Twisted Diagrams

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We introduce generalized diagram categories, construct Kan extensions, and establish various model category structures. Using these, we define "homotopy sheaves" and show that a twisted diagram is a homotopy sheaf if and only if it gives rise to a "sheaf in the homotopy category".

Contents:

1. Introduction

One often encounters constructions which look like diagrams in some category but cannot be described with that formalism. A prominent example is the notion of (naïve) spectra, a sequence of pointed spaces $X_0, X_1, \ldots$ and structure maps $\Sigma X_n \longrightarrow X_{n+1}$. This almost determines a diagram indexed over $\mathbf{N}$ (regarded as a category), and in fact can be described by a "twisted diagram" with "twists" given by iterated suspension functors. Another example is the category of quasi-coherent sheaves on projective spaces as defined in [Hü]. The new formalism also applies, as a special case, to diagram categories in the usual sense (*i.e.*, functor categories).

This paper is divided into three parts. §§ 2–4 are devoted to the definition of twisted diagrams and the development of basic machinery. In §§ 5–8 we prove the existence of several Quillen closed model category structures on categories of twisted diagrams. This part is based on model category structures for diagram categories as in [Ho]. Finally, in § 9 we propose definitions of sheaves and homotopy sheaves and show how they are related.

Required prerequisites are elementary category theory as presented in [ML] and basic model category theory ([DS] or [Ho]).

A special case of the results on model structures has been used by the first author to study the algebraic K-theory of projective spaces ([Hü]). Twisted diagrams and their model structures also appear implicitly in [HKVWW]. As the authors learned recently, Hirschowitz and Simpson obtained related results ([HS]).

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2. Basic Definitions

Let $\mathcal{I}$ be a small category. It will serve as the index category for our diagrams.

Definition 2.1. An *adjunction bundle* $\mathfrak{B} = (\mathcal{C},\ F,\ U)$ *over* $\mathcal{I}$, or $\mathcal{I}$-bundle, consists of the following data:

- for each object $i \in \mathcal{I}$ a category $\mathcal{C}_i$,

- for each morphism $\sigma : i \longrightarrow j$ in $\mathcal{I}$ a pair of adjoint functors $F_\sigma : \mathcal{C}_i \longrightarrow C_j$ and $U_\sigma : \mathcal{C}_j \longrightarrow \mathcal{C}_i$ (with F_σ being the left adjoint),

such that U determines a functor $\mathcal{I}^{\mathrm{op}} \longrightarrow Cat$, *i.e.*, $U_{\mathrm{id}_i} = \mathrm{Id}_{\mathcal{C}_i}$, and for each pair of composable arrows $i \xrightarrow{\sigma} j \xrightarrow{\tau} k$, the equality $U_{\tau\circ\sigma} = U_\sigma \circ U_\tau$ holds. In addition, we require $F_{\mathrm{id}_i} = \mathrm{Id}_{\mathcal{C}_i}$. The properties of adjunctions guarantee that there is a canonical isomorphism $F_{\tau\circ\sigma} \cong F_\tau \circ F_\sigma$ (which will be referred to as *uniqueness isomorphism*), since both functors are left adjoint to $U_{\tau\circ\sigma} = U_\sigma \circ U_\tau$ ([ML, IV.1, corollary 1, p. 83]).

Example 2.2. Any category $\mathcal{C}$ gives rise to a *trivial* $\mathcal{I}$*-bundle* with $\mathcal{C}_i = \mathcal{C}$ for all i, and all adjunctions being the identity adjunction.

Example 2.3. *(The projective line.)*

If M is a monoid, denote by M-$k\,Top_*$ the category of pointed, compactly generated topological spaces having a basepoint-preserving action of M. A map of monoids $f : M \longrightarrow M'$ determines an adjoint functor pair

$$\cdot \wedge_M M' : M\text{-}k\,Top_* \rightleftarrows M'\text{-}k\,Top_* : R_f$$

with R_f being restriction along f, and $\cdot \wedge_M M'$ being its left adjoint (inducing up). The integers $\mathbf{Z}$ form a monoid under addition, and we have submonoids $\mathbf{N}_+$ (non-negative integers) and $\mathbf{N}_-$ (non-positive integers). Hence we can form the adjunction bundle $\mathfrak{P}^1$ over $\mathcal{I} = (+ \xrightarrow{\alpha} 0 \xleftarrow{\beta} -)$, consisting of the categories $\mathbf{N}_+$-$k\,Top_*$, $\mathbf{Z}$-$k\,Top_*$ and $\mathbf{N}_-$-$k\,Top_*$, and the adjoint pairs "inducing up" and "restriction" along the inclusions $\mathbf{N}_+ \subseteq \mathbf{Z}$ and $\mathbf{N}_- \subseteq \mathbf{Z}$.

Definition 2.4. *(Inverse image of bundles.)*

Given a functor $\Phi : \mathcal{I} \longrightarrow \mathcal{J}$ and a $\mathcal{J}$-bundle $\mathfrak{B} = (\mathcal{D},\ G,\ V)$, we define the *inverse image of* $\mathfrak{B}$ *under* Φ, denoted $\Phi^*\mathfrak{B}$, to be the $\mathcal{I}$-bundle $(\mathcal{C},\ F,\ U)$ given by $\mathcal{C}_i := \mathcal{D}_{\Phi(i)}$, $U_i := V_{\Phi(i)}$ and $F_i := G_{\Phi(i)}$.

If $\Phi : \mathcal{I} \longrightarrow \mathcal{J}$ is the inclusion of a subcategory, we write $\mathfrak{B}|_{\mathcal{I}}$ instead of $\Phi^*\mathfrak{B}$ and call the resulting $\mathcal{I}$-bundle the *restriction* of $\mathfrak{B}$ to $\mathcal{I}$.

Forming inverse images is functorial, *i.e.*, $\mathrm{id}_{\mathcal{C}}^*\mathfrak{B} = \mathfrak{B}$ and $(\Phi \circ \Theta)^*\mathfrak{B} = \Theta^*\Phi^*\mathfrak{B}$. The inverse image of a trivial bundle is a trivial bundle.

Definition 2.5. *(Morphisms of bundles.)*

Suppose $\mathfrak{A} = (\mathcal{C},\ F,\ U)$ and $\mathfrak{B} = (\mathcal{D},\ G,\ V)$ are $\mathcal{I}$-bundles. An $\mathcal{I}$*-morphism* $\Psi : \mathfrak{A} \longrightarrow \mathfrak{B}$ consists of two families of functors $\rho_i : \mathcal{C}_i \longrightarrow \mathcal{D}_i$ and $\lambda_i : \mathcal{D}_i \longrightarrow \mathcal{C}_i$ where i ranges over the objects of $\mathcal{I}$ such that λ_i is left adjoint to ρ_i, and such that for each morphism $\sigma : i \longrightarrow j$ in $\mathcal{I}$ we have $V_\sigma \circ \rho_j = \rho_i \circ U_\sigma$.

Given an $\mathcal{I}$-bundle $\mathfrak{A}$ and a $\mathcal{J}$-bundle $\mathfrak{B}$, a *morphism of bundles* $\Xi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a pair $\Xi = (\Phi,\ \Psi)$ where $\Phi : \mathcal{I} \longrightarrow \mathcal{J}$ is a functor and $\Psi : \mathfrak{A} \longrightarrow \Phi^*\mathfrak{B}$ is an $\mathcal{I}$-morphism of $\mathcal{I}$-bundles.

Definition 2.6. *(Twisted diagrams.)*

Let $\mathfrak{B}$ be an adjunction bundle over $\mathcal{I}$. A *twisted diagram Y with coefficients in $\mathfrak{B}$* consists of the following data:

- for each object $i \in \mathcal{I}$ an object $Y_i \in \mathcal{C}_i$,
- for each morphism $\sigma : i \longrightarrow j$ in $\mathcal{I}$ a map $y^\flat_\sigma : Y_i \longrightarrow U_\sigma(Y_j)$ in $\mathcal{C}_i$

such that Y behaves like a functor, *i.e.*, $y^\flat_{\mathrm{id}_i} = \mathrm{id}_{Y_i}$ and $y^\flat_{\tau\circ\sigma} = U_\sigma(y^\flat_\tau) \circ y^\flat_\sigma$ for each pair $i \xrightarrow{\sigma} j \xrightarrow{\tau} k$ of composable arrows in $\mathcal{I}$. (A reformulation using the left adjoints will be given below.)

A *map $f : Y \longrightarrow Z$ of twisted diagrams* is a collection of maps $f_i : Y_i \longrightarrow Z_i$ in $\mathcal{C}_i$, one for each object $i \in \mathcal{I}$, such that for each morphism $\sigma : i \longrightarrow j$ in $\mathcal{I}$ the equality $U_\sigma(f_j) \circ y^\flat_\sigma = z^\flat_\sigma \circ f_i$ holds. (A reformulation using the left adjoints will be given below.) The category of twisted diagrams and their maps is denoted $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$.

For each of the structure maps $y^\flat_\sigma : Y_i \longrightarrow U_\sigma(Y_j)$ there is a corresponding adjoint map $y^\sharp_\sigma : F_\sigma(Y_i) \longrightarrow Y_j$. The idea is to think of the (meaningless) symbol $y_\sigma : Y_i \multimap Y_j$ as a kind of "structure map" having two incarnations as a $\flat$-type map (a morphism in $\mathcal{C}_i$) and a $\sharp$-type map (a morphism in $\mathcal{C}_j$).

The definition of twisted diagrams does not make explicit use of the left adjoints provided by the adjunction bundle. However, the properties of adjunctions will play a crucial rôle for the discussion of limits and colimits in $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$.

Example 2.7.

(1) If $\mathfrak{B}$ is a trivial $\mathcal{I}$-bundle (2.2), we recover the usual functor category: $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B}) = Fun(\mathcal{I},\ \mathcal{C})$.

(2) If $\mathcal{I}$ is a discrete category (*i.e.*, contains no non-identity morphisms), an adjunction bundle over $\mathcal{I}$ is simply a collection of categories $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$, and the category of twisted diagrams is the product category $\prod_{i\in\mathcal{I}} \mathcal{C}_i$.

(3) Suppose $\mathfrak{B}_\nu = (\mathcal{C}^\nu,\ F^\nu,\ U^\nu)$ is a family of adjunction bundles indexed by $\mathcal{I}_\nu$. Then we can form the following adjunction bundle $\prod_\nu \mathfrak{B}_\nu =: \mathfrak{B} = (\mathcal{C},\ F,\ U)$ indexed by the disjoint union $\mathcal{I} := \amalg_\nu \mathcal{I}_\nu$: for each $i \in \mathcal{I}$ there is exactly one ν with $i \in \mathcal{I}_\nu$, and we define $\mathcal{C}_i = \mathcal{C}^\nu_i$ (and similarly for the F and U). It is easy to see that $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B}) = \prod_\nu \mathbf{Tw}\,(\mathcal{I}_\nu, \mathfrak{B}_\nu)$ in this case.

Example 2.8. *(Spectra.)*

Let $\mathbf{N}$ denote the ordered set of natural numbers, considered as a category. For each $n \in \mathbf{N}$, define $\mathcal{C}_n$ to be the category $\mathcal{S}$ of pointed simplicial sets. If $n \le m$, we have an adjunction $\Sigma^{m-n} : \mathcal{S} \rightleftarrows \mathcal{S} : \Omega^{m-n}$ of iterated loop space and suspension functors. It is clear that this defines an adjunction bundle Sp over $\mathbf{N}$. A twisted diagram X with coefficients in Sp, graphically represented by the "diagram"

$$X_0 \multimap X_1 \multimap X_2 \multimap \ \dots\ ,$$

is nothing but a spectrum in the sense of [BF].

Given twisted diagrams $Y, Z \in \mathbf{Tw}(\mathcal{I}, \mathfrak{B})$ and a collection of maps $f_i \colon Y_i \longrightarrow Z_i$ in $\mathcal{C}_i$, we can form two squares for each morphism $\sigma \colon i \longrightarrow j$ in $\mathcal{I}$

$$\begin{tikzcd} Y_i \arrow[r, "f_i"] \arrow[d, "y^\flat_\sigma"'] & Z_i \arrow[d, "z^\flat_\sigma"] \\ U_\sigma(Y_j) \arrow[r, "U_\sigma(f_j)"'] & U_\sigma(Z_j) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F_\sigma(Y_i) \arrow[r, "F_\sigma(f_i)"] \arrow[d, "y^\sharp_\sigma"'] & F_\sigma(Z_i) \arrow[d, "z^\sharp_\sigma"] \\ Y_j \arrow[r, "f_j"'] & Z_j \end{tikzcd}$$

and the definition of adjunctions imply that the left square commutes if and only if the right square commutes. Thus the family $(f_i)_{i\in\mathcal{I}}$ determines a map of twisted diagrams if and only if $z^\sharp_\sigma \circ F_\sigma(f_i) = f_j \circ y^\sharp_\sigma$.

For later use, we record the following fact:

Lemma 2.9. *Suppose we have a map $y^\flat_\sigma \colon Y_i \longrightarrow U_\sigma(Y_j)$ in $\mathcal{C}_i$ for each morphism $\sigma \colon i \longrightarrow j$ in $\mathcal{I}$ satisfying $y^\flat_{\mathrm{id}} = \mathrm{id}$, and denote by $y^\sharp_\sigma$ the adjoint map $F_\sigma(Y_i) \longrightarrow Y_j$. Let $\tau \colon j \longrightarrow k$ be another morphism in $\mathcal{I}$. Then if one of the squares*

$$\begin{tikzcd} F_\tau \circ F_\sigma(Y_i) \arrow[r, "\cong"] \arrow[d, "F_\tau(y^\sharp_\sigma)"'] & F_{\tau\circ\sigma}(Y_i) \arrow[d, "y^\sharp_{\tau\circ\sigma}"] \\ F_\tau(Y_j) \arrow[r, "y^\sharp_\tau"'] & Y_k \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} Y_i \arrow[r, equal] \arrow[d, "y^\flat_\sigma"'] & Y_i \arrow[d, "y^\flat_{\tau\circ\sigma}"] \\ U_\sigma(Y_j) \arrow[r, "U_\sigma(y^\flat_\tau)"'] & U_{\tau\circ\sigma}(Y_k) \end{tikzcd}$$

commutes so does the other (the upper horizontal map in the left square is the uniqueness isomorphism). In other words, if one of the squares commutes, the objects Y_i together with the maps $y^\flat_\sigma$ form a twisted diagram.

Proof. Assume that the square on the left commutes. We want to show that the square on the right is commutative. The strategy is to divide the square into smaller pieces which are known to commute.

For each morphism $\sigma \colon i \longrightarrow j$ in $\mathcal{I}$, there exists a natural transformation of functors $\eta^\sigma \colon \mathrm{Id} \longrightarrow U_\sigma \circ F_\sigma$ called *unit* of the adjunction of F_σ and U_σ. Given the structure map $y^\sharp_\sigma \colon F_\sigma(Y_i) \longrightarrow Y_j$, we obtain the corresponding adjoint map $y^\flat_\sigma \colon Y_i \longrightarrow U_\sigma(Y_j)$ as the composite

$$Y_i \xrightarrow{\eta^\sigma_{Y_i}} U_\sigma \circ F_\sigma(Y_i) \xrightarrow{U_\sigma(y^\sharp_\sigma)} U_\sigma(Y_j)$$

(cf. [ML], IV.1, p. 80). In particular, the functors $F_{\tau\circ\sigma}$ and $U_{\tau\circ\sigma}$ are adjoint with unit $\eta^{\tau\circ\sigma} \colon \mathrm{Id} \longrightarrow U_{\tau\circ\sigma} \circ F_{\tau\circ\sigma}$. But $U_{\tau\circ\sigma} = U_\sigma \circ U_\tau$ has another left adjoint $F_\tau \circ F_\sigma$, and we denote the corresponding unit by $\bar\eta^{\tau\circ\sigma} \colon \mathrm{Id} \longrightarrow U_\sigma \circ U_\tau \circ F_\tau \circ F_\sigma$.

Now we redraw the square on the right with some extra data added:

$$\begin{tikzcd}
Y_i \arrow[rr, equal] \arrow[d, "\eta^{\sigma}_{Y_i}"'] \arrow[drr, phantom, "1"] & & Y_i \arrow[rr, equal] \arrow[d, "\bar\eta^{\tau\circ\sigma}_{Y_i}"'] \arrow[drr, phantom, "2"] & & Y_i \arrow[d, "\eta^{\tau\circ\sigma}_{Y_i}"] \\
U_\sigma \circ F_\sigma(Y_i) \arrow[rr, "U_\sigma(\eta^{\tau}_{F_\sigma(Y_i)})"] \arrow[d, "U_\sigma(y^\sharp_\sigma)"'] \arrow[drr, phantom, "3"] & & U_\sigma \circ U_\tau \circ F_\tau \circ F_\sigma(Y_i) \arrow[rr, "\cong"] \arrow[d, "U_\sigma \circ U_\tau \circ F_\tau(y^\sharp_\sigma)"'] \arrow[drr, phantom, "4"] & & U_{\tau\circ\sigma} \circ F_{\tau\circ\sigma}(Y_i) \arrow[d, "U_{\tau\circ\sigma}(y^\sharp_{\tau\circ\sigma})"] \\
U_\sigma(Y_j) \arrow[rr, "U_\sigma(\eta^{\tau}_{Y_j})"'] & & U_\sigma \circ U_\tau \circ F_\tau(Y_j) \arrow[rr, "U_{\tau\circ\sigma}(y^\sharp_\tau)"'] & & U_{\tau\circ\sigma}(Y_k)
\end{tikzcd}$$

The outer square is the right hand square of the lemma. Square 1 commutes by the composition rules for adjunctions and units ([ML], IV.8.1, p. 101). Square 2 commutes by definition of the uniqueness isomorphism. Square 3 commutes since η^τ is a natural transformation of functors, and since U_σ is a functor. Finally, square 4 commutes by hypothesis (apply $U_\sigma \circ U_\tau = U_{\tau\circ\sigma}$ to the left diagram of the lemma).

The other direction of the lemma is proved using similar techniques. We omit the details. □

The next lemma says that $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$ *is as complete and cocomplete as all the* $\mathcal{C}_i$, and limits *resp.* colimits can be computed "pointwise" in the categories $\mathcal{C}_i$. For $i \in \mathcal{I}$, let $Ev_i : \mathbf{Tw}\,(\mathcal{I}, \mathfrak{B}) \longrightarrow \mathcal{C}_i$ denote the ith evaluation functor which maps a twisted diagram Y to its ith term Y_i.

Lemma 2.10. *Let* $G : \mathcal{D} \longrightarrow \mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$ *be a functor, and suppose that for all* i *the limit of* $Ev_i \circ G$ *exists. Then* $\lim G$ *exists and the canonical map*

$$Ev_i(\lim G) \longrightarrow \lim(Ev_i \circ G)$$

is an isomorphism, and similarly for colimits.

Proof. The proof relies on the compatibility of left (*resp.* right) adjoint functors with colimits (*resp.* limits): if F is a left adjoint, and D is a functor, then there is a unique natural isomorphism $\operatorname{colim}(F \circ D) \longrightarrow F(\operatorname{colim} D)$, and similarly for right adjoints and limits ([ML, V.5, theorem 1, p. 114]).

To prove the lemma, we treat the case of colimits only. (For limits one has to use similar techniques. Since U is supposed to be functorial, this is slightly easier.) Let $G_i := Ev_i \circ G$, and define $C_i := \operatorname{colim} G_i$. We claim that the objects C_i assemble to a twisted diagram C, and it is almost obvious that C is "the" colimit of G.

Let $\sigma : i \longrightarrow j$ denote a morphism in $\mathcal{I}$. The $\sharp$-type structure maps of the twisted diagrams $G(d)$ (for objects $d \in \mathcal{D}$) assemble to a natural transformation

$$G^\sharp_\sigma : F_\sigma \circ G_i \longrightarrow G_j$$

of functors $\mathcal{D} \longrightarrow \mathcal{C}_j$. Hence we can define the $\sharp$-type structure map $c^\sharp_\sigma$ as the composite

$$F_\sigma(C_i) = F_\sigma(\operatorname{colim} G_i) \cong \operatorname{colim}(F_\sigma \circ G_i) \xrightarrow{f} \operatorname{colim} G_j = C_j$$

with f induced by $G^\sharp_\sigma$. By lemma 2.9 we are left to show that the following square commutes for all composable morphisms σ and τ in $\mathcal{I}$:

$$\begin{tikzcd} F_\tau \circ F_\sigma(C_i) \arrow[r, "\cong"] \arrow[d, "F_\tau(c^\sharp_\sigma)"'] & F_{\tau\circ\sigma}(C_i) \arrow[d, "c^\sharp_{\tau\circ\sigma}"] \\ F_\tau(C_j) \arrow[r, "C^\sharp_\tau"'] & C_k \end{tikzcd} \tag{$*$}$$

We replace the symbols C_ℓ and the structure maps by their definition and obtain the following bigger diagram:

$$\begin{tikzcd} F_\tau \circ F_\sigma(\operatorname{colim} G_i) \arrow[r, equal] \arrow[d, "\cong"'] \arrow[dr, phantom, "1"] & F_\tau \circ F_\sigma(\operatorname{colim} G_i) \arrow[r, "\cong"] \arrow[d, "\cong"'] \arrow[dr, phantom, "2"] & F_{\tau\circ\sigma}(\operatorname{colim} G_i) \arrow[d, "\cong"] \\ F_\tau\big(\operatorname{colim}(F_\sigma \circ G_i)\big) \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "3"] & \operatorname{colim}(F_\tau \circ F_\sigma \circ G_i) \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "4"] & \operatorname{colim}(F_{\tau\circ\sigma} \circ G_i) \arrow[d] \\ F_\tau\big(\operatorname{colim}(G_j)\big) \arrow[r, "\cong"'] & \operatorname{colim}(F_\tau \circ G_j) \arrow[r] & \operatorname{colim}(G_k) \end{tikzcd} \tag{$**$}$$

All the small squares commute: for 1 this is true by uniqueness of the isomorphisms for commuting left adjoints with colimits. The horizontal maps of 2 are induced by the uniqueness isomorphism, the vertical maps are induced by the isomorphism for commuting left adjoints with colimits. By uniqueness, 2 commutes. Both horizontal maps of 3 are induced by the isomorphism for commuting colimits with F_τ, and both vertical maps are induced by the natural transformation $G^\sharp_\sigma : F_\sigma \circ G_i \longrightarrow G_j$. Hence 3 commutes. Finally, square 4 commutes by lemma 2.9, applied componentwise to the diagrams G_ℓ, and by functoriality of colim.

Hence the diagram $(**)$ commutes. But the square $(*)$ is contained in there as the outer square, thus is commutative as claimed. □

Definition 2.11. *(Inverse image of twisted diagrams.)*

Suppose we have a functor $\Phi : \mathcal{I} \longrightarrow \mathcal{J}$, a $\mathcal{J}$-bundle $\mathfrak{B}$, and a twisted diagram $Y \in \mathbf{Tw}(\mathcal{J}, \mathfrak{B})$. We define the *inverse image of Y under Φ*, denoted Φ^*Y, as the twisted diagram over $\mathcal{I}$ with coefficients in $\Phi^*\mathfrak{B}$ given by $(\Phi^*Y)_i := Y_{\Phi(i)}$ and $(\Phi^*y)^\flat_\sigma := y^\flat_{\Phi(\sigma)}$ for all objects $i \in \mathcal{I}$ and all morphisms $\sigma \in \mathcal{I}$. We obtain a functor $\Phi^* : \mathbf{Tw}(\mathcal{J}, \mathfrak{B}) \longrightarrow \mathbf{Tw}(\mathcal{I}, \Phi^*\mathfrak{B})$.

Now suppose we have $\mathcal{I}$-bundles $\mathfrak{A} = (\mathcal{C}, F, U)$ and $\mathfrak{B} = (\mathcal{D}, G, V)$, and an $\mathcal{I}$-morphism $\Psi = (\rho, \lambda) : \mathfrak{A} \longrightarrow \mathfrak{B}$. The functor *inverse image under* Ψ, denoted $\Psi^* : \mathbf{Tw}(\mathcal{I}, \mathfrak{B}) \longrightarrow \mathbf{Tw}(\mathcal{I}, \mathfrak{A})$, assigns to a twisted diagram $Y \in \mathbf{Tw}(\mathcal{I}, \mathfrak{B})$ the object $\Psi^*Y \in \mathbf{Tw}(\mathcal{I}, \mathfrak{A})$ given by $(\Psi^*Y)_i := \lambda_i(Y_i)$ with $\sharp$-type structure maps $(\Psi^*y)^\sharp_\sigma$ given by the composition

$$F_\sigma((\Psi^*Y)_i) = F_\sigma(\lambda_i(Y_i)) \cong \lambda_j(G_\sigma(Y_i)) \xrightarrow{\lambda_i(y^\sharp_\sigma)} \lambda_j(Y_j) = (\Psi^*Y)_j$$

for all objects $i \in \mathcal{I}$ and morphisms $\sigma : i \longrightarrow j$. (We will prove in the next lemma that Ψ^* is well-defined, *i.e.*, that $\Psi^* Y$ is a twisted diagram.)

More generally, a morphism $\Xi = (\Phi,\ \Psi) : \mathfrak{A} \longrightarrow \mathfrak{B}$ of bundles induces an inverse image functor $\Xi^* = \Psi^* \circ \Phi^* : \mathbf{Tw}\,(\mathcal{J}, \mathfrak{B}) \longrightarrow \mathbf{Tw}\,(\mathcal{I}, \mathfrak{A})$.

If $\Phi : \mathcal{I} \longrightarrow \mathcal{J}$ is the inclusion of a subcategory, we write $Y|_{\mathcal{I}}$ instead of $\Phi^* Y$ and call the resulting twisted diagram with coefficients in $\mathfrak{B}|_{\mathcal{I}}$ the *restriction of* Y *to* $\mathcal{I}$. This defines the restriction functor $\mathbf{Tw}\,(\mathcal{J}, \mathfrak{B}) \longrightarrow \mathbf{Tw}\,(\mathcal{I}, \mathfrak{B}|_{\mathcal{I}})$. As a special case of restriction (if $\mathcal{I} = \{i\}$ is the trivial subcategory consisting of i), we obtain the evaluation functors Ev_i as defined above.

Lemma 2.12. *Given $\mathcal{I}$-bundles $\mathfrak{A} = (\mathcal{C},\ F,\ U)$ and $\mathfrak{B} = (\mathcal{D},\ G,\ V)$, an $\mathcal{I}$-morphism $\Psi = (\lambda, \rho) : \mathfrak{A} \longrightarrow \mathfrak{B}$, and a twisted diagram $Y \in \mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$, the object $\Psi^* Y$ defined in 2.11 is a twisted diagram with coefficients in $\mathfrak{A}$.*

Proof. Let $\sigma : i \longrightarrow j$ and $\tau : j \longrightarrow k$ be morphisms in $\mathcal{I}$ and consider the diagram

$$\begin{tikzcd}
F_\tau \circ F_\sigma \circ \lambda_i(Y_i) \arrow[rr, "\cong"] \arrow[d, "\cong"'] & & F_{\tau\circ\sigma} \circ \lambda_i(Y_i) \arrow[d, "\cong"] \\
F_\tau \circ \lambda_j \circ G_\sigma(Y_i) \arrow[r, "\cong"] \arrow[d, "F_\tau \circ \lambda_j(y^\sharp_\sigma)"'] & \lambda_k \circ G_\tau \circ G_\sigma(Y_i) \arrow[r, "\cong"] \arrow[d, "\lambda_k \circ G_\tau(y^\sharp_\sigma)"'] & \lambda_k \circ G_{\tau\circ\sigma}(Y_i) \arrow[d, "\lambda_k(y^\sharp_{\tau\circ\sigma})"] \\
F_\tau \circ \lambda_j(Y_j) \arrow[r, "\cong"'] & \lambda_k \circ G_\tau(Y_j) \arrow[r, "\lambda_k(y^\sharp_\tau)"'] & \lambda_k(Y_k)
\end{tikzcd}$$

in which all arrows labeled with "$\cong$" denote uniqueness isomorphisms. Recall that the compositions of functors appearing in the upper rectangle are left adjoints to the functor $U_\sigma \circ U_\tau \circ \rho_k$. Thus the upper rectangle commutes by uniqueness. The lower left square commutes by naturality. The lower right square commutes since Y is a twisted diagram (lemma 2.9) and λ_k is a functor. Hence the whole diagram commutes and $\Psi^* Y$ is a twisted diagram by another application of lemma 2.9. $\square$

Definition 2.13. *(Direct image of twisted diagrams.)*

Suppose we have a bundle morphism $\Xi = (\Phi, \Psi) : \mathfrak{A} \longrightarrow \mathfrak{B}$, where $\mathfrak{A} = (\mathcal{C},\ F,\ U)$ is an $\mathcal{I}$-bundle, $\mathfrak{B} = (\mathcal{D},\ G,\ V)$ is a $\mathcal{J}$-bundle, Φ is a functor $\mathcal{I} \longrightarrow \mathcal{J}$, and $\Psi = \{(\lambda_i, \rho_i)\}_{i \in \mathcal{I}}$ is an $\mathcal{I}$-morphism $\mathfrak{A} \longrightarrow \Phi^* \mathfrak{B}$. Let Y be a twisted diagram with coefficients in $\mathfrak{A}$. It is straightforward to check that the definition $\Psi_*(Y)_i := \rho_i(Y_i)$ yields a twisted diagram with coefficients in $\Phi^* \mathfrak{B}$ having the structure maps

$$\Psi_*(y)^\flat_\alpha : \rho_i(Y_i) \xrightarrow{\rho_i(y^\flat_\alpha)} \rho_i \circ U_\alpha(Y_j) = V_{\Phi(\alpha)} \circ \rho_j(Y_j)$$

for $\alpha : i \longrightarrow j$. In this way we obtain a functor $\Psi_* \mathbf{Tw}\,(\mathcal{I}, \mathfrak{A}) \longrightarrow \mathbf{Tw}\,(\mathcal{I}, \Phi^* \mathfrak{B})$.

Suppose the right adjoint $R\Phi$ of Φ^* exists. The composition

$$\Xi_* := R\Phi \circ \Psi_* : \mathbf{Tw}\,(\mathcal{I}, \mathfrak{A}) \longrightarrow \mathbf{Tw}\,(\mathcal{J}, \mathfrak{B})$$

is called the *direct image* functor.

We will see that if the bundle $\mathfrak{B}$ consists of complete categories, the functor $R\Phi$ exists and can be constructed by twisted KAN extension.

Corollary 2.14. *Let $\Xi = (\Phi, \Psi)\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ be a bundle morphism, with $\mathfrak{B}$ consisting of complete categories. Then the functor Ξ^* (inverse image under Ξ) has a right adjoint Ξ_* (direct image under Ξ).*

Proof. Since $R\Phi$ is right adjoint to Φ^* by assumption, it remains to show that Ψ_* is right adjoint to Ψ^*. However, this is true, because Ψ_* is pointwise right adjoint to Ψ^*, and it can be checked that adjoining pointwise respects maps of twisted diagrams. We omit the details. $\square$

3. Twisted KAN Extensions

Assume that $\mathfrak{B}$ is a trivial bundle over $\mathcal{J}$, consisting of the category $\mathcal{C}$ (and identity functors), and $\Phi\colon \mathcal{I} \longrightarrow \mathcal{J}$ is a functor. In this case, the inverse image of $\mathfrak{B}$ under Φ is the trivial bundle over $\mathcal{I}$ (consisting of $\mathcal{C}$ and identity functors), and Φ^* is the functor $Fun(\mathcal{J}, \mathcal{C}) \longrightarrow Fun(\mathcal{I}, \mathcal{C})$ mapping Y to $Y \circ \Phi$. If $\mathcal{C}$ is complete and cocomplete, the functor Φ^* has a left and a right adjoint given by left and right KAN extension respectively ([ML, X.3, corollary 2]).

It is possible to lift this construction to our framework. We consider only left KAN extensions, the other case being similar (and easier).

Let $\Phi\colon \mathcal{I} \longrightarrow \mathcal{J}$ be a functor, $\mathfrak{B} = (\mathcal{C},\ F,\ U)$ a $\mathcal{J}$-bundle, and Y a twisted diagram (over $\mathcal{I}$) with coefficients in $\Phi^*\mathfrak{B} = (\mathcal{D},\ G,\ U)$. First, we have to define a twisted diagram $L(Y)$ over $\mathcal{J}$ with coefficients in $\mathfrak{B}$. (Later, we will convince ourselves that the assignment $Y \longmapsto L(Y)$ is a functor which is left adjoint to Φ^*.) Let $j \in \mathcal{J}$ be given, and let $\Phi \downarrow j$ denote the category of objects Φ-over j. Its objects are maps of the form $\sigma\colon \Phi(i) \longrightarrow j \in \mathcal{J}$ (for i an object of $\mathcal{I}$). The morphisms from $\sigma\colon \Phi(i) \longrightarrow j$ to $\tau\colon \Phi(i') \longrightarrow j$ are morphisms $\alpha\colon i \longrightarrow i' \in \mathcal{I}$ satisfying $\tau \circ \Phi(\alpha) = \sigma$. Consider the assignment

$$D_j^Y : \Phi \downarrow j \longrightarrow \mathcal{C}_j, \qquad (\Phi(i) \xrightarrow{\sigma} j) \mapsto F_\sigma(Y_i)$$

This is well-defined because Y_i is an object of $\mathcal{D}_i = \mathcal{C}_{\Phi(i)}$ by definition of $\Phi^*\mathfrak{B}$, so $F_\sigma(Y_i)$ is an object of $\mathcal{C}_j$.

The assignment D_j^Y is in fact a functor, as one can deduce as follows. Let $\mathrm{pr}_{\mathcal{I}}$ denote the obvious projection functor $\Phi \downarrow j \longrightarrow \mathcal{I}$ mapping the object $\Phi(i) \longrightarrow j$ to i, and define $\mathrm{pr}_{\mathcal{J}} := \Phi \circ \mathrm{pr}_{\mathcal{I}}$. Using the equality $\mathrm{pr}_{\mathcal{J}}^*\mathfrak{B} = \mathrm{pr}_{\mathcal{I}}^*(\Phi^*\mathfrak{B})$, we get a functor $\mathrm{pr}_{\mathcal{I}}^*\colon \mathbf{Tw}\,(\mathcal{I}, \Phi^*\mathfrak{B}) \longrightarrow \mathbf{Tw}\,(\Phi \downarrow j, \mathrm{pr}_{\mathcal{J}}^*\mathfrak{B})$. Let $\{j\}$ denote the subcategory of $\mathcal{J}$ given by the object j (and no non-identity morphism) and consider the category $\mathcal{C}_j$ as a (trivial) bundle over $\{j\}$. Then we have a morphism of bundles $\Xi\colon \mathcal{C}_j \longrightarrow \mathrm{pr}_{\mathcal{J}}^*\mathfrak{B}$ consisting of the functor $\Phi \downarrow j \longrightarrow \{j\}$ and the $(\Phi \downarrow j)$-morphism Ψ from $\mathrm{pr}_{\mathcal{J}}^*\mathfrak{B}$ to the trivial bundle with σ-component the adjunction $F_\sigma\colon \mathcal{C}_{\Phi(i)} \rightleftarrows \mathcal{C}_j : U_\sigma$ (for $\sigma\colon \Phi(i) \longrightarrow j$).

The inverse image under Ψ is a functor $\Psi^*\colon \mathbf{Tw}\,(\Phi\downarrow j, \mathrm{pr}_{\mathcal{J}}^*\mathfrak{B}) \longrightarrow Fun(\Phi\downarrow j, \mathcal{C}_j)$. Tracing the definitions shows $D_j^Y = \Psi^*\mathrm{pr}_{\mathcal{I}}^*(Y)$.

Now assume that the bundle $\mathfrak{B}$ consists of cocomplete categories. Define $L(Y)_j$ as the colimit of D_j^Y. To prove that the $L(Y)_j$ assemble to a twisted diagram, we construct for each $\alpha\colon j \longrightarrow k$ a structure map $l_\alpha^\sharp\colon F_\alpha(L(Y)_j) \longrightarrow L(Y)_k$ and apply lemma 2.9.

Since F_α is a left adjoint, we have a unique isomorphism

$$u_\alpha\colon F_\alpha(\operatorname{colim} D_j^Y) \cong \operatorname{colim}\,(F_\alpha \circ D_j^Y)\ .$$

Let $\Phi(i) \xrightarrow{\sigma} j$ be an object of $\Phi\downarrow j$. Then $\alpha\circ\sigma$ is an object of $\Phi\downarrow k$, and there is a canonical map $F_{\alpha\circ\sigma}(Y_i) \longrightarrow \operatorname{colim} D_k^Y = L(Y)_k$ (since $F_{\alpha\circ\sigma}(Y_i)$ appears in the diagram D_k^Y). The composition with a uniqueness isomorphism yields a map

$$t_\sigma\colon F_\alpha\circ F_\sigma(Y_i) \longrightarrow L(Y)_k\ .$$

The t_σ's assemble to a natural transformation from $F_\alpha\circ D_j^Y$ to the constant diagram with value $L(Y)_k$ (a proof involves the uniqueness of the uniqueness isomorphisms and the naturality of the canonical maps mentioned above; we omit the details). By taking colimits, this determines a map

$$v_\alpha\colon \operatorname{colim}\,(F_\alpha\circ D_j^Y) \longrightarrow L(Y)_k\ ,$$

and we set $l_\alpha^\sharp := v_\alpha\circ u_\alpha$.

Now we have to check that, for $j \xrightarrow{\alpha} k \xrightarrow{\beta} l \in \mathcal{J}$, the square

$$\begin{tikzcd}
F_\beta\circ F_\alpha(L(Y)_j) \arrow[r, "\cong"] \arrow[d, "F_\beta(l_\alpha^\sharp)"'] & F_{\beta\circ\alpha}(L(Y)_j) \arrow[d, "l_{\beta\circ\alpha}^\sharp"] \\
F_\beta(L(Y)_k) \arrow[r, "l_\beta^\sharp"'] & L(Y)_l
\end{tikzcd} \tag{$*$}$$

commutes. First of all, the diagram

$$\begin{tikzcd}
F_\beta\circ F_\alpha(L(Y)_j) \arrow[r, "\cong"] \arrow[d, "\cong"'] & F_{\beta\circ\alpha}(L(Y)_j) \arrow[d, "\cong"] \\
\operatorname{colim}\,(F_\beta\circ F_\alpha\circ D_j^Y) \arrow[r, "\cong"'] & \operatorname{colim}\,(F_{\beta\circ\alpha}\circ D_j^Y)
\end{tikzcd}$$

consisting of uniqueness isomorphisms commutes because of their uniqueness. By the universal property of the colimit and the definition of the structure maps, we are left to show that, for every $\sigma\colon \Phi(i) \longrightarrow j$, the diagram

$$\begin{tikzcd}
F_\beta \circ F_\alpha \circ F_\sigma(Y_i) \arrow[r, "\cong"] \arrow[d, "\cong"'] & F_{\beta\circ\alpha} \circ F_\sigma(Y_i) \arrow[d, "\cong"] \\
F_\beta \circ F_{\alpha\circ\sigma}(Y_i) \arrow[r, "\cong"'] \arrow[d, "c_{\alpha\circ\sigma}"'] & F_{\beta\circ\alpha\circ\sigma}(Y_i) \arrow[dd, "c_{\beta\circ\alpha\circ\sigma}"] \\
\operatorname{colim}(F_\beta \circ D_k^Y) \arrow[d, "\cong"'] & \\
F_\beta(L(Y)_k) \arrow[r, "l^\sharp_\beta"'] & L(Y)_l
\end{tikzcd}$$

commutes, where the maps $c_{\alpha\circ\sigma}$ and $c_{\beta\circ\alpha\circ\sigma}$ are canonical maps to the colimit, and all maps labeled with '$\cong$' are uniqueness isomorphisms. The upper square commutes by uniqueness, and the lower square commutes by definition of $l^\sharp_\beta$. This implies that the square $(*)$ commutes, and 2.9 shows that $L(Y)$ is a twisted diagram as claimed.

Theorem 3.1. *(Left* Kan *extensions.)*

Let $\mathfrak{B}$ be a $\mathcal{J}$-bundle consisting of cocomplete categories, $\Phi\colon \mathcal{I} \longrightarrow \mathcal{J}$ a functor and Y a twisted diagram with coefficients in $\Phi^*\mathfrak{B}$. The assignment $Y \mapsto L(Y)$ described above is the object function of a functor $L\Phi\colon \mathbf{Tw}(\mathcal{I}, \Phi^*\mathfrak{B}) \longrightarrow \mathbf{Tw}(\mathcal{J}, \mathfrak{B})$ which is left adjoint to Φ^*.

Proof. Abbreviate $L\Phi$ by L and keep the notation used in the construction of $L(Y)$.

We start by describing the effect of L on morphisms. Let $f\colon Y \longrightarrow Z$ be a map of twisted diagrams with coefficients in $\Phi^*\mathfrak{B}$, and fix an object $j \in \mathcal{J}$. For each $\sigma\colon \Phi(i) \longrightarrow j$, the maps $F_\sigma(f_i)$ form a natural transformation from D_j^Y to D_j^Z, because the uniqueness isomorphisms are natural, f is a map of twisted diagrams and F_σ is a functor. This defines a map on the colimits $L(f)_j\colon L(Y)_j \longrightarrow L(Z)_j$.

We claim that the maps $L(f)_j$ assemble to a map $L(f)$ of twisted diagrams. For $\alpha\colon j \longrightarrow k$ in $\mathcal{J}$, consider the diagram

$$\begin{tikzcd}
F_\alpha(L(Y)_j) \arrow[r, "F_\alpha(L(f)_j)"] \arrow[d, "l^\sharp_\alpha"'] & F_\alpha(Z_j) \arrow[d, "m^\sharp_\alpha"] \\
L(Y)_k \arrow[r, "L(f)_k"'] & L(Z)_k
\end{tikzcd}$$

where l and m denote the structure maps of $L(Y)$ and $L(Z)$. It commutes if and only if for each object $\sigma\colon \Phi(i) \longrightarrow j$ of $\Phi \downarrow j$, the diagram

$$\begin{tikzcd}
F_\alpha \circ F_\sigma(Y_i) \arrow[r, "F_\alpha \circ F_\sigma(f_i)"] \arrow[d, "\cong"'] & F_\alpha \circ F_\sigma(Z_i) \arrow[d, "\cong"] \\
F_{\alpha\circ\sigma}(Y_i) \arrow[r, "F_{\alpha\circ\sigma}(f_i)"'] \arrow[d] & F_{\alpha\circ\sigma}(Z_i) \arrow[d] \\
L(Y)_k \arrow[r, "L(f)_k"'] & L(Z)_k
\end{tikzcd}$$

commutes. The isomorphisms are uniqueness isomorphisms, which are natural, hence the upper square commutes. The lower vertical arrows denote the canonical map to the colimit, and the naturality of these make the lower square commute.

Having checked that $L(f)$ is indeed a map of twisted diagrams, it is clear that L is a functor. To prove that L is left adjoint to Φ^*, we construct natural transformations $\eta\colon \mathrm{Id} \longrightarrow \Phi^* \circ L$ and $\epsilon\colon L \circ \Phi^* \longrightarrow \mathrm{Id}$ satisfying the triangular identities ([ML, IV.1], theorem 2 (v)).

For $Y \in \mathbf{Tw}(\mathcal{I}, \Phi^*\mathfrak{B})$, the Y-component η_Y is given (pointwise) as the canonical map to the colimit $Y_i \longrightarrow \Phi^*(L(Y))_i = L(Y)_{\Phi(i)}$ which corresponds to the identity $\mathrm{id}\colon \Phi(i) \longrightarrow \Phi(i)$ (an object of $\Phi \downarrow \Phi(i)$). We check that η_Y is a map of twisted diagrams. Let $\alpha\colon i \longrightarrow j \in \mathcal{I}$ be given and consider the diagram

$$\begin{tikzcd}
F_{\Phi(\alpha)}(Y_i) \arrow[r, "F_{\Phi(\alpha)}((\eta_Y)_i)"] \arrow[d, "y^\sharp_\alpha"'] & F_{\Phi(\alpha)}(L(Y)_{\Phi(i)}) \arrow[d, "l^\sharp_{\Phi(\alpha)}"] \\
Y_j \arrow[r, "(\eta_Y)_j"'] & L(Y)_{\Phi(j)}
\end{tikzcd}$$

with the structure map $y^\sharp_\alpha$ starting from $G_\alpha(Y_i) = F_{\Phi(\alpha)}(Y_i)$ by definition of $\Phi^*\mathfrak{B}$. Since the structure map $l^\sharp_{\Phi(\alpha)}$ is defined via the canonical maps to the colimit

$$F_{\Phi(\alpha)} \circ F_\sigma(Y_k) \cong F_{\Phi(\alpha)\circ\sigma}(Y_k) \longrightarrow L(Y)_{\Phi(j)}$$

(for $\sigma\colon \Phi(k) \longrightarrow \Phi(i)$ an object of $\Phi \downarrow \Phi(i)$), the composition $l^\sharp_{\Phi(\alpha)} \circ F_{\Phi(\alpha)}((\eta_Y)_i)$ coincides with the canonical map to the colimit $c\colon F_{\Phi(\alpha)}(Y_i) \longrightarrow L(Y)_{\Phi(j)}$ (the special case $\sigma = id_{\Phi(i)}$). Hence we have to show that the triangle

$$\begin{tikzcd}
F_{\Phi(\alpha)}(Y_i) \arrow[d, "y^\sharp_\alpha"'] \arrow[dr, "c"] & \\
Y_j \arrow[r, "(\eta_Y)_i"'] & L(Y)_{\Phi(j)}
\end{tikzcd}$$

commutes. But this is true by the definition of $L(Y)_{\Phi(j)}$ as the colimit of $D^Y_{\Phi(j)}$.

The naturality of η follows easily from the naturality of the canonical maps to the colimit.

We turn to the definition of ϵ. For $Z \in \mathbf{Tw}\,(\mathcal{J}, \mathfrak{B})$, the map ϵ_Z is given pointwise as follows: for every $j \in \mathcal{J}$ and every $\sigma\colon \Phi(i) \longrightarrow j$ in $\Phi \downarrow j$, the structure maps $F_\sigma(\Phi^*(Z)_i) = F_\sigma(Z_{\Phi(i)}) \xrightarrow{y^\sharp_\sigma} Z_j$ assemble to a natural transformation from $D_j^{\Phi^* Z}$ to the constant diagram with value Z_j (this follows from lemma 2.9 and the fact that Z is a twisted diagram). By taking the colimit, we obtain a map

$$(\epsilon_Z)_j \colon L(\Phi^*(Z))_j \longrightarrow Z_j \ .$$

To prove that ϵ_Z is a map of twisted diagrams, let $\alpha\colon j \longrightarrow k \in \mathcal{J}$ and consider the following diagram:

$$\begin{tikzcd} F_\alpha(L(\Phi^*(Z))_j) \arrow[r, "F_\alpha((\epsilon_Z)_j)"] \arrow[d, "m^\sharp_\alpha"'] & F_\alpha(Z_j) \arrow[d, "z^\sharp_\alpha"] \\ L(\Phi^*(Z))_k \arrow[r, "(\epsilon_Z)_k"'] & Z_k \end{tikzcd}$$

Using the universal property of the colimit, the definition of ϵ_Z and the definition of the structure map $m^\sharp_\alpha$, we are left to show that, for each $\sigma\colon \Phi(i) \longrightarrow j$, the diagram

$$\begin{tikzcd} F_\alpha(F_\sigma(Z_i)) \arrow[r, "F_\alpha(z^\sharp_\alpha)"] \arrow[d] & F_\alpha(Z_j) \arrow[d, "z^\sharp_\alpha"] \\ L(\Phi^*(Z))_k \arrow[r, "(\epsilon_Z)_k"'] & Z_k \end{tikzcd}$$

commutes, where the left vertical map is the composition of the uniqueness isomorphism and the canonical map to the colimit $F_{\alpha\circ\sigma}(Z_i) \longrightarrow L(\Phi^*(Z))_k$. However, the definition of ϵ_Z implies that the diagram above commutes since Z is a twisted diagram.

The naturality of ϵ is obvious.

It remains to prove that the composites

$$L \xrightarrow{L\eta} L \circ \Phi^* \circ L \xrightarrow{\epsilon L} L \qquad \text{and} \qquad \Phi^* \xrightarrow{\eta\Phi^*} \Phi^* \circ L \circ \Phi^* \xrightarrow{\Phi^*\epsilon} \Phi^*$$

are identity natural transformations.

In the first case, let $Y \in \mathbf{Tw}\,(\mathcal{I}, \Phi^*\mathfrak{B})$ and $j \in \mathcal{J}$. The map

$$L(\eta_Y)_j \colon L(Y)_j \longrightarrow L(\Phi^*(L(Y)))_j$$

is defined via the canonical maps to the colimit $F_\sigma(Y_i) \longrightarrow F_\sigma(L(Y)_{\Phi(i)})$ (for morphisms $\sigma\colon \Phi(i) \longrightarrow j$). The definition of ϵ then implies that it suffices to prove the commutativity of the triangle

$$\begin{tikzcd} F_\sigma(Y_i) \arrow[r] \arrow[dr] & F_\sigma(L(Y)_{\Phi(i)}) \arrow[d, "l^\sharp_\sigma"] \\ & L(Y)_j \end{tikzcd}$$

for each $\sigma\colon \Phi(i) \longrightarrow j$, where the two arrows in the middle denote canonical maps to the colimit. The definition of $l^\sharp_\sigma$ gives the desired result.

In the second case, let $Z \in \mathbf{Tw}(\mathcal{J}, \mathfrak{B})$ and $i \in \mathcal{I}$. We have to show that the triangle

$$\begin{tikzcd} Z_{\Phi(i)} \arrow[r] \arrow[dr, "id"'] & L(\Phi^*(Z))_{\Phi(i)} \arrow[d, "(\epsilon_Z)_{\Phi(i)}"] \\ & Z_{\Phi(i)} \end{tikzcd}$$

commutes, where the upper horizontal map is the canonical map to the colimit (corresponding to $\mathrm{id}_{\Phi(i)}$). But this is obvious from the definition of ϵ. $\square$

The right adjoint of Φ^*, obtained by the corresponding twisted version of right Kan extension along Φ, will be denoted $R\Phi$. By the dual of theorem 3.1 it exists if $\mathfrak{B}$ consists of complete categories.

Recall the functor Ev_i defined as the restriction along $\{i\} \longrightarrow \mathcal{J}$, and suppose its left adjoint $Fr_i\colon \mathcal{C}_i \longrightarrow \mathbf{Tw}(\mathcal{J}, \mathfrak{B})$ exists. Then we call $Fr_i(K)$ the *free twisted diagram generated by* $K \in \mathcal{C}_i$.

Example 3.2. *(Spectra, continued.)*

Let Sp be the bundle defined in 2.8 which leads to ordinary spectra. The nth evaluation functor maps a spectrum to its nth term, and the corresponding nth free twisted diagram of a pointed simplicial set K is the spectrum

$$* \multimap * \multimap \dots \multimap * \multimap K \multimap \Sigma K \multimap \Sigma^2 K \multimap \dots$$

with K appearing at the nth spot and all $\sharp$-type structure maps being identities except for the map $\Sigma(*) = * \longrightarrow K$.

4. Construction of Adjunction Bundles

Twisted diagrams were introduced as generalized diagrams. However, there is a different (but equivalent) approach using fibred and cofibred categories in the sense of Grothendieck. For definitions and notation the reader may wish to consult [Q].

Let us recall the Grothendieck construction $\widetilde{Gr}(U)$ of a contravariant functor U defined on $\mathcal{I}$ with values in the category of (small) categories. The objects of $\widetilde{Gr}(U)$ are the pairs (i, Y) with i an object of $\mathcal{I}$ and Y an object of $U(i)$. A morphism $(i, Y) \longrightarrow (j, Z)$ consists of a morphism $i \xrightarrow{\sigma} j$ in $\mathcal{I}$ and a morphism $Y \xrightarrow{A} U(\sigma)(Z)$ in $U(i)$. Composition is given by the rule

$$(\tau, B) \circ (\sigma, A) := \big(\tau \circ \sigma, U(\sigma)(B) \circ A\big) .$$

This construction comes equipped with a functor $\widetilde{Gr}(U) \longrightarrow \mathcal{I}$.

Remark 4.1. An adjunction bundle determines a functor $U\colon \mathcal{I}^{\mathrm{op}} \longrightarrow \mathit{Cat}$, hence a functor $\widetilde{Gr}(U) \longrightarrow \mathcal{I}$. The existence of the left adjoints F_σ make $\widetilde{Gr}(U)$ a cofibred category over $\mathcal{I}^{\mathrm{op}}$, even a bifibred bundle in the sense of the next definition.

Definition 4.2. Given a functor $\pi\colon \mathcal{E} \longrightarrow \mathcal{A}$, we call $\mathcal{E}$ a *bifibred bundle over* $\mathcal{A}$ if the following conditions are satisfied (using notation from [Q]):

(1) The functor π is fibred, and for all composable morphisms α and β in $\mathcal{A}$, the natural isomorphism $\alpha^* \circ \beta^* \longrightarrow (\beta \circ \alpha)^*$ is the identity.

(2) The functor π is cofibred, and for all morphisms $\alpha \in \mathcal{A}$ the functor α^* is right adjoint to α_*.

In this situation, a functor $f : \mathcal{I} \longrightarrow \mathcal{A}$ determines an $\mathcal{I}$-indexed adjunction bundle $f \bowtie \pi = \mathcal{I} \bowtie_{\mathcal{A}} \mathcal{E}$ which sends the object $i \in \mathcal{I}$ to the category $\pi^{-1}(f(i))$ and the morphism $\mu \in \mathcal{I}$ to the adjoint pair $f(\mu)_*$ and $f(\mu)^*$.

Remark 4.3. *(*M. Brun*'s reformulation of twisted diagrams.)*

Recall from remark 4.1 the functor $\pi\colon \widetilde{Gr}(U) \longrightarrow \mathcal{I}$ associated to an adjunction bundle. A straightforward calculation which we omit shows that $\mathbf{Tw}(\mathcal{I}, \mathfrak{B})$ is the category of sections of π.

More generally, given a bifibred bundle π and an adjunction bundle $f \bowtie \pi$ as in 4.2, the category of twisted diagrams $\mathbf{Tw}(\mathcal{I}, f \bowtie \pi)$ is the category of lifts of f to $\mathcal{E}$, *i.e.*, the category of functors $g\colon \mathcal{I} \longrightarrow \mathcal{E}$ satisfying $\pi \circ g = f$.

Example 4.4. Let $\mathit{Mod} \longrightarrow \mathit{Rng}$ denote the canonical functor from the category of all modules over all rings to the category of rings. (The objects of Mod are pairs (R, M) with R a ring and M an R-module. A morphism $(R, M) \longrightarrow (S, N)$ consists of a ring map $f : R \longrightarrow S$ and an f-semilinear additive map $M \longrightarrow N$.) This defines a bifibred bundle.

A toric variety determines a functor into Rng, hence (by 4.2) an adjunction bundle. In fact, a fan Σ of a toric variety can be regarded as a poset, hence as a category, and we obtain a functor

$$\Sigma^{\mathrm{op}} \longrightarrow \mathit{Rng},\ \sigma \mapsto \mathbf{C}[\check{\sigma} \cap M]$$

where $\check{\sigma}$ is the dual cone of σ and M is the dual lattice (see [O] for details). Thus the toric variety $X(\Sigma)$ determines the adjunction bundle $\Sigma^{\mathrm{op}} \bowtie_{\mathit{Rng}} \mathit{Mod}$.

This example can be generalized to obtain an adjunction bundle from a diagram of monoids and a cocomplete category $\mathcal{D}$. We proceed with a construction.

It is well known that we can consider any monoid M as a category with one object and morphisms corresponding to the elements of M. A morphism of monoids then is a functor between two such categories. Suppose that $\mathcal{D}$ is a cocomplete category. We define the category of M-equivariant objects in $\mathcal{D}$, denoted M-$\mathcal{D}$, as the category of functors $M \longrightarrow \mathcal{D}$. A monoid homomorphism $f : M \longrightarrow M'$ induces the "restriction" functor $f^* = R_f\colon M'\text{-}\mathcal{D} \longrightarrow M\text{-}\mathcal{D}$ (given by precomposing with f). Since $\mathcal{D}$ is cocomplete, this functor has a left adjoint $f_* = \cdot \wedge_M M'\colon M\text{-}\mathcal{D} \longrightarrow M'\text{-}\mathcal{D}$. For composable monoid homomorphisms we have the relations $(g \circ f)^* = f^* \circ g^*$ and $(g \circ f)_* \cong g_* \circ f_*$. Moreover $\mathrm{id}^* = \mathrm{id}$, and we choose $\mathrm{id}_* = \mathrm{id}$.

Let $Eq\mathcal{D}$ denote the category of equivariant objects in $\mathcal{D}$. Objects are the pairs $(M,\ D)$ where M is a monoid and D is a functor $M \longrightarrow \mathcal{D}$. A morphism from $(M,\ D)$ to $(M',\ D')$ is a pair (α, ν) where $\alpha\colon M \longrightarrow M'$ is a monoid homomorphism and ν is a natural transformation of functors $D \longrightarrow D' \circ \alpha$. The forgetful functor $\pi\colon Eq\mathcal{D} \longrightarrow Mon$ into the category of monoids make $Eq\mathcal{D}$ into a bifibred bundle in the sense of 4.2. The fibre over the monoid M is the category M-$\mathcal{D}$ of M-equivariant objects in $\mathcal{D}$.

Definition 4.5. Suppose we have a (small) category $\mathcal{I}$ and an $\mathcal{I}$-indexed diagram G of monoids, *i.e.*, a functor $G\colon \mathcal{I} \longrightarrow Mon$. For a cocomplete category $\mathcal{D}$ we define the $\mathcal{I}$-indexed adjunction bundle $\mathrm{Ad}_{\mathcal{D}}G = (\mathcal{C},\ F,\ U)$ by

$$\mathrm{Ad}_{\mathcal{D}}G := \mathcal{I} \bowtie_{Mon} Eq\mathcal{D}\ .$$

Explicitly, for an object $i \in \mathcal{I}$ we let $\mathcal{C}_i := G(i)$-$\mathcal{D}$, the category of $G(i)$-equivariant objects in $\mathcal{D}$, and for a morphism $\sigma \in \mathcal{I}$ we define $F_\sigma := G(\sigma)_*$ and $U_\sigma := G(\sigma)^*$.

This definition is clearly natural in G, *i.e.*, given a natural transformation of diagrams of monoids $G \longrightarrow G'$ we obtain an $\mathcal{I}$-morphism of adjunction bundles $\mathrm{Ad}_{\mathcal{D}}G' \longrightarrow \mathrm{Ad}_{\mathcal{D}}G$.

Example 4.6. *(Non-linear projective spaces.)*

This generalizes example 2.3. Let $[n]$ denote the set $\{0,\ 1,\ \ldots,\ n\}$, and write $\langle n\rangle$ for the subcategory of non-empty subsets of $[n]$. For $A \subseteq [n]$, define the (additive) monoid

$$M^A := \left\{ (a_0,\ \ldots, a_n) \in \mathbf{Z}^{n+1} \,\middle|\, \sum_0^n a_i = 0 \text{ and } \forall i \notin A\colon a_i \geq 0 \right\}\ .$$

These monoids assemble to a functor $G\colon \langle n\rangle \longrightarrow Mon$. Let G denote a topological monoid, and write $\mathcal{D} = G$-$k\,Top_*$ for the category of G-equivariant, based, compactly generated topological spaces. We are now in the situation of definition 4.5 (with $\mathcal{I} = \langle n\rangle$); call the resulting adjunction bundle $\mathfrak{P}^n(G)$. The category of twisted diagrams $\mathbf{Tw}\,(\langle n\rangle, \mathfrak{P}^n(G))$ is nothing but the category $\mathbf{pP}^n(G)$ of G-equivariant quasi-coherent presheaves as defined in [Hü, 6.1].

5. Remarks on Model Structures

The terminology concerning model categories is taken from [DS] and [Ho], the proofs are mostly modifications of the corresponding proofs in [Ho]. The term "model category" is always to be understood in the sense of [DS], which is slightly more general than the definition given in [Ho]. The differences are the following: In [Ho], it is required that a model category has all small limits and colimits (instead of just finite ones), and the factorizations have to be functorial and are part of the structure (instead of assuming that they simply exist).

Definition 5.1. Let $\mathfrak{B} = (\mathcal{C},\ F,\ U)$ be an adjunction bundle over $\mathcal{I}$. We call $\mathfrak{B}$ an *adjunction bundle of model categories* if all the $\mathcal{C}_i$ are model categories, and all the F_σ preserve cofibrations and acyclic cofibrations. In other words, we require the pair (F_σ, U_σ) to form a QUILLEN adjoint pair.—If in addition all the $\mathcal{C}_i$ are proper model categories, $\mathfrak{B}$ is called *proper*. Note that the inverse image of a (proper) adjunction bundle of model categories is again a (proper) adjunction bundle of model categories.

Example 5.2. The projective space bundles $\mathfrak{P}^n(G)$ (for G a cofibrant topological monoid, 2.3) and spectra Sp (cf. 2.8) are examples of proper adjunction bundles of model categories. The model structure defined on $M\text{-}kTop_*$ (for M a monoid) has weak equivalences and fibrations on underlying spaces, the model structure on the category of pointed simplicial sets is the usual one.

Before defining the model structures on twisted diagrams, we make a technical observation.

Remark 5.3. Suppose $\mathcal{C} = \prod_\nu \mathcal{C}_\nu$ is the product of model categories $\mathcal{C}_\nu$. Then there is a product model structure on $\mathcal{C}$ where a map is a weak equivalence (*resp.* fibration, *resp.* cofibration) if its image under the canonical projection is a weak equivalence (*resp.* fibration, *resp.* cofibration) in $\mathcal{C}_\nu$ for all ν (see [Ho, 1.1.6]). If all the $\mathcal{C}_\nu$ are proper, $\mathcal{C}$ is a proper model category.

6. The c-Structure

Definition 6.1. A *category with degree function* is a (small) category $\mathcal{I}$ together with a $\mathbf{Z}$-valued function d, defined on the objects, such that whenever there is a non-identity morphism $i \longrightarrow j$ we have $d(i) \neq d(j)$. (We say that all non-identity arrows change the degree. In particular, objects have no non-trivial endomorphisms.) The category is called *bounded* if d is bounded below, and it is called *locally bounded* if each connected component is bounded. If non-identity arrows always increase the degree and the category is (locally) bounded, we say that $\mathcal{I}$ is a *(locally) direct category*.

Remark 6.2. It is possible to allow more general degree functions with arbitrary ordinals as values (cf. [Ho]). The two inductive proofs appearing in this section just have to be completed with a discussion of the “limit ordinal case”.

All finite dimensional categories (*i.e.*, categories with finite dimensional nerve) admit degree functions and can be made into direct categories. A disjoint union of locally direct categories is locally direct. If $\mathcal{I}$ is (locally) direct, so are subcategories, under and over categories formed with $\mathcal{I}$. In particular, the full subcategory $\mathcal{I}_n$ of objects of degree less than or equal to n is (locally) direct. A finite product of direct categories is direct (with degree given by sum of partial degrees).

In what follows, $\mathfrak{B} = (\mathcal{C}, F, U)$ is an adjunction bundle of cocomplete model categories over $\mathcal{I}$. Let Y be a twisted diagram with coefficients in $\mathfrak{B}$ and i an object of $\mathcal{I}$. To describe the cofibrations in the model structure we are going to construct, we have to introduce the latching object of Y at i. Roughly speaking, it is given by the colimit over all components of Y mapping to Y_i; the colimit is to be taken with respect to the $\sharp$-type structure maps.—For each object $i \in \mathcal{I}$, let $\mathcal{I} \downarrow i$ denote the category of objects over i. Let $\mathcal{I} \Downarrow i$ denote the full subcategory of $\mathcal{I} \downarrow i$ which consists of all objects $\sigma : j \longrightarrow i$ with $\sigma \neq \mathrm{id}_i$. There are canonical functors $\iota : \mathcal{I} \Downarrow i \hookrightarrow \mathcal{I} \downarrow i$ (the inclusion) and $\mathrm{pr} : \mathcal{I} \downarrow i \longrightarrow \mathcal{I}$ (the projection $(\sigma : j \longrightarrow i) \mapsto j$). Set $P_{\mathcal{I}\Downarrow i} := \mathrm{pr} \circ \iota$ and denote the trivial bundle over $\mathcal{I} \Downarrow i$ with value $\mathcal{C}_i$ by $\mathcal{C}_i$ again. We define an $\mathcal{I} \Downarrow i$-morphism of bundles $\Psi : \mathcal{C}_i \longrightarrow (P_{\mathcal{I}\Downarrow i})^*\mathfrak{B}$ as follows: For $\sigma : j \longrightarrow i$, the adjoint pair

$$F_\sigma : \mathcal{C}_j \rightleftarrows \mathcal{C}_i : U_\sigma$$

is the σ-component of Ψ, and it is obvious from the definitions that Ψ is in fact a bundle morphism. Hence we have a functor $\Psi^* : \mathbf{Tw}\,(\mathcal{I} \Downarrow i, (P_{\mathcal{I}\Downarrow i})^*\mathfrak{B}) \longrightarrow Fun(\mathcal{I} \Downarrow i, \mathcal{C}_i)$. Define $G_i : \mathbf{Tw}\,(\mathcal{I} \Downarrow i, (P_{\mathcal{I}\Downarrow i})^*\mathfrak{B}) \longrightarrow \mathcal{C}_i$ as the composition $\operatorname{colim} \circ \Psi^*$.

Definition 6.3. The *latching object of* Y *at* i is defined as $L_iY := G_i \circ (P_{\mathcal{I}\Downarrow i})^*(Y)$. It is an object of $\mathcal{C}_i$.

Remark 6.4. The structure maps $y^\sharp_\sigma : F_\sigma(Y_j) \longrightarrow Y_i$ for $\sigma : j \longrightarrow i$ define a natural transformation $L_i \longrightarrow Ev_i$. If a map $L_iY \longrightarrow Y_i$ is mentioned, it is always this natural map.

Example 6.5. If X is a spectrum and $n > 0$, the latching object of X at n is the pointed simplicial set ΣX_{n-1}, and the natural map $\Sigma X_{n-1} \longrightarrow X_n$ of 6.4 is the structure map of the spectrum.

Example 6.6. Let $Y = (Y_+ \xrightarrow{y_\alpha} Y_0 \xleftarrow{y_\beta} Y_-)$ be a twisted diagram with coefficients in the projective line bundle $\mathfrak{P}^1$ (cf. 2.3). The latching objects of Y at $+$ and at $-$ are the initial objects in $\mathbf{N}_+$-$k\,Top_*$ and $\mathbf{N}_-$-$k\,Top_*$, respectively. The latching object at 0 is the $\mathbf{Z}$-equivariant pointed space $(Y_+ \wedge_{\mathbf{N}_+} \mathbf{Z}) \vee (Y_- \wedge_{\mathbf{N}_-} \mathbf{Z})$. The $\sharp$-type structure maps induce a map to Y_0.

Definition 6.7. *(c-structure)*

Let $f : Y \longrightarrow Z$ be a map in $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$. We call f a *weak equivalence* if f_i is a weak equivalence in $\mathcal{C}_i$ for every object $i \in \mathcal{I}$. We call f a *c-cofibration* if for all objects i of I, the induced map $Y_i \cup_{L_iY} L_iZ \longrightarrow Z_i$ is a cofibration. We call f a *c-fibration* if all f_i are fibrations in $\mathcal{C}_i$.

To prove that the c-structure is a model structure, we have to concentrate on the lifting axiom first. Call a map $f \in \mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$ a *good acyclic c-cofibration* if for all objects i of I, the induced map $Y_i \cup_{L_iY} L_iZ \longrightarrow Z_i$ is an acyclic cofibration. Later, we will prove that the class of good acyclic c-cofibrations coincides with the class of acyclic c-cofibrations.

Lemma 6.8. *Let $\mathcal{I}$ be a direct category, and let $\mathfrak{B}$ be an adjunction bundle of cocomplete model categories over $\mathcal{I}$. Good acyclic c-cofibrations have the left lifting property with respect to c-fibrations. Similarly, c-cofibrations have the left lifting property with respect to acyclic c-fibrations.*

Proof. We treat the first case only, the other is similar. Let

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & X \arrow[d, "p"] \\ B \arrow[r, "h"'] & Y \end{tikzcd}$$

be a commutative diagram in $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$ such that f is a good acyclic c-cofibration and p is a c-fibration. We will construct the desired lift by induction on the degree of objects of $\mathcal{I}$.

Since $\mathcal{I}$ is direct, the degree function d has a minimum k. If i is an object in $\mathcal{I}$ of degree k, then L_i is the constant functor with the initial object as value. By definition of a good acyclic cofibration, the map f_i is an acyclic cofibration in $\mathcal{C}_i$. Hence we can find a lift l_i in the following diagram:

$$\begin{tikzcd} A_i \arrow[r, "g_i"] \arrow[d, "f_i"', "\sim"] & X_i \arrow[d, "p_i"] \\ B_i \arrow[r, "h_i"'] \arrow[ur, dotted, "l_i"'] & Y_i \end{tikzcd}$$

Since the full subcategory $\mathcal{I}_k$ of objects of degree k is discrete, the lifts l_i for the various $i \in \mathcal{I}_k$ assemble to a map $l|_{\mathcal{I}_k} : B|_{\mathcal{I}_k} \longrightarrow X|_{\mathcal{I}_k}$ in $\mathbf{Tw}\,(\mathcal{I}_k, \mathfrak{B}|_{\mathcal{I}_k})$.

Now let $n > k$, and assume that we have constructed a lift in the diagram

$$\begin{tikzcd} A|_{\mathcal{I}_{n-1}} \arrow[r, "g|_{\mathcal{I}_{n-1}}"] \arrow[d, "f|_{\mathcal{I}_{n-1}}"'] & X|_{\mathcal{I}_{n-1}} \arrow[d, "p|_{\mathcal{I}_{n-1}}"] \\ B|_{\mathcal{I}_{n-1}} \arrow[r, "h|_{\mathcal{I}_{n-1}}"'] \arrow[ur, dotted, "l|_{\mathcal{I}_{n-1}}"'] & Y|_{\mathcal{I}_{n-1}} \end{tikzcd}$$

making it a commutative diagram in $\mathbf{Tw}\,(\mathcal{I}|_{n-1}, \mathfrak{B}|_{\mathcal{I}_{n-1}})$. If i is an object of degree n and $\sigma : j \longrightarrow i$ an object of $\mathcal{I} \Downarrow i$, the map $F_\sigma(B_j) \xrightarrow{F_\sigma(l_j)} F_\sigma(X_j) \xrightarrow{x_\sigma^\sharp} X_i$ is part of a natural transformation $\phi\colon L_i B \longrightarrow X_i$ such that the diagram

$$\begin{tikzcd} L_i A \arrow[r] \arrow[d, "L_i f"'] & A_i \arrow[d, "g_i"] \\ L_i B \arrow[r, "\phi"'] & X_i \end{tikzcd}$$

commutes. Hence we get a diagram

$$\begin{tikzcd} B_i \cup_{L_i B} L_i A \arrow[r] \arrow[d, "\sim"'] & X_i \arrow[d, "p_i"] \\ B_i \arrow[r, "h_i"'] & Y_i \end{tikzcd}$$

in which, by hypothesis, the left vertical map is an acyclic cofibration and the right vertical map is a fibration. Thus a lift $l_i \colon B_i \longrightarrow X_i$ exists, and it is straightforward to check that these maps l_i, together with the morphism $l|_{\mathcal{I}_{n-1}}$, define a map of twisted diagrams $l|_{\mathcal{I}_n} : B|_{\mathcal{I}_n} \longrightarrow X|_{\mathcal{I}_n}$ such that the diagram

$$\begin{tikzcd} A|_{\mathcal{I}_n} \arrow[r, "g|_{\mathcal{I}_n}"] \arrow[d, "f|_{\mathcal{I}_n}"'] & X|_{\mathcal{I}_n} \arrow[d, "p|_{\mathcal{I}_n}"] \\ B|_{\mathcal{I}_n} \arrow[r, "h|_{\mathcal{I}_n}"'] \arrow[ur, dotted, "l|_{\mathcal{I}_n}"'] & Y|_{\mathcal{I}_n} \end{tikzcd}$$

commutes. This completes the proof. □

Let $\Phi\colon \mathcal{I} \longrightarrow \mathcal{J}$ be a functor and $\mathfrak{A}$ an adjunction bundle of cocomplete model categories over $\mathcal{J}$. Obviously, the functor $\Phi^* : \mathbf{Tw}\,(\mathcal{J}, \mathfrak{A}) \longrightarrow \mathbf{Tw}\,(\mathcal{I}, \Phi^*\mathfrak{A})$ preserves weak equivalences and c-fibrations. The question is whether Φ^* also preserves c-cofibrations. Under certain conditions (which are satisfied in the case of interest) we can give a positive answer.

Suppose the functor $\Phi\colon \mathcal{I} \longrightarrow \mathcal{J}$ is *injective at identities*, *i.e.*, whenever $\Phi(\sigma)$ is an identity morphism, so is σ. (For example, a faithful functor is injective at identities.) Then Φ induces a functor

$$\Phi \Downarrow i \colon \mathcal{I} \Downarrow i \longrightarrow \mathcal{J} \Downarrow \Phi(i)$$

which sends $\sigma\colon k \longrightarrow i$ to $\Phi(\sigma)\colon \Phi(k) \longrightarrow \Phi(i)$. This construction is compatible with the projection functors, *i.e.*, we have $\Phi \circ P_{\mathcal{I}\Downarrow i} = P_{\mathcal{J}\Downarrow\Phi(i)} \circ \Phi \Downarrow i$.

Recall that a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is called *final* if for each $A \in \mathcal{D}$ the category $A \downarrow F$ of objects F-under A is non-empty and connected.

We say that the functor Φ satisfies the *finality condition* if it is injective at identities, and the functor $\Phi \Downarrow i$ is final for all objects $i \in \mathcal{I}$.

Lemma 6.9. *Let $\Phi\colon \mathcal{I} \longrightarrow \mathcal{J}$ be a functor, $\mathfrak{B}$ an adjunction bundle of cocomplete model categories over $\mathcal{I}$ and i an object of $\mathcal{I}$. Denote by L_i the i-th latching object functor of $\mathbf{Tw}\,(\mathcal{I}, \Phi^*B)$, and by $L'_{\Phi(i)}$ the $\Phi(i)$-th latching object functor of $\mathbf{Tw}\,(\mathcal{J}, \mathfrak{B})$. If Φ satisfies the finality condition, then there is a natural isomorphism $L_i \circ \Phi^* \cong L'_{\Phi(i)}$.*

Proof. The functor L_i is defined as the composition $\operatorname{colim} \circ \Psi^* \circ P^*_{\mathcal{I}\Downarrow i}$, with Ψ being an $\mathcal{I} \Downarrow i$-morphism with σ-component given by the adjunction

$$F_{\Phi(\sigma)} : C_{\Phi(j)} \rightleftarrows \mathcal{C}_{\Phi(i)} : U_{\Phi(\sigma)}$$

where $\sigma : j \longrightarrow i$ is an object of $\mathcal{I} \Downarrow i$. On the other hand, $L'_{\Phi(i)}$ is the composition $L'_{\Phi(i)} = \operatorname{colim} \circ \Theta^* \circ P^*_{J\Downarrow\Phi(i)}$, with Θ having the τ-component given by the adjunction

$$F_\tau : C_j \rightleftarrows \mathcal{C}_{\Phi(i)} : U_\tau$$

where $\tau : j \longrightarrow \Phi(i)$ is an object of $\mathcal{J} \Downarrow \Phi(i)$. It is straightforward to check that the equality $L_i \circ \Phi^* = \operatorname{colim} \circ (\Phi \Downarrow i)^* \circ \Theta^* \circ P^*_{J\Downarrow\Phi(i)}$ holds. Hence the i-th latching object of $\Phi^*(A)$ is given by

$$L_i(\Phi^*(A)) = \operatorname{colim} \circ (\Theta^* \circ P^*_{J\Downarrow\Phi(i)}(A)) \circ (\Phi \Downarrow i) \ .$$

The functor $\Phi \Downarrow i$ induces a map $L_i(\Phi^*(A)) \longrightarrow L'_{\Phi(i)}(A)$ which is an isomorphism by [ML, IX.3.1] since $\Phi \Downarrow i$ is final. $\square$

Corollary 6.10. *If Φ satisfies the finality condition, then Φ^* preserves c-cofibrations and good acyclic c-cofibrations.*

Proof. This follows immediately from 6.9 since the maps $L_i(\Phi^* A) \longrightarrow A_{\Phi(i)}$ and $L'_{\Phi(i)} A \longrightarrow A_{\Phi(i)}$ correspond under the isomorphism. $\square$

Remark 6.11. The functor $P_{\mathcal{I}\Downarrow i}$ satisfies the finality condition because $(P_{\mathcal{I}\Downarrow i}) \Downarrow \alpha$ is an isomorphism of categories for each object $\alpha \in \mathcal{I} \Downarrow i$.

Lemma 6.12. *Let $\mathcal{I}$ be direct. For each $i \in \mathcal{I}$, the latching object functor L_i maps c-cofibrations to cofibrations and good acyclic c-cofibrations to acyclic cofibrations.*

Proof. Recall that L_i was defined as the composite $G_i \circ (P_{\mathcal{I}\Downarrow i})^*$. By remark 6.11 and corollary 6.10, we are left to show that G_i maps c-cofibrations to cofibrations and good acyclic c-cofibrations to acyclic cofibrations. However, G_i has a right adjoint $V_i := \Psi_* \circ \delta$, where $\delta : \mathcal{C}_i \longrightarrow Fun(\mathcal{I} \Downarrow i, \mathcal{C}_i)$ denotes the constant diagram functor and Ψ_* is the direct image under Ψ. It is easy to see that V_i maps (acyclic) fibrations to (acyclic) c-fibrations. Hence the statement follows from lemma 6.8 and the fact that $\mathcal{C}_i$ is a model category. $\square$

Corollary 6.13. *If f is a (good acyclic) c-cofibration, all its components are (acyclic) cofibrations in their respective categories. In particular, a good acyclic c-cofibration is an acyclic c-cofibration.*

Proof. Let $f\colon A \longrightarrow B$ be a c-cofibration. By 6.12, the map $L_i f\colon L_i A \longrightarrow L_i B$ is a cofibration in $\mathcal{C}_i$, hence its cobase change $A_i \longrightarrow A_i \cup_{L_i A} L_i B$ is a cofibration. Observe that f_i factors as this last map followed by $A_i \cup_{L_i A} L_i B \longrightarrow B_i$. Since the latter is a cofibration by hypothesis, we conclude that f_i is a cofibration.—The other case is similar. □

Theorem 6.14. *Suppose $\mathcal{I}$ is a locally direct category, and $\mathfrak{B}$ is an adjunction bundle of cocomplete model categories over $\mathcal{I}$.*
(1) The c-structure is a model structure.
(2) A map f of twisted diagrams is an acyclic c-cofibration if and only if it is a good acyclic c-cofibration
(3) If $\mathfrak{B}$ is a proper bundle, the c-structure is proper.

Proof. Let $(\mathcal{I}_\nu)$ denote the family of path components of $\mathcal{I}$. Then $\mathcal{I} = \coprod \mathcal{I}_\nu$, and each of the $\mathcal{I}_\nu$ is a direct category. Since $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B}) = \prod_\nu \mathbf{Tw}\,(\mathcal{I}_\nu, \mathfrak{B}|_{\mathcal{I}_\nu})$, it is enough to show that the c-structure is a model structure for each of the categories $\mathbf{Tw}\,(I_\nu, \mathfrak{B}|_{\mathcal{I}_\nu})$; by 5.3 we can equip $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$ with the product model structure. Consequently, we can assume that $\mathcal{I}$ is direct.

We use the axioms for model categories as given in [DS]. First we note that the class of weak equivalences is closed under composition since weak equivalences are defined pointwise. Similarly, the composition of two c-fibrations is a c-fibration again.

Now assume we have two composable c-cofibrations $A \xrightarrow{f} B \xrightarrow{g} C$. To show that $g \circ f$ is a c-cofibration, we have to prove that for all objects $i \in \mathcal{I}$ the induced map

$$A_i \cup_{L_i A} L_i C \longrightarrow C_i$$

is a cofibration in $\mathcal{C}_i$. But we can factor this map as

$$\begin{aligned} A_i \cup_{L_i A} L_i C \cong A_i \cup_{L_i A} L_i B &\cup_{L_i B} L_i C \\ &\xrightarrow{x} B_i \cup_{L_i B} L_i C \\ &\xrightarrow{y} C_i \end{aligned}$$

where x is induced by f, and y is induced by g. But both of these maps are cofibrations (since they are cobase changes of cofibrations), hence so is their composite.

It is obvious that each of the classes above contains all identities.

Axiom **MC1**: existence of finite limits and colimits is guaranteed by 2.10 since they exist in all $\mathcal{C}_i$.

Axiom **MC2**: the "2-of-3" property for weak equivalences is satisfied since weak equivalences are defined pointwise and **MC2** holds in all the categories $\mathcal{C}_i$.

Axiom **MC3**: the class of weak equivalences is closed under retracts since weak equivalences are defined pointwise, and in each category $\mathcal{C}_i$ a retract of a weak equivalence is a weak equivalence. Similarly, the class of fibrations is closed under retracts.

Suppose $g\colon Y \longrightarrow Z$ is a retract of $f\colon A \longrightarrow B$ and f is a c-cofibration. We have to show that for all objects $i \in \mathcal{I}_n$, the map $L_i Z \cup_{L_i Y} Y_i \longrightarrow Z_i$ induced by g is a cofibration in $\mathcal{C}_i$. But by functoriality of pushouts and latching objects, this map is a retract of the map $L_i B \cup_{L_i A} A_i \longrightarrow B_i$ induced by f, which is a cofibration by hypothesis. Since **MC3** is valid in $\mathcal{C}_i$, the former map is a cofibration. Hence g is a c-cofibration as claimed. This argument also shows that the class of good acyclic c-cofibrations is closed under retracts.

Axiom **MC5**: let $f\colon A \longrightarrow X$ be a map in $\mathbf{Tw}(\mathcal{I}, \mathfrak{B})$. We will construct inductively a factorization of f as a good acyclic c-cofibration followed by a c-fibration. (The other factorization axiom is proved in a similar manner). Let k be the minimum of the degree function on $\mathcal{I}$, and let i be of degree k. Then f_i factors in $\mathcal{C}_i$ as $A_i \underset{\sim}{\overset{g_i}{\rightarrowtail}} T_i \overset{p_i}{\twoheadrightarrow} X_i$, with g_i being an acyclic cofibration and p_i being a fibration. The collection of these factorizations (where i ranges through all objects of degree k) yields a factorization of $f|_{\mathcal{I}_k}$ in $\mathbf{Tw}(\mathcal{I}_k, \mathfrak{B}|_{\mathcal{I}_k})$ as $g|_{\mathcal{I}_k} : A|_{\mathcal{I}_k} \longrightarrow T|_{\mathcal{I}_k}$ followed by $p|_{\mathcal{I}_k} : T|_{\mathcal{I}_k} \longrightarrow X|_{\mathcal{I}_k}$.

Let $n > k$, and assume we have already constructed a factorization of $f|_{\mathcal{I}_{n-1}}$ in $\mathbf{Tw}(\mathcal{I}_{n-1}, \mathfrak{B}|_{\mathcal{I}_{n-1}})$ as the composite $A|_{\mathcal{I}_{n-1}} \xrightarrow{g|_{\mathcal{I}_{n-1}}} T|_{\mathcal{I}_{n-1}} \xrightarrow{p|_{\mathcal{I}_{n-1}}} X|_{\mathcal{I}_{n-1}}$. Let i be of degree n. The canonical functor $P_{\mathcal{I}\Downarrow i}\colon \mathcal{I} \Downarrow i \longrightarrow \mathcal{I}$ factors through the inclusion $\Phi\colon \mathcal{I}_{n-1} \hookrightarrow \mathcal{I}$ as $\Theta\colon \mathcal{I} \Downarrow i \longrightarrow \mathcal{I}_{n-1}$ since $\mathcal{I}$ is direct. Recall the functor

$$G_i\colon \mathbf{Tw}(\mathcal{I} \Downarrow i, (P_{\mathcal{I}\Downarrow i})^* \mathfrak{B}) \longrightarrow \mathcal{C}_i$$

appearing in the definition of the i-th latching object functor L_i (6.3). By definition, $L_i = G_i \circ P_{\mathcal{I}\Downarrow i} = G_i \circ \Theta^* \circ \Phi^*$, hence $G_i \circ \Theta^*(A|_{\mathcal{I}_{n-1}}) = L_i A$. The maps $F_\sigma(T_j) \xrightarrow{F_\sigma(p_j)} F_\sigma(X_j) \xrightarrow{x_\sigma^\sharp} X_i$ for the different objects $\sigma\colon j \longrightarrow i$ of $\mathcal{I} \Downarrow i$ induce a map $G_i \circ \Theta^*(T|_{\mathcal{I}_{n-1}}) \longrightarrow X_i$ which makes the diagram

$$\begin{tikzcd}
G_i \circ \Theta^*(A|_{\mathcal{I}_{n-1}}) = L_i A \arrow[r] \arrow[d, "G_i\circ\Theta^*(g|_{\mathcal{I}_{n-1}})"'] & A_i \arrow[d, "f_i"] \\
G_i \circ \Theta^*(T|_{\mathcal{I}_{n-1}}) \arrow[r] & X_i
\end{tikzcd}$$

commute. Now factor the induced map $A_i \cup_{L_i(A)} (G_i \circ \Theta^*)(T|_{\mathcal{I}_{n-1}}) \longrightarrow X_i$ as an acyclic cofibration $h_i\colon A_i \cup_{L_i(A)} (G_i \circ \Theta^*)(T|_{\mathcal{I}_{n-1}}) \overset{\sim}{\rightarrowtail} T_i$ followed by a fibration $p_i\colon T_i \twoheadrightarrow X_i$ in $\mathcal{C}_i$. The collection of the T_i's for the different objects i of degree n, together with $T|_{\mathcal{I}_{n-1}}$ define a twisted diagram in $\mathbf{Tw}(\mathcal{I}_n, \mathfrak{B}|_{\mathcal{I}_n})$. The new structure maps for $\sigma\colon j \longrightarrow i$ are the compositions

$$F_\sigma(T_j) \longrightarrow G_i \circ \Theta^*(T|_{\mathcal{I}_{n-1}}) \longrightarrow A_i \cup_{L_i(A)} (G_i \circ \Theta^*)(T|_{\mathcal{I}_{n-1}}) \underset{\sim}{\overset{h_i}{\rightarrowtail}} T_i$$

where the first two maps are the canonical ones. If we define g_i as the composition of the canonical map $A_i \longrightarrow A_i \cup_{L_i(A)} (G_i \circ \Theta^*)(T|_{\mathcal{I}_{n-1}})$ with h_i, it is straightforward to check that we get a factorization $f|_{\mathcal{I}_n} = p|_{\mathcal{I}_n} \circ g|_{\mathcal{I}_n}$ in $\mathbf{Tw}(\mathcal{I}_n, \mathfrak{B}|_{\mathcal{I}_n})$. This completes the induction.

We end up with a factorization of f as $A \xrightarrow{g} T \xrightarrow{p} X$. The object $T|_{\mathcal{I}_n}$ we constructed in the induction step coincides with the restriction of T, and similarly for the maps g and p. It is clear that p is a c-fibration in $\mathbf{Tw}(\mathcal{I}, \mathfrak{B})$. To complete the proof of axiom **MC5**, it remains to show that the map g is a good acyclic c-cofibration. However, if i is of degree $k = \min d$, the map $A_i \cup_{L_i A} L_i T = A_i \xrightarrow{g_i} T_i$ is an acyclic cofibration in $\mathcal{C}_i$, and if i is of degree $n > k$, the map $A_i \cup_{L_i A} L_i T \longrightarrow T_i$ coincides with the map $h_i : A_i \cup_{L_i(A)} (G_i \circ \Theta^*)(T|_{\mathcal{I}_{n-1}}) \longrightarrow T_i$ which is an acyclic cofibration in $\mathcal{C}_i$. Hence g is a good acyclic c-cofibration.

We prove part (2) of the theorem. We have already seen that every good acyclic c-cofibration is an acyclic c-cofibration (6.13). To prove the converse, let $f\colon A \longrightarrow X$ be an acyclic c-cofibration. Factor f as a good acyclic c-cofibration $g\colon A \longrightarrow T$ followed by a c-fibration $p\colon T \longrightarrow X$, and note that p is an acyclic c-fibration by axiom **MC2**. The map f is in particular a c-cofibration, so we can find a lift in the diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & T \arrow[d, "p"] \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}$$

which expresses f as a retract of g. Since good acyclic c-cofibrations are closed under retracts, we are done.

Knowing (2), we see that axiom **MC4** is an immediate consequence of lemma 6.8. This finishes the proof of (1).

Finally, recall from lemma 2.10 that pushouts and pullbacks are calculated pointwise. Since the components of a weak equivalence (c-fibration, c-cofibration) are weak equivalences (fibrations, cofibrations) in the respective categories (use corollary 6.13 for the c-cofibrations), assertion (3) follows. $\square$

7. The f-Structure

The construction of the c-structure can be dualized. There is a notion of a (locally) inverse category, and matching objects allow us to define an f-structure with pointwise cofibrations and weak equivalences.

In the following, let $\mathfrak{B} = (\mathcal{C},\ F,\ U)$ be an adjunction bundle of complete model categories over $\mathcal{I}$. Denote by $i \Downarrow \mathcal{I}$ the full subcategory of the under category $i \downarrow \mathcal{I}$ consisting of objects $\sigma : i \longrightarrow j$ with $\sigma \neq \mathrm{id}_i$. Again we have a canonical functor $\Phi : i \Downarrow \mathcal{I} \longrightarrow \mathcal{I}$. Consider $\mathcal{C}_i$ as a trivial bundle over $i \Downarrow \mathcal{I}$, and let $\Psi : \Phi^*\mathfrak{B} \longrightarrow \mathcal{C}_i$ be the $i \Downarrow \mathcal{I}$-morphism of bundles with σ-component given by the adjunction

$$F_\sigma : \mathcal{C}_i \rightleftarrows \mathcal{C}_j : U_\sigma$$

for $\sigma\colon i \longrightarrow j$. Define $H_i\colon \mathbf{Tw}\,(i \Downarrow \mathcal{I}, \Phi^*\mathfrak{B}) \longrightarrow \mathcal{C}_i$ as the composition $\lim \circ \Psi_*$. In fact, H_i coincides with the direct image functor Ξ_* where Ξ is the bundle morphism given by the pair $(\Psi, i \Downarrow \mathcal{I} \longrightarrow \{i\})$ (here $\{i\}$ is the trivial category).

Definition 7.1. Let Y be a twisted diagram with coefficients in $\mathfrak{B}$. The *matching object of* Y *at* i is defined as $M_iY := H_i \circ \Phi^*(Y)$.

Remark 7.2. The structure maps $y_\sigma^\flat\colon Y_i \longrightarrow U_\sigma(Y_j)$ for $\sigma\colon i \longrightarrow j$ define a natural transformation $Ev_i \longrightarrow M_i$. If a map $Y_i \longrightarrow M_iY$ is mentioned, it is always this natural map.

Definition 7.3. *(f-structure)*

Let $f\colon Y \longrightarrow Z$ be a map in $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$. We call f a *weak equivalence* if f_i is a weak equivalence in $\mathcal{C}_i$ for every object $i \in \mathcal{I}$. We call f an *f-fibration* if for all objects $i \in I$, the induced map $Y_i \longrightarrow Z_i \times_{M_iZ} M_iY$ is a fibration. We call f an *f-cofibration* if all f_i are cofibrations in $\mathcal{C}_i$.

Definition 7.4. A category with degree function is called a *(locally) inverse category* if its opposite category (with the same degree function) is (locally) direct.

Theorem 7.5. *Suppose $\mathcal{I}$ is a locally inverse category, and $\mathfrak{B}$ is an adjunction bundle of complete model categories over $\mathcal{I}$.*

(1) The f-structure is a model structure.

(2) If f is an f-fibration, all its components are fibrations in their respective categories.

(3) A map $f\colon Y \longrightarrow Z$ of twisted diagrams is an acyclic f-fibration if and only if for all objects $i \in \mathcal{I}$, the induced map $Y_i \longrightarrow Z_i \times_{M_iZ} M_iY$ is an acyclic fibration in $\mathcal{C}_i$.

(4) If $\mathfrak{B}$ is a proper bundle, the f-structure is proper. □

Remark 7.6. In fact, it is possible to construct a model structure on $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$ if $\mathcal{I}$ is a Reedy category and $\mathfrak{B}$ consists of complete and cocomplete model categories. One has to combine the construction of the c-structure and the f-structure. The weak equivalences are pointwise weak equivalences, the fibrations and cofibrations are more complicated to define. In the case of diagram categories, this is done in section 5.2 of [Ho], and the proof given there applies to our situation as well. The authors decided to restrict attention to the simpler case.

8. The g-Structure

In this section we consider cofibrantly generated model structures. (The dual case of fibrantly generated structures seems to be irrelevant in practice, hence is omitted from the discussion.) Terminology is taken from [Ho].

Definition 8.1. An $\mathcal{I}$-bundle $\mathfrak{B}$ of cocomplete model categories is called a *cofibrantly generated adjunction bundle* if for all objects $i \in \mathcal{I}$ the model category $\mathcal{C}_i$ is cofibrantly generated.

Examples of cofibrantly generated adjunction bundles include the spectrum bundle Sp of example 2.8 and the projective space bundle $\mathfrak{P}^n(G)$ of 4.6. The inverse image of a cofibrantly generated adjunction bundle is cofibrantly generated.

Since $\mathcal{C}_i$ has all colimits, the i-th evaluation functor $Ev_i \colon \mathbf{Tw}(\mathcal{I}, \mathfrak{B}) \longrightarrow \mathcal{C}_i$ has a left adjoint $Fr_i \colon \mathcal{C}_i \longrightarrow \mathbf{Tw}(\mathcal{I}, \mathfrak{B})$, the i-th free twisted diagram functor obtained by twisted left KAN extension (theorem 3.1). Explicitly, for an object A of $\mathcal{C}_i$ the j-component of $Fr_i(A)$ is given by the coproduct

$$\coprod_{\alpha \in \hom_{\mathcal{I}}(i,j)} F_\alpha(A)$$

and the structure maps are given in the following way: if $\beta \colon j \longrightarrow k$ is a morphism in $\mathcal{I}$, the map $Fr_i(A)^\sharp_\beta$ is the composition

$$\begin{aligned} F_\beta\big(Fr_i(A)_j\big) = F_\beta\Big(\coprod_{\alpha \in \hom_{\mathcal{I}}(i,j)} F_\alpha(A)\Big) \cong \coprod_{\alpha \in \hom_{\mathcal{I}}(i,j)} F_\beta \circ F_\alpha(A) &\cong \coprod_{\alpha \in \hom_{\mathcal{I}}(i,j)} F_{\beta\circ\alpha}(A) \\ &\longrightarrow \coprod_{\gamma \in \hom_{\mathcal{I}}(i,k)} F_\gamma(A) \end{aligned}$$

where the last map is the canonical map induced by the identity on each summand, mapping the α-summand of the source into the $\beta \circ \alpha$-summand of the target.

Define M to be the set of maps in $\mathbf{Tw}(\mathcal{I}, \mathfrak{B})$ of the form $Fr_i(f)$ with i some object of $\mathcal{I}$ and f a generating cofibration in $\mathcal{C}_i$. Define N to be the set of maps in $\mathbf{Tw}(\mathcal{I}, \mathfrak{B})$ of the form $Fr_i(f)$, with i some object of $\mathcal{I}$ and f a generating acyclic cofibration in $\mathcal{C}_i$. Note that M and N are sets because $\mathcal{I}$ is small.

Definition 8.2. *(g-structure)*

Let $f \colon Y \longrightarrow Z$ be a map in $\mathbf{Tw}(\mathcal{I}, \mathfrak{B})$. We call f a *weak equivalence* if f_i is a weak equivalence in $\mathcal{C}_i$ for every object $i \in \mathcal{I}$. We call f a *g-fibration* if f has the right lifting property with respect to the set N. We call f a *g-cofibration* if f has the left lifting property with respect to every g-fibration which is also a weak equivalence.

Lemma 8.3. *A map has the right lifting property with respect to the set N (resp. M) if and only if all its components are fibrations (resp. acyclic fibrations).*

Proof. This follows from the adjointness of Fr_i and Ev_i, and the fact that $\mathfrak{B}$ is cofibrantly generated. □

Lemma 8.4. *The domains of the maps of M are small relative to M-cell. The domains of the maps of N are small relative to N-cell.*

Proof. This follows from the adjointness of Fr_i and Ev_i, and the fact that $\mathfrak{B}$ is cofibrantly generated. We give a detailed argument for the case of M. Let A be the domain of a map in M, so A is of the form $Fr_i(X)$ for some $i \in \mathcal{I}$, with X being the domain of a generating cofibration in $\mathcal{C}_i$. Denote the set of generating cofibrations in $\mathcal{C}_i$ by J and recall that X is κ-small relative to the class J-cell for some cardinal κ, because $\mathcal{C}_i$ is cofibrantly generated. We will prove that $A = Fr_i(X)$ is κ-small relative to the class M-cell.

Let λ be a κ-filtered ordinal and $B\colon \lambda \longrightarrow \mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$ be a functor such that the map $B_\beta \longrightarrow B_{\beta+1}$ is in M-cell for all β with $\beta + 1 < \lambda$. We have to prove that the canonical map

$$\operatorname{colim} \mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})(A, B_\beta) \longrightarrow \mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})(A, \operatorname{colim} B)$$

is an isomorphism. The adjointness of Fr_i and Ev_i provides that this map is isomorphic to the composite

$$\operatorname{colim} \mathcal{C}_i(X, Ev_i(B_\beta)) \longrightarrow \mathcal{C}_i(X, Ev_i \circ \operatorname{colim} B) \cong \mathcal{C}_i(X, \operatorname{colim} Ev_i \circ B)$$

(where the isomorphism is the one from lemma 2.10). This composite is the canonical map, and X is κ-small relative to J-cell. By [Ho, 2.1.16], X is then even κ-small relative to the class of cofibrations in $\mathcal{C}_i$. Hence we are done if for all β with $\beta + 1 < \lambda$ the map $Ev_i(b)\colon Ev_i(B_\beta) \longrightarrow Ev_i(B_{\beta+1})$ is a cofibration. However, since the maps in M are in particular pointwise cofibrations, and the class of pointwise cofibrations is closed under cobase changes and transfinite compositions, every map in M-cell is a pointwise cofibration. This finishes the proof. $\square$

Theorem 8.5. *Let $\mathfrak{B}$ be a cofibrantly generated bundle over $\mathcal{I}$. The g-structure is a model structure on $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$ which is cofibrantly generated by the sets M and N.*

Proof. We use Theorem 2.1.19 of [Ho], which applies also for model categories in the sense of [DS]. The weak equivalences clearly define a subcategory which is closed under retracts and satisfies **MC2**, so condition 1 holds. Lemma 8.4 implies conditions 2 and 3, and lemma 8.3 implies conditions 5 and 6, and one half of condition 4. It remains to prove that every map in N-cell is a weak equivalence. Since every map in N is pointwise an acyclic cofibration, and the class of pointwise acyclic cofibrations is closed under pushouts and transfinite compositions, every map in N-cell is pointwise an acyclic cofibration, so in particular a weak equivalence. $\square$

Remark 8.6. The g-structure coincides with the c-structure provided both are defined. This is true because both have the same classes of weak equivalences and fibrations.

9. Sheaves and Homotopy Sheaves

Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. If $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a left Quillen functor (*i.e.*, F preserves cofibrations and acyclic cofibrations, and F has a right adjoint U), we obtain ([DS, 9.5 and 9.7]) adjoint functors $\mathbf{L}F\colon \operatorname{Ho}\mathcal{C} \longrightarrow \operatorname{Ho}\mathcal{D}$ and $\mathbf{R}U\colon \operatorname{Ho}\mathcal{D} \longrightarrow \operatorname{Ho}\mathcal{C}$, called *total left derived of* F and *total right derived of* U, respectively.

Let $\operatorname{Ho}_f \mathcal{C}$ denote the full subcategory of $\operatorname{Ho}\mathcal{C}$ spanned by the fibrant objects. The inclusion $\iota_{\mathcal{C}}$ is an equivalence of categories. Since U is a right Quillen functor, the image of $\operatorname{Ho}_f \mathcal{D}$ under $\mathbf{R}U$ is contained in $\operatorname{Ho}_f \mathcal{C}$. Thus we obtain, by restriction, a functor $\operatorname{Ho}_f \mathcal{D} \longrightarrow \operatorname{Ho}_f \mathcal{C}$ which will be called $\mathbf{R}_f U$. Define $\mathbf{L}_f F := (\iota_{\mathcal{D}})^{-1} \circ \mathbf{L}F \circ \iota_{\mathcal{C}}$ where $(\iota_{\mathcal{D}})^{-1}$ is an inverse of the equivalence $\iota_{\mathcal{D}}$. In particular, $\mathbf{L}_f F$ is left adjoint to $\mathbf{R}_f U$.

By the construction of total derived functors as given in proposition 9.3 of [DS], we have $\mathbf{R}_f(U_1 \circ U_2) = \mathbf{R}_f U_1 \circ \mathbf{R}_f U_2$ and $\mathbf{R}_f(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}_{\operatorname{Ho}_f \mathcal{C}}$ (for this it is important to choose the Dwyer-Spalinski model for total derived functors and to restrict to fibrant objects). Thus the following definition makes sense:

Definition 9.1. *(Associated homotopy bundle.)*
If $\mathfrak{B} = (\mathcal{C},\ F,\ U)$ is an $\mathcal{I}$-indexed adjunction bundle of model categories, we define its *associated homotopy bundle of fibrant objects* $\operatorname{Ho}_f \mathfrak{B} = (\operatorname{Ho}_f \mathcal{C},\ \mathbf{L}_f F,\ \mathbf{R}_f U)$ as the $\mathcal{I}$-indexed adjunction bundle given by $i \mapsto \operatorname{Ho}_f \mathcal{C}_i$ for objects $i \in \mathcal{I}$ and $\sigma \mapsto \mathbf{L}_f F_\sigma$ and $\sigma \mapsto \mathbf{R}_f U_\sigma$ for morphisms $\sigma \in \mathcal{I}$.

The idea of this definition is that a twisted diagram Y with coefficients in $\mathfrak{B}$ gives rise to a twisted diagram $h(Y)$ with coefficients in $\operatorname{Ho}_f \mathfrak{B}$ having the same components, but structure maps of $h(Y)$ corresponding to homotopy classes of structure maps of Y. In detail, we can prove:

Proposition 9.2. *Let $\mathfrak{B}$ denote an $\mathcal{I}$-indexed adjunction bundle of model categories. Assume that $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$ can be equipped with a model structure with pointwise weak equivalences (this is certainly the case if $\mathcal{I}$ is locally direct or locally inverse, or if $\mathfrak{B}$ is cofibrantly generated).*

(1) Suppose that for all $i \in \mathcal{I}$, all objects of $\mathcal{C}_i$ are fibrant. Then there exists a functor $h\colon \mathbf{Tw}\,(\mathcal{I}, \mathfrak{B}) \longrightarrow \mathbf{Tw}\,(\mathcal{I}, \operatorname{Ho}_f \mathfrak{B})$ which maps weak equivalences to isomorphisms, hence descends to a functor $\hbar\colon \operatorname{Ho}\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B}) \longrightarrow \mathbf{Tw}\,(\mathcal{I}, \operatorname{Ho}_f \mathfrak{B})$.

(2) Suppose that fibrant objects of $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$ are pointwise fibrant (i.e., c-fibrant), and that $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$ admits functorial factorization of morphisms. Then there is a functor $h\colon \mathbf{Tw}\,(\mathcal{I}, \mathfrak{B}) \longrightarrow \mathbf{Tw}\,(\mathcal{I}, \operatorname{Ho}_f \mathfrak{B})$ which maps weak equivalences to isomorphisms, hence descends to a functor $\hbar\colon \operatorname{Ho}\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B}) \longrightarrow \mathbf{Tw}\,(\mathcal{I}, \operatorname{Ho}_f \mathfrak{B})$.

(3) Suppose that fibrant objects of $\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B})$ are pointwise fibrant (i.e., c-fibrant). Then there is a functor $\hbar\colon \operatorname{Ho}\mathbf{Tw}\,(\mathcal{I}, \mathfrak{B}) \longrightarrow \mathbf{Tw}\,(\mathcal{I}, \operatorname{Ho}_f \mathfrak{B})$.

Proof. (1). We know that $\operatorname{Ho}_f \mathcal{C}_i = \operatorname{Ho}\mathcal{C}_i$ and $\mathbf{R}_f U = \mathbf{R}U$ in this case. Let $\gamma_i : \mathcal{C}_i \longrightarrow \operatorname{Ho}\mathcal{C}_i$ denote the canonical functor. Let $\sigma : i \longrightarrow j$ be a morphism in $\mathcal{I}$.

Since U_σ preserves all weak equivalences by Brown's lemma ([DS, 9.9]), we have the equality $\mathbf{R}U_\sigma \circ \gamma_j = \gamma_i \circ U_\sigma$ by the remark preceding [DS, 9.3]. Hence given an object $Y \in \mathbf{Tw}(\mathcal{I}, \mathfrak{B})$, the assignment

$$i \mapsto \gamma_i(Y_i), \quad (\sigma : i \longrightarrow j) \mapsto \gamma_i(y_\sigma^\flat)$$

defines a twisted diagram $h(Y) \in \mathbf{Tw}(\mathcal{I}, \mathrm{Ho}_f \mathfrak{B})$. Clearly h is a functor. Moreover, by construction it maps weak equivalences to isomorphisms. By the universal property of the homotopy category, h induces a functor $\hbar : \mathrm{Ho}\, \mathbf{Tw}(\mathcal{I}, \mathfrak{B}) \longrightarrow \mathbf{Tw}(\mathcal{I}, \mathrm{Ho}_f \mathfrak{B})$.

(2). For a map $\sigma : i \longrightarrow j$ in $\mathcal{I}$, we know that $U_\sigma : \mathcal{C}_j \longrightarrow \mathcal{C}_i$ preserves weak equivalences between fibrant objects. By the construction of total derived functors, and by the remark preceding [DS, 9.3] we conclude that

$$\mathbf{R}_f U_\sigma \circ \gamma_j|_{\mathcal{C}_j^f} = \gamma_i \circ U_\sigma|_{\mathcal{C}_j^f} \tag{$*$}$$

where $\mathcal{C}_j^f$ denotes the full subcategory of fibrant objects of $\mathcal{C}_j$.

Let $(\,\cdot\,)^f$ denote the fibrant replacement functor in $\mathbf{Tw}(\mathcal{I}, \mathfrak{B})$. It preserves all weak equivalences, and maps twisted diagrams to pointwise fibrant twisted diagrams. Using $(*)$, we conclude that the assignment

$$i \mapsto \gamma_i(Y_i^f), \quad (\sigma : i \longrightarrow j) \mapsto \gamma_i((y^f)_\sigma^\flat)$$

defines an object of $\mathbf{Tw}(\mathcal{I}, \mathrm{Ho}_f \mathfrak{B})$ (here γ_i is the localization functor $\mathcal{C}_i \longrightarrow \mathrm{Ho}\, \mathcal{C}_i$). This construction is functorial and maps weak equivalences to isomorphisms, thus induces a functor $\hbar$ on the homotopy category of $\mathbf{Tw}(\mathcal{I}, \mathfrak{B})$.

(3). Let $\mathfrak{T}$ denote the category with objects the fibrant and cofibrant twisted diagrams in $\mathbf{Tw}(\mathcal{I}, \mathfrak{B})$, and morphisms the homotopy classes of maps between such objects. By [DS, 5.6] the map $\nu : \mathfrak{T} \longrightarrow \mathrm{Ho}\, \mathbf{Tw}(\mathcal{I}, \mathfrak{B})$ is an equivalence of categories. Thus it suffices to construct a functor $\phi : \mathfrak{T} \longrightarrow \mathbf{Tw}(\mathcal{I}, \mathrm{Ho}_f \mathfrak{B})$; then we can define $\hbar$ by the composition of an inverse of ν with ϕ.

Given an object $Y \in \mathfrak{T}$, we define $\phi(Y)$ by the assignment

$$i \mapsto \gamma_i(Y_i), \quad (\sigma : i \longrightarrow j) \mapsto \gamma_i(y_\sigma^\flat)\ .$$

This yields an object of $\mathbf{Tw}(\mathcal{I}, \mathrm{Ho}_f \mathfrak{B})$ by an argument similar to the one used in (2) (restrict the functors $\mathbf{R}U_\sigma$ to fibrant objects).

A morphism $f : Y \longrightarrow Z$ in $\mathfrak{T}$ can be represented by a map $\bar{f} : Y \longrightarrow Z$ in $\mathbf{Tw}(\mathcal{I}, \mathfrak{B})$ by [DS, 5.7], and $\bar{f}$ induces a morphism $\phi(f) : \phi(Y) \longrightarrow \phi(Z)$ with components $\phi(f)_i = \gamma_i \bar{f}_i$ using the notation of (2). We have to show that this definition is independent of the choice of $\bar{f}$. Recall that homotopy is an equivalence relation for maps $Y \longrightarrow Z$ by [DS, 4.22]. Moreover, the evaluation functors Ev_i commute with products and preserve weak equivalences. Hence they preserve path objects and right homotopies. Thus if $\bar{f}$ and $\bar{g}$ are homotopic, so are $\bar{f}_i$ and $\bar{g}_i$ which proves that $\phi(f)$ is well defined. Obviously $\phi(f)$ is a map of twisted diagrams.

Since homotopy is compatible with composition ([DS, 4.11, 4.19]), and since the identity morphisms in $\mathfrak{T}$ are represented by identity maps, ϕ is a functor as required.

□

Definition 9.3. *(Left strict sheaves.)*

Given an $\mathcal{I}$-indexed adjunction bundle $\mathfrak{B}$, we call an object $Y \in \mathbf{Tw}(\mathcal{I}, \mathfrak{B})$ a *left strict sheaf* if the $\sharp$-type structure map $y^\sharp_\sigma : F_\sigma(Y_i) \longrightarrow Y_j$ is an isomorphism for all morphisms $\sigma : i \longrightarrow j$ of $\mathcal{I}$. We write $\mathfrak{Shv}(\mathcal{I}, \mathfrak{B})$ for the full subcategory of $\mathbf{Tw}(\mathcal{I}, \mathfrak{B})$ generated by left strict sheaves.

There is also a dual notion of a *right strict sheaf* requiring that all $\flat$-type structure maps are isomorphisms.

Example 9.4. *(Quasi-coherent sheaves on toric varieties.)*

Recall the adjunction bundle $\Sigma^{\mathrm{op}} \bowtie_{Rng} Mod$ associated to a toric variety X with fan Σ, cf. 4.4. We claim that the category $\mathfrak{Shv}(\Sigma^{\mathrm{op}}, \Sigma^{\mathrm{op}} \bowtie_{Rng} Mod)$ is equivalent to the category of quasi-coherent sheaves on X. To see this, recall that a cone $\sigma \in \Sigma$ corresponds to an open affine subscheme U_σ of X. Given a quasi-coherent sheaf $\mathcal{F}$, the associated twisted diagram is given by $\sigma \mapsto \mathcal{F}(U_\sigma)$ with $\flat$-type structure maps given by restriction maps. Conversely, a left strict sheaf Y defines quasi-coherent sheaves $\tilde{Y}_\sigma$ on the subschemes U_σ which can be glued via the $\sharp$-type structure maps to give a quasi-coherent sheaf on X. The details are left to the reader.

Definition 9.5. *(Left homotopy sheaves.)*

Suppose that $\mathfrak{B}$ is an adjunction bundle of model categories. We call an object $Y \in \mathbf{Tw}(\mathcal{I}, \mathfrak{B})$ a *left homotopy sheaf* if for all morphisms $\sigma : i \longrightarrow j$ of $\mathcal{I}$ there is an acyclic fibration $\bar{Y}_i \xrightarrow{\sim} Y_i$ in $\mathcal{C}_i$ with $\bar{Y}_i$ cofibrant such that the adjoint to the composite

$$\bar{Y}_i \xrightarrow{\sim} Y_i \xrightarrow{y^\flat_\sigma} U_\sigma(Y_j)$$

is a weak equivalence in $\mathcal{C}_j$. We write $\mathfrak{hShv}(\mathcal{I}, \mathfrak{B})$ for the full subcategory of $\mathbf{Tw}(\mathcal{I}, \mathfrak{B})$ generated by left homotopy sheaves.

Proposition 9.6. *(Comparison of strict sheaves and homotopy sheaves.)*

Let $\mathfrak{B}$ denote an $\mathcal{I}$-indexed adjunction bundles of model categories. Assume that we have a map $\hbar$ as given by one of the cases of 9.2. An object $Y \in \mathbf{Tw}(\mathcal{I}, \mathfrak{B})$ is a left homotopy sheaf if and only if $\hbar(Y) \in \mathbf{Tw}(\mathcal{I}, \mathrm{Ho}_f\, \mathfrak{B})$ is a left strict sheaf. In particular, if $Y \xrightarrow{\sim} Z$ is a weak equivalence of twisted diagrams, Y is a left homotopy sheaf if and only if Z is.

Proof. Fix a morphism $\sigma : i \longrightarrow j$ of $\mathcal{I}$, and define $Z := \hbar(Y)$. By construction, $z^\flat_\sigma$ is a morphism in $\mathrm{Ho}_f\, \mathcal{C}_i$ which is isomorphic, in $\mathrm{Ho}\, \mathcal{C}_i$, to a morphism $k^\flat : Y_i \longrightarrow \mathbf{R}U_\sigma(Y_j)$. Since we have a commutative diagram of categories

$$\begin{tikzcd}
\mathrm{Ho}_f\, \mathcal{C}_i \arrow[d] & \mathrm{Ho}_f\, \mathcal{C}_j \arrow[l, "\mathbf{R}_f U_\sigma"'] \arrow[d] \\
\mathrm{Ho}\, \mathcal{C}_i & \mathrm{Ho}\, \mathcal{C}_j \arrow[l, "\mathbf{R}U_\sigma"]
\end{tikzcd}$$

where the vertical arrows are equivalences of categories, we know that $z^\sharp_\sigma$ is isomorphic (in $\mathrm{Ho}\,\mathcal{C}_j$) to the adjoint $k^\sharp : \mathbf{L}F_\sigma(Y_i) \longrightarrow Y_j$ of $k^\flat$. In particular, $z^\sharp_\sigma$ is an isomorphism if and only if $k^\sharp$ is.

Choose a cofibrant replacement $q_i : Y_i^c \xrightarrow{\sim} Y_i$ of Y_i and a fibrant replacement $p_j : Y_j \xrightarrow{\sim} Y_j^f$ of Y_j. Let $\ell^\flat$ denote the composite map

$$Y_i^c \xrightarrow[q_i]{\sim} Y_i \xrightarrow{y^\flat_\sigma} U_\sigma(Y_j) \xrightarrow[U_\sigma(p_j)]{} U_\sigma(Y_j^f)\ .$$

By the proof of [DS, 9.7] we know that $k^\flat$ is isomorphic to $\gamma_i(\ell^\flat)$ where $\gamma_i : \mathcal{C}_i \longrightarrow \mathrm{Ho}\,\mathcal{C}_i$ denotes the canonical functor. Similarly, $k^\sharp$ is isomorphic to $\gamma_j(\ell^\sharp)$, where γ_j denotes the canonical functor $\mathcal{C}_j \longrightarrow \mathrm{Ho}\,\mathcal{C}_j$, and $\ell^\sharp$ is adjoint to $\ell^\flat$. In particular, $k^\sharp$ is an isomorphism if and only if $\ell^\sharp$ is a weak equivalence. But $\ell^\sharp$ factors as $F_\sigma(Y_i^c) \longrightarrow Y_j \xrightarrow[p_j]{\sim} Y_j^f$ which shows that $\ell^\sharp$ is a weak equivalence if and only if the homotopy sheaf condition ("at σ") holds for Y.

The second assertion follows immediately since $\hbar$ maps weak equivalences to isomorphisms and the property of being a left strict sheaf is clearly invariant under isomorphism. $\square$

Example 9.7. Recall the adjunction bundle $\mathfrak{P}^n(G)$ from 4.6. This is an adjunction bundle of model categories. The resulting category $\mathfrak{hShv}\,(\langle n\rangle, \mathfrak{P}^n(G))$ is the category $\mathbf{P}^n(G)$ of G-equivariant quasi-coherent sheaves as defined in [Hü, 6.3].

The index category $\langle n\rangle$ is direct with degree function $d(A) := \#A$. Hence the c-structure exists. Moreover, all objects of $\mathbf{Tw}\,(\langle n\rangle, \mathfrak{P}^n(G))$ are c-fibrant. Thus 9.2 (1) applies, and 9.6 shows that the notion of a homotopy sheaf is homotopy invariant ([Hü], corollary 6.5).

10. References

[BF] A. K. Bousfield, E. M. Friedlander: *Homotopy theory of Γ-spaces, spectra, and bisimplicial sets*, Springer Lecture Notes in Mathematics 658, pp. 80–130, Springer-Verlag 1978

[DS] W. Dwyer, J. Spalinski: *Homotopy Theory and Model Categories*, in: Handbook of Algebraic Topology (edited by I. M. James) Elsevier Science B. V., 1995

[HKVWW] T. Hüttemann, J. Klein, W. Vogell, F. Waldhausen, B. Williams: *The "Fundamental Theorem" for the Algebraic K-Theory of Spaces: I*, to appear in J. Pure Appl. Algebra

[Ho] M. Hovey: *Model Categories*, Mathematical Surveys and Monographs 63, American Mathematical Society 1999

[HS] A. Hirschowitz, C. Simpson: *Descentes pour les n-champs*, Preprint arXiv:math.AG/9807049; available via internet from xxx.lanl.gov

[Hü] T. Hüttemann: *Algebraic K-Theory of Non-Linear Projective Spaces*, Preprint 00–045, Universität Bielefeld, SFB 343 (submitted to J. Pure Appl. Algebra)

[ML] S. MacLane: *Categories for the Working Mathematician*, Springer Graduate Texts in Mathematics 5, Springer-Verlag 1971

[O] T. Oda: *Convex Bodies and Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete 15, Springer-Verlag 1988

[Q] D. G. Quillen: *Higher algebraic K-theory: I*, in: Springer Lecture Notes 341, Springer-Verlag 1973, pp. 85-147

[T] P. Taylor: *Commutative Diagrams in TEX*, available via internet (anonymous FTP) from theory.doc.ic.ac.uk as `/tex/contrib/Taylor/tex/diagrams.tex`

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