\alpha x_a^+, K_\beta x_b^-) = \varphi(K_\alpha x_{a_1}^+\cdots x_{a_{p-1}}^+ \otimes x_{a_p}^+, \Delta_{A^-}^{\mathrm{opp}}(K_\beta x_{b_1}^-\cdots x_{b_{q-1}}^- x_{b_q}^-)) \\ = & \varphi(K_\alpha x_{a_1}^+\cdots x_{a_{p-1}}^+ \otimes x_{a_p}^+, (K_\beta\otimes K_\beta)(K_{-\delta_{b_1}}\otimes x_{b_1}^- + x_{b_1}^-\otimes 1)\cdots \\ & \qquad\qquad (K_{-\delta_{b_q}}\otimes x_{b_q}^- + x_{b_q}^-\otimes 1)) \end{array}$$

By induction on p, we finally get

$$\varphi(K_\alpha x_a^+, K_\beta x_b^-) = \begin{cases} v^{-(\alpha,\beta)-(\nu,\beta)+(\alpha,\nu)}\varphi(x_a^+, x_b^-) & \text{if } \mu=\mu' \\ 0 & \text{if } \mu\neq\mu'. \end{cases}$$

Remark. The Hopf algebra $\tilde{F}^+$ defined in 3.3 admits an $\mathbb{N}[I\cup J]$–gradation. In fact, for each $\mu\in\mathbb{N}[I\cup J]$, we denote by $\tilde{F}^+_\mu$ the $\mathcal{T}$–submodule spanned by all monomials $\Theta_{i_1}^+\Theta_{i_2}^+\cdots\Theta_{i_s}^+$ such that for each $i\in I\cup J$, the occurrence of i in the sequence $(i_1,i_2,\cdots,i_s)$ is equal to μ_i. In view of Lemma 3.4, the Hopf ideal $\mathcal{I}^+$ of $\tilde{F}^+$ defined in 3.3 is compatible with the $\mathbb{N}[I\cup J]$–gradation of $\tilde{F}^+$. Dually, $\tilde{F}^-$ admits also $\mathbb{N}[I\cup J]$–gradation, and the Hopf ideal $\mathcal{I}^-$ is compatible with the $\mathbb{N}[I\cup J]$–gradation of $\tilde{F}^-$. Therefore, both F^+ and F^- have an $\mathbb{N}[I\cup J]$–gradation which is a refined weight decomposition of the original weight decomposition of F^+ and F^-.

3.5 Lemma *Given a triple $\tilde{C}=(I,(\ ,\ ),\{\delta_j : j\in J\}$ in 3.1. Then, for any $\mu\in\mathbb{N}[I\cup J]$ and $a,b\in I(\mu)$, there exists an element $M_{a,b}(t)\in\mathbb{Z}[t,t^-]$ (t indeterminate) such that for any skew–Hopf pairing (A^+,A^-,φ) in $\mathcal{L}(\tilde{C})$, it holds that*

$$\varphi(x_a^+, x_b^-) = M_{a,b}(v)\prod_{i\in I\cup J}\varphi(x_i^+, x_i^-)^{\mu_i}.$$

For the proof of the lemma we refer to [G, Proposition 3.2a] (see also [X, Proposition 3.3].

3.6 Two skew–Hopf pairings (A^+, A^-, φ) and (A'^+, A'^-, φ) in $\mathcal{L}(\tilde{C})$ are said to be canonically isomorphic if there are Hopf algebra isomorphisms $f : A^+ \to A'^+$ and $f : A^- \to A'^-$ such that $f(x_i^\pm) = x_i'^\pm$ for all $i \in I \cup J$ and f preserves $\mathcal{T} = A_0 = A'_0$ elementwise.

Analogously to [G] (see also [X,Theorem 3.6]), we have the following

Proposition. *Let $\tilde{C} = (I, (\ ,\ ), \{\delta_j : j \in J\}$ be a triple in 3.1. Then any two restricted nondegenerate skew–Hopf pairings in $\mathcal{L}(\tilde{C})$ are canonically isomorphic.*

3.7 Let $\mathcal{S}$ be a k–species, Λ the tensor algebra of $\mathcal{S}$, and $\mathcal{D}(\Lambda)$ the double Hall algebra of Λ. By $(I, (\ ,\ ))$ we denote the associated Cartan datum of Λ. In 3.3, we have seen that $(\mathcal{D}(m)^+, \mathcal{D}(m)^-, \varphi)$ is a restricted nondegenerate skew–Hopf pairing belonging to $\tilde{C}_m = (I, (\ ,\ ), \{\delta_j : j \in J_m\})$ for $m \in \mathbb{N} \cup \{\infty\}$.

Now let i be a sink of Γ, $\sigma_i\Lambda$ the tensor algebra of $\sigma_i\mathcal{S}$. By 2.1, we also obtain subalgebras $\mathcal{D}_m(\sigma_i\Lambda)$ of $\mathcal{D}(\sigma_i\Lambda)$ for $m \geq 0$. It is well known that both $(\mathcal{D}_0^+(\Lambda), \mathcal{D}_0^-(\Lambda), \varphi)$ and $(\mathcal{D}_0^+(\sigma_i\Lambda), \mathcal{D}_0^-(\sigma_i\Lambda), \varphi)$ are nondegenerate skew–Hopf pairings belonging to $\tilde{C}_0$ and are canonically isomorphic. By [H], this then implies that $(\mathcal{D}_1^+(\sigma_i\Lambda), \mathcal{D}_1^-(\sigma_i\Lambda), \varphi)$ is also a nondegenerate skew–Hopf pairing belonging to $\tilde{C}_1$. Thus, by Proposition 3.6, $(\mathcal{D}_1^+(\Lambda), \mathcal{D}_1^-(\Lambda), \varphi)$ and $(\mathcal{D}_1^+(\sigma_i\Lambda), \mathcal{D}_1^-(\sigma_i\Lambda), \varphi)$ are canonically isomorphic. Inductively, we obtain that, for each $m \geq 1$, the pairings $(\mathcal{D}_m^+(\Lambda), \mathcal{D}_m^-(\Lambda), \varphi)$ and $(\mathcal{D}_m^+(\sigma_i\Lambda), \mathcal{D}_m^-(\sigma_i\Lambda), \varphi)$ are restricted nondegenerate members in $\mathcal{L}(\tilde{C}_m)$ and thus are canonically isomorphic. Finally, we have that $(\mathcal{H}(\Lambda)^+, \mathcal{H}(\Lambda)^-, \varphi)$ and $(\mathcal{H}(\sigma_i\Lambda)^+, \mathcal{H}(\sigma_i\Lambda)^-, \varphi)$ are restricted nondegenerate members in $\mathcal{L}(\tilde{C}_\infty)$ and are canonically isomorphic. As a consequence, we have the following theorem.

Theorem. *There exist canonical Hopf algebra isomorphisms $\Phi_m : \mathcal{D}_m(\Lambda) \to \mathcal{D}_m(\sigma_i\Lambda)$ for each $m \geq 0$ and $\Phi : \mathcal{D}(\Lambda) \to \mathcal{D}(\sigma_i\Lambda)$ such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{D}_m(\Lambda) & \subseteq & \mathcal{D}_{m+1}(\Lambda) & \subseteq & \mathcal{D}(\Lambda) \\ \Phi_m \downarrow & & \downarrow \Phi_{m+1} & & \downarrow \Phi \\ \mathcal{D}_m(\sigma_i\Lambda) & \subseteq & \mathcal{D}_{m+1}(\sigma_i\Lambda) & \subseteq & \mathcal{D}(\sigma_i\Lambda) \end{array}$$

In particular, the Hall algebras $\mathfrak{h}(\Lambda)$ and $\mathfrak{h}(\sigma_i\Lambda)$ are canonically isomorphic.

Remark. 1. In case Λ is a path algebra of a finite quiver Q, Lusztig [L, 13] states that by using a notion of Fourier transform one can prove that the Ringel–Hall algebra $\mathfrak{h}(\Lambda)$ is independent of the orientation of Q. A complete proof of this statement has been provided by Sevenhant and van den Bergh in [SV1, Theorem 7.1].

2. The canonical isomorphisms Φ_m ($m \geq 1$) and Φ in the theorem are not unique and depend on the choice of basis elements $E_p(m)$ of $L^+(\Lambda)_{\pi_m}$ and $L^+(\sigma_i\Lambda)_{\pi_m}$.

3.8 Let $C = (I, (\, , \,))$ be the Cartan datum of the finite dimesional hereditary algebra Λ over k and $\tilde{C} = (I, (\, , \,), \{\delta_j : j \in J\})$ be the triple corresponding to $\mathcal{D}(\Lambda)$, where $J = \{(m,p) : m \geq 1, 1 \leq p \leq \eta_m\}$ and $\delta_j = \pi_m$ for each $j = (m,p) \in J$ (see 3.1 and 3.3). Then $(\mathcal{H}^+(\Lambda), \mathcal{H}^-(\Lambda), \varphi)$ is a restricted nondegenerate member in $\mathcal{L}(\tilde{C})$. From $\tilde{C}$ we define a new datum $\tilde{C}' = (I \cup J, (\, , \,)')$, where $(i,j)' = (\delta_i, \delta_j)$ for all $i, j \in I \cup J$. Note that $\delta_i = i$ if $i \in I$. We have the reduced Drinfeld double $\mathcal{D}(\Lambda)$ and $\mathcal{D}'(\Lambda)$ of the restricted nondegenerate members in $\mathcal{L}(\tilde{C})$ and $\mathcal{L}(\tilde{C}')$, respectively. The proposition 3.6 can then be restated as follows.

Proposition. *There exits a Hopf algebra epimorphism $F : \mathcal{D}'(\Lambda) \to \mathcal{D}(\Lambda)$ such that $F(x_i^{\pm}) = x_i^{\pm}$ and $F(K_i) = K_{\delta_i}$ for $i \in I \cup J$, and $\ker F$ is the ideal generated by $\{K_j - K_{\delta_j} : j \in J\}$.*

4 R–matrices and quantum Yang-Baxter relation

This section is devoted to giving an isomorphism from $M \otimes M'$ to $M' \otimes M$ for certain modules over a double Ringel–Hall algebra $\mathcal{D}(\Lambda)$. The construction for such an isomorphism is standard (see for example, [J]), but here certain extra computations are involved. We keep all notations in Section 2.

4.1. The double Ringel–Hall algebra admits two operators ω and τ which are defined respectively by

$$\begin{array}{l} \omega(u_\lambda^+) = u_\lambda^-, \ \omega(u_\lambda^-) = u_\lambda^+ \ \text{ for all } \ \lambda \in \mathcal{P} \\ \omega(K_\alpha) = K_{-\alpha} \ \text{ for all } \ \alpha \in \mathbb{Z}[I] \end{array}$$

and by

$$\begin{array}{l} \tau(x_\alpha) = (-1)^{\mathrm{tr}\alpha+1} v^{-\tau(\alpha)} K_\alpha \sigma(x_\alpha) \ \text{ for } \ x_\alpha \in \mathfrak{h}^+(\Lambda)_\alpha, \alpha \in \mathbb{N}[I], \\ \tau(y_\alpha) = (-1)^{\mathrm{tr}\alpha+1} v^{\tau(\alpha)} \sigma(y_\alpha) K_{-\alpha} \ \text{ for } \ y_\alpha \in \mathfrak{h}^-(\Lambda)_\alpha, \alpha \in \mathbb{N}[I], \\ \tau(K_\alpha) = K_{-\alpha} \ \text{ for } \ \alpha \in \mathbb{Z}[I], \end{array}$$

where $\mathrm{tr}\alpha = \sum_i k_i$ and $\tau(\alpha) = \sum_i k_i(i,i)/2$ for $\alpha = \sum_i k_i i \in \mathbb{Z}[I]$. The operators ω and τ are respectively an involution and an anti–automorphism of $\mathcal{D}(\Lambda)$, and both of them respect the bilinear form φ, that is,

$$\varphi(x,y) = \varphi(\omega(y), \omega(x)) = \varphi(\tau(x), \tau(y)) \ \text{ for any } \ x \in \mathfrak{h}^+(\Lambda), y \in \mathfrak{h}^-(\Lambda)$$

By Lemma 2.2(1), we get easily the following

$$\begin{aligned}\tau(E_p(m)) &= (-1)^{\mathrm{tr}(\pi_m)} v^{-\tau(\pi_m)} E_p(m),\\ \tau(F_p(m)) &= (-1)^{\mathrm{tr}(\pi_m)} v^{\tau(\pi_m)} F_p(m)\end{aligned}$$

for $m \geq 1$, $1 \leq p \leq \eta_m$.

4.2. According to the comultiplication formula of Green, for any $x \in \mathfrak{h}^+(\Lambda)_\alpha$, we may assume that

$$\Delta(x) = x \otimes 1 + r_{p,m}(x) K_{\pi_m} \otimes E_p(m) + \text{ "other terms"},$$

and

$$\Delta(x) = K_\alpha \otimes x + E_p(m) K_{\alpha - \pi_m} \otimes r'_{p,m}(x) + \text{ "other terms"},$$

where $r_{p,m}(x), r'_{p,m}(x) \in R$. It then follows that

$$r_{p,m}(E_q(n)) = \delta_{mn}\delta_{pq} = r'_{p,m}(E_q(n)).$$

For all $x \in \mathfrak{h}^+(\Lambda)_\alpha$ and $x' \in \mathfrak{h}^+(\Lambda)_\beta$, we have

$$r_{p,m}(xx') = x r_{p,m}(x') + v^{(\pi_m,\beta)} r_{p,m}(x) x', \quad r'_{p,m}(xx') = v^{(\pi_m,\alpha)} x r'_{p,m}(x') + r'_{p,m}(x) x'.$$

Further, for all $x \in \mathfrak{h}^+(\Lambda)_\alpha$ and $y \in \mathfrak{h}^-(\Lambda)$, we have

$$\begin{aligned}\varphi(x, F_p(m) y) &= \tfrac{-1}{v^m - v^{-m}} \varphi(r'_{p,m}(x), y),\\ \varphi(x, y F_p(m)) &= \tfrac{-1}{v^m - v^{-m}} \varphi(r_{p,m}(x), y),\\ r'_{p,m}(x) &= (-1)^{\mathrm{tr}\pi_m} v^{\tau(\pi_m)} \tau r_{p,m} \tau(x).\end{aligned}$$

Dually, for any $y \in \mathfrak{h}^-(\Lambda)_\alpha$, we may assume that

$$\Delta(y) = y \otimes K_{-\alpha} + r_{p,m}(y) \otimes F_p(m) K_{-(\alpha - \pi_m)} + \text{ "other terms"}.$$

and

$$\Delta(y) = 1 \otimes y + F_p(m) \otimes r'_{p,m}(y) K_{-\pi_m} + \text{ "other terms"}.$$

It then follows that

$$r_{p,m}(F_q(n)) = \delta_{mn}\delta_{pq} = r'_{p,m}(F_q(n)).$$

For all $y \in \mathfrak{h}^-(\Lambda)_\alpha$ and $y' \in \mathfrak{h}^-(\Lambda)_\beta$, we get

$$r_{p,m}(yy') = v^{(\pi_m,\alpha)} y r_{p,m}(y') + r_{p,m}(y) y', \quad r'_{p,m}(yy') = y r'_{p,m}(y') + v^{(\pi_m,\beta)} r'_{p,m}(y) y'.$$

Further, for all $x \in \mathfrak{h}^+(\Lambda)$ and $y \in \mathfrak{h}^-(\Lambda)_\alpha$, we obtain

$$\begin{aligned}\varphi(E_p(m) x, y) &= \tfrac{-1}{v^m - v^{-m}} \varphi(x, r_{p,m}(y)),\\ \varphi(x E_p(m), y) &= \tfrac{-1}{v^m - v^{-m}} \varphi(x, r'_{p,m}(y)),\\ r_{p,m}(y) &= \omega r'_{p,m}(\omega(y)), \quad r'_{p,m}(y) = \omega r_{p,m}(\omega(y)).\end{aligned}$$

Lemma. *For any $x \in \mathfrak{h}^+(\Lambda)$, $y \in \mathfrak{h}^-(\Lambda)$ and $m \geq 1$, $1 \leq p \leq \eta_m$, we have*

$$E_p(m)y - yE_p(m) = \frac{K_{\pi_m} r_{p,m}(y) - r'_{p,m}(y)K_{-\pi_m}}{v^m - v^{-m}},$$

$$xF_p(m) - F_p(m)x = \frac{r_{p,m}(x)K_{\pi_m} - K_{-m\pi_m} r'_{p,m}(x)}{v^m - v^{-m}}.$$

Proof. We prove the first equality only.

It is obvious that the equality holds for y to be the generators $F_i(0)$, $i \in I$, and $F_p(m)$, $m \geq 1$, $1 \leq p \leq \eta_m$. Inductively, for $y \in \mathfrak{h}^-(\Lambda)_\alpha$, $y' \in \mathfrak{h}^-(\Lambda)_\beta$, we obtain

$$\begin{aligned}
& E_p(m)(yy') - (yy')E_p(m) \\
= \; & (E_p(m)y - yE_p(m))y' + y(E_p(m)y' - y'E_p(m)) \\
= \; & \tfrac{K_{\pi_m} r_{p,m}(y) - r'{}_{r,p}(y)K_{-\pi_m}}{v^m - v^{-m}} y' + y \tfrac{K_{\pi_m} r_{p,m}(y') - r'{}_{r,p}(y')K_{-\pi_m}}{v^m - v^{-m}} \\
= \; & \tfrac{(K_{\pi_m} r_{r,p}(y)y' + yK_{\pi_m} r_{p,m}(y')) - (r'_{p,m}(y)K_{-\pi_m} y' + yr'_{p,m}(y')K_{-\pi_m}}{v^m - v^{-m}} \\
= \; & \tfrac{K_{\pi_m} r_{p,m}(yy') - r'_{p,m}(yy')K_{-\pi_m}}{v^m - v^{-m}}.
\end{aligned}$$

4.3. For each $\alpha \in \mathbb{N}[I]$, we choose an R-basis $x_\alpha^1, \cdots, x_\alpha^{\gamma(\alpha)}$ of $\mathfrak{h}^+(\Lambda)_\alpha$. By the Ringel pairing, there exists an R-basis $y_\alpha^1, \cdots, y_\alpha^{\gamma(\alpha)}$ of $\mathfrak{h}^-(\Lambda)_\alpha$ such that $\varphi(x_\alpha^s, y_\alpha^t) = \delta_{st}$ for all $1 \leq s, t \leq \gamma(\alpha)$. Set

$$\Theta_\alpha = \sum_{s=1}^{\gamma(\alpha)} y_\alpha^s \otimes x_\alpha^s \in \mathcal{D}(\Lambda) \otimes \mathcal{D}(\Lambda).$$

By linear algebra, Θ_α does not depend on the choice of the basis $(x_\alpha^s)_s$. By (4.1), we have

$$(\tau \otimes \tau)\Theta_\alpha = \Theta_\alpha \quad \text{and} \quad (\omega \otimes \omega)\Theta_\alpha = \Theta_\alpha^{op}.$$

where $\Theta_\alpha^{op} = \sum_s x_\alpha^s \otimes y_\alpha^s$.

Lemma. *Let $\alpha \in \mathbb{N}[I]$. For each $i \in I$, we have*

(1) $(E_i(0) \otimes 1)\Theta_\alpha + (K_i \otimes E_i(0))\Theta_{\alpha-i} = \Theta_\alpha(E_i(0) \otimes 1) + \Theta_{\alpha-i}(K_{-i} \otimes E_i(0))$,
(2) $(1 \otimes F_i(0))\Theta_\alpha + (F_i(0) \otimes K_{-i}^{-1})\Theta_{\alpha-i} = \Theta_\alpha(1 \otimes F_i(0)) + \Theta_{\alpha-i}(F_i(0) \otimes K_i)$.

For each $m \geq 1$ and each $1 \leq p \leq \eta_m$, we have

(3)

$$\begin{aligned}
& (E_p(m) \otimes 1)\Theta_\alpha + (K_{\pi_m} \otimes E_p(m))\Theta_{\alpha-\pi_m} \\
= \; & \Theta_\alpha(E_p(m) \otimes 1) + \Theta_{\alpha-\pi_m}(K_{-\pi_m} \otimes E_p(m)),
\end{aligned}$$

(4)
$$\begin{aligned} & (1\otimes F_p(m))\Theta_\alpha + (F_p(m)\otimes K_{-\pi_m})\Theta_{\alpha-\pi_m} \\ = \ & \Theta_\alpha(1\otimes F_p(m)) + \Theta_{\alpha-\pi_m}(F_p(m)\otimes K_{\pi_m}). \end{aligned}$$

The proof of this lemma is similar to that of Lemma 7.1 in [Ja] (see also [HX, Lemma 3.3.1]).

Now set ${}^\tau\Delta = (\tau\otimes\tau)\circ\Delta\circ\tau^{-1}$. Then we have

$${}^\tau\Delta(K_\alpha) = K_\alpha\otimes K_\alpha \text{ for all } \alpha\in\mathbb{Z}[I].$$

For each $i\in I$, we have

$$\begin{aligned} {}^\tau\Delta(E_i(0)) = \ & (\tau\otimes\tau)\Delta(v^{(i,i)/2}E_i(0)) \\ = \ & v^{(i,i)/2}(\tau\otimes\tau)(E_i(0)\otimes 1 + K_i\otimes E_i(0)) \\ = \ & E_i(0)\otimes 1 + K_{-i}\otimes E_i(0) \end{aligned}$$

and

$${}^\tau\Delta(F_i(0)) = F_i(0)\otimes K_i + 1\otimes F_i(0).$$

For $m\geq 1$, $1\leq p\leq \eta_m$, we have

$$\begin{aligned} {}^\tau\Delta(E_p(m)) = \ & (\tau\otimes\tau)\Delta((-1)^{\mathrm{tr}\pi_m)}v^{-\tau(\pi_m)}E_p(m)) \\ = \ & (-1)^{\mathrm{tr}\pi_m)}v^{-\tau(\pi_m)}(\tau\otimes\tau)(E_p(m)\otimes 1 + K_{\pi_m}\otimes E_p(m)) \\ = \ & E_p(m)\otimes 1 + K_{-\pi_m}\otimes E_p(m) \end{aligned}$$

and

$${}^\tau\Delta(F_p(m)) = F_p(m)\otimes K_{\pi_m} + 1\otimes F_p(m).$$

Now we formally define $\Theta = \sum_{\alpha\in\mathbb{N}[I]}\Theta_\alpha$ and set $\Theta_{\leq p} = \sum_{\alpha\in\mathbb{N}[I],\mathrm{tr}\alpha\leq p}\Theta_\alpha$. By the lemma above, we obtain

$$\Delta(a)\circ\Theta = \Theta\circ{}^\tau\Delta(a) \text{ for all } a\in\mathcal{D}(\Lambda).$$

(Note that the equality holds in the completion of $\mathcal{D}(\Lambda)\otimes\mathcal{D}(\Lambda)$).

4.4. Let X be the weight lattice of $\Gamma = (I, (\ ,\ ))$ and X_+ the set of dominant weights. A $\mathcal{D}(\Lambda)$-module M is called a weight module if M admits a decomposition $M = \oplus_{\lambda\in X}M_\lambda$ (as R-vector space) such that $K_\alpha x = v^{(\alpha,\lambda)}x$ for all $\alpha\in\mathbb{Z}[I]$ and $x\in M_\lambda$. Finally, we denote by $\mathcal{O}$ the category consisting of weight modules M which satisfy: (1) every weight space M_λ is finite dimensional, (2) for every $x\in M$ there exists an $n_0\geq 0$ such that $\mathfrak{h}^+_\alpha x = 0$ whenever $\mathrm{tr}\alpha\geq n_0$.

Let M and M' be weight modules in $\mathcal{O}$. We have

$$\Theta_\alpha(M_\lambda\otimes M'_{\lambda'})\subset M_{\lambda-\alpha}\otimes M'_{\lambda'+\alpha} \text{ for all } \lambda,\lambda'\in X, \alpha\in\mathbb{N}[I].$$

If, moreover, M and M' lie in the category $\mathcal{O}$, it then holds that $\Theta_\alpha(M\otimes M') = 0$ for almost all $\alpha \in \mathbb{N}[I]$. Therefore, Θ induces indeed a linear transformation

$$\Theta = \Theta_{M,M'} : M \otimes M' \to M' \otimes M.$$

By choosing a basis of $M \otimes M'$ suitably, the matrix of Θ with respect to the chosen basis is unipotent. This implies that Θ is invertible.

4.5 Consider a map $f : X \times X \to R^\times = R\backslash\{0\}$ given by

$$f(\lambda+\nu,\mu) = v^{-(\nu,\mu)} f(\lambda,\mu) \ \text{ and } \ f(\lambda,\mu+\nu) = v^{-(\nu,\lambda)} f(\lambda,\mu)$$

for all $\lambda, \mu \in X$ and $\nu \in \mathbb{Z}[I]$. Note that such a map does exist: Choose a system $\lambda_1, \lambda_2, \cdots, \lambda_r$ for $X/\mathbb{Z}\Phi$, choose $f(\lambda_i, \lambda_j)$ arbitrarily, and set

$$f(\lambda_i+\mu, \lambda_j+\nu) = v^{-(\lambda_i,\nu)-(\mu,\lambda_j)-(\mu,\nu)} f(\lambda_i,\lambda_j)$$

for all i, j and all $\mu, \nu \in \mathbb{Z}\Phi$. Such a map f induces for weight modules M and M' a bijective linear map $\tilde{f} : M \otimes M' \to M \otimes M'$ defined by

$$\tilde{f}(m \otimes m') = f(\lambda,\mu) m \otimes m' \ \text{ for all } \ m \in M_\lambda \ \text{ and } \ m' \in M'_\mu.$$

We finally set $\Theta^f = \Theta \circ \tilde{f}$. By P we denote the map $M \otimes M' \to M' \otimes M$ with $P(m \otimes m') = m' \otimes m$.

Theorem. *For all $\mathcal{D}(\Lambda)$–modules M and M' in $\mathcal{O}$, the map*

$$\Theta^f \circ P : M \otimes M' \to M' \otimes M$$

is an isomorphism of $\mathcal{D}(\Lambda)$–modules.

In order to prove the proposition, we first show the following lemma.

Lemma. *For all $a \in \mathcal{D}(\Lambda)$, it holds*

$$\Delta(a) \circ \Theta^f = \Theta^f \circ (P \circ \Delta)(a).$$

Proof. By 4.4, we have for all $a \in \mathcal{D}(\Lambda)$ that

$$\Delta(a) \circ \Theta^f = \Delta(a) \circ \Theta \circ \tilde{f} = \Theta \circ {}^\tau\Delta(a) \circ \tilde{f}.$$

It suffices to show that

$${}^\tau\Delta(a) \circ \tilde{f} = \tilde{f} \circ P \circ \Delta(a) \ \text{ for all } \ a \in \mathcal{D}(\Lambda).$$

We check this equality only for the generators $E_i(0), F_i(0), i \in I$ and $E_p(m)$, $F_p(m), m \geq 1, 1 \leq r \leq \xi_m, 1 \leq p \leq \eta_m$. In fact, for all $u \otimes u' \in M \otimes M'$ with $u \in M_\lambda, u' \in M_\mu$, we have

$$\begin{aligned} & {}^\tau\Delta(E_i(0)) \circ \tilde{f}(u \otimes u') = (E_i(0) \otimes 1 + (K_{-i} \otimes E_i(0)) \circ \tilde{f}(u \otimes u') \\ = \ & (E_i(0) \otimes 1)(f(\lambda, \mu)u \otimes u') + (K_{-i} \otimes E_i(0))(f(\lambda, \mu)u \otimes u') \\ = \ & f(\lambda, \mu)E)_i(0)u \otimes u' + v^{-(i,\lambda)} f(\lambda, \mu)u \otimes E_i(0)u' \end{aligned}$$

and

$$\begin{aligned} & \tilde{f} \circ P \circ \Delta(E_i(0))(u \otimes u') = \tilde{f}(E_i(0)u \otimes K_i u') + \tilde{f}(u \otimes E_i(0)u' \\ = \ & v^{(i,\mu)} f(\lambda + i, \mu)E_i(0)u \otimes u' + f(\lambda, \mu + i)u \otimes E_i(0)u'. \end{aligned}$$

From the definition of $\tilde{f}$ it follows that ${}^\tau\Delta(E_i(0)) \circ \tilde{f} = \tilde{f} \circ P \circ \Delta(E_i(0))$.

For the generators $F_i(0)$, $E_p(m)$ and $F_p(m)$, the equality can be shown in a similar way.

Proof of the theorem. Since $\Theta, \tilde{f}$ and P are all bijective, so is $\Theta^f \circ P$. Further, for all $a \in \mathcal{D}(\Lambda)$ and $u \in M \otimes M'$, we have

$$\begin{aligned} \Theta^f \circ P(a(u)) \ & = \Theta^f \circ P(\Delta(a)u) \\ & = \Theta \circ \tilde{f} \circ (P \circ \Delta)(a) \circ P(u) = \Delta(a) \circ \Theta^f \circ P(u), \end{aligned}$$

that is, $\Theta^f \circ P$ is a $\mathcal{D}(\Lambda)$–module homomorphism. This finishes the proof.

Remark. The isomorphism $\Theta^f \circ P$ is functorial, that is, if M, M', N, N' are modules in $\mathcal{O}$, and $g : M \to M', g' : N \to N'$ are homomorphisms, then we have the following commutative diagram

$$\begin{tikzcd} M \otimes M' \arrow[r, "\Theta^f \circ P"] \arrow[d, "g \otimes g'"'] & M' \otimes M \arrow[d, "g' \otimes g"] \\ N \otimes N' \arrow[r, "\Theta^f \circ P"] & N' \otimes N \end{tikzcd}$$

4.6 For each $x \in \mathfrak{h}_\alpha^+$, $\alpha \in \mathbb{N}[I]$, we have by Green formula that

$$\Delta(x) = \sum_{\beta \leq \alpha} \sum_{i=1}^{\gamma(\alpha-\beta)} \sum_{j=1}^{\gamma(\beta)} \varphi(x, y_{\alpha-\beta}^i y_\beta^j) x_{\alpha-\beta}^i K_\beta \otimes x_\beta^j.$$

Similarly, for each $y \in \mathfrak{h}_\alpha^-$, we have

$$\Delta(y) = \sum_{\beta \leq \alpha} \sum_{i=1}^{\gamma(\alpha-\beta)} \sum_{j=1}^{\gamma(\beta)} \varphi(x_{\alpha-\beta}^i x_\beta^j, y) y_\beta^j \otimes y_{\alpha-\beta}^i K_{-\beta}.$$

For each vector space V, there are 6 natural embeddings $V\otimes V \to V\otimes V\otimes V$ given by $z \longmapsto z_{ij}$ for distinct pairs $i,j \in \{1,2,3\}$. For example, if $z = \sum_s a_s \otimes b_s$, then $z_{12} = \sum_s a_s \otimes b_s \otimes 1$ and $z_{31} = \sum_s b_s \otimes 1 \otimes a_s$.

Lemma. *We have for all $\alpha \in \mathbb{N}[I]$*

$$(\Delta \otimes 1)\Theta_\alpha = \sum_{0\le\beta\le\alpha} (\Theta_{\alpha-\beta})_{23}(1\otimes K_{-\beta}\otimes 1)(\Theta_\beta)_{13}$$

and

$$(1 \otimes \Delta)\Theta_\alpha = \sum_{0\le\beta\le\alpha} (\Theta_{\alpha-\beta})_{12}(1\otimes K_{\beta}\otimes 1)(\Theta_\beta)_{13}.$$

Proof. We prove the first equality. The proof of the second one is similar. Indeed, we have

$$\begin{aligned}
& (\Delta \otimes 1)\Theta_\alpha = \textstyle\sum_l \Delta(y^l_\alpha)\otimes x^l_\alpha \\
= & \textstyle\sum_l \sum_{\beta,i,j} \varphi(x^i_{\alpha-\beta}x^j_\beta, y^l_\alpha) y^j_\beta \otimes y^i_{\alpha-\beta}K_{-\beta}\otimes x^l_\alpha \\
= & \textstyle\sum_{\beta,i,j} y^j_\beta \otimes y^i_{\alpha-\beta}K_{-\beta} \otimes (\sum_l)\varphi(x^i_{\alpha-\beta}x^j_\beta, y^l_\alpha)x^l_\alpha) \\
= & \textstyle\sum_{\beta,i,j} y^j_\beta \otimes y^i_{\alpha-\beta}K_{-\beta}\otimes x^i_{\alpha-\beta}x^j_\beta \\
= & \textstyle\sum_{\beta,i,j} (y^i_{\alpha-\beta}\otimes 1 \otimes x^i_{\alpha-\beta})(1\otimes K_{-\beta}\otimes 1)(y^j_\beta \otimes 1 \otimes x^j_\beta) \\
= & \textstyle\sum_\beta (\Theta_{\alpha-\beta})_{23}(1\otimes K_{-\beta}\otimes 1)(\Theta_\beta)_{13}.
\end{aligned}$$

From this lemma it also follows for all $\alpha \in \mathbb{N}[I]$ that

$$({}^\tau\Delta \otimes 1)\Theta_\alpha = \sum_{0\le\beta\le\alpha} (\Theta_{\alpha-\beta})_{13}(1\otimes K_{\beta}\otimes 1)(\Theta_\beta)_{23}$$

and

$$(1 \otimes {}^\tau\Delta)\Theta_\alpha = \sum_{0\le\beta\le\alpha} (\Theta_{\alpha-\beta})_{13}(1\otimes K_{\beta}\otimes 1)(\Theta_\beta)_{12}.$$

4.7 Given three modules M, M', M'' in $\mathcal{O}$, we can define three automorphisms $\Theta^f_{12}, \Theta^f_{23}, \Theta^f_{13}$ of $M\otimes M'\otimes M''$ respectively by $\Theta^f\otimes 1$, $1\otimes\Theta^f$, and the composition $(1\otimes P)\circ(\Theta^f\otimes 1)\circ(1\otimes P)$.

Theorem. *Let M, M', M'' be $\mathcal{D}(\Lambda)$–modules in $\mathcal{O}$. Then it holds that*

$$\Theta^f_{12}\circ\Theta^f_{13}\circ\Theta^f_{23} = \Theta^f_{23}\circ\Theta^f_{13}\circ\Theta^f_{12}.$$

Proof. *For every distinct pair $i,j\in\{1,2,3\}$, we define $\tilde f_{ij} : M\otimes M'\otimes M'' \to M\otimes M'\otimes M''$ by $\tilde f_{ij}(m\otimes m'\otimes m'') = f(\lambda_i,\lambda_j)m\otimes m'\otimes m''$ for $m\otimes m'\otimes m'' \in M_{\lambda_1}\otimes M'_{\lambda_2}\otimes M''_{\lambda_3}$.*

We claim first that

$$\tilde{f}_{12} \circ (\Theta_\alpha)_{13} = (\Theta_\alpha)_{13} \circ (1 \otimes K_\alpha \otimes 1) \circ \tilde{f}_{12}$$

and

$$\tilde{f}_{12} \circ \tilde{f}_{13} \circ (\Theta_\alpha)_{23} = (\Theta_\alpha)_{23} \circ \tilde{f}_{12} \circ \tilde{f}_{13}.$$

Indeed, for $m \otimes m' \otimes m'' \in M_\lambda \otimes M'_\mu \otimes M''_\nu$, we have

$$\begin{array}{rl} & \tilde{f}_{12} \circ (\Theta_\alpha)_{13}(m \otimes m' \otimes m'') = \tilde{f}_{12} \sum_s (y^s_\alpha \otimes 1 \otimes x^s_\alpha)(m \otimes m' \otimes m'') \\ = & \tilde{f}_{12}(\sum_s y^s_\alpha m \otimes m' \otimes x^s_\alpha m'') = f(\lambda - \alpha, \mu) \sum_s y^s_\alpha m \otimes m' \otimes x^s_\alpha m'' \end{array}$$

and

$$\begin{array}{rl} & (\Theta_\alpha)_{13} \circ (1 \otimes K_\alpha \otimes 1) \circ \tilde{f}_{12}(m \otimes m' \otimes m'') \\ = & \sum_s (y^s_\alpha \otimes 1 \otimes x^s_\alpha)(1 \otimes K_\alpha \otimes 1) f(\lambda, \mu)(m \otimes m' \otimes m'') \\ = & \sum_s (y^s_\alpha \otimes 1 \otimes x^s_\alpha) f(\lambda, \mu)(m \otimes K_\alpha m' \otimes m'') \\ = & v^{(\alpha,\mu)} f(\lambda, \mu) \sum_s y^s_\alpha m \otimes m' \otimes x^s_\alpha m''. \end{array}$$

This implies that $\tilde{f}_{12} \circ (\Theta_\alpha)_{13} = (\Theta_\alpha)_{13} \circ (1 \otimes K_\alpha \otimes 1) \circ \tilde{f}_{12}$.

Further, we have

$$\begin{array}{rl} & \tilde{f}_{12} \circ \tilde{f}_{13} \circ (\Theta_\alpha)_{23}(m \otimes m' \otimes m'') \\ = & \tilde{f}_{12} \circ \tilde{f}_{13} \sum_s (1 \otimes y^s_\alpha \otimes x^s_\alpha)(m \otimes m' \otimes m'') \\ = & f(\lambda, \mu - \alpha) f(\lambda, \nu + \alpha) \sum_s (m \otimes y^s_\alpha m' \otimes x^s_\alpha m'') \end{array}$$

and

$$\begin{array}{rl} & (\Theta_\alpha)_{23} \circ \tilde{f}_{12} \circ \tilde{f}_{13}(m \otimes m' \otimes m'') \\ = & \sum_s (1 \otimes y^s_\alpha \otimes x^s_\alpha)(f(\lambda, \mu) f(\lambda, \nu) m \otimes m' \otimes m'') \\ = & f(\lambda, \mu) f(\lambda, \nu) \sum_s (m \otimes y^s_\alpha m' \otimes x^s_\alpha m''). \end{array}$$

Thus, we get $\tilde{f}_{12} \circ \tilde{f}_{13} \circ (\Theta_\alpha)_{23} = (\Theta_\alpha)_{23} \circ \tilde{f}_{12} \circ \tilde{f}_{13}$. It then follows that

$$\Theta^f_{12} \circ \Theta^f_{13} \circ \Theta^f_{23} = \Theta_{12} \circ (\sum_\alpha (\Theta_\alpha)_{13})(1 \otimes K_\alpha \otimes 1) \circ \Theta_{23} \circ \tilde{f}_{12} \circ \tilde{f}_{13} \circ \tilde{f}_{23}.$$

Similarly, we can prove

$$\tilde{f}_{23} \circ (\Theta_\alpha)_{13} = (\Theta_\alpha)_{13} \circ (1 \otimes K_{-\alpha} \otimes 1) \circ \tilde{f}_{23}$$

and

$$\tilde{f}_{23} \circ \tilde{f}_{13} \circ (\Theta_\alpha)_{12} = (\Theta_\alpha)_{12} \circ \tilde{f}_{23} \circ \tilde{f}_{13}.$$

This then implies

$$\Theta^f_{23} \circ \Theta^f_{13} \circ \Theta^f_{12} = \Theta_{23} \circ (\sum_\alpha (\Theta_\alpha)_{13})(1 \otimes K_{-\alpha} \otimes 1) \circ \Theta_{12} \circ \tilde{f}_{23} \circ \tilde{f}_{13} \circ \tilde{f}_{12}.$$

By 4.6, we have

$$\Theta_{12}\circ(\sum_{\alpha}(\Theta_\alpha)_{13})(1\otimes K_\alpha\otimes 1)\circ\Theta_{23}=\Theta_{12}\circ((^{\tau}\Delta\otimes 1)\Theta)$$

and

$$\Theta_{23}\circ(\sum_{\alpha}(\Theta_\alpha)_{13})(1\otimes K_{-\alpha}\otimes 1)\circ\Theta_{12}=((\Delta\otimes 1)\Theta)\circ\Theta_{12}.$$

Further, it holds

$$\begin{array}{rl} & ((\Delta\otimes 1)\Theta)\circ\Theta_{12}=(\sum_{\alpha,l}\Delta(y^l_\alpha)\otimes x^l_\alpha)(\sum_{\beta,m})y^m_\beta\otimes x^m_\beta\otimes 1) \\ = & \sum_{\alpha,l}\sum_{\beta,m}\Delta(y^l_\alpha)(y^m_\beta\otimes x^m_\beta)\otimes=\sum_{\alpha,l}(\Delta(y^l_\alpha)\Theta)\otimes x^l_\alpha \\ = & \sum_{\alpha,l}(\Theta\circ^{\tau}\Delta(y^l_\alpha)\otimes x^l_\alpha=\Theta_{12}\sum_{\alpha,l}{}^{\tau}\Delta(y^l_\alpha)\otimes x^l_\alpha \\ = & \Theta_{12}\circ((^{\tau}\Delta\otimes 1)\Theta). \end{array}$$

As a conclusion, we obtain

$$\Theta^f_{12}\circ\Theta^f_{13}\circ\Theta^f_{23}=\Theta^f_{23}\circ\Theta^f_{13}\circ\Theta^f_{12}.$$

4.8 Remark. We consider a particular case of Theorem 4.7. Take $M=M'=M''=V$ to be a $\mathcal{D}(\Lambda)$–module in category $\mathcal{O}$. Then each permutation σ of $\{1,2,3\}$ defines a linear automorphism P_σ of $V\otimes V\otimes V$ by

$$P_\sigma(v_1\otimes v_2\otimes v_3)=v_{\sigma^{-1}(1)}\otimes v_{\sigma^{-1}(2)}\otimes v_{\sigma^{-1}(3)}.$$

For a transposition $\sigma=(ij)$, we simply set $P_{ij}=P_\sigma$. By Theorem 4.5, the maps $R_{12}=\Theta^f_{12}\circ P_{12}$ and $R_{23}=\Theta^f_{23}\circ P_{23}$ are automorphisms of the $\mathcal{D}(\Lambda)$–module $V\otimes V\otimes V$. It is easy to see that the equality in Theorem 4.7 implies the following one

$$R_{12}\circ R_{23}\circ R_{12}=R_{23}\circ R_{12}\circ R_{23},$$

that is, Θ^f gives rise to a solution of the quantum Yang–Baxter equation. Thus one could say that Hall algebras behave very much like quantum groups.

5 Integrable modules and complete reducibility

5.1 Let Λ be a hereditary algebra and $\mathcal{D}(\Lambda)$ its double Ringel–Hall algebra.

Throughout this section, we suppose that all π_m, $m\geq 1$, arising in the decomposition of $\mathcal{D}(\Lambda)$ lie in the fundamental set.

A weight $\mathcal{D}(\Lambda)$–module M is said to be integrable if, for any $x \in M$ and any $i \in I$, there exists an $n_0 \geq 1$ such that

$$(u_i^+)^n x = 0 = (u_i^-)^n x, \quad \text{for all } n \geq n_0.$$

Lemma. *Given an $i \in I$, an $\alpha \in \mathcal{P}$ and an integer N, there exists an $m_0 \geq 1$ such that, for all $m \geq m_0$, it holds that*

$$u_i^m u_\alpha \in \mathfrak{h}(\Lambda) u_i^N.$$

Proof. For each $\lambda \in \mathcal{P}$, we set

$$l_\lambda = \dim_k \mathrm{Ext}_\Lambda^1(V_i, V_\lambda) + \dim_k \mathrm{Ext}_\Lambda^1(V_\lambda, V_i).$$

We prove this lemma by induction on l_α.

In case $l_\alpha = 0$, that is, $\mathrm{Ext}_\Lambda^1(V_i, V_\alpha) = 0 = \mathrm{Ext}_\Lambda^1(V_\alpha, V_i)$, we have that

$$u_i u_\alpha = v^{\langle i,\alpha\rangle} g_{i\,\alpha}^{i\oplus\alpha} u_{i\oplus\alpha} \quad \text{and} \quad u_\alpha u_i = v^{\langle \alpha,i\rangle} g_{\alpha\,i}^{i\oplus\alpha} u_{i\oplus\alpha}.$$

This implies that

$$u_i u_\alpha = v^{\langle i,\alpha\rangle - \langle \alpha,i\rangle} g_{i\,\alpha}^{i\oplus\alpha} (g_{\alpha\,i}^{i\oplus\alpha})^{-1} u_\alpha u_i.$$

Thus, in this case we may take $m_0 = N$.

Let $n \geq 1$. Suppose that the statement of the lemma holds for all $\lambda \in \mathcal{P}$ with $l_\lambda < n$. Let $\alpha \in \mathcal{P}$ with $l_\alpha = n$. Then it holds

$$u_i^N u_\alpha = \sum_{\lambda \in \mathcal{P}} c_\lambda u_\lambda = v^{\langle Ni,\alpha\rangle} g_{i\cdots i\,\alpha}^{\alpha\oplus Ni} u_{\alpha\oplus Ni} + \sum_{\beta \in \mathcal{P},\, \beta \neq \alpha\oplus Ni} c_\beta u_\beta.$$

If $c_\beta \neq 0$ for some β, then there exists a nonsplit sequence

$$0 \to V_\alpha \to V_\beta \to V_i^N \to 0.$$

Since $\mathrm{Ext}_\Lambda^1(V_i, V_i) = 0$, we obtain the following exact sequences

$$\begin{aligned} 0 &\to \mathrm{Hom}_\Lambda(V_i, V_\alpha) \to \mathrm{Hom}_\Lambda(V_i, V_\beta) \to \mathrm{Hom}_\Lambda(V_i, V_i^N) \\ &\to \mathrm{Ext}_\Lambda^1(V_i, V_\alpha) \to \mathrm{Ext}_\Lambda^1(V_i, V_\beta) \to 0 \end{aligned}$$

and

$$0 \to \mathrm{Ext}_\Lambda^1(V_\beta, V_i) \to \mathrm{Ext}_\Lambda^1(V_\alpha, V_i) \to 0.$$

This implies that $l_\beta < l_\alpha = n$. Similarly, we have

$$u_\alpha u_i^N = v^{\langle \alpha,Ni\rangle} g_{\alpha\,i\cdots i}^{\alpha\oplus Ni} u_{\alpha\oplus Ni} + \sum_{\beta \in \mathcal{P},\, \beta \neq \alpha\oplus Ni} c'_\beta u_\beta$$

such that $l_\beta < n$ if $c'_\beta \neq 0$. Therefore, we get

$$u_i^N u_\alpha = d_1 u_\alpha u_i^N + \sum_{\lambda \in \mathcal{P},\, \lambda \neq \alpha \oplus Ni} d_\lambda u_\lambda,$$

where $d_1 = v^{N(\langle i,\alpha\rangle - \langle \alpha, i\rangle)} g_{i\cdots i\,\alpha}^{\alpha\oplus Ni} (g_{\alpha\, i\cdots i}^{\alpha\oplus Ni})^{-1}$ and all λ are such that $l_\lambda < n$. By induction hypothesis, for each λ, there exists an $m_\lambda^0 \geq 1$ such that $u_i^m u_\lambda \in \mathfrak{h}(\Lambda) u_i^N$ for $m \geq m_\lambda^0$. Note that there are only finitely many u_λ with $d_\lambda \neq 0$. Set $m_0 = N + \max_\lambda \{m_\lambda^0 : d_\lambda \neq 0\}$. Then, for all $m \geq m_0$, we have $u_i^m u_\lambda \in \mathfrak{h}(\Lambda) u_i^N$. This finishes the proof.

5.2 By Proposition 3.8, we can extend the torus $\mathcal{T}$ of $C = (I, (\ ,\ ))$ to the torus $\mathcal{T}'$ of $\tilde{C} = (I \cup J, (\ ,\ ))$ and view $\mathcal{D}(\Lambda)$ as a $\mathbb{Z}[I \cup J]$–graded Hopf algebra. We denote by $\mathcal{D}'(\Lambda)$ the extended form of $\mathcal{D}(\Lambda)$ as defined in 3.8. As usual, we set $x_i = E_i(0)$ and $y_i = F_i(0)$ for $i \in I$, $x_j = E_p(m)$ and $y_j = F_p(m)$ for $j = (m,p) \in J$. Thus $\mathcal{D}'(\Lambda)$ admits a triangular decomposition

$$\mathcal{D}'(\Lambda) = \mathfrak{h}'^{-}(\Lambda) \otimes \mathcal{T}' \otimes \mathfrak{h}'^{+}(\Lambda)$$

where $\mathfrak{h}'^{+}(\Lambda)$ (resp. $\mathfrak{h}'^{-}(\Lambda)$) is generated by x_i (resp. y_i) for $i \in I \cup J$. Then $\mathcal{H}'^{+}(\Lambda) = \mathcal{T}' \otimes \mathfrak{h}'^{+}(\Lambda)$ and $\mathcal{H}'^{-}(\Lambda) = \mathcal{T}' \otimes \mathfrak{h}'^{-}(\Lambda)$ are naturally $\mathbb{N}[I \cup J]$–graded. Let

$$\varphi' : \mathcal{H}'^{+}(\Lambda) \times \mathcal{H}'^{-}(\Lambda) \to R$$

be the resitricted pairing induced by $\varphi : \mathcal{H}^{+}(\Lambda) \times \mathcal{H}^{-}(\Lambda) \to R$, which satisfies that $\varphi'(x_i, y_j) \neq 0$ if and only if $i = j$ for $i, j \in I \cup J$.

Let X' be the weight system of $\mathbb{Z}[I \cup J]$ (or of $\tilde{C}' = (I \cup J, (\ ,\ )')$). Then the bilinear form $(\ ,\ )'$ can be extended on X'. A weight $\mathcal{D}'(\Lambda)$–module (with respect to X') M is said to be integrable if for any $m \in M$ and any $i \in I$, there exists an $n_0 \geq 1$ such that

$$x_i^n m = 0 = y_i^n m \qquad \text{for all } n \geq n_0.$$

For $\alpha = \sum_{i\in I} k_i i \in \mathbb{Z}[I]$, we define $\mathrm{tr}\alpha = \sum_{i \in I} k_i$, but for $\alpha = \sum_{i\in I} k_i i + \sum_{j \in J} k_j j \in \mathbb{Z}[I \cup J]$, we set $\bar{\alpha} = \sum_{i \in I} k_i i + \sum_{j\in J} k_j \delta_j \in \mathbb{Z}[I]$ and define $\mathrm{tr}\alpha = \mathrm{tr}\bar{\alpha}$. Finally, we define $\mathcal{O}'$ to be the category cosisting of weight $\mathcal{D}'(\Lambda)$–modules M which satisfy: For each $m \in M$, there exists an $n_0 \geq 0$ such that $\mathfrak{h}'^{+}(\Lambda)_\alpha m = 0$ whenever $\alpha \in \mathbb{N}[I \cup J]$ with $\mathrm{tr}\alpha \geq n_0$.

5.3 For each $\alpha \in \mathbb{N}[I \cup J]$, we choose an R–basis $x_\alpha^1, \cdots, x_\alpha^r$ of $\mathfrak{h}'^{+}(\Lambda)_\alpha$. By the pairing φ', there exists an R–basis $y_\alpha^1, \cdots, y_\alpha^r$ of $\mathfrak{h}'^{-}(\Lambda)_\alpha$ such that

$$\varphi'(x_\alpha^s, y_\alpha^t) = \delta_{st} \qquad \text{for all } 1 \leq s, t \leq r.$$

We now set

$$\Theta'_\alpha = \sum_{s=1}^{r} y^s_\alpha \otimes x^s_\alpha \in \mathcal{D}'(\Lambda) \otimes \mathcal{D}'(\Lambda).$$

The definition of τ and ω on $\mathcal{D}(\Lambda)$ can be easily extended to $\mathcal{D}'(\Lambda)$. We then have

$$(\tau \otimes \tau)\Theta'_\alpha = \Theta'_\alpha \qquad \text{and}(\omega \otimes \omega)\Theta'_\alpha = \Theta'^{\text{op}}_\alpha.$$

We may define $\Theta' = \sum_{\alpha \in \mathbb{N}[I\cup J]} \Theta'_\alpha$ in the sense in the completion of $\mathcal{D}'(\Lambda) \otimes \mathcal{D}'(\Lambda)$ and $\Theta'_{\leq p} = \sum_{\alpha \in \mathbb{N}[I\cup J]\, \mathrm{tr}\alpha \leq p} \Theta'_\alpha$ for all $p \geq 0$. Then by Lemma 4.3, we obtain

$$\Delta(a)\Theta' = \Theta'\, {}^\tau\Delta(a) \qquad \text{for all } a \in \mathcal{D}'(\Lambda)$$

and

(1)

$$\begin{aligned} & (x_i \otimes 1 + K_i \otimes x_i)\Theta'_{\leq p} - \Theta'_{\leq p}(x_i \otimes 1 + K_{-i} \otimes x_i) \\ = \ & \textstyle\sum_{\alpha \in \mathbb{N}[I\cup J]\, \mathrm{tr}\alpha = p}((K_i \otimes x_i)\Theta'_\alpha - \Theta'_\alpha(K_{-i} \otimes x_i)), \end{aligned}$$

(2)

$$\begin{aligned} & (1 \otimes y_i + y_i \otimes K_{-i})\Theta'_{\leq p} - \Theta'_{\leq p}(1 \otimes y_i + y_i \otimes K_i) \\ = \ & \textstyle\sum_{\alpha \in \mathbb{N}[I\cup J]\, \mathrm{tr}\alpha = p}((y_i \otimes K_{-i})\Theta'_\alpha - \Theta'_\alpha(y_i \otimes K_i)) \end{aligned}$$

for all $i \in I \cup J$.

Applying $\mu(\sigma \otimes 1)$ to the equality (1), where $\mu : \mathcal{D}'(\Lambda) \otimes \mathcal{D}'(\Lambda) \to \mathcal{D}'(\Lambda)$ is the multiplication, we get the following (for all $i \in I \cup J$ and $p \geq 0$):

$$\begin{aligned} & \textstyle\sum_{\mathrm{tr}\alpha \leq p} \sum_s (\sigma(x_i y^s_\alpha)x^s_\alpha + \sigma(K_i y^s_\alpha)x_i x^s_\alpha - \sigma(y^s_\alpha x_i)x^s_\alpha - \sigma(y^s_\alpha K_{-i})x^s_\alpha x_i) \\ = \ & \textstyle\sum_{\mathrm{tr}\alpha = p} \sum_s (\sigma(K_i y^s_\alpha)x_i x^s_\alpha - \sigma(y^s_\alpha K_{-i})x^s_\alpha x_i) \end{aligned}$$

We finally set

$$\Omega = \sum_{\alpha \in \mathbb{N}[I\cup J]} \sum_s \sigma(y^s_\alpha)x^s_\alpha \quad \text{and} \quad \Omega_{\leq p} = \sum_{\mathrm{tr}\alpha \leq p} \sum_s \sigma(y^s_\alpha)x^s_\alpha.$$

Since $\sigma(x_i) = -K_{-i}x_i$ and $\sigma(K_i) = K_{-i}$ for all $i \in I \cup J$, we obtain

$$K_{-i}x_i\Omega_{\leq p} - K_i\Omega_{\leq p}x_i = \sum_{\mathrm{tr}\alpha = p} \sum_s (\sigma(K_i y^s_\alpha)x_i x^s_\alpha - \sigma(y^s_\alpha K_{-i})x^s_\alpha x_i).$$

For each $M \in \mathcal{O}'$ and each $m \in M$, we have that $\Omega(m) = \Omega_{\leq p}(m)$ for large enough p. It then holds that

$$K_{-i}x_i\Omega = K_i\Omega x_i \qquad \text{for all } i \in I \cup J$$

as operators on M. Similarly, we get

$$\Omega y_i = y_i K_i \Omega K_i \qquad \text{for all } i \in I \cup J$$

and

$$\Omega K_\alpha = K_\alpha \Omega \qquad \text{for all } \alpha \in \mathbb{Z}[I \cup J]$$

as operaors on M.

5.4 There exists a $\rho \in X'$ such that $(\rho, i)' = \frac{1}{2}(i,i)'$ for all $i \in I \cup J$. We define an operator $\tilde{\Omega}$ on modules $M \in \mathcal{O}'$ by

$$\tilde{\Omega}(m) = v^{(\lambda+\rho,\lambda+\rho)'}\Omega(m)$$

for each weight λ of M and each $m \in M_\lambda$. For each $\mathbb{Z}[I \cup J]$–coset C in X', we define $M_C = \oplus_{\lambda \in C} M_\lambda$, it is easy to see that M_C is a $\mathcal{D}'(\Lambda)$–submodule of M and $M = \oplus_{C \in X'/\mathbb{Z}[I\cup J]} M_C$.

For each $\lambda \in X'$, consider the left ideal

$$\begin{aligned} J &= \textstyle\sum_{i \in I \cup J} \mathcal{D}'(\Lambda) x_i + \sum_{\alpha \in \mathbb{Z}[I \cup J]} \mathcal{D}'(\Lambda)(K_\alpha - v^{(\lambda,\alpha)'} 1) \\ &= \textstyle\sum_{\mu \in \mathcal{P}_1} \mathcal{D}(\Lambda) u_\mu^+ + \sum_{\alpha \in \mathbb{Z}[I \cup J]} \mathcal{D}'(\Lambda)(K_\alpha - v^{(\lambda,\alpha)'} 1). \end{aligned}$$

Then $M(\lambda) := \mathcal{D}'(\Lambda)/J$ becomes naturally a $\mathcal{D}'(\Lambda)$–module which lies in $\mathcal{O}'$ and called Verma module of weigth λ. Using the triangular decomposition, we see that the R–linear map $\mathfrak{h}'^-(\Lambda) \to \mathcal{D}'(\Lambda)/J$, $y \longmapsto y + J$ is a bijection. Via this bijection, $\mathfrak{h}'^-(\Lambda)$ can be endowed with a $\mathcal{D}'(\Lambda)$–module structure which is isomorphic to the Verma module $M(\lambda)$. The module structure is in fact given by $K_\alpha \cdot y = v^{(\alpha,\lambda-\beta)'} y$, $y_i \cdot y = y_i y$, $x_i \cdot 1 = 0$ for all $y \in \mathfrak{h}'^-(\Lambda)_\beta$, $\alpha \in \mathbb{Z}[I \cup J]$, $i \in I \cup J$.

Lemma. *Let $M \in \mathcal{O}'$. Then*

(1) $\tilde{\Omega}$ is central to the action of all elements of $\mathcal{D}'(\Lambda)$ on M, i.e. for all $x \in \mathcal{D}'(\Lambda)$, it holds that $\tilde{\Omega} x = x \tilde{\Omega}$ as operators on M;

(2) The action of $\tilde{\Omega}$ on M is locally finite;

(3) If M is a quotient of a Verma module $M(\lambda)$, then $\tilde{\Omega}$ acts on M as the scarlar multiplication by $v^{(\lambda+\rho,\lambda+\rho)'}$;

(4) If $M = M_C$ for some coset $C \in X'/\mathbb{Z}[I \cup J]$, then the eigenvalues of $\tilde{\Omega} : M \to M$ are of the form v^c for various integers c.

Proof. (1) For each weight λ and each $m \in M_\lambda$, it holds that

$$\begin{aligned} \tilde{\Omega} x_i(m) &= v^{(\lambda+i+\rho,\lambda+i+\rho)'} \Omega x_i(m) = v^{(\lambda+i+\rho,\lambda+i+\rho)'} K_{-2i} x_i \Omega(m) \\ &= v^{(\lambda+\rho,\lambda+\rho)' + 2(\lambda+\rho,i)' + (i,i)' - 2(\lambda+i,i)'} x_i \Omega(m) = v^{(\lambda+\rho,\lambda+\rho)'} x_i \Omega(m) \\ &= x_i \tilde{\Omega}(m), \end{aligned}$$

since $(\rho, i)' = \frac{1}{2}(i,i)'$ and (5.3). Similarly, we have

$$\tilde{\Omega} y_i = y_i \tilde{\Omega} \quad \text{and} \quad \tilde{\Omega} K_\alpha = K_\alpha \tilde{\Omega}$$

viewed as operators acting on M for $i \in I \cup J$ and $\alpha \in \mathbb{Z}[I \cup J]$.

(2) For each $m \in M_\lambda$, let $M' = \mathfrak{h}'^+(\Lambda)m$, then $\dim M' < \infty$. Further, let M'' be the $\mathcal{D}'(\Lambda)$–submodule of M generated by m. Then $M'' = \mathfrak{h}'^-(\Lambda)M'$. It follows easily that all weight spaces of M'' are finite dimensional. Thus M''_λ is finite dimensional and stable under the action of $\tilde{\Omega}$.

The statements in (3) and (4) are obvious.

5.5 A weight λ of a module M is called primitive if there is an $m \in M_\lambda$ and a $\mathcal{D}'(\Lambda)$–submodule M' of M such that $m \notin M'$, but $\mathfrak{h}'^+(\Lambda)_\alpha m \subseteq M'$ for $\alpha > 0$.

We now define the dominant weight system to be

$$X'_+ = \{\lambda \in X' : (\lambda, i)' \geq 0 \text{ for all } i \in I \cup J\}.$$

For each $\lambda \in X'_+$, let $J_\lambda = \{j \in J : (\lambda, j)' = 0\} \subset J$ and set

$$Y(\lambda) = M(\lambda)/ \sum_{i \in I \cup J_\lambda} \mathcal{D}'(\Lambda) y_i^{(\lambda,i)'+1} = M(\lambda)/(\sum_{i \in I} \mathcal{D}'(\Lambda) y_i^{(\lambda,i)'+1} + \sum_{j \in J_\lambda} \mathcal{D}'(\Lambda) y_j).$$

According to Lemma 5.1, $Y(\lambda)$ is an integrable $\mathcal{D}'(\Lambda)$–module. We denote by W the unique maxiaml submodule of $Y(\lambda)$. Then $L(\lambda) := Y(\lambda)/W$ is the irreducible integrable module of highest weight λ. In fact, we will see that $W = 0$.

Lemma. *For $\lambda \in X'_+$, it holds that $Y(\lambda) = L(\lambda)$.*

Proof. Suppose that M is a proper submodule of $Y(\lambda)$. It is clear that $M_\lambda = 0$. Let $\mu \in X'$ be maxiaml such that $M_\mu \neq 0$, then $\lambda - \mu \in \mathbb{N}[I \cup J]$. Thus there is a non–zero homomorphism $\phi : M(\mu) \to M$, where $M(\mu)$ denotes the Verma module of weight μ. Then $M' := \operatorname{Im} \phi$ is integrable, thus $(\mu, i) \geq 0$ for $i \in I$. Applying $\tilde{\Omega}$ on $Y(\lambda)$ and M', we get $(\lambda + \rho, \lambda + \rho) = (\mu + \rho, \mu + \rho)$. Let $\lambda - \mu = \sum_{i \in I \cup J} a_i i$ with $a_i \geq 0$ and set $\gamma = \sum_{i \in I \cup J} a_i i$, $\gamma_1 = \sum_{i \in I} a_i i$ and $\gamma_2 = \sum_{j \in J} a_j j$. Form the equaility $(\lambda + \rho, \lambda + \rho) = (\mu + \rho, \mu + \rho)$ it follows that $2(\lambda + \rho, \gamma) - (\gamma, \gamma) = 0$. Further, we have

$$\begin{aligned} & 2(\lambda + \rho, \gamma) - (\gamma, \gamma) = 2(\lambda + \rho, \gamma_1 + \gamma_2) - (\gamma_1 + \gamma_2, \gamma_1 + \gamma_2) \\ = \;& 2(\lambda + \rho, \gamma_1) - (\gamma_1, \gamma_1) - 2(\gamma_1, \gamma_2) + 2(\lambda + \rho, \gamma_2) - (\gamma_2, \gamma_2). \end{aligned}$$

On one hand, we have

$$\begin{aligned} & 2(\lambda + \rho, \gamma_1) - (\gamma_1, \gamma_1) = (\lambda + (\lambda - \gamma_1) + 2\rho, \gamma_1) \\ = \;& (\lambda, \gamma_1) + (\lambda - \gamma_1, \gamma_1) + 2(\rho, \gamma_1) = (\lambda, \gamma_1) + (\mu + \gamma_2, \gamma_1) + 2(\rho, \gamma_1) \\ = \;& (\lambda, \gamma_1) + (\mu, \gamma_1) + 2(\rho, \gamma_1) + (\gamma_1, \gamma_2), \end{aligned}$$

where $(\lambda, \gamma_1) \geq 0$, $(\mu, \gamma_1) \geq 0$ since λ, μ are primitive weights of an integrable module and $(\rho, \gamma_1) \geq 0$ with that the equlity holds if and only if $\gamma_1 = 0$.

On the other hand, it holds that

$$2(\lambda+\rho,\gamma_2)-(\gamma_2,\gamma_2)=2(\lambda,\gamma_2)+2(\rho,\gamma_2)-(\gamma_2,\gamma_2),$$

where $(\lambda,\gamma_2)=\sum_{j\in J}a_j(\lambda,j)'\geq 0$ and $2(\rho,\gamma_2)-(\gamma_2,\gamma_2)=\sum_{j\in J}a_j(j,j-\gamma_2)'\geq 0$ by Theorem 2.4. Thus, the equality $2(\lambda+\rho,\gamma)-(\gamma,\gamma)=0$ implies that $\gamma_1=0$. We then infer that $(\lambda,j)'=0$ whenever $a_j>0$.

Further, each $0\neq x\in M'_\mu$ is a linear combination of monomials of the form $y_{i_1}y_{i_2}\cdots y_{i_n}\cdot 1$ for some $i_1,\cdots,i_n\in I\cup J$. We may assume that each of these i_k's and, in particular, i_n, is such that that $a_{i_k}>0$, that is, $(\lambda,i_k)=0$. By the definition of $Y(\lambda)$, we get that $y_{i_n}\cdot 1=0$. Thus each monomial appearing in the expression of x is zero, so $x=0$. This contradiction shows that $Y(\lambda)$ is irreducible.

5.6 Lemma. *Let M be an integrable $\mathcal{D}'(\Lambda)$–module in the category $\mathcal{O}'$. Suppose that for every two primitive weights λ and μ with $\lambda-\mu=\sum_{i\in I\cup J}a_i i\in\mathbb{N}[I\cup J]$, it holds that $(\lambda,i)=0$ if $a_i>0$. Then M is completely irreducible and is decomposed into a direct sum of $L(\lambda)$ with $(\lambda,i)\geq 0$ for $i\in I$.*

Proof. By the proof of Lemma 5.5, we get the following fact: Let W be a highest weight module with highest weight $\lambda\in X$. If λ satisfies that, for any primitive weight μ of W with $\lambda-\mu=\sum_{i\in I\cup J}a_i i\mathbb{N}[I\cup J]$, $a_i>0$ implies $(\lambda,i)>0$. Then W is irreducible.

We may assume that $0\neq M=M_C$ for some $\mathbb{Z}[I\cup J]$–coset C in X. By Lemma 5.4, M is decomposed into a direct sum of the generalized eigenspaces of $\tilde{\Omega}$. Since the action of $\tilde{\Omega}$ is central, each generalized eigenspaces of $\tilde{\Omega}$ in M is in fact a submodule. Then M is the sum of those generalized eigenspaces as submodules. Hence, we may furhter assume that there is a $c\in\mathbb{Z}$ such that $(\tilde{\Omega}-v^c):M\to M$ is locally nilpotent.

Let $N=\{m\in M: x_i m=0 \text{ for all } i\in I\cup J\}$, then $N=\sum_{\lambda\in C}N_\lambda$, where $N_\lambda=N\cap M_\lambda$. For each $0\neq m\in N_\lambda$, the $\mathcal{D}'(\Lambda)$–module of M generated by m is irreducible by the fact above. Thus the $\mathcal{D}'(\Lambda)$–submodule M' of M generated by N is a sum of irreducible submodules.

We claim that $M'=M$. Indeed, suppose that $M'':=M/M'\neq 0$. Then we have a maximal $\lambda_1\in C$ such that $M''_{\lambda_1}\neq 0$. Then each $0\neq m_1\in M''_{\lambda_1}$ satisfies that $x_i m_1=0$ for all $i\in I\cup J$. This implies $\tilde{\Omega}(m_1)=v^{(\lambda_1+\rho,\lambda_1+\rho)}m_1$ and $c=(\lambda_1+\rho,\lambda_1+\rho)$. Let $w_1\in M_{\lambda_1}$ be such that $\pi(w_1)=m_1$, where $\pi:M\to M/M'=M''$ denotes the canonical projection. Then $\tilde{M}:=\mathfrak{h}'^{+}(\Lambda)w_1$ is a finite dimensional space. We can choose a maximal $\lambda_2\in C$ such that $\tilde{M}\cap M_{\lambda_2}\neq 0$. Let $0\neq w_2\in\tilde{M}\cap M_{\lambda_2}$, then $x_i w_2=0$ for all $i\in I\cup J$. This implies that $\lambda_2-\lambda_1\in\mathbb{N}[I\cup J]$ and $(\lambda_2+\rho,\lambda_2+\rho)=(\lambda_1+\rho,\lambda_1+\rho)$. Hence $\lambda_1=\lambda_2$ by the assumption. It follows that $\tilde{M}$ is one–dimensional and $x_i w_1=0$ for all $i\in I\cup J$, thus $w_1\in M'$ and $m_1=0$. This is a contradiction.

As a result, we obtain that $M = M'$ is a (direct) sum of irreducible highest weight modules $M(\lambda)$. Moreover, those λ's satisfy $(\lambda, i) \geq 0$ for $i \in I$ since M is integrable.

5.7 Now we consider the integrable modules in the category $\mathcal{O}$ of $\mathcal{D}(\Lambda)$–modules.

Let X be the integral weight system of $\mathbb{Z}[I]$. We denote by X^{++} the strong dominant weight system of X, i.e.

$$X^{++} = \{\lambda \in X : (\lambda, i) > 0 \text{ for all } i \in I\}.$$

Theorem. *The category $\mathcal{O}^{++}$ consisting of integrable highest weight $\mathcal{D}(\Lambda)$–modules with highest weights $\lambda \in X^{++}$ is completely irreducible, that is, each integrable highest weight module with weight $\lambda \in X^{++}$ is irreducible and isomorphic to $L(\lambda)$, where $L(\lambda) = M(\lambda)/\sum_{i\in I}\mathcal{D}(\Lambda)y_i^{(\lambda,i)+1}$; and any integrable module with composition factors isomorphic to these $L(\lambda)$'s is a direct sum of these irreducible modules.*

Proof. Let $F : \mathcal{D}'(\Lambda) \to \mathcal{D}(\Lambda)$ be the canonical epimorphism in 3.8. Then each $\mathcal{D}(\Lambda)$–module M can be naturally viewed as a $\mathcal{D}'(\Lambda)$–module, denoted by M'. Given any $\lambda \in X^{++}$, let $L(\lambda)$ be the integrable highest weight $\mathcal{D}(\Lambda)$–module with highest weight λ. Then $L'(\lambda)$ is still an integrable highest weight $\mathcal{D}'(\Lambda)$–module with highest weight $\lambda' \in X'^{+}$, where $(\lambda', i)' = (\lambda, \delta_i)$ for any $i \in I \cup J$. Thus $L'(\lambda)$ is irreducible and of the form $L'(\lambda) = M'(\lambda)/\sum_{i\in I}\mathcal{D}'(\Lambda)y_i^{(\lambda',i)'+1}$ according to Lemma 5.5. Therefore, we obtain that

$$L(\lambda) = M(\lambda)/\sum_{i\in I}\mathcal{D}(\Lambda)y_i^{(\lambda,i)+1}$$

is irreducible.

Further, let

$$0 \longrightarrow L(\mu) \longrightarrow M \longrightarrow L(\lambda) \longrightarrow 0 \tag{1}$$

be an exact sequence in $\mathcal{O}^{++}$ with $\lambda, \mu \in X^{++}$. This induces an exact sequence

$$0 \longrightarrow L'(\mu) \longrightarrow M' \longrightarrow L'(\lambda) \longrightarrow 0 \tag{2}$$

in $\mathcal{O}'$. It is easy to see that (1) splits if and only if (2) splits. But (2) must split since Lemma 5.5 and $(\lambda', i)' > 0$ for all $i \in I \cup J$. This finishes the proof.

5.8 Proposition. *Let $x \in \mathcal{D}(\Lambda)$ be such that $xM = 0$ for all modules M in $\mathcal{O}^{++}$. Then $x = 0$.*

Proof. For $m, n \in \mathbb{N}$, $\lambda \in X^{++}$, we consider the left ideal of $\mathcal{D}(\Lambda)$:

$$\mathcal{L}_{m,n,\lambda} = \sum_{\mathrm{tr}\alpha > m} \mathcal{D}(\Lambda) u_\alpha^+ + \sum_{\mathrm{tr}\beta > n} \mathcal{D}(\Lambda) u_\beta^- + \sum_{i \in I} \mathcal{D}(\Lambda)(K_i - v^{(\lambda, i)})$$

By Lemma 5.1 and the relations in (2.3), the quotient module $\mathcal{D}(\Lambda)/\mathcal{L}_{m,n,\lambda}$ is an integrable $\mathcal{D}(\Lambda)$-module. Let $M(\lambda + \alpha)$ be the Verma module with the highest weight $\lambda + \alpha$. The morphism

$$\bigoplus_{\alpha \in \mathbb{N}[I], \mathrm{tr}\alpha = m} M(\lambda + \alpha) \longrightarrow \mathcal{D}(\Lambda)/\mathcal{L}_{m,n,\lambda}$$

induced by maps $w_{\lambda+\alpha} \mapsto u_\alpha^+ 1$ is an epimorphism, where $w_{\lambda+\alpha}$ is the highest weight vector of $M(\lambda+\alpha)$. We may require that $\lambda+\alpha \in X^{++}$ for any $\alpha \in \mathbb{N}[I]$ with $\mathrm{tr}\alpha = m$. Therefore, by Theorem 5.7, $\mathcal{D}(\Lambda)/\mathcal{L}_{m,n,\lambda} \in \mathcal{O}^{++}$. Then $x \in \mathcal{L}_{m,n,\lambda}$. By the triangular decomposition of $\mathcal{D}(\Lambda)$, one sees that the intersection $\bigcap \mathcal{L}_{m,n,\lambda}$ is zero by letting $m, n \gg 0$ and $(\lambda, i) \gg 0$ for all $i \in I$. This implies that $x = 0$.

6 The proof of Theorem 2.4

6.1 In order to prove Theorem 2.4, we first define Lusztig's symmetries of the double Ringel–Hall algebras (see [L2]). We shall follow the construction in [Ja, Sect. 8] to produce such symmetries. However, we will deal with integrable weight modules instead of finite dimensional ones in [Ja].

Now we recall some notation and relations in quantum groups which will be needed later on. For each $n \in \mathbb{Z}$, we set

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}} = v^{n-1} + v^{n-3} + \cdots + v^{-n+1},$$

$$[n]! = \Pi_{r=1}^n [r] \qquad \begin{bmatrix} n \\ t \end{bmatrix} = \frac{[n]!}{[r]![n-r]!}.$$

Then we have

$$\sum_{t=0}^{n} (-1)^t v^{t(n-1)} \begin{bmatrix} n \\ t \end{bmatrix} = 0$$

and

$$(a+b)^n = \sum_{t=0}^{n} (-1)^t v^{t(n-t)} \begin{bmatrix} n \\ t \end{bmatrix} b^t a^{n-t},$$

where $ab = v^{-2}ba$.

Further, we set

$$|n] = \frac{q^n - 1}{q - 1}(q^{n-1} + q^{n-2} + \cdots + q + 1),$$

$$|n]! = \Pi_{r=1}^{n}|r] \qquad \left|\begin{matrix} n \\ t \end{matrix}\right] = \frac{|n]!}{|t]!|n-t]!}.$$

Then for $q = v^2$ we have

$$|n] = v^{n-1}[n], \ \ |n]! = v^{\frac{n(n+1)}{2}}[n]!, \ \ \left|\begin{matrix} n \\ t \end{matrix}\right] = \frac{|n]!}{|r]!|n-r]!}$$

and

$$\sum_{t=0}^{n}(-1)^t v^{t(t-1)} \left|\begin{matrix} n \\ t \end{matrix}\right] = 0.$$

For each $i \in I$ and a rational function $f(v)$ of v, we write $f(v)_i$ for $f(v_i)$. Note that $v_i = v^{\langle i,i \rangle}$.

6.2 Let Λ be a hereditary algebra associated with a k–species $\mathcal{S}$, and $\mathcal{D}(\Lambda)$ the double Ringel–Hall algebra of Λ. By 3.3, $\mathcal{D}(\Lambda)$ gives rise to a restricted nondegenerate skew–Hopf pairing belonging to $\tilde{C}_\infty := (I, (\ ,\ ), \{\delta_j : j \in J\})$, where $J = \{(m,p) : m \geq 1, 1 \leq p \leq \eta_m\}$ and $\delta_j = \pi_m$ for all $j = (m,p) \in J$.

Throughout 6.2–6.8, we assume that all π_m, $m \geq 1$, lie in the fundamental set.

As in 5.1, we set $x_i = E_i(0)$ and $y_i = F_i(0)$ for $i \in I$, $x_j = E_p(m)$ and $y_j = F_p(m)$ for $j = (m,p) \in J$. Further, for each $p \geq 0$, we set $x_i^{(p)} = x_i^p/[p]!_i$ and $y_i^{(p)} = y_i^p/[p]!_i$. Note that $[p]!_i$ is obtained from $[p]!$ by replacing v by v_i (see 6.1). Then we have the following lemma.

Lemma. *(1) For each $p \geq 1$ and each $i \in I \cup J$, it holds that*

$$\Delta(x_i^{(p)}) = \sum_{t+t'=p} v^{-tt'} K_{\delta_i}^t x_i^{(t')} \otimes x_i^{(t)},$$

(2) For each $p \geq 0$ and each $i \in I \cup J$, it holds that

$$\varphi(x_i^{(p)}, y_i^{(p)}) = \prod_{s=1}^{n}(1 - v_i)^{-2s} = v^{p(p+1)/2}(v_i - v_i^{-1})^{-p}([p]_i!)^{-1}.$$

(3) Let $n, p, p', q, q' \in \mathbb{N}$ be such that $p+p' = q+q' = n$ and $i \neq j$ in $I \cup J$. Then it holds that

$$\begin{aligned}
&\varphi(x_i^{(p)} x_j x_i^{(p')}, y_i^{(q')} y_j y_i^{(q)}) \\
= \; & \tfrac{v_i^{q(q+1)/2+q'(q'+1)/2}}{(v_i - v_i^{-1})^n} \varphi(x_j, y_j) \\
& \times \textstyle\sum_{t+s=q, t+t'=p, t'+s'=q', s+s'=p'} \tfrac{v_i^{-t(q-1)-s'(q'-1)} v^{(t+s')((\delta_i,\delta_j)+(n-1)(\delta_i,\delta_i)/2)}}{[t]_i![t']_i![s]_i![s']_i!}.
\end{aligned}$$

Proof. (1) This formula follows from $\Delta(x_i) = x_i \otimes 1 + K_{\delta_i} \otimes x_i$ and $(x_i \otimes 1)(K_{\delta_i} \otimes x_i) = v_i^{-2}(K_{\delta_i} \otimes x_i)(x_i \otimes 1)$.

(2) This statement can be proved by an inductive argument.

(3) By (1), we get

$$\begin{aligned}
&\Delta(x_i^{(p)} x_j x_i^{(p')}) \\
= \; & (\textstyle\sum_{t+t'=p} v_i^{-tt'} K_{\delta_i}^t x_i^{(t')} \otimes x_i^{(t)})(x_j \otimes 1 + K_{\delta_j} \otimes x_j)(\sum_{s+s'=p'} v_i^{-ss'} K_{\delta_i}^s x_i^{(s')} \otimes x_i^{(s)}) \\
= \; & \textstyle\sum v_i^{-tt'} v_i^{-ss'} K_{\delta_i}^t x_i^{(t')} x_j K_{\delta_i}^s x_i^{(s')} \otimes x_i^{(t)} x_i^{(s)} \\
& + \textstyle\sum v_i^{-tt'} v_i^{-ss'} K_{\delta_i}^t x_i^{(t')} K_{\delta_j} K_{\delta_i}^s x_i^{(s')} \otimes x_i^{(t)} x_j x_i^{(s)} \\
= \; & \textstyle\sum v_i^{-tt'} v_i^{-ss'} v^{-t's(\delta_i,\delta_i)-s(\delta_i,\delta_j)} K_{\delta_i}^{t+s} x_i^{(t')} x_j x_i^{(s')} \otimes x_i^{(t)} x_i^{(s)} \\
& + \textstyle\sum v_i^{-tt'} v_i^{-ss'} v^{-t'(\delta_i,\delta_j)-t's(\delta_i,\delta_i)} K_{\delta_i}^{t+s} K_{\delta_j} x_i^{(t')} x_i^{(s')} \otimes x_i^{(t)} x_j x_i^{(s)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
& \varphi(x_i^{(p)} x_j x_i^{(p')}, y_i^{(q')} y_j y_i^{(q)}) = \varphi(\Delta(x_i^{(p)} x_j x_i^{(p')}), y_i^{(q')} y_j \otimes y_i^{(q)}) \\
= \; & \textstyle\sum v_i^{-tt'} v_i^{-ss'} v^{-t's(\delta_i,\delta_i)-s(\delta_i,\delta_j)} \varphi(K_{\delta_i}^q x_i^{(t')} x_j x_i^{(s')}, y_i^{(q')} y_j) \varphi(x_i^{(t)} x_i^{(s)}, y_i^{(q)}),
\end{aligned}$$

where the sum is taken over all s, s', t, t' such that $t+s = q, t+t' = p, t'+s' = q', s+s' = p'$. In case $t+s = q$ and $t'+s' = q'$, we have

$$\begin{aligned}
& \varphi(K_{\delta_i}^q x_i^{(t')} x_j x_i^{(s')}, y_i^{(q')} y_j) = v^{qq'(\delta_i,\delta_i)+q(\delta_i,\delta_j)} \varphi(x_i^{(t')} x_j x_i^{(s')}, y_i^{(q')} y_j) \\
= \; & v^{qq'(\delta_i,\delta_i)+q(\delta_i,\delta_j)} \varphi(\Delta(x_i^{(t')} x_j x_i^{(s')}), y_i^{(q')} \otimes y_j) \\
= \; & v^{qq'(\delta_i,\delta_i)+q(\delta_i,\delta_j)} v^{q'(\delta_i,\delta_j)} v^{-t'(\delta_i,\delta_j)} \varphi(\Delta(x_i^{(t')} x_i^{(s')}), y_i^{(q')} \varphi(x_j, y_j).
\end{aligned}$$

Further, we have

$$x_i^{(t)} x_i^{(s)} = \begin{bmatrix} q \\ t \end{bmatrix}_i x_i^{(q)} \text{ and } x_i^{(t')} x_i^{(s')} = \begin{bmatrix} q \\ t \end{bmatrix}_i x_i^{(q')}.$$

Thus we obtain

$$
\begin{aligned}
& \varphi(x_i^{(p)} x_j x_i^{(p')}, y_i^{(q')} y_j y_i^{(q)}) \\
= & \sum v_i^{-tt'} v_i^{-ss'} v^{-t's(\delta_i,\delta_i)-s(\delta_i,\delta_j)} v^{qq'(\delta_i,\delta_i)+q(\delta_i,\delta_j)+q'(\delta_i,\delta_j)-t'(\delta_i,\delta_j)} \\
& \begin{bmatrix} q \\ t \end{bmatrix}_i \begin{bmatrix} q' \\ t' \end{bmatrix}_i \varphi(x_i^{(q')}, y_i^{(q')}) \varphi(x_i^{(q)}, y_i^{(q)}) \varphi(x_j, y_j) \\
= & \sum v_i^{-tt'} v_i^{-ss'} v^{-t's(\delta_i,\delta_i)-s(\delta_i,\delta_j)} v^{qq'(\delta_i,\delta_i)+q(\delta_i,\delta_j)+q'(\delta_i,\delta_j)-t'(\delta_i,\delta_j)} \\
& \begin{bmatrix} q \\ t \end{bmatrix}_i \begin{bmatrix} q' \\ t' \end{bmatrix}_i \frac{v_i^{q'(q'+1)/2}}{(v_i - v_i)^{-1})^{q'} [q']_i!} \frac{v_i^{q(q+1)/2}}{(v_i - v_i)^{-1})^{q} [q]_i! \varphi(x_j, y_j)}.
\end{aligned}
$$

The formula to be proved is then follows from the equalities

$$
\begin{aligned}
& q(\delta_i, \delta_j) + q'(\delta_i, \delta_j) - t'(\delta_i, \delta_j) - s(\delta_i, \delta_j) = (t + s')(\delta_i, \delta_j), \\
& 2qq' - 2t's - tt' - ss' = q't + qs', \\
\text{and} \quad & -t(q-1) - s'(q'-1) + (t+s')(n-1) = tq' + s'q.
\end{aligned}
$$

6.3 Proposition. *(1) For any $i \in I$ and $j \in I \cup J$, if $(\delta_i, \delta_j) = (i, \delta_j) < 0$, then*

$$
\sum_{p+p'=1-a_{ij}} (-1)^p x_i^{(p)} x_j x_i^{(p')} = 0 \quad \textit{and} \quad \sum_{p+p'=1-a_{ij}} (-1)^p y_i^{(p)} y_j y_i^{(p')} = 0
$$

hold respectively in $\mathfrak{h}^+(\Lambda)$ and $\mathfrak{h}^-(\Lambda)$, where $a_{ij} = 2(\delta_i, \delta_j)/(\delta_i, \delta_i)$,

(2) For any $i, j \in I \cup J$, if $(\delta_i, \delta_j) = 0$, we have that

$$
x_i x_j = x_j x_i \quad \textit{and} \quad y_i y_j = y_j y_i.
$$

Proof. (1) We prove the first equality only. The second one can be proved similarly.

By the non–degeneracy of φ, it suffices to show that under the pairing φ, $\sum_{p+p'=1-a_{ij}} (-1)^p x_i^{(p)} x_j x_i^{(p')}$ is orthogonal to $y_i^{(q')} y_j y_i^{(q)}$, where $q + q' = 1 - a_{ij}$. According to Lemma 6.2(3), it is enough to verify the following equality

$$
\sum_{t+s=q, t'+s'=q'} (-1)^{s+s'} v_i^{-t(q-1)-s'(q'-1)} ([t]_i! [t']_i! [s]_i! [s']_i!)^{-1} = 0
$$

for $n = 1 - a_{ij}$ since $(\delta_i, \delta_j) + (n-1)(\delta_i, \delta_i)/2 = 0$. The left hand side of the equality above is decomposed into

$$
\Big(\sum_{t+s=q} (-1)^s v_i^{-t(q-1)} ([t]_i! [s]_i!)^{-1}\Big) \Big(\sum_{t'+s'=q'} (-1)^{s'} v_i^{-s'(q'-1)} ([t']_i! [s']_i!)^{-1}\Big).
$$

If $q > 0$, the first factor is zero by 6.1, and the second factor vanishes if $q' > 0$. However, we have $q + q' = 1 - a_{ij} > 0$ since $(\delta_i, \delta_j) \le 0$ and $(\delta_i, \delta_i) > 0$. Hence, either q or q' is positive. This implies the required equality.

(2) This statement is obvious.

Remark. Combining Propositon 6.3 together with 2.3, we obtain that $\mathcal{D}(\Lambda)$ is generated by $\{x_j, y_j : j \in I \cup J\} \cup \{K_i, K_{-i} : i \in I\}$ satisfying the following relations:

(1)

$$x_i y_j - y_j x_i = \frac{K_{\delta_i} - K_{-\delta_i}}{v' - v'^{-1}} \delta_{ij} \quad \text{for } \ i, j \in I \cup J,$$

where $v' = v^{\langle i,i \rangle}$ if $i \in I$ and $v' = v^m$ if $i = (m, p) \in J$,

(2)

$$K_0 = 1, \ K_\alpha K_\beta = K_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \mathbb{Z}[I],$$

(3)

$$K_\alpha x_i = v^{(\alpha,\delta_i)} x_i K_\alpha, \ K_\alpha y_i = v^{-(\alpha,\delta_i)} y_i K_\alpha \ \text{ for } i \in I \cup J, \alpha \in \mathbb{Z}[I].$$

(4)

$$\sum_{p+p'=1-a_{ij}} (-1)^p x_i^{(p)} x_j x_i^{(p')} = 0 \ \text{ and } \ \sum_{p+p'=1-a_{ij}} (-1)^p y_i^{(p)} y_j y_i^{(p')} = 0$$

for $i \in I$ and $j \in I \cup J$ with $i \ne j$, where $a_{ij} = 2(\delta_i, \delta_j)/(\delta_i, \delta_i)$,

(5)

$$x_i x_j = x_j x_i \quad \text{and} \quad y_i y_j = y_j y_i$$

for any $i, j \in I \cup J$ with $(\delta_i, \delta_j) = 0$.

6.4 We now fix an $i \in I$ and define the symmetry T_i of $\mathcal{D}(\Lambda)$. All $\mathcal{D}(\Lambda)$–modules M considered in the following lie in the category $\mathcal{O}$, that is, M admits a weight decomposition $M = \oplus_{\lambda \in X} M_\lambda$ satisfying: (1) every weight space M_λ is finite dimensional, (2) for every $x \in M$ there exists an $n_0 \ge 0$ such that $\mathfrak{h}^+_\alpha x = 0$ whenever $\mathrm{tr}\alpha \ge n_0$.

Let M be an integrable $\mathcal{D}(\Lambda)$–module. For each $\xi \in M_\lambda$, $\lambda \in X$, we define

$$T_i(\xi) = \sum_{a,b,c \ge 0;\, -a+b-c=m} (-1)^b v_i^{b-ac} x_i^{(a)} y_i^{(b)} x_i^{(c)} \xi,$$

$$T_i'(\xi) = \sum_{a,b,c \ge 0;\, -a+b-c=m} (-1)^b v_i^{ac-b} x_i^{(a)} y_i^{(b)} x_i^{(c)} \xi,$$

$${}^\omega T_i(\xi) = \sum_{a,b,c \ge 0;\, a-b+c=m} (-1)^b v_i^{b-ac} y_i^{(a)} x_i^{(b)} y_i^{(c)} \xi,$$

$$^{\omega}T_i'(\xi) = \sum_{a,b,c\geq 0;\, a-b+c=m} (-1)^b v_i^{ac-b} y_i^{(a)} x_i^{(b)} y_i^{(c)} \xi,$$

where $m = 2(\lambda, i)/(i,i)$ and $v_i = v^{(i,i)/2}$.

By $\mathcal{D}_i$ we denote the subalgebra of $\mathcal{D}(\Lambda)$ generated by x_i, y_i, K_i and K_{-i}, which is obviously isomorphic to $U_{v_i}(sl_2)$. By [L2], the integrable $\mathcal{D}(\Lambda)$–module M considered as a $\mathcal{D}_i$–module is a direct sum of finite dimensional simple $\mathcal{D}_i$–modules. Thus, by the arguments in [Ja, 8.2–8.7], we have the following lemma.

Lemma. *(1) The operators T_i, T_i', $^{\omega}T_i$, and $^{\omega}T_i'$ are bijective on each integrable $\mathcal{D}(\Lambda)$–module M in $\mathcal{O}$. Moreover, we have*

$$T_i^{-1} = {}^{\omega}T_i' \quad \textit{and} \quad T'^{-1}_i = {}^{\omega}T_i.$$

For all $\lambda \in X$ and all $\xi \in M_\lambda$, we get

$$^{\omega}T_i(\xi) = (-v_i)^{-2(\lambda,i)/(i,i)} T_i(\xi) \quad \textit{and} \quad {}^{\omega}T_i'(\xi) = (-v_i)^{2(\lambda,i)/(i,i)} T_i'(\xi).$$

(2) For all ξ in each integrable $\mathcal{D}(\Lambda)$–module M and all $\mu \in \mathbb{Z}[I]$, we have

$$\begin{array}{ll} T_i(K_\mu \xi) = K_{s_i\mu} T_i(\xi), & K_\mu T_i(\xi) = T_i(K_{s_i\mu}\xi), \\ T_i(x_i\xi) = (-y_iK_i)T_i(\xi), & x_iT_i(\xi) = T_i((-K_{-i}y_i)\xi), \\ T_i(y_i\xi) = (-K_{-i}x_i)T_i(\xi), & y_iT_i(\xi) = T_i((-x_iK_i)\xi). \end{array}$$

(3) Suppose that $x, x' \in \mathcal{D}(\Lambda)$ are such that

$$T_i(x\xi) = x' T_i(\xi)$$

for all ξ in all integrable $\mathcal{D}(\Lambda)$–modules. Then it holds

$$^{\omega}T_i(\omega(x)\xi) = \omega(x')\, {}^{\omega}T_i(\xi).$$

If $x \in \mathcal{D}(\Lambda)_\mu$ for some $\mu \in \mathbb{Z}[I]$, then we have

$$T_i(\omega(x)\xi) = (-v_i)^{-2(\mu,i)/(i,i)} \omega(x') T_i(\xi).$$

6.5 We now want to have formulae for $x_jT_i(\xi)$ and $T_i(x_j\xi)$ for all $i \neq j \in I \cup J$; similarly for y_j instead of x_j.

The Hopf algebra structure of $\mathcal{D}(\Lambda)$ gives rise to an adjoint representation of $\mathcal{D}(\Lambda)$ on itself such that

$$\mathrm{ad}\,(z)(x) = \sum_s z_s x \sigma(z_s'), \quad \text{if} \quad \Delta(z) = \sum_s z_s \otimes z_s',$$

where $x, z \in \mathcal{D}(\Lambda)$. In paticular, we have

$$\begin{aligned}
&\mathrm{ad}\,(x_i)x = x_i x - K_i x K_{-i} x_i,\\
&\mathrm{ad}\,(y_i)x = y_i x K_i - x y_i K_i,\\
&\mathrm{ad}\,(K_i)x = K_i x K_{-i}.
\end{aligned}$$

Inductively, we have

$$\mathrm{ad}\,(x_i^{(m)})x = \sum_{s=0}^{m} (-1)^s v_i^{s(m-1)} x_i^{(m-s)} K_i^s x K_{-i}^s x_i^{(s)}$$

and

$$\mathrm{ad}\,(y_i^{(m)})x = \sum_{s=0}^{m} (-1)^{m-s} v_i^{-(m-s)(m-1)} y_i^{(s)} x y_i^{(m-s)} K_i^m$$

for all $x \in \mathcal{D}(\Lambda)$ and all $m \geq 0$. Further, for a fixed $j \in I \cup J$ with $j \neq i$, from $K_i x_j = v_i^{-a_{ij}} x_j K_i$ it follows

$$\mathrm{ad}\,(x_i^{(m)})x_j = \sum_{s=0}^{m} (-1)^s v_i^{s(m-1+a_{ij})} x_i^{(m-s)} x_j x_i^{(s)},$$

where $a_{ij} = 2(\delta_j, i)/(i, i)$. Proposition 6.3 then implies that

$$\mathrm{ad}\,(x_i^{(m)})x_j = 0 \quad \text{for all } \ m > -a_{ij}.$$

Since x_i and y_j commute, we have $\mathrm{ad}\,(y_j)x_i = 0$. A further calculation yields for all $m, t \geq 0$

$$\mathrm{ad}\,(y_j^{(t)})\mathrm{ad}\,(x_i^{(m)})x_j = \begin{cases} \begin{bmatrix} -a_{ij} + t - m \\ t \end{bmatrix}_i \mathrm{ad}\,(x_i^{(m-t)})x_j & \text{if} \quad t \leq m, \\ 0 & \text{if} \quad t > m. \end{cases}$$

For each $m \geq 0$, we set $a(m) = \mathrm{ad}\,(x_i^{(m)})x_j$. Using similar arguments in [Ja, 8.9, 8.10], we have the following results.

Lemma. *For all integers $m, t \geq 0$, we obtain*

$$a(m)x_i^{(t)} = \sum_{s=0}^{t} \begin{bmatrix} m+s \\ s \end{bmatrix}_i v_i^{t(-a_{ij}-2m)-s(t-1)} x_i^{(t-s)} a(m+s)$$

and

$$a(m)y_i^{(t)} = \sum_{s=0}^{t} \begin{bmatrix} -a_{ij} - m + s \\ s \end{bmatrix}_i v_i^{s(t-1)} y_i^{(t-s)} a(m-s) K_{-i}^s.$$

Remark. Since $\mathrm{ad}\,(x_i^{(-a_{ij}+s)})x_j = 0$ for all $s > 0$, we have from the lemma above that

$$(\mathrm{ad}\,(x_i^{(-a_{ij})})x_j)x_i^{(t)} = v_i^{ta_{ij}} x_i^{(t)}(\mathrm{ad}\,(x_i^{(-a_{ij})})x_j).$$

Proposition. *For all ξ in all integrable $\mathcal{D}(\Lambda)$–modules M, we have*

$$T_i(x_j\xi) = (ad\,(x_i^{(-a_{ij})})x_j)T_i(\xi)$$

and

$$T_i(y_j\xi) = (\sum_{s=0}^{-a_{ij}}(-1)^s v_i^s y_i^{(s)} y_j y_i^{(-a_{ij}-s)})T_i(\xi).$$

This proposition is an analogue of [Ja, 8.10].

6.6 We set for all $z \in \mathcal{D}(\Lambda)$

$$^{\tau}\mathrm{ad}\,(z) = \tau \circ \mathrm{ad}\,(z) \circ \tau^{-1} : \mathcal{D}(\Lambda) \longrightarrow \mathcal{D}(\Lambda),$$

where τ is the anti–automorphism of $\mathcal{D}(\Lambda)$ defined in 4.1. Then we have for all $x \in \mathcal{D}(\Lambda)$

$$\begin{aligned} &^{\tau}\mathrm{ad}\,(x_i)x = v_i^{-1}(x_iK_ixK_{-i} - xx_i), \\ &^{\tau}\mathrm{ad}\,(y_i)x = v_iK_{-i}(y_ix - xy_i), \\ &^{\tau}\mathrm{ad}\,(K_i)x = K_ixK_{-i} \end{aligned}$$

since $\tau(x_i) = -v_i^{-1}x_i$ and $\tau(y_i) = -v_iy_i$. Note that the formulae above are slightly different from those in [Ja, 8.11(2)]. Further, one has the following lemma.

Lemma. *Suppose that $x, x' \in \mathcal{D}(\Lambda)$ are such that*

$$T_i(x\xi) = x'T_i(\xi)$$

for all ξ in all integrable $\mathcal{D}(\Lambda)$–modules. Then we have

$$T_i((^{\tau}ad\,(x_i)x)\xi) = -v_i^{-1}(ad\,(y_i)x')T_i(\xi)$$

and

$$T_i((^{\tau}ad\,(y_i)x)\xi) = -v_i(ad\,(x_i)x')T_i(\xi).$$

From 6.5, we obtain

$$^{\tau}\mathrm{ad}\,(x_i^{(m)})x_j = (-1)^m v_i^{-m}\sum_{s=0}^{m}(-1)^s v_i^{s(m-1+a_{ij})} x_i^{(s)} x_j x_i^{(m-s)}.$$

This together with the lamma above and Propositon 6.5 implies

$$T_i((^\tau \mathrm{ad}\,(x_i^{(m)}x_j)\xi) = (-1)^m v_i^{-m}(\mathrm{ad}\,(y_i^{(m)})\mathrm{ad}\,(x_i^{(-a_{ij})}x_j)T_i(\xi).$$

Again, by 6.5, we further get

$$T_i((^\tau \mathrm{ad}\,(x_i^{(m)}x_j)\xi) = (-1)^m v_i^{-m}(\mathrm{ad}\,(x_i^{(-a_{ij}-m)})x_j)T_i(\xi).$$

Finally, we have the equality

$$\begin{array}{rl} & T_i(\sum_{s=0}^m (-1)^s v_i^{s(m-1+a_{ij})} x_i^{(s)} x_j x_i^{(m-s)}\xi) \\ = & (\sum_{s=0}^{-a_{ij}-m} (-1)^s v_i^{-s(m+1)} x_i^{(-a_{ij}-m-s)} x_j x_i^{(s)}) T_i(\xi). \end{array}$$

This together with Lemma 6.3(3) implies the equality

$$\begin{array}{rl} & T_i(\sum_{s=0}^m (-1)^s v_i^{-s(m-1+a_{ij})} y_i^{(m-s)} y_j y_i^{(s)}\xi) \\ = & (\sum_{s=0}^{-a_{ij}-m} (-1)^s v_i^{s(m+1)} y_i^{(s)} y_j y_i^{(-a_{ij}-m-s)}) T_i(\xi). \end{array}$$

6.7 As a conclusion, Proposition 5.8 leads to the following result.

Proposition. *Let $i \in I$. Then for all $x \in \mathcal{D}(\Lambda)$, there is a unique element $x' \in \mathcal{D}(\Lambda)$ such that*

$$T_i(x\xi) = x'T_i(\xi)$$

for all ξ in an integrable $\mathcal{D}(\Lambda)$–module M. Moreover, the map $x \longmapsto x'$ is an automorphism of $\mathcal{D}(\Lambda)$.

By T_i we also denote the isomorphism $\mathcal{D}(\Lambda) \to \mathcal{D}(\Lambda), x \longmapsto x'$ in the proposition. Then we have

$$T_i(x\xi) = T_i(x)T_i(\xi)$$

for all $x \in \mathcal{D}(\Lambda)$ and all ξ in an integrable $\mathcal{D}(\Lambda)$–module. Moreover, we have the explicit formulae of $T_i(x)$ and $T_i^{-1}(x)$ for the generators $\{x_j, y_j : j \in I \cup J\} \cup \{K_i, K_{-i} : i \in I\}$ of $\mathcal{D}(\Lambda)$:

$$T_i(K_\mu) = K_{s_i\mu} = T_i^{-1}(K_\mu),$$

$$T_i(x_i) = -y_iK_i, \qquad T_i^{-1}(x_i) = -K_{-i}y_i,$$

$$T_i(y_i) = -K_{-i}x_i, \qquad T_i^{-1}(y_i) = -x_iK_i,$$

$$T_i(x_j) = \sum_{s=0}^{-a_{ij}} (-1)^s v_i^{-s} x_i^{(-a_{ij}-s)} x_j x_i^{(s)},$$

$$T_i^{-1}(x_j) = \sum_{s=0}^{-a_{ij}} (-1)^s v_i^{-s} x_i^{(s)} x_j x_i^{(-a_{ij}-s)},$$

$$T_i(y_j) = \sum_{s=0}^{-a_{ij}} (-1)^s v_i^s y_i^{(s)} y_j y_i^{(-a_{ij}-s)},$$

$$T_i^{-1}(y_j) = \sum_{s=0}^{-a_{ij}} (-1)^s v_i^s y_i^{(-a_{ij}-s)} y_j y_i^{(s)}$$

for all $\mu \in \mathbb{Z}[I]$ and $i \neq j \in I \cup J$. Furhter, we have

$$T_i^{-1} = \tau \circ T_i \circ \tau^{-1}.$$

6.8 For each $i \in I$, by $\mathfrak{h}^+(\Lambda)[i]$ we denote the subalgebra of $\mathfrak{h}^+(\Lambda)$ generated by $\tau(\mathrm{ad}\,(x_i^{(m)})x_j)$ for all $i \neq j \in I \cup J$ and all $m \geq 0$, and by ${}^{\tau}\mathfrak{h}^+(\Lambda)[i]$ the subalgebra of $\mathfrak{h}^+(\Lambda)$ generated by all $\mathrm{ad}\,(x_i^{(m)})x_j$ with $j \neq i$. Thus we have ${}^{\tau}\mathfrak{h}^+(\Lambda)[i] = \tau(\mathfrak{h}^+(\Lambda)[i])$. By 6.5 and 6.6, we can easily show

$$\mathfrak{h}^+(\Lambda) = \sum_{s\geq 0} x_i^s(\mathfrak{h}^+(\Lambda)[i]) = \sum_{s\geq 0} x_i^s({}^{\tau}\mathfrak{h}^+(\Lambda)[i]).$$

Moreover, T_i induces an isomorphism $\mathfrak{h}^+(\Lambda)[i] \to {}^{\tau}\mathfrak{h}^+(\Lambda)[i]$. By using the analogous arguments in [L2, 38.1.1–38.1.6], we get the following proposition.

Proposition. *We have*

$$\mathfrak{h}^+(\Lambda)[i] = \{x \in \mathfrak{h}^+(\Lambda) : T_i(x) \in \mathcal{H}^+(\Lambda)\} = \{x \in \mathfrak{h}^+(\Lambda) : r_i'(x) = 0\}$$

and

$${}^{\tau}\mathfrak{h}^+(\Lambda)[i] = \{x \in \mathfrak{h}^+(\Lambda) : T_i^{-1}(x) \in \mathcal{H}^+(\Lambda)\} = \{x \in \mathfrak{h}^+(\Lambda) : r_i(x) = 0\}.$$

Remark. If i is a sink of $\mathcal{S}$, where $\mathcal{S}$ is the k–species of Λ, we obtain from the lemma and 1.7 that $\mathfrak{h}^+(\Lambda)[i] = \mathfrak{h}^+(\Lambda)\langle i\rangle$. Dually, if i is a source, then ${}^{\tau}\mathfrak{h}^+(\Lambda)[i] = \mathfrak{h}^+(\Lambda)\langle i\rangle$

6.9 In Section 2 a family of subalgebras $\mathcal{D}_m(\Lambda)$, $m \geq 1$, of $\mathcal{D}(\Lambda)$ has been constructed. By 3.3, each $\mathcal{D}_m(\Lambda)$ gives rise to a restricted nondegenerate skew–Hopf pairing $(\mathcal{D}_m^+(\Lambda), \mathcal{D}_m^-(\Lambda), \varphi)$ belonging to $\tilde{C}_m := (I, (\ ,\ ), \{\delta_j : j \in J_m\})$, where $J_m = \{(t,p) : 1 \leq t \leq m, 1 \leq p \leq \eta_m\}$ and $\delta_j = \pi_t$ for all $j = (t,p) \in J_m$.

Let $m \geq 1$. Under the assumption that all π_t, $1 \leq t \leq m$, lie in the fundamental set, by using similar arguments one can show that all results in

Section 5 and in 6.2–6.8 can be formulated for $\mathcal{D}_m(\Lambda)$. In particular, one has that $\mathcal{D}_m(\Lambda)$ is generated by $x_j, y_j, j \in I \cup J_m$ and $K_i, K_{-i}, i \in I$ satisfying the corresponding relations (1)–(5) listed in Remark 6.3. Further, one obtains that $\mathcal{D}_m(\Lambda)$ admits an automorphism defined in a similiar way to the T_i of $\mathcal{D}(\Lambda)$, which will be denoted by T_i, too.

6.10 Proof of Theorem 2.4. For each $i \in I$ and each $m \geq 0$, we set $\mathfrak{d}_m^{\pm}(\Lambda)[i] = \mathfrak{d}_m^{\pm}(\Lambda) \cap \mathfrak{h}^{\pm}(\Lambda)[i]$.

By the choice of each π_m, we see that π_m is a dimension vector of some indecomposable representation in rep $\mathcal{S}$. We use induction on m to show that all π_m lie in the fundamental set.

In case $m = 0$, we have $\mathcal{D}_0(\Lambda) = \mathcal{C}(\Lambda)$, and nothing to prove. Further, if i is a sink, T_i^R induces an isomorphism $\mathcal{D}_0(\Lambda) \to \mathcal{D}_0(\sigma_i\Lambda)$ (see [SV1] and [XY]).

Let $m \geq 1$. Suppose that all π_t, $t \leq m-1$, lie in the fundamental set and that T_i^R induces an isomorphism from $\mathcal{D}_{m-1}(\Lambda)$ to $\mathcal{D}_{m-1}(\sigma_i\Lambda)$ (in case i is a sink). We shall prove that π_m lies in the fundamental set and that T_i^R induces an isomorphism from $\mathcal{D}_m(\Lambda)$ to $\mathcal{D}_m(\sigma_i\Lambda)$.

Since all π_t, $t \leq m-1$, lie in the fundamental set, by 6.9 we have that, for each $i \in I$, T_i is well defined on $\mathcal{D}_{m-1}(\Lambda)$.

Suppose that π_m does not lie in the fundamental set. Then there is some $i \in I$ such that $(i, \pi_m) > 0$, that is, $s_i\pi_m < \pi_m$. In view of Theorem 3.7, we may suppose that i is a sink of Ω. By 1.8, T_i^R induces an R–linear bijective map

$$\mathfrak{h}^+(\Lambda)\langle i\rangle_{\pi_m} \to \mathfrak{h}^+(\sigma_i\Lambda)\langle i\rangle_{s_i\pi_m} = ({}^{\tau}\mathfrak{h}^+(\sigma_i\Lambda)[i])_{s_i\pi_m}.$$

On the other hand, by the induction hypothesis, T_i^R induces an R–linear bijective map

$$\mathfrak{d}_{m-1}^+(\Lambda)[i]_{\pi_m} \to ({}^{\tau}\mathfrak{d}_{m-1}^+(\sigma_i\Lambda)[i])_{s_i\pi_m}.$$

Since $s_i\pi_m < \pi_m$, we get $\mathfrak{d}_{m-1}^+(\sigma_i\Lambda)_{s_i\pi_m} = \mathfrak{h}^+(\sigma_i\Lambda)_{s_i\pi_m}$ by the minimality of π_m. Thus we have $\mathfrak{d}_{m-1}^+(\sigma_i\Lambda)[i]_{s_i\pi_m} = \mathfrak{h}^+(\sigma_i\Lambda)[i]_{s_i\pi_m}$. By comparing dimensions, we have

$$\mathfrak{d}_{m-1}^+(\Lambda)[i]_{\pi_m} = \mathfrak{h}^+(\Lambda)\langle i\rangle_{\pi_m} = \mathfrak{h}^+(\Lambda)[i]_{\pi_m}.$$

This then implies $\mathfrak{d}_{m-1}^+(\Lambda)_{\pi_m} = \mathfrak{h}^+(\Lambda)_{\pi_m}$. This contradicts the choice of π_m. Hence, π_m lies in the fundamental set.

By 8.9 we have that both T_i and T_i^{-1} are well defined on $\mathcal{D}_m(\Lambda)$ for each $i \in I$. Further, T_i^{-1} induces a bijective R–linear map from ${}^{\tau}\mathfrak{d}_m^+(\Lambda)[i]_{\pi_m}$ to $\mathfrak{d}_m^+(\Lambda)[i]_{s_i\pi_m}$. As a result, we have the induced maps

$$\begin{aligned}\mathfrak{h}^+(\Lambda)\langle i\rangle_{\pi_m} &\overset{\tau}{\cong} ({}^{\tau}\mathfrak{h}^+(\Lambda)\langle i\rangle)_{\pi_m} = ({}^{\tau}\mathfrak{d}_m^+(\Lambda)[i])_{\pi_m} \xrightarrow{T_i^{-1}} \mathfrak{d}_m^+(\Lambda)[i]_{s_i\pi_m}\\ &\subseteq \mathfrak{h}^+(\Lambda)\langle i\rangle_{s_i\pi_m} \xrightarrow{T_i^R} \mathfrak{h}^+(\sigma_i\Lambda)\langle i\rangle_{\pi_m} = ({}^{\tau}\mathfrak{d}_m^+(\sigma_i\Lambda)[i])_{\pi_m}\end{aligned}$$

whose composition is obviously injective, thus bijective since $\dim {}_R\mathfrak{h}^+(\Lambda)\langle i\rangle_{\pi_m} = \dim {}_R\mathfrak{h}^+(\sigma_i\Lambda)\langle i\rangle_{\pi_m}$ by [H]. It then follows that

$$\mathfrak{d}_m^+(\Lambda)[i]_{s_i\pi_m} = \mathfrak{h}^+(\Lambda)\langle i\rangle_{s_i\pi_m}.$$

Thus, we have $\mathfrak{d}_m^+(\sigma_i\Lambda)[i]_{s_i\pi_m} = \mathfrak{h}^+(\sigma_i\Lambda)\langle i\rangle_{s_i\pi_m}$, and hence

$$T_i^R(\mathfrak{d}_m^+(\Lambda)[i])_{\pi_m}) \subseteq \mathfrak{h}^+(\sigma_i\Lambda)[i]_{s_i\pi_m} = \mathfrak{d}_m^+(\sigma_i\Lambda)[i]_{s_i\pi_m}.$$

By 2.1, it is easy to see that $L_{\pi_m}^+ \subseteq \mathfrak{d}_m^+(\Lambda)[i]_{\pi_m}$. This implies $T_i^R(L_{\pi_m}^+) \subseteq \mathfrak{d}_m^+(\sigma_i\Lambda)[i]_{s_i\pi_m}$. Dually, we have $T_i^R(L_{\pi_m}^-) \subseteq \mathfrak{d}_m^-(\sigma_i\Lambda)[i]_{s_i\pi_m}$. But $\mathcal{D}_m(\Lambda)$ is generated by $\mathcal{D}_{m-1}(\Lambda)$, $L_{\pi_m}^{\pm}$. We conclude that $T_i^R(\mathcal{D}_m(\Lambda)) \subseteq \mathcal{D}_m(\sigma_i\Lambda)$, thus T_i^R induces an isomorphism from $\mathcal{D}_m(\Lambda)$ to $\mathcal{D}_m(\sigma_i\Lambda)$.

This finishes the proof of the theorem 2.4

Remark. Under the canonical isomorphism of $\mathcal{C}(\Lambda)$ and $\mathcal{C}(\sigma_i\Lambda)$, it is shown in [XY] that T_i coincides with T_i^R as operators on $\mathcal{C}(\Lambda)$. The comparison of T_i and T_i^R on the whole double Ringel–Hall algebra $\mathcal{D}(\Lambda)$ will be given in [DX].

7 Weyl–Kac character formula

7.1 Let Φ^+ be the set of the dimension vectors of all indecomposable modules of Λ, and set $\Phi^- = -\Phi^+$ and $\Phi = \Phi^- \cup \Phi^+$. Let W be the Weyl group corresponding to the Cartan datum $C = (I, (\, , \,))$. Let $\mathcal{S}$ be the k–species of Λ. If i is a sink of $\mathcal{S}$, we have the BGP–reflection functor $\sigma_i^+ : \text{rep } \mathcal{S}\langle i\rangle \to \text{rep } \sigma_i\mathcal{S}\langle i\rangle$. By [H], the number of isomorphism classes of indecomposable representations of $\mathcal{S}$ with a fixed dimension vector is independent on the orientation of $\mathcal{S}$. Therefore, for each $i \in I$, the fundamental reflection γ_i is well–defined on Φ by $\gamma_i(\alpha) = \alpha - \frac{2(\alpha,i)}{(i,i)}i$. Thus the action of W on Φ is also well–defined.

Let M be a $\mathcal{D}(\Lambda)$–module in the category $\mathcal{O}$ and $M = \oplus_{\lambda\in X} M_\lambda$ be its weight space decomposition. The formal character of M is defined by

$$\text{ch}\,(M) = \sum_{\lambda\in X} (\dim M_\lambda) e(\lambda).$$

For $\alpha \in \mathbb{N}[I]$, we denote by $m(\alpha)$ the number of isomorphism classes of Λ–modules with dimension vector α, and by $I_{\mathcal{P}}(\alpha)$ the number of isomorphism classes of indecomposable Λ–modules with dimension vector α in case $\alpha \in \Phi^+$.

For the Verma module $M(\lambda)$, if $M(\lambda)_\beta \neq 0$, then $\lambda - \beta \in \mathbb{N}[I]$ and $\dim M(\lambda)_\beta = m(\lambda - \beta)$. Thus

$$\text{ch}\, M(\lambda) = \sum_\beta m(\lambda - \beta) e(\beta) = e(\lambda) \sum_{\alpha\in\mathbb{N}[I]} m(\alpha) e(-\alpha).$$

Since all monomials on the set $\{u_\beta^- : V_\beta$ is indecomposable Λ–modules$\}$ in a fixed order provide a universal PBW–basis of $\mathfrak{h}^-(\Lambda)$ by [GP], we have

$$\mathrm{ch}\, M(\lambda) = e(\lambda) \prod_{\alpha\in\Phi^+} (1 + e(-\alpha)) + e(-2\alpha) + \cdots)^{I_{\mathcal{P}}(\alpha)}.$$

This implies

$$\mathrm{ch}\, M(\lambda) = e(\lambda) \prod_{\alpha\in\Phi^+} (1 - e(-\alpha))^{-I_{\mathcal{P}}(\alpha)}.$$

Let $\lambda \in X^{++}$ and $L(\lambda)$ the corresponding irreducible module. It is easy to see that $\dim L(\lambda)_\beta = \dim L(\lambda)_{w(\beta)}$ for all $w \in W$ and all weights β of $L(\lambda)$. The action of the Weyl group W on the formal characters is given by

$$w(\sum_\beta c_\beta e(\beta)) = \sum_\beta c_\beta e(w(\beta)) \qquad \text{for } w \in W.$$

It then follows that $w(\mathrm{ch}\, L(\lambda)) = \mathrm{ch}\, L(\lambda)$ for $w \in W$ and $\lambda \in X^{++}$.

7.2 Consider now the element

$$Q = \prod_{\alpha\in\Phi^+} (1 - e(-\alpha))^{I_{\mathcal{P}}(\alpha)}.$$

For each $w \in W$, set $\varepsilon(w) = (-1)^{l(w)}$.

Lemma. *It holds that*

$$w(e(\rho)Q) = \varepsilon(w)e(\rho)Q \quad \text{for } w \in W,$$

that is, we have the W–skew invariant property.

The proof follows easily from [K, 10.2] and the fact that $I_{\mathcal{P}}(\alpha)$ is independent on the orientation of $\mathcal{S}$ (see [H]). The main result of this section is as follows.

7.3 Theorem. *Let $L(\lambda)$ be an irreducible integrable highest weight $\mathcal{D}(\Lambda)$–module with highest weight $\lambda \in X^{++}$. Then*

$$ch\, L(\lambda) = \frac{e(\lambda)\sum_{w\in W} \varepsilon(w)e(w(\lambda+\rho) - (\lambda+\rho))}{\prod_{\alpha\in\Phi^+}(1 - e(-\alpha))^{I_{\mathcal{P}}(\alpha)}}.$$

Note that we have a map $\delta : \mathbb{Z}[I \cup J] \to \mathbb{Z}[I]$ by sending $\sum_{i\in I\cup J} a_i i$ to $\sum_{i\in I\cup J} a_i\delta_i$. For $\lambda \in X^{++}$, we fix a $\lambda' \in X'^{++}$ such that $(\lambda', i)' = (\lambda, \delta_i)$ for all $i \in I \cup J$. For each $\beta \le \lambda$, we define

$$A(\beta) = \{\beta' \in X' : \lambda' - \beta' \in \mathbb{N}[I \cup J], (\lambda' + \rho, \lambda' + \rho) = (\beta' + \rho, \beta' + \rho) \\ \text{and } \lambda - \beta = \delta(\lambda' - \beta') \ \}.$$

In order to prove Theorem 7.3, we first prove the following lemma.

7.4 Lemma. *Let $\lambda \in X^{++}$ and M be a highest weight $\mathcal{D}(\Lambda)$–module with highest weight λ. Then*

$$ch\, M = \sum_{\beta \leq \lambda} c_{A(\beta)}\, ch\, M(\beta),$$

where $c_{A(\lambda)} = 1$ and $c_{A(\beta)} = \sum_{\beta' \in A(\beta)} c_{\beta'}$ for certain $c_{\beta'} \in \mathbb{Z}$ ($\beta < \lambda$).

Proof. All finitely generated $\mathcal{D}(\Lambda)$–modules in the category $\mathcal{O}$ have a filtration by irreducible highest weight modules $L(\beta)$ (see [Kac, 9.6]). It suffices to show the formula for $M = L(\lambda)$. Let $\lambda' \in X'$ be given by $(\lambda', i)' = (\lambda, \delta_i)$ for all $i \in I \cup J$. Consider a Verma $\mathcal{D}'(\Lambda)$–module $M'(\beta')$ with $\beta' \leq \lambda'$, i.e. $\lambda' - \beta' \in \mathbb{N}[I \cup J]$. Then it holds

$$\mathrm{ch}\, M'(\beta') = \sum_{\mu' \leq \beta'} [M'(\beta') : L'(\mu')] \mathrm{ch}\, L'(\mu').$$

By Lemma 5.3, the action of $\tilde{\Omega}$ on $L'(\mu')$ is the scarlar mulitiplication by $v^{(\mu'+\rho, \mu'+\rho)}$. Moreover, if $[M'(\beta') : L'(\mu')] \neq 0$, then $v^{(\mu'+\rho,\mu'+\rho)} = v^{(\beta'+\rho,\beta'+\rho)}$. This implies that $(\mu' + \rho, \mu' + \rho) = (\beta' + \rho, \beta' + \rho)$ sinve v is not a root of unity. Set

$$B(\lambda') = \{\beta' : \beta' \leq \lambda' \text{ and } (\beta' + \rho, \beta' + \rho) = (\lambda' + \rho, \lambda' + \rho)\}$$

and order the elements $\beta'_1, \beta'_2, \cdots$ in $B(\lambda')$ such that $\beta'_i \geq \beta'_j$ implies $i \leq j$. By the filtration of $M'(\beta'_i)$, we have the following system of linear equations

$$\mathrm{ch}\, M'(\beta'_i) = \sum_j c_{ij} \mathrm{ch}\, L'(\beta'_j).$$

The coefficient matrix $(c_{ij})_{i,j}$ of this system is triangular over integers with ones on the diagonal. By solving this system, we obtain

$$\mathrm{ch}\, L'(\lambda') = \sum_{\beta' \in B(\lambda')} c_{\beta'} \mathrm{ch}\, M'(\beta'),$$

where $c_{\beta'} \in \mathbb{Z}$. We set $\beta = \lambda - \sum_{i \in I \cup J} a_i \delta_i$ if $\lambda' - \beta' = \sum_{i \in I \cup J} a_i i \in \mathbb{N}[I \cup J]$. Then we have

$$\mathrm{ch}\, L(\lambda) = \sum_{\beta \leq \lambda} c_{A(\beta)} \mathrm{ch}\, M(\beta),$$

where $c_{A(\beta)} = \sum_{\beta' \in A(\beta)} c_{\beta'}$.

7.5 The proof of Theorem 7.3. From 7.1 and 7.4 it follows that

$$e(\rho) Q \mathrm{ch}\, L(\lambda) = \sum_{\beta \leq \lambda} c_{A(\beta)} e(\beta + \rho),$$

where $c_{A(\beta)} \in \mathbb{Z}$ and $c_{A(\lambda)} = 1$. The left–hand side of the equality above is W–skew invaraint, so the coefficients in the right–hand side satisfy

$$c_{A(\beta)} = \varepsilon(w)c_{A(\mu)} \quad \text{if } w(\beta+\rho) = \mu+\rho \ \text{ for some } \ w \in W.$$

Let β be such that $c_{A(\beta)} \neq 0$. Then $c_{A(w(\beta+\rho)-\rho)} \neq 0$ for each $w \in W$. This implies that $w(\beta+\rho) \leq \lambda+\rho$. Let $\mu \in \{w(\beta+\rho)-\rho : w \in W\}$ be such that $\lambda-\mu$ is minimal. Then $(\mu+\rho, i) \geq 0$ for each $i \in I$. Otherwise, there would exist an $i \in I$ such that $\lambda > \gamma_i(\mu+\rho)-\rho > \mu$ and $\gamma_i(\mu+\rho)-\rho \in \{w(\beta+\rho)-\rho : w \in W\}$.

Let $\mu' \in A(\mu)$ and $\mu' = \lambda' - \sum_{i\in I\cup J} a_i i$ with $a_i \geq 0$. Set $\gamma = \sum_{i\in I\cup J} a_i i$. Then the equality $(\lambda'+\rho, \lambda'+\rho) = (\mu'+\rho, \mu'+\rho)$ implies the equality $(\lambda', \gamma) + (\mu', \gamma) + 2(\rho, \gamma) = 0$. It follows that

$$\sum_{i\in I\cup J} (a_i(\lambda', i)' + a_i(\mu'+2\rho, i)') = 0.$$

But we have $(\lambda', i)' > 0$ for all $i \in I \cup J$ and $(\mu'+2\rho, i) > 0$ for all $i \in I$. For each $j \in J$, $a_j > 0$ implies that

$$(\mu'+2\rho, j)' = (\mu', j)' + (j, j)' = (\lambda', j') - \sum_{i\in I\cup J,\, i\neq j} a_i(i,j)' + (1-a_j)(j,j)' > 0.$$

From the equality $(\lambda', \gamma) + (\mu', \gamma) + 2(\rho, \gamma) = 0$ it follows that $\gamma = 0$, that is, $\mu' = \lambda'$ and $\mu = \lambda$. Thus $c_{A(\beta)} \neq 0$ implies that $w(\beta+\rho) = \lambda+\rho$ for some $w \in W$. In this case it holds $c_{A(\beta)} = \varepsilon(w)$. It is easy to see that $w(\lambda+\rho) = \lambda+\rho$ implies that $w = 1$ (see [K, 3.12(b)]). Finally, we obtain

$$e(\rho)Q\operatorname{ch} L(\lambda) = \sum_{w\in W} \varepsilon(w)e(w(\lambda+\rho)).$$

As a result, we have

$$\operatorname{ch} L(\lambda) = \frac{e(\lambda)\sum_{w\in W}\varepsilon(w)e(w(\lambda+\rho)-(\lambda+\rho))}{\prod_{\alpha\in\Phi^+}(1-e(-\alpha))^{I_{\mathcal{P}}(\alpha)}}.$$

8 A theorem of Sevenhant and Van den Bergh

8.1 Let $\mathcal{D}(\Lambda)$ be the double Ringel–Hall algebra of a finite dimensional hereditary algebra Λ. As before, we set $x_i = E_i(0)$ and $y_i = F_i(0)$ for $i \in I$, $x_j = E_p(m)$ and $y_j = F_p(m)$ for $j = (m,p) \in J$. From Remark 6.3 we see that $\mathcal{D}(\Lambda)$ is generated by the elements x_i, y_i, $i \in I \cup J$, and K_α, $\alpha \in \mathbb{Z}[I]$

which satisfy the relations (1)–(5). The purpose of this section is to prove these relations are generating ones of $\mathcal{D}(\Lambda)$.

8.2 Let $U = U^- \otimes U^0 \otimes U^+$ be the quantized enveloping algebra in the sense of Drinfeld and Jimbo with the generators $\{E_i, F_i, K_i, K_{-i} : i \in I \cup J\}$ subject to the relations
(1)
$$E_iF_j - F_jE_i = \frac{K_i - K_{-i}}{v' - v'^{-1}}\delta_{ij} \quad \text{for } i, j \in I \cup J,$$
where $v' = v^{\langle i,i\rangle}$ if $i \in I$ and $v' = v^m$ if $i = (m,p) \in J$,
(2)
$$K_0 = 1,\ K_\alpha K_\beta = K_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \mathbb{Z}[I \cup J],$$
(3)
$$K_\alpha E_i = v^{(\alpha,\delta_i)}E_iK_\alpha,\ K_\alpha F_i = v^{-(\alpha,\delta_i)}F_iK_\alpha \ \text{ for } i \in I \cup J, \alpha \in \mathbb{Z}[I \cup J].$$
(4)
$$\sum_{p+p'=1-a_{ij}} (-1)^p E_i^{(p)}E_jE_i^{(p')} = 0 \ \text{ and } \sum_{p+p'=1-a_{ij}} (-1)^p F_i^{(p)}F_jF_i^{(p')} = 0$$
for $i \in I$ and $j \in I \cup J$, where $a_{ij} = 2(j,i)'/(i,i) = 2(\delta_i, \delta_j)/(\delta_i, \delta_i)$,
(5)
$$E_iE_j = E_jE_i \quad \text{and} \quad F_iF_j = F_jF_i$$
for any $i, j \in I \cup J$ with $(i,j)' = (\delta_i, \delta_j) = 0$.

Then it is well–known that U is a Hopf algebra with the comultiplication defined by
$$\begin{aligned} &\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,\\ &\Delta(F_i) = F_i \otimes K_{-i} + 1 \otimes F_i,\\ \text{and}\quad &\Delta(K_i) = K_i \otimes K_i \end{aligned}$$
for all $i \in I \cup J$ (see [L2]). Using the arguments in 6.4–6.7, for each $i \in I$, there exists an automorphism T_i of U such that
$$\begin{aligned} &T_i(E_i) = -F_iK_i,\ T_i(F_i) = -K_{-i}E_i,\ T_i(K_\alpha) = K_{s_i(\alpha)},\ \ i \in I, \alpha \in \mathbb{Z}[I]\\ &T_i(E_j) = \textstyle\sum_{r=0}^{-a_{ij}} (-1)^r v_i^{-r} E_i^{(-a_{ij}-r)}E_jE_i^{(r)},\\ &T_i(F_j) = \textstyle\sum_{r=0}^{-a_{ij}} (-1)^r v_i^{r} F_i^{(r)}F_jF_i^{(-a_{ij}-r)} \ \text{ for } i \neq j \in I \cup J \end{aligned}$$
where s_i is the fundamental reflection at i.

8.3 Let $U^{\geq 0} = U^0 \otimes U^+$, $U^{\leq 0} = U^0 \otimes U^-$ and $\varphi : U^{\geq 0} \times U^{\leq 0} \to R$ be the skew Hopf pairing given as in 3.3. Then we see that $U = U^- \otimes U^0 \otimes U^+$ is a member in $\mathcal{L}(\tilde{C})$, too.

We have a canonical surjection $\Pi : U \to \mathcal{D}'(\Lambda)$ defined by $\Pi(E_i) = x_i$, $\Pi(F_i) = y_i$ and $\Pi(K_i) = K_i$ for all $i \in I \cup J$. Our aim next is to prove that Π is in fact an isomorphism. By the uniqueness of the restricted nondegenerated skew–Hopf pairings due to Green, it suffices to show that $J^- = 0 = J^+$, where

$$\begin{aligned} & J^- = \{y \in U^- : \varphi(x, y) = 0 \text{ for all } x \in U^+\} \\ \text{and} \quad & J^+ = \{x \in U^+ : \varphi(x, y) = 0 \text{ for all } y \in U^-\}. \end{aligned}$$

Lemma. *It holds that $J^- = 0$ (resp. $J^+ = 0$) in U^- (resp. U^+).*

Proof. We only prove that $J^- = 0$. The proof of $J^+ = 0$ is dual.

Note that $J^- = \mathrm{Ker}\,(\Pi : U^- \to \mathfrak{d}'^-(\Lambda))$, where $\mathfrak{d}'^-(\Lambda)$ denotes the subalgebra of $\mathcal{D}'(\Lambda)$ generated by y_i, $i \in I \cup J$. It is clear that J^- is a graded ideal of U^- generated by homogeneous elements. Suppose that $J^- \neq 0$. Let $-\beta$ be the weight of the maximal (with respect to weights) non–zero generator of J^-, where $\beta \in \mathbb{N}[I \cup J]$.

Let $M(-i)$ be the Verma $\mathcal{D}'(\Lambda)$–module with highest weight $-i$. We then have the natural $\mathcal{D}'(\Lambda)$–module homomorphism

$$\omega_i : M(-i) \to M(0), \quad y_{i_1} \cdots y_{i_m} \eta_i \longmapsto y_{i_1} \cdots y_{i_m} y_i \eta_0,$$

where η_i and η_0 are respectively the highest weight vectors of $M(-i)$ and $M(0)$. Let further $K = \mathrm{Ker}\,((\omega_i)_i : \oplus_{i \in I \cup J} M(-i) \to M(0))$. It is obvious that the weights of the maximal generators of K are the same as the weights of the maximal generators of J^-.

Since K is in the category $\mathcal{O}'$ of $\mathcal{D}'(\Lambda)$–modules, K is generated by the primitive vectors. Let $-\alpha$ be the weight of a primitive vector of K. Consider the action of $\tilde{\Omega}$ on K and $M(-i)$. Then we have by Lemma 5.4

$$(\rho - \alpha, \rho - \alpha)' = (\rho - i, \rho - i)' \ \text{ for some } i \in I \cup J.$$

This implies that $(\alpha, \alpha)' = 2(\rho, \alpha)'$ since $(\rho, i)' = (i, i)'/2$. Since $-\beta$ is the weight of a maximal generator of K and K is generated by primitive vectors, $-\beta$ is the weight of a primitive vector of K. In particular, we have $(\beta, \beta) = 2(\rho, \beta)$.

For each $i \in I$, it is clear that $\beta \neq i$. Let $U_i^- = \{y \in U^- : T_i(y) \in U^-\}$. Since T_i preserves $J = J^- \otimes U^{\geq 0} + U^{\leq 0} \otimes J^+$, we have $T_i(J \cap U_i^-) \subseteq J \cap U_i^-$. This implies that $J \cap U_i^- = J^- \cap U_i^-$, thus T_i induces an automorphism of $J^- \cap U_i^-$. On the other hand, each $y \in (J^-)_{-\beta}$ has the form $\sum_{p \geq 0} F_i^p y_p$ with $y_p \in U_i^-$ (see 6.8). This implies $y_p \in J^-$ since $\varphi(U^+, y_p) = 0$. But $-\beta$ is the weight of the maximal non–zero generator of J^-, we have $y_p = 0$ for $p \geq 1$, that is, $y = y_0 \in U_i^-$. This then implies that $(J^- \cap U_i^-)_{-\beta} = J^-_{-\beta} \neq 0$ and that $(J^- \cap U_i^-)_{-s_i(\beta)} = T_i((J^- \cap U_i^-)_{-\beta}) \neq 0$. Hence, we have $(\beta, i) \leq 0$ for each $i \in I$. Let $\beta = \sum_{i \in I \cup J} a_i i$ with $a_i \in \mathbb{N}$. By [K, Lemma 11.13.2], we have $(i,j) = 0$ if $i \neq j$ and $a_i a_j \neq 0$ and $(i,i) = 0$ if $a_i > 1$. It follows that $(J^- \cap U^-)_{-\beta}$ contains a monomial of the form $y = F_{i_1}^{n_1} F_{i_2}^{n_2} \cdots F_{i_m}^{n_m}$ for distinct and orthogonal $i_1, i_2, \cdots, i_m \in I \cup J$. By the definition of the pairing φ, it follows that

$$\varphi(E_{i_1}^{n_1} E_{i_2}^{n_2} \cdots E_{i_m}^{n_m}, F_{i_1}^{n_1} F_{i_2}^{n_2} \cdots F_{i_m}^{n_m}) \neq 0,$$

thus $\varphi(-, y) \neq 0$. This is a contradiction and finishes the proof.

From this lemma, we get the following corollary and theorem.

Corollary. *The canonical map* $\Pi : U \to \mathcal{D}'(\Lambda)$ *is an ismorphism.*

Theorem. (Sevenhant–Van den Bergh) The algebra $\mathcal{D}(\Lambda)$ is generated by the generators $\{x_j, y_j : j \in I \cup J\} \cup \{K_i, K_{-i} : i \in I\}$ with the generating relations:

(1)

$$x_i y_j - y_j x_i = \frac{K_{\delta_i} - K_{-\delta_i}}{v' - v'^{-1}} \delta_{ij} \quad \text{for } i, j \in I \cup J,$$

where $v' = v^{\langle i,i \rangle}$ if $i \in I$ and $v' = v^m$ if $i = (m,p) \in J$,

(2)

$$K_0 = 1, \; K_\alpha K_\beta = K_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \mathbb{Z}[I],$$

(3)

$$K_\alpha x_i = v^{(\alpha,\delta_i)} x_i K_\alpha, \; K_\alpha y_i = v^{-(\alpha,\delta_i)} y_i K_\alpha \; \text{ for } i \in I \cup J, \alpha \in \mathbb{Z}[I].$$

(4)

$$\sum_{p+p'=1-a_{ij}} (-1)^p x_i^{(p)} x_j x_i^{(p')} = 0 \; \text{ and } \sum_{p+p'=1-a_{ij}} (-1)^p y_i^{(p)} y_j y_i^{(p')} = 0$$

for $i \in I$ and $j \in I \cup J$, where $a_{ij} = 2(\delta_i, \delta_j)/(\delta_i, \delta_i)$,

(5)

$$x_i x_j = x_j x_i \quad \text{and} \quad y_i y_j = y_j y_i$$

for any $i, j \in I \cup J$ with $(\delta_i, \delta_j) = 0$.

8.4 Remark. Finally, we remark that Theorem 8.3 can be formulated for all subalgebras $\mathcal{D}_m(\Lambda)$ of $\mathcal{D}(\Lambda)$, $m \geq 0$.

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