LONG-TIME BEHAVIOUR AND REGULARITY PROPERTIES OF TRANSITION SEMIGROUPS OF FLEMING-VIOT PROCESSES

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Abstract. Let A be the Feller generator on a compact space and L be the corresponding Fleming-Viot (FV) operator with no selection and no recombination. In this paper we give conditions on A implying that the semigroup (T_t) generated by L (i) converges towards equilibrium with exponential rate (moreover, we determine explicit bounds on the rate of convergence in terms of A), (ii) is hypercontractive, (iii) is strong Feller, and (iv) is compact. We give applications of the last result to the existence of invariant measures for FV-operators with interactive selection.

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1. Introduction

Let S be a compact space and $E := \mathcal{M}_1(S)$ be the space of all probability measures on S equipped with the weak topology. Let A with domain $D(A)$ be the generator of a Feller-semigroup on the space $C(S)$ of continuous functions on S. Throughout the whole paper we will assume that A has a unique invariant probability measure ν and that supp $(\nu) \equiv S$. The Fleming–Viot (FV) operator associated with A with no selection and no recombination is defined as

$$LF(\mu) := \frac{1}{2} \int \int \mu(dx) \left(\delta_x(dy) - \mu(dy)\right) \frac{\partial^2 F}{\partial \delta_x \partial \delta_y}(\mu) + \int \mu(dx) A \left(\frac{\partial F}{\partial \delta_\cdot}\right)(x) \ , F \in \mathcal{F}C_b^\infty(D(A)) \ ,$$

where

$$\frac{\partial F}{\partial \delta_x}(\mu) = \frac{dF}{ds}(\mu + s\delta_x)|_{s=0} \ , F \in \mathcal{F}C_b^\infty(D(A)) \ ,$$

and

$$\mathcal{F}C_b^\infty(D(A)) := \{F = \varphi(\langle f_1, \cdot \rangle, \dots, \langle f_d, \cdot \rangle) | f_i \in D(A), \varphi \in C_b^\infty(\mathbb{R}^d), d \in \mathbb{N}\} \ .$$

Here $\langle f, \mu \rangle := \int f \, d\mu$. It is quite easy to see that the closure of this operator in the space of continuous functions on E generates a Feller semigroup (T_t). (Moreover, the $C_E([0, \infty))$–martingale problem is well–posed (cf. [EK2]).) Consequently, L has a stationary distribution $\Pi \in \mathcal{M}_1(E)$. Since the invariant measure of A is assumed to be unique, it follows that Π is unique too (cf. [EK2]). By general semigroup theory, (T_t) induces uniquely strongly continuous semigroups on all L^p–spaces $L^p(\Pi)$ with corresponding L^p–generators $(L_p, D(L_p))$ extending $(L, \mathcal{F}C_b^\infty(D(A)))$, $p \in [1, \infty)$.

The purpose of this paper now is to investigate the long-time behaviour and regularity properties of (T_t) (and hence of the associated FV-process) in the corresponding L^p–spaces. We will study the following problems:

1) Determine explicit bounds on the exponential rate of speed of convergence towards the unique invariant measure in terms of the mutation (Section 2).

2) Determine regularity properties of the FV–semigroup in dependence of the mutation. We are in particular interested in the question under what assumptions on the mutation A, the FV-semigroup is (i) hypercontractive (Section 3), (ii) strong Feller (Section 4) and (iii) compact (Section 5)).

Concerning Problem (1) we will show that the FV-transition semigroup converges to equilibrium with exponential rate if the same is true in the variational norm for the semigroup generated by the mutation operator (cf. 2.7). This condition is also necessary (cf. 2.8). Moreover, in contrast to previous results (cf. [EK4] and [DK]) we also determine explicit bounds on the rate of convergence for general mutation operators. The main feature of our result is that the rate of convergence w.r.t. the L^∞–norm is precisely the same as the rate of convergence of the semigroup genrated by the corresponding mutation. We emphasize that we use the invariant measure Π to obtain this more refined result.

The example of parent independent mutation (PIM) (i.e., $Af(x) = \frac{\theta}{2} \int f(y) - f(x) \, \nu(dy)$, $\theta > 0$, (cf. Example 6.b))) shows that FV-transition semigroups do not have a regularizing effect in general. In fact, it was the result obtained in [St2] concerning the hypercontractivity of FV-transition semigroups in the PIM-case that was the starting point for a further detailed analysis which led to this paper. It was shown in [St2] that the semigroup (T_t) generated by the FV-operator with parent-independent mutation is hypercontractive if and only if $|\mathrm{supp}(\nu)| = \infty$. We will generalize one half of this result: We prove in 3.1 that the transition semigroups generated by FV-operators with bounded mutation have no regularizing effect whatsoever if $|\mathrm{supp}(\nu)| = \infty$ (here ν denotes again the invariant measure of A).

As a positive result we will show in 3.1, in the case where the semigroup (p_t) generated by the mutation is ultracontrative (U) and the corresponding densities satisfy a certain lower bound (L), the existence of positive constants c_1 and c_2 for which

$$(0.1) \qquad \|T_t F\|_{L^\infty} \le e^{c_1(1+\frac{1}{t^3})} \|e^{|F|}\|_{L^1(\Pi)} \ , t > 0 \ , F \in L^\infty(\Pi) \ ,$$

and

$$(0.2) \qquad \int (T_t F)^2 \log \left(\frac{(T_t F)^2}{\|T_t F\|^2_{L^2(\Pi)}} \right) d\Pi \le e^{c_2(1+\frac{1}{t^3})} \|F\|^2_{L^2(\Pi)} \ , t > 0, F \in L^2(\Pi) \ .$$

Clearly, (0.1) is a weak form of ultracontractivity, whereas (0.2) is a weak form of hypercontractivity of (T_t) (cf. [Ba], [Da]).

In Section 4 we study the strong Feller property of (T_t) in the case where S is a compact subset of $\mathbb{R}^d$. Again the example of the parent independent mutation shows that some additional assumption on the mutation is clearly needed. Assuming (U) and (L) again (cf. Section 3) and in addition a certain regularity assumption (F) on the density of (p_t), which implies that p_t, $t > 0$, maps bounded measurable functions into Lipschitz continuous functions, we will prove in 4.2 that (T_t) is strong Feller. Moreover, we will show that under these assumptions (T_t) maps bounded functions into functions that are Lipschitz w.r.t. a metric that is compatible with the weak topology on E. All three assumptions are for example satisfied if A is the Laplacian on $[0,1]^d$ with periodic boundary conditions (cf. 4.1 below). We will consider in Example 6.c) also general diffusion operators on the d–dimensional torus.

The result on Lipschitz continuity, together with the uniform integrability of the set $\{(T_t F)^2 \,|\, \|F\|_{L^2(\Pi)} \le 1\}$, which follows from (0.2), easily implies the compactness of (T_t) on $L^2(\Pi)$ (cf. 5.1), so that Fredholm perturbation theory is available to prove existence of invariant probability measures for FV-operators with general bounded interactive selection (cf. 5.5). FV-operators with interactive selection are given as first-order perturbations of the "neutral" model (i.e., the model without selection and recombination) (cf. Section 5). We emphasize that, since the underlying "neutral" FV-operators are not symmetrizable in the corresponding L^2-spaces in general, and since the selection need not to be of gradient-type, the existence of such measures cannot be deduced from previous results (cf. [EK3], [ORS]). Moreover, even in the case where the selection is of gradient type, an explicit formula for the invariant measure, as it is known in the case where the "neutral" FV-operator is symmetrizable (cf. [EK3]), is not known in the general case.

In Section 6 we will discuss three particular classes of FV-processes in more detail: FV-processes with

a) first-order mutation,

b) parent-independent mutation,

c) mutations induced by diffusions on $\mathbb{T}^d$.

Finally, let us make two remarks concerning the techniques used in our paper. First, the main tool in our analysis of the FV-transition semigroup is the analysis of the family of corresponding Moran-processes (cf. Section 2). By the law of large numbers, every common invariant of the family of n–particle Moran processes can be lift up to some analogous invariant of the corresponding FV-process (cf. 2.4 and 3.3). The n–particle Moran process associated with A is just an n–independent particle process, with each particle undergoing an A–motion, together with a pair–interaction called "sampling–replacement". It is essentially the influence of this pair–interaction that is responsible for new effects concerning the regularity properties of the corresponding FV-transition semigroups, which are different from the known results on hypercontractivity for transition semigroups of Ornstein–Uhlenbeck type processes (cf. [Ba]). Second, in Section 5 we present an abstract method of modelling first–order perturbations of (non-symmetric) FV-operators in L^2–spaces w.r.t. an invariant measure. We emphasize that, using the theory of generalized Dirichlet forms (cf. [St1]), it is possible to construct associated strong Markov processes with cadlag paths (in fact standard processes), which are solutions of the corresponding $D_E([0,\infty))$–martingale problem. We emphasize again at this point that our methods depend crucially on the invariant measure for the FV-transition semigroup.

2. Exponential rate of convergence towards equilibrium

Let X be a compact space and L be a Feller generator on X, i.e., the generator of a Markovian C_0–semigroup (T_t) on the space $C(X)$ of continuous functions on X. Since X is compact, (T_t) can be uniquely represented by a semigroup (p_t) of probability kernels on X, i.e., $T_t f(x) = \int f(y) p_t(x,dy)$, $x \in X$, and therefore, $f \mapsto T_t f$ can be uniquely extended to a linear operator on $\mathcal{B}_b(X)$, i.e., the space of all bounded measurable functions on X. We will denote this extension again by T_t. If m is an invariant probability measure for (T_t) it follows from Jensen's inequality that

$$\left(\int |T_t f|^p \, dm\right)^{\frac{1}{p}} \le \left(\int T_t(|f|^p)\, dm\right)^{\frac{1}{p}} = \left(\int |f|^p \, dm\right)^{\frac{1}{p}} , p \in [1,\infty) ,$$

so that (T_t) uniquely induces C_0–semigroups of contractions on the corresponding L^p–spaces $L^p(m)$, $p \in [1,\infty)$. Denote by $(L_p, D(L_p))$ the corresponding generators. We will sometimes call L_p also the *L^p-realization of L.*

Definition 2.1. *Let $p \in [1,\infty]$. The semigroup (T_t) converges to the invariant distribution m in $L^p(m)$ with exponential rate $\lambda_0 > 0$ if there exists $M_p \ge 1$ such that $\|T_t F - \langle F\rangle\|_{L^p(m)} \le M_p e^{-\lambda_0 t}$, $t \ge 0$, for all $F \in L^p(m)$. Here, $\langle F\rangle = \int F\, dm$.*

Remark. Let $L_0^p(m) := \{f \in L^p(m) | f \perp 1\}$. In the case where (T_t) is symmetric it follows that $\|T_t\|_{L_0^2(m)} = \lim_{n\to\infty} \|T_t^n\|_{L_0^2(m)}^{\frac{1}{n}}$ (cf. [ReSiI, VI.6]). Consequently,

$$\|T_t\|_{L_0^2(m)} = \lim_{n\to\infty} \|T_{nt}\|_{L_0^2(m)}^{\frac{1}{n}} = \lim_{n\to\infty} \|T_{nt}\|_{L_0^2(m)}^{\frac{1}{n}} \le M_2^{\frac{1}{n}} e^{-\lambda_0 t} = e^{-\lambda_0 t} ,$$

which implies that in the symmetric case the constant M_2 in the above definition can always be chosen to be 1.

The purpose of this section now is to calculate explicit bounds on the exponential rate of speed of convergence of semigroups generated by FV-operators with no selection and no recombination in terms of the corresponding quantities of the underlying mutation operator. To this end let S be a compact space and A be the generator of a Feller semigroup on $C(S)$. Denote by L the associated FV-operator.

The n–particle Moran process associated with A.

Let (p_t) be the semigroup generated by $(A, D(A))$. For $n \in \mathbb{N}$ denote by $(A^{(n)}, D(A^{(n)}))$ the generator of the n-fold product $(p_t^{(n)})$ of (p_t). Clearly, $(p_t^{(n)})$ is just the transition semigroup of an n–particle process on S^n, with each particle undergoing an A–motion independently of each other. The n–particle Moran process is then obtained by introducing a pair-interaction which is called "sampling replacement". Its generator $(L^{(n)}, D(A^{(n)}))$ is given as follows:

$$L^{(n)}f(x) := A^{(n)}f(x) + \sum_{1\le i<j\le n} \phi_{ij}^{(n)} f(x_1, \dots, x_{n-1}) - f(x) \ , f \in D(A^{(n)}) \ ,$$

where $\phi_{ij}^{(n)} : \mathcal{B}(S^n) \to \mathcal{B}(S^{n-1})$, $\phi_{ij}^{(n)} f(x_1, \dots, x_{n-1}) = f(x_1, \dots, x_{j-1}, x_i, x_j, \dots, x_{n-1})$. (Note that $L^{(1)} = A$.) Consequently, the operator $(L^{(n)}, D(A^{(n)}))$ can be seen as the sum of the operators $A^{(n)}f - \binom{n}{2}f$, $f \in D(A^{(n)})$, and $\sum_{1\le i<j\le n} \phi_{ij}^{(n)} f(x_1, \dots, x_{n-1})$, $f \in C(S^n)$, which depends on one variable less. Since $f \mapsto \sum_{1\le i<j\le n} \phi_{ij}^{(n)} f(x_1, \dots, x_{n-1})$ is bounded on $C(S^n)$ it follows that $(L^{(n)}, D(A^{(n)}))$ generates a C_0–semigroup $(q_t^{(n)})$ on $C(S^n)$ as well. Clearly, $(q_t^{(n)})$ is Markovian again, since $L^{(n)}1 = 0$, hence $q_t^{(n)}1 = 1$, and $L^{(n)}f(x_0) \le 0$, where $f(x_0) = \max_{x\in S^n} f(x)$.

Lemma 2.2. *Let $(q_t^{(n)})$ be the transition semigroup of the n–particle Moran process associated with A. Then:*

(i) $q_t^{(n)} f = q_t^{(n-1)} f$, $f \in \mathcal{B}_b(S^{n-1})$.

(ii) The following recursive formula holds:

$q_t^{(1)} f = p_t^{(1)} f$, $f \in \mathcal{B}_b(S)$, $t \ge 0$,

$q_t^{(n)} f = e^{-\binom{n}{2}t} p_t^{(n)} f + \sum_{1\le i<j\le n} \int_0^t e^{-\binom{n}{2}s} q_{t-s}^{(n-1)} \left(\phi_{ij}^{(n)} p_s^{(n)} f\right) ds$, $f \in \mathcal{B}_b(S^n)$, $t \ge 0$.

Proof. (i) First ssuppose that $g \in D(A^{(n-1)})$. Then $g \in D(A^{(n)})$ too and $A^{(n)}g(x_1, \dots, x_n) = A^{(n-1)}g(x_1, \dots, x_{n-1})$. Since $\phi_{ij}^{(n)} g(x_1, \dots, x_n) = \phi_{ij}^{(n-1)} g(x_1, \dots, x_{n-1})$ if $1 \le i < j \le n-1$, and $= g(x_1, \dots, x_{n-1})$ if $j = n$, it follows that $L^{(n)}g(x_1, \dots, x_n) = L^{(n-1)}g(x_1, \dots, x_{n-1})$. In particular, if $f \in C(S^{n-1})$, hence $(\lambda - L^{(n-1)})^{-1} f \in D(A^{(n-1)})$ for all $\lambda > 0$, and $(\lambda - L^{(n)})(\lambda - L^{(n-1)})^{-1} f = f$, it follows that

$$\int_0^\infty e^{-\lambda t} q_t^{(n-1)} f \, dt = (\lambda - L^{(n-1)})^{-1} f = (\lambda - L^{(n)})^{-1} f = \int_0^\infty e^{-\lambda t} q_t^{(n)} f \, dt$$

for all $\lambda > 0$ and thus $q_t^{(n)} f = q_t^{(n-1)} f$ for all t by the uniqueness of the Laplace-transform. Clearly, the last equality now extends to all $f \in \mathcal{B}_b(S^n)$.

(ii) Clearly, $L^{(1)} = A$ implies $q_t^{(1)} f = p_t f$. For the proof of the second equality fix $f \in \mathcal{B}_b(S^n)$. By Duhamels formula

$$q_t^{(n)} f = e^{-t\binom{n}{2}} p_t^{(n)} f + \sum_{1\le i<j\le n} \int_0^t e^{-s\binom{n}{2}} q_{t-s}^{(n)} \phi_{ij}^{(n)} p_s^{(n)} f \, ds \ .$$

Since $\phi_{ij}^{(n)} f \in \mathcal{B}_b(S^{n-1})$ (1) now implies that $q_{t-s}^{(n)}\phi_{ij}^{(n)} p_s^{(n)} f = q_{t-s}^{(n-1)}\phi_{ij}^{(n)} p_s^{(n)} f$ which implies the asssertion. $\square$

For simplicity let us introduce the following notation: If $f \in \mathcal{B}_b(S^n)$ let

$$\ell_f : \mathcal{M}_1(S) \to \mathbb{R}, \mu \mapsto \langle f, \mu^n \rangle \ .$$

The semigroup generated by the associated FV-operator with mutation A can be calculated with the help of the transition semigroups of the n–particle Moran processes as follows:

Lemma 2.3. *Let (T_t) be the semigroup generated by the FV-operator L with mutation A. Then $T_t \ell_f(\mu) = \langle q_t^{(n)} f, \mu^n \rangle$, $\mu \in \mathcal{M}_1(S)$, $t \geq 0$, $f \in \mathcal{B}_b(S^n)$, $n \geq 1$.*

Proof. By [DawM, Th. 3.B.2]

$$(\lambda - L)^{-1} F(\mu) = \sum_{k=1}^{n} \langle (\lambda - \binom{k}{2} - A^{(k)})^{-1} S_{k,\lambda}^n f, \mu^k \rangle \ ,$$

for $\lambda > 0$, where

$$S_{k,\lambda}^n f = \bigotimes_{l=k+1}^{n} \left(\sum_{1 \leq i < j \leq l} \phi_{ij}^{(l)} \left(\lambda + \binom{l}{2} - A^{(l)} \right)^{-1} \right) f \ , 1 \leq k \leq n-1 \, , \lambda > 0 \ ,$$

and $S_{n,\lambda}^n f = f$, $\lambda > 0$.

By the uniqueness of the Laplace–transform it is therefore sufficient to show that for $\lambda > 0$

$$\int_0^\infty e^{-\lambda t} q_t^{(n)} f \, dt = \sum_{k=1}^{n} (\lambda + \binom{k}{2} - A^{(k)})^{-1} S_{k,\lambda}^n f \ .$$

We proceed by induction: If $n = 1$ then clearly

$$\int_0^\infty e^{-\lambda t} q_t^{(1)} f \, dt = \int_0^\infty e^{-\lambda t} p_t^{(1)} f \, dt = (\lambda - A^{(1)})^{-1} f \ .$$

Fix now $n \geq 2$ and suppose that the assertion is proved for all $g \in \mathcal{B}_b(S^{n-1})$.

Note that

$$\sum_{k=1}^{n} (\lambda + \binom{k}{2} - A^{(k)})^{-1} S_{k,\lambda}^n f = (\lambda + \binom{n}{2} - A^{(n)})^{-1} f + \sum_{k=1}^{n-1} (\lambda + \binom{k}{2} - A^{(k)})^{-1} S_{k,\lambda}^{n-1} (S_{n-1,\lambda}^n f) \ .$$

It is therefore enough to show that

$$\int_0^\infty e^{-\lambda t} q_t^{(n)} f \, dt = (\lambda + \binom{n}{2} - A^{(n)})^{-1} f + \int_0^\infty e^{-\lambda t} q_t^{(n-1)} (S_{n-1,\lambda}^n f) \, dt \ .$$

But clearly

$$\int_0^\infty e^{-\lambda t} q_t^{(n)} f\, dt = \int_0^\infty e^{-(\lambda+\binom{n}{2})t} p_t^{(n)} f\, dt + \sum_{1\le i<j\le n} \int_0^\infty e^{-\lambda t} \int_0^t e^{-\binom{n}{2}s} q_{t-s}^{(n-1)} (\phi_{ij}^{(n)} p_s^{(n)} f)\, ds\, dt$$

$$= (\lambda + \binom{n}{2} - A^{(n)})^{-1} f + \sum_{1\le i<j\le n} \int_0^\infty \int_0^\infty e^{-\lambda(r+s)} e^{-\binom{n}{2}s} q_r^{(n-1)} (\phi_{ij}^{(n)} p_s^{(n)} f)\, ds\, dr$$

$$= (\lambda + \binom{n}{2} - A^{(n)})^{-1} f + \int_0^\infty e^{-\lambda r} q_r^{(n-1)} \left(\sum_{1\le i<j\le n} \phi_{ij}^{(n)} \left(\int_0^\infty e^{-(\lambda+\binom{n}{2})s} p_s^{(n)} f)\, ds \right)\right) dr$$

$$= (\lambda + \binom{n}{2} - A^{(n)})^{-1} f + \int_0^\infty e^{-\lambda r} q_r^{(n-1)} (S_{n-1,\lambda}^n f)\, dr\ . \quad \square$$

Let Π be an invariant probability measure for the semigroup (T_t) generated by the FV-operator with mutation A. Since

$$\int \langle f, \mu^n\rangle\, \Pi(d\mu) = \int \ell_f\, d\Pi = \int T_t \ell_f\, d\Pi = \int \langle q_t^{(n)} f, \mu^n\rangle\, \Pi(d\mu)$$

for all $f \in \mathcal{B}_b(S^n)$, it follows that the measure $\varrho_n \in \mathcal{M}_1(S^n)$ defined by

$$\int f\, d\varrho_n = \int \langle f, \mu^n\rangle\, \Pi(d\mu)\ ,\ f \in \mathcal{B}_b(S^n)\,,$$

is an invariant measure for the n–particle Moran process associated with A. Consequently, $(q_t^{(n)})$ induces C_0–semigroups on the corresponding L^p–spaces, $p \in [1,\infty)$. The following Proposition is one example that demonstrates how to transfer common properties of the n–particle Moran processes to a similar property of the FV-process with the help of the law of large numbers. For similar transfer procedures see the proof of 3.3 below.

Proposition 2.4. *Let $p \in [1,\infty)$ and M. Suppose that $\|q_t^{(n)} f - \langle f\rangle\|_{L^p(\varrho_n)} \le M\|f\|_{L^p(\varrho_n)}$ for all $f \in L^p(\varrho_n)$. Then*

$$\|T_t F - \langle F\rangle_\Pi\|_{L^p(\Pi)} \le M\|F\|_{L^p(\Pi)}\ ,$$

for all $F \in L^p(\Pi)$.

Proof. Fix $f \in \mathcal{B}_b(S^n)$ and let $S_m f := \frac{1}{m}\sum_{k=1}^m f\circ p_k \in \mathcal{B}_b(S^{nm})$, where $p_k : (S^n)^m \to S^n$ denotes the projection onto the k–th coordinate. Note that $\langle S_m f, \mu^{nm}\rangle = \langle f, \mu^n\rangle$ for all $\mu \in \mathcal{M}_1(S)$ but $\langle |S_m f|^p, \mu^{nm}\rangle \ne \langle |f|^p, \mu^n\rangle$ in general. On the other hand, by the law of large numbers $|\langle |S_m f|^p, \mu^{nm}\rangle - |\langle f, \mu^n\rangle|^p| \to 0$, $m\to\infty$, for all $\mu \in \mathcal{M}_1(S)$. Since $\langle |S_m f|^p, \mu^{nm}\rangle$, $m \ge 1$, is uniformly bounded it follows that $\int \langle |S_m f|^p, \mu^{nm}\rangle\, \Pi(d\mu) \to \int |\langle f, \mu^n\rangle|^p\, \Pi(d\mu)$, $m\to\infty$.

Fix $m \ge 1$. Then

$$\int |T_t \ell_{S_m f} - \langle \ell_{S_m f}\rangle|^p\, d\Pi = \int |\langle q_t^{(nm)} S_m f - \langle \ell_{S_m f}\rangle, \mu^{nm}\rangle|^p\, \Pi(d\mu)$$

$$\le \int \langle |q_t^{(nm)} S_m f - \langle \ell_{S_m f}\rangle|^p, \mu^{nm}\rangle\, \Pi(d\mu) = \int |q_t^{(nm)} S_m f - \langle \ell_{S_m f}\rangle|^p\, d\varrho_{nm}$$

$$\le M^p \int |S_m f|^p\, d\varrho_{nm} = M^p \int \langle |S_m f|^p, \mu^{nm}\rangle\, d\Pi(d\mu)\ .$$

Finally, using Lebesgue's theorem, we obtain that

$$\begin{aligned}\int |T_t \ell_f - \langle \ell_f \rangle|^p \, d\Pi &= \lim_{m\to\infty} \int |T_t \ell_{S_m f} - \langle \ell_{S_m f}\rangle|^p \, d\Pi \\ &\le \lim_{m\to\infty} M^p \int \langle |S_m f|^p, \mu^{nm}\rangle \, \Pi(d\mu) \\ &= M^p \int |\langle f, \mu^n \rangle|^p \, \Pi(d\mu) \, .\end{aligned}$$

Since $\{\ell_f | f \in \mathcal{B}_b(S^n), n \ge 1\} \subset L^p(\Pi)$ dense, the last inequality extends to all $F \in L^p(\Pi)$. $\square$

The following Proposition now gives a sufficient condition under which we can verify the assumptions of 2.4.

Proposition 2.5. *Suppose that $\|p_t g - \langle g \rangle\|_\infty \le M e^{-\lambda_0 t} \|g\|_\infty$ for all $g \in \mathcal{B}_b(S)$ and some $M \ge 1$ and $\lambda_0 \in (0, \infty)$. Let $n_0 = n(\lambda_0)$ be the largest integer n with $\binom{n}{2} \le \lambda_0$. (Note that $n_0 \ge 1$.) Let $f \in \mathcal{B}_b(S^n)$. Then*

$$\|q_t^{(n)} f - \langle \ell_f \rangle\|_\infty \le (2M)^{n \wedge n_0} (n \wedge n_0)! \, e^{\sum_{k=n_0+1}^n \frac{\lambda_0}{\binom{k}{2} - \lambda_0}} e^{-\lambda_0 t} \|f\|_\infty \, , t \ge 0 \, .$$

For the Proof of the Proposition we will need the following Lemma:

Lemma 2.6. *Let (p_t) be as in 2.5 and $g \in \mathcal{B}_b(S^n)$. Then*

$$\|p_t^{(n)} g - \langle g \rangle_{\nu^n}\|_\infty \le n M e^{-\lambda_0 t} \|g\|_\infty \ , t \ge 0 \, .$$

Proof. We will prove the assertion using induction. The statement is clear if $n = 1$. If $n > 1$ let $\hat{g} \in \mathcal{B}_b(S^{n-1})$ be defined by $\hat{g}(x_1, \dots, x_{n-1}) = \int g(x_1, \dots, x_{n-1}, y) \nu(dy)$. Then

$$\begin{aligned}&|p_t^{(n)} g(x_1, \dots, x_n) - p_t^{(n-1)} \hat{g}(x_1, \dots, x_{n-1})| \\ &\quad \le \int \dots \left| \int g(y_1, \dots, y_n) - \int g(y_1, \dots, y_{n-1}, x) \, \nu(dx) \, p_t(x_n, dy_n) \right| \dots p_t(x_1, dy_1) \\ &\quad \le M e^{-\lambda_0 t} \|g\|_\infty \, .\end{aligned}$$

Since $\langle g \rangle_{\nu^n} = \langle \hat{g} \rangle_{\nu^{n-1}}$ and by assumption $\|p_t^{(n-1)} \hat{g} - \langle \hat{g} \rangle_{\nu^{n-1}}\|_\infty \le (n-1) M e^{-\lambda_0 t} \|\hat{g}\|_\infty \le (n-1) M e^{-\lambda_0 t} \|g\|_\infty$, the assertion now follows. $\square$

Proof (of 2.5). For simplicity define

$$d_n(t) := (2M)^{n \wedge n_0} (n \wedge n_0)! \, e^{\sum_{k=n_0+1}^n \frac{\lambda_0}{\binom{k}{2} - \lambda_0}} e^{-\lambda_0 t} \ , t \ge 0.$$

We will prove the assertion using induction. If $n = 1$ then clearly,

$$\|q_t^{(1)} f - \langle \ell_f \rangle\|_\infty = \|p_t^{(1)} f - \langle f, \nu \rangle\|_\infty \le M e^{-\lambda_0 t} \|f\|_\infty \, .$$

Suppose now that the statement is proved for all $g \in \mathcal{B}_b(S^{n-1})$ and $t \geq 0$. The Π–invariance of (T_t) implies that

$$\langle \ell_f \rangle = \langle T_t \ell_f \rangle = e^{-\binom{n}{2}t} \langle \ell_{p_t^{(n)} f} \rangle + \sum_{1 \leq i < j \leq n} \int_0^t e^{-\binom{n}{2}s} \langle \ell_{\phi_{ij}^{(n)} p_s^{(n)} f} \rangle \, ds \; .$$

Hence

$$\begin{aligned}\| q_t^{(n)} f - \langle \ell_f \rangle \|_\infty &\leq e^{-\binom{n}{2}t} \| p_t^{(n)} f - \langle \ell_{p_t^{(n)} f} \rangle \|_\infty \\ &\quad + \sum_{1 \leq i < j \leq n} \int_0^t e^{-\binom{n}{2}s} \| q_{t-s}^{(n-1)} (\phi_{ij}^{(n)} p_s^{(n)} f) - \langle \ell_{\phi_{ij}^{(n)} p_s^{(n)} f} \rangle \|_\infty \, ds \; .\end{aligned}$$

In the following we will consider the two cases $n \leq n_0$ and $n > n_0$ separately. If $n \leq n_0$ then, using 2.6,

$$\begin{aligned}&\| q_{t-s}^{(n-1)} (\phi_{ij}^{(n)} p_s^{(n)} f) - \langle \ell_{\phi_{ij}^{(n)} p_s^{(n)} f} \rangle \|_\infty \\ &\qquad = \| q_{t-s}^{(n-1)} (\phi_{ij}^{(n)} p_s^{(n)} f - \langle f, \nu^n \rangle) - \langle \ell_{\phi_{ij}^{(n)} p_s^{(n)} f - \langle f, \nu^n \rangle} \rangle \|_\infty \\ &\qquad \leq d_{n-1}(t-s) \| \phi_{ij}^{(n)} p_s^{(n)} f - \langle f, \nu^n \rangle \|_\infty \\ &\qquad \leq d_{n-1}(t-s) M n e^{-\lambda_0 s} \| f \|_\infty\end{aligned}$$

for all $s \in [0, t]$ and $1 \leq i < j \leq n$. Consequently,

$$\begin{aligned}\| q_t^{(n)} f - \langle \ell_f \rangle \|_\infty &\leq e^{-\binom{n}{2}t} \| p_t^{(n)} f - \langle \ell_{p_t^{(n)} f} \rangle \|_\infty \\ &\quad + \sum_{1 \leq i < j \leq n} \int_0^t e^{-\binom{n}{2}s} \| q_{t-s}^{(n-1)} (\phi_{ij}^{(n)} p_s^{(n)} f) - \langle \ell_{\phi_{ij}^{(n)} p_s^{(n)} f} \rangle \|_\infty \, ds \\ &\leq e^{-\binom{n}{2}t} \| p_t^{(n)} f - \langle f, \nu^n \rangle \|_\infty + e^{-\binom{n}{2}t} \| \int \langle \langle f, \nu^n \rangle - p_t^{(n)} f, \mu^n \rangle \, \Pi(d\mu) \|_\infty \\ &\quad + (2M)^n n! e^{-\lambda_0 t} \binom{n}{2} \int_0^t e^{-\binom{n}{2}s} \, ds \| f \|_\infty \\ &\leq 2Mn e^{-\lambda_0 t} e^{-\binom{n}{2}t} \| f \|_\infty + (2M)^n n! e^{-\lambda_0 t} (1 - e^{-\binom{n}{2}t}) \| f \|_\infty \\ &\leq d_n(t) \| f \|_\infty \; .\end{aligned}$$

If $n > n_0$ then

$$\begin{aligned}\| q_t^{(n)} f - \langle \ell_f \rangle \|_\infty &\leq e^{-\binom{n}{2}t} \| p_t^{(n)} f - \langle \ell_{p_t^{(n)} f} \rangle \|_\infty \\ &\quad + \sum_{1 \leq i < j \leq n} \int_0^t e^{-\binom{n}{2}s} \| q_{t-s}^{(n-1)} (\phi_{ij}^{(n)} p_s^{(n)} f) - \langle \ell_{\phi_{ij}^{(n)} p_s^{(n)} f} \rangle \|_\infty \, ds \\ &\leq 2 e^{-\binom{n}{2}t} \| f \|_\infty + (2M)^{n_0} n_0! e^{\sum_{k=n_0+1}^{n-1} \frac{\lambda_0}{\binom{k}{2} - \lambda_0}} e^{-\lambda_0 t} \binom{n}{2} \int_0^t e^{-(\binom{n}{2} - \lambda_0)s} \, ds \, \| f \|_\infty \\ &\leq 2 e^{-\binom{n}{2}t} \| f \|_\infty + (2M)^{n_0} n_0! e^{\sum_{k=n_0+1}^{n-1} \frac{\lambda_0}{\binom{k}{2} - \lambda_0}} e^{-\lambda_0 t} \frac{\binom{n}{2}}{\binom{n}{2} - \lambda_0} (1 - e^{-(\binom{n}{2} - \lambda_0)t}) \| f \|_\infty \\ &\leq (2M)^{n_0} n_0! e^{\sum_{k=n_0+1}^{n} \frac{\lambda_0}{\binom{k}{2} - \lambda_0}} e^{-\lambda_0 t} \| f \|_\infty \; ,\end{aligned}$$

where we used $\frac{\binom{n}{2}}{\binom{n}{2}-\lambda_0} = 1 + \frac{\lambda_0}{\binom{n}{2}-\lambda_0} \le e^{\frac{\lambda_0}{\binom{n}{2}-\lambda_0}}$ in the last inequality. $\square$

The following is the main result of this section:

Theorem 2.7. *Let A be the generator of a Feller semigroup (p_t) on $C(S)$ with unique invariant probability measure ν satisfying $supp(\nu) \equiv S$. Let (T_t) be the semigroup generated by the corresponding FV-operator and let Π be its unique invariant probability measure. If $\|p_t g - \langle g\rangle\|_\infty \le Me^{-\lambda_0 t}\|g\|_\infty$ for all $g \in \mathcal{B}_b(S)$ and some $M \ge 1$ and $\lambda_0 \in (0,\infty)$ then*

$$\|T_tF - \langle F\rangle\|_{L^p(\Pi)} \le 2^{\frac{1}{p}} \left((2M)^{n_0}(n_0)!\right)^{\frac{p-1}{p}} e^{\frac{p-1}{p}\sum_{k=n_0+1}^{\infty}\frac{\lambda_0}{\binom{k}{2}-\lambda_0}} e^{-\frac{p-1}{p}\lambda_0 t}\|F\|_{L^p(\Pi)}$$

for all $F \in L^p(\Pi)$, $t \ge 0$, and $p > 1$. Here, as before, $n_0 = n(\lambda_0)$ is the largest integer n with $\binom{n}{2} \le \lambda_0$.

Remark 2.8. (i) Note that the assumption of an exponential rate of convergence to equilibrium in variational norm on the semigroup generated by the mutation operator A is clearly necessary to obtain the same statement for the corresponding FV transition semigroup, since for $f \in \mathcal{B}_b(S)$ $\|\ell_f\|_{L^\infty(\Pi)} = \|f\|_{L^\infty(\nu)}$ and

$$\|T_t\ell_f - \langle \ell_f\rangle\|_{L^\infty(\Pi)} = \|p_tf - \langle f\rangle_\nu\|_{L^\infty(\nu)} .$$

(ii) 2.7 does not follow immediately from 2.5 since in general the supremum norm $\|f\|_\infty$ for $f \in \mathcal{B}_b(S^n)$ is strictly bigger than $\|\ell_f\|_{L^\infty(\Pi)}$ even in the case where Π has full support. As an example consider $\otimes_{i=1}^n 1_{A_i}$, $A_i \in \mathcal{B}(S)\setminus\{\emptyset\}$, pairwise disjoint. Clearly, $\|f\|_\infty = 1$, but $\langle f, \mu^n\rangle \le n^{-n}$ for all $\mathcal{M}_1(S)$, hence $\|\ell_f\|_{L^\infty(\Pi)} \le n^{-n}$. On the other hand, if $f \in C_b^+(S^n)$ is symmetric under permutations, and $x_0 \in S^n$ such that $f(x_0) = \|f\|_\infty$, let $\mu = \frac{1}{n}\sum_{k=1}^n \delta_{x_0^k} \in \mathcal{M}_1(S)$. If Π has full support then

$$\begin{aligned}\|\ell_f\|_{L^\infty(\Pi)} \ge \langle f, \mu^n\rangle = n^{-n} & \sum_{\sigma:\{0,\dots,n\}\to\{0,\dots,n\}} f(x_0^{\sigma(1)},\dots,x_0^{\sigma(n)}) \\ & \ge n!\, n^{-n}\|f\|_\infty \sim \sqrt{2\pi n}e^{-n}\|f\|_\infty\end{aligned}$$

by Sterling's formula.

For the proof of 2.7 we need two Propositions on the general connection between convergence in variational norm and convergence in L^p–spaces.

Proposition 2.9. *Let (X,m) be a probability space, $T: L^1(X,m) \to L^1(X,m)$ be a Markovian contraction and assume that $\|Tf - \langle f\rangle\|_{L^\infty(m)} \le \lambda_0\|f\|_{L^\infty(m)}$ for all $f \in L^\infty(m)$.*

(i) Let $p \ge 1$. Then

$$\|Tf - \langle f\rangle\|_{L^p(m)} \le 2^{\frac{1}{p}}\lambda_0^{\frac{p-1}{p}}\|f\|_{L^p(m)}$$

for all $f \in L^p(m)$.

(ii) Let T be symmetric and $p \ge 1$. Then

$$\|Tf - \langle f\rangle\|_{L^p(m)} \le \lambda_0\|f\|_{L^p(m)}$$

for all $f \in L^p(m)$.

Proof:. Define the linear operator $U : L^1(X,m) \to L^1(X,m)$ by $Uf = Tf - \langle f \rangle$.

(i) Since T is a contraction on $L^1(m)$ it follows that $\|Uf\|_1 \le 2\|f\|_1$. By the Riesz–Thorin Interpolation theorem (cf. [ReSiII, Theorem IX.17]) U can be restricted to a bounded linear operator on $L^p(m)$ with operator norm less than $2^{\frac{1}{p}} \lambda_0^{\frac{p-1}{p}}$ for all $p \ge 1$.

(ii) Let $f \in L^\infty(m)$ and $h := 1_{\{Tf > \langle f \rangle\}} - 1_{\{Tf < \langle f \rangle\}}$. Since T is symmetric it follows that

$$\begin{aligned}\int |Tf - \langle f \rangle| \, dm &= \int (Tf - \langle f \rangle) h \, dm \\ &= \int Tfh \, dm - \langle f \rangle \langle h \rangle = \int f(Th) \, dm - \langle f \rangle \langle h \rangle \\ &= \int f(Th - \langle h \rangle) \, dm \le \|f\|_{L^1(m)} \|Th - \langle h \rangle\|_{L^\infty(m)} \le \lambda_0 \|f\|_{L^1(m)} \, .\end{aligned}$$

It follows that $\|Uf\|_{L^1(m)} \le \lambda_0 \|f\|_{L^1(m)}$ for all $f \in L^1(m)$. By the Riesz–Thorin Interpolation theorem U can now be restricted to a bounded linear operator on $L^p(m)$ with operator norm less than λ_0. $\square$

2.9 and the Remark following 2.1 now immediately imply:

Proposition 2.10. *Let (X,m) be a probability space, (T_t) be a semigroup of Markovian contractions (T_t) on $L^1(m)$. Assume that $\|T_t f - \langle f \rangle\|_{L^\infty(m)} \le M e^{-\lambda t} \|f\|_{L^\infty(m)}$ for all $f \in L^\infty(m)$.*

(i) Let $p \ge 1$. Then

$$\|T_t f - \langle f \rangle\|_p \le 2^{\frac{1}{p}} M^{\frac{p-1}{p}} e^{-\frac{p-1}{p}\lambda t} \|f\|_p$$

for all $f \in L^p(m)$.

(ii) Let (T_t) be symmetric on $L^2(m)$. Then:

(a) $\|T_t f - \langle f \rangle\|_{L^p(m)} \le M e^{-\lambda t} \|f\|_{L^p(m)}$ for all $f \in L^p(m)$, $p \ge 1$.

(b) $\|T_t f - \langle f \rangle\|_{L^2(m)} \le e^{-\lambda t} \|f\|_{L^2(m)}$ for all $f \in L^2(m)$.

Proof (of 2.7). For simplicity let $M_1 := (2M)^{n_0} (n_0)! e^{\sum_{k=n_0+1}^\infty \frac{\lambda_0}{\binom{k}{2} - \lambda_0}}$. Let $\varrho_n \in \mathcal{M}_1(S^n)$ be the invariant measure for the n–particle Moran process associated with A. ϱ_n has full support, since $\int f \, d\varrho_n = 0$, $f \in C(S^n)$, ≥ 0, implies that

$$0 = \lim_{t \to \infty} e^{t\binom{n}{2}} \int q_t^{(n)} f \, d\varrho_n \ge \lim_{t \to \infty} \int p_t^{(n)} f \, d\varrho_n = \int f \, d\nu^n \, ,$$

hence $f = 0$, since ν^n has full support. Since $\|q_t^{(n)} f - \langle f \rangle\|_\infty \le M_1 e^{-\lambda_0 t} \|f\|_\infty$ for all $f \in \mathcal{B}_b(S^n)$ it follows that $\|q_t^{(n)} f - \langle f \rangle\|_{L^\infty(\varrho_n)} \le M_1 e^{-\lambda_0 t} \|f\|_{L^\infty(\varrho_n)}$ for all $f \in C(S^n)$ and subsequently for all $f \in L^\infty(\varrho_n)$. 2.10 now implies that

$$\|q_t^{(n)} f - f\|_{L^p(\varrho_n)} \le 2^{\frac{1}{p}} M_1^{\frac{p-1}{p}} e^{-\frac{p-1}{p}\lambda_0 t} \|f\|_{L^p(\varrho_n)}$$

for all $f \in L^p(\varrho_n)$, $p \in [1,\infty]$. By 2.4 $\|T_tF - \langle F\rangle\|_{L^p(\Pi)} \le 2^{\frac{1}{p}} M_1^{\frac{p-1}{p}} e^{-\frac{(p-1)}{p}\lambda_0 t}\|F\|_{L^p(\Pi)}$ for all $F \in L^p(\Pi)$, $p \in [1,\infty)$.

Finally, let $F \in L^\infty(\Pi)$. Then

$$\begin{aligned}\|T_tF - \langle F\rangle\|_{L^\infty(\Pi)} &= \lim_{p\to\infty} \|T_tF - \langle F\rangle\|_{L^p(\Pi)} \le \lim_{p\to\infty} 2^{\frac{1}{p}} M_1^{\frac{p-1}{p}} e^{-\frac{p-1}{p}\lambda_0 t}\|F\|_{L^p(\Pi)} \\ &= M_1 e^{-\lambda_0 t}\|F\|_{L^\infty(\Pi)} \,. \quad \square\end{aligned}$$

3. Hypercontractivity

The purpose of this section is to study contractive properties of the FV–semigroup in dependence on the mutation operator. Two important concepts have been studied in many cases of Stochastic Analysis: ultracontractivity and hypercontractivity of transition semigroups (cf. [Ba], [Da]) . Recall that ultracontractivity of a Markovian transition semigroup (T_t) on $L^1(m)$, m being a probability measure, means that $T_t : L^1(m) \to L^\infty(m)$ is bounded, $t > 0$, whereas the common use of the notion of hypercontractivity just implies that $T_t : L^2(m) \to L^{2+\varepsilon}(m)$ is bounded for $t > 0$ and for some $\varepsilon > 0$ (which may depend additionally on t). It turns out that even this weaker concept is still too strong for FV-transition semigroups (cf. 3.3 below).

Throughout the whole section let S be a compact space, A a Feller generator on S with unique invariant measure ν and supp $(\nu) \equiv S$. Let (T_t) be the transition semigroup of the corresponding FV-operator. We will state below sufficient conditions on A implying a weak form of ultracontractivity and a weak form of hypercontractivity for the semigroup (T_t) (cf. 3.3). The latter result implies in particular the uniform integrability of the set $\{(T_tF)^2 | \|F\|_{L^2(\Pi)} \le 1\}$ for all $t > 0$. That this cannot be true for general mutation operators is implied by the following negative result:

Theorem 3.1. *Suppose that $|supp\,\nu| = +\infty$. If A is a bounded operator on $L^2(\nu)$ it follows that for all $t > 0$ the set $\{(T_tF)^2 | \|F\|_{L^2(\Pi)} \le 1\}$ is not uniformly integrable.*

Proof. Let (p_t) be the semigroup generated by A and $f \in \mathcal{B}_b(S)$. It follows from [EK2, proof of 5.2] that $\int \langle f,\mu\rangle \Pi(d\mu) = \int f\,d\nu$ and $\int \langle f,\mu\rangle^2 \Pi(d\mu) = \frac{1}{2}\int_0^\infty e^{-t}\|p_tf\|^2_{L^2(\nu)}\,dt$. By Jensen's inequality,

$$\frac{1}{2}\int_0^\infty e^{-t}\|p_tf\|^2_{L^2(\nu)}\,dt \ge \frac{1}{2}\|\int_0^\infty e^{-t}p_tf\,dt\|^2_{L^2(\nu)} = \frac{1}{2}\|(1-A)^{-1}f\|^2_{L^2(\nu)} \,.$$

Since A, and thus $1 - A$ too, is a bounded operator we have that

$$\int f^2\,d\nu = \|(1-A)(1-A)^{-1}f\|^2_{L^2(\nu)} \le \|1-A\|^2\|(1-A)^{-1}f\|^2_{L^2(\nu)} \,.$$

Combining the last two estimates we obtain that

$$\int \langle f,\mu\rangle^2 \Pi(d\mu) \ge \frac{1}{2\|1-A\|^2}\int f^2\,d\nu \,.$$

Since A is bounded on $L^2(\nu)$ e^{tA} can be defined as a bounded operator on $L^2(\nu)$ for all $t \in \mathbb{R}$ and thus

$$\|f\|_{L^2(\nu)} = \|e^{-tA}e^{tA}f\|_{L^2(\nu)} \le e^{t\|A\|}\|e^{tA}f\|_{L^2(\nu)} = e^{t\|A\|}\|p_t f\|_{L^2(\nu)} .$$

Again combining the last two inequalities, we obtain that

$$\int \langle p_t f, \mu\rangle \Pi(d\mu) \ge \frac{e^{-2t\|A\|}}{2\|1-A\|^2}\int f^2\, d\nu . \tag{3.1}$$

Since $|\mathrm{supp}\,\nu| = +\infty$ it follows that $L^2(\nu) \ne L^1(\nu)$. Consequently, there exist $(f_n) \subset L^2(\nu)$ with $\int f_n^2\, d\nu = 1$ but $\int |f_n|\, d\nu \to 0$, $n \to \infty$. Let $F_n(\mu) := \langle f_n, \mu\rangle$, $n \ge 1$. Then $\int (T_t F_n)^2\, d\Pi \le \int \langle (p_t f_n)^2, \mu\rangle \Pi(d\mu) = \int (p_t f_n)^2\, d\nu \le 1$, $n \ge 1$. However, $((T_t F_n)^2)$ is not uniformly integrable, since

$$\int |T_t F_n|\, d\Pi \le \int \langle |p_t f_n|, \mu\rangle\, \Pi(d\mu) = \int |p_t f_n|\, d\nu \le \int |f_n|\, d\nu \to 0, n \to \infty ,$$

whereas on the other hand (3.1) implies that

$$\inf_{n\ge 1} \int (T_t F_n)^2\, d\Pi \ge \inf_{n \ge 1} \frac{e^{-2t\|A\|}}{2\|1-A\|^2}\int f_n^2\, d\nu > 0 . \quad \square$$

Remark 3.2.

(i) The proof of 3.1 shows that the same conclusion still holds true if we can find $f_n \in D(A)$ such that

(a) $\int f_n^2\, d\nu \le 1$, $n \ge 1$,

(b) $\int |f_n|\, d\nu \to 0$ if $n \to \infty$, and

(c) $\inf_{n \ge 1} \int_0^\infty e^{-t}\|p_{t+s} f_n\|^2_{L^2(\nu)}\, dt > 0$, $s > 0$, where (p_t) denotes the semigroup generated by A.

(ii) The last result extends a result obtained in [St2] on the hypercontractivity of the semigroups corresponding to the FV-operator with parent–independent mutation (cf. Example 6.b) below).

Let us now specify conditions on A implying boundedness of $T_t : L(L^2 \log L^2, \Pi) \to L(L^2, \Pi)$, $t > 0$. Denote by (p_t) the semigroup generated by A and let ν be its unique invariant measure. Suppose that (p_t) is ultracontractive. More precisely,

(U) There exist $M, \alpha > 0$ such that $\|p_t f\|_\infty \le M t^{-\alpha}\|f\|_{L^1(\nu)}$ for all $t \in (0, 1]$.

It follows from (U) in particular that $p_t(x, \cdot)$ is absolutely continuous w.r.t. ν. We suppose in addition that its density $p_t(x, y)$ satisfies a lower bound for small t of the following type:

(L) There exist $\delta, m, \beta > 0$ such that $p_t(x, y) \ge m e^{-\frac{\beta}{t}}$ for all $t \in (0, \delta)$, $x, y \in S$.

Theorem 3.3. *Assume that the semigroup (p_t) generated by A satisfies (U) and (L). Let (T_t) be the transition semigroup generated by the associated FV-operator and Π its unique invariant measure. Then there exist positive constants c_1, c_2 such that:*

(i)

$$\|T_t F\|_{L^\infty(\Pi)} \le e^{c_1(1+\frac{1}{t^3})}\|e^{|F|}\|_{L^1(\Pi)} \ , t > 0, F \in L^\infty(\Pi),$$

(ii)

$$\int (T_tF)^2 \log\left(\frac{(T_tF)^2}{\|T_tF\|^2_{L^2(\Pi)}}\right) d\Pi \le e^{c_2(1+\frac{1}{t^3})}\|F\|^2_{L^2(\Pi)}, t>0, F\in L^2(\Pi) \ .$$

For the proof of 3.3 we will state a series of Propositions and Lemmas first.

Let $\varrho_n \in \mathcal{M}_1(S^n)$ be the invariant measure for the n–particle Moran process associated with A.

Proposition 3.4. *(Absolute continuity and lower bound on the density)* $\varrho_n \ll \nu^n$ *and for its density* $h_n := \frac{d\varrho_n}{d\nu^n}$ *w.r.t.* ν^n *we have that*

(i) $h_1 \equiv 1$, $h_n(x) = \sum_{1\le i<j\le n} \int_0^\infty e^{-t\binom{n}{2}} \int p_t(\phi_{ij}^{(n)} z, x) h_{n-1}(z)\nu^{n-1}(dz)\, dt$, $n \ge 2$.

(ii) (Lower bound) $\inf_{x\in S^n} h_n(x) \ge \left(\frac{\delta m}{2^{\frac{3}{2}}}\right)^{n-1} (n-1)! e^{-(\delta+\frac{2\beta}{\delta})\frac{n(n+1)}{2}}$.

Proof. (i) Clearly, $\varrho_1 = \nu$, hence $h+1 \equiv 1$ (cf. the proof of 3.1). Since $\int\langle f,\mu^n\rangle\,\Pi(d\mu) = \sum_{1\le i<j\le n}\int_0^\infty e^{-t\binom{n}{2}} \int \langle \phi_{ij}^{(n)} p_t^{(n)} f, \mu^{n-1}\rangle\,\Pi(d\mu)\, dt$, $n\ge 2$ (cf. [EK2, proof of 5.2]), we obtain that

$$\int f\, d\varrho_n = \sum_{1\le i<j\le n} \int_0^\infty e^{-t\binom{n}{2}} \int \phi_{ij}^{(n)} p_t^{(n)} f\, d\varrho_{n-1}\, dt \ .$$

Hence, if ϱ_{n-1} is absolutely continuous w.r.t. $\nu^{(n-1)}$ with density h_{n-1} it follows that

$$\int f\, d\varrho_n = \sum_{1\le i<j\le n} \int_0^\infty e^{-t\binom{n}{2}} \int p_t^{(n)} f(\bar\phi_{ij}^{(n)}(z)) h_{n-1}(z)\, \nu^{n-1}(dz)\, dt$$

$$= \int f(x) \left(\sum_{1\le i<j\le n} \int_0^\infty e^{-t\binom{n}{2}} \int p_t^{(n)}(\bar\phi_{ij}^{(n)}(z), x) h_{n-1}(z)\, \nu^{n-1}(dz)\, dt\right) \nu^n(dx) \ , \tag{3.2}$$

where $\bar\phi_{ij}^{(n)}(z) = (z_1, \dots, z_{j-1}, z_i, z_j, \dots, z_{n-1})$. In particular, ϱ_n is absolutely continuous w.r.t. ν^n and its density is given by (3.2).

(ii) For the proof of the lower bound note that (L) and (i) now imply that for $n\ge 2$

$$\begin{aligned}
h_n(x) &\ge \left(\inf_{z\in S^{n-1}} h_{n-1}(z)\right)\binom{n}{2}\int_0^\infty e^{-t\binom{n}{2}}\int p_t^{(n)}(\bar\phi_{ij}^{(n)}(z),x)\,\nu^{n-1}(dz)\,dt\\
&\ge \left(\inf_{z\in S^{n-1}} h_{n-1}(z)\right) m\binom{n}{2}\int_0^\delta e^{-t\binom{n}{2}} e^{-\frac{\beta}{t}}\,dt\\
&\ge \left(\inf_{z\in S^{n-1}} h_{n-1}(z)\right) m\int_0^{\delta\binom{n}{2}} e^{-t} e^{-\frac{\beta\binom{n}{2}}{t}}\,dt\\
&\ge \left(\inf_{z\in S^{n-1}} h_{n-1}(z)\right) m\int_{\frac{1}{2}\delta\sqrt{\binom{n}{2}}}^{\delta\sqrt{\binom{n}{2}}} e^{-t} e^{-\frac{\beta\binom{n}{2}}{t}}\,dt\\
&\ge \left(\inf_{z\in S^{n-1}} h_{n-1}(z)\right) m\frac{1}{2}\delta\sqrt{\binom{n}{2}}\, e^{-\delta\sqrt{\binom{n}{2}}} e^{-\frac{2\beta}{\delta}\sqrt{\binom{n}{2}}}\\
&\ge \left(\inf_{z\in S^{n-1}} h_{n-1}(z)\right) \frac{(n-1)m\delta}{2^{\frac{3}{2}}} e^{-(\delta+\frac{2\beta}{\delta})n} \ .
\end{aligned}$$

Iterating the inequality above and using the fact that $h_1 = 1$ we obtain that

$$\inf_{x \in S^n} h_n(x) \geq (\frac{m\delta}{2^{\frac{3}{2}}})^{n-1}(n-1)! e^{-(\delta + \frac{2\beta}{\delta})\frac{n(n+1)}{2}} . \quad \square$$

Theorem 3.3 (i) clearly follows from the next Proposition.

Proposition 3.5. *There exists a positive constant c such that*

$$\|q_t^{(n)} f\|_{L^\infty(\nu^n)} \leq e^{c(1+\frac{1}{t^3})} \|e^{|\ell_f|}\|_{L^1(\Pi)} , t > 0,$$

for all $f \in \mathcal{B}_b(S^n)$. In particular, $\|T_t \ell_f\|_{L^\infty(\Pi)} \leq e^{c(1+\frac{1}{t^3})} \|e^{|\ell_f|}\|_{L^1(\Pi)}$, $t > 0$, for all $f \in \mathcal{B}_b(S^n)$, and thus $\|T_t F\|_{L^\infty(\Pi)} \leq e^{c(1+\frac{1}{t^3})} \|e^{|F|}\|_{L^1(\Pi)}$, $t > 0$, for all $F \in L^\infty(\Pi)$.

Proof. Step 1: Note that (U) implies $p_t(x,y) \leq M t^{-\alpha}$ for all $x, y \in S$, $t \in (0,1]$. Hence we obtain that $p_t^{(n)}(x,y) \leq (M t^{-\alpha})^n$ for all $x, y \in S^n$ and thus

$$\|p_t^{(n)} f\|_\infty \leq (M t^{-\alpha})^n \|f\|_{L^1(\nu^n)} . \tag{3.3}$$

Since $\|p_t^{(n)} f\|_{L^\infty(\nu^n)} \leq \|f\|_{L^\infty(\nu^n)}$ too, the Riesz-Thorin Interpolation theorem now implies that $\|p_t^{(n)}\|_{L^\infty(\nu^n)} \leq (M t^{-\alpha})^{\frac{n}{p}} \|f\|_{L^p(\nu^n)}$, $t \in (0,1]$. In particular, if $p_n := 4\bar{\alpha} n$, $n \geq 1$, where $\bar{\alpha}$ is the smallest integer $> \alpha$, it follows that $\|p_t^{(n)} f\|_{L^\infty(\nu^n)} \leq M^{-\frac{1}{4\bar{\alpha}}} t^{-\frac{1}{4}} \|f\|_{L^{p_n}(\nu^n)}$ for $t \in (0,1]$. Since by 3.4 (ii) $\|f\|_{L^{p_n}(\nu^n)} \leq e^{\tilde{c} n} \|f\|_{L^{p_n}(\varrho_n)}$ for some positive constant $\tilde{c}$ it follows that

$$\|p_t^{(n)} f\|_{L^\infty(\nu^n)} \leq e^{cn} t^{-\frac{1}{4}} \|f\|_{L^{p_n}(\varrho_n)} , t \in (0,1] , \tag{3.4}$$

for some positive constant c independent of n.

Step 2: $\|q_t^{(n)} f\|_{L^\infty(\nu^n)} \leq e^{cn - t\binom{n}{2}} t^{-\frac{1}{4}} \|f\|_{L^{p_n}(\varrho_n)} + \sum_{k=1}^{n-1} c_k^n e^{-t\binom{k}{2}} \|f\|_{L^{p_k}(\varrho_n)}$, $t \in (0,1]$, where

$$c_k^n = 2e^{ck} \binom{k+1}{2} \exp\left(\sum_{\ell=k+2}^{n} \frac{\binom{k}{2}}{\binom{l}{2} - \frac{p_k}{p_k - 1}\binom{k}{2}} \right) , 1 \leq k \leq n-1 .$$

Proof: We will prove this step using induction. If $n = 1$ then clearly, $\|q_t f\|_{L^\infty(\nu)} = \|p_t\|_{L^\infty(\nu)} \leq e^c t^{-\frac{1}{4}} \|f\|_{L^{p_1}(\varrho_1)}$ by (3.4).

Suppose now that the assertion is proved for $n-1$. Then

$$\begin{aligned}
&\sum_{i<j}\int_0^t e^{-s\binom{n}{2}}\|q_{t-s}^{(n-1)}\phi_{ij}^{(n)}p_s^{(n)}f\|_{L^\infty(\nu^{n-1})}\,ds \\
&\quad\le \sum_{i<j}\int_0^t e^{-s\binom{n}{2}}e^{c(n-1)-(t-s)\binom{n-1}{2}}(t-s)^{-\frac14}\|\phi_{ij}^{(n)}p_s^{(n)}f\|_{L^{p_{n-1}}(\varrho_{n-1})}\,ds \\
&\qquad+\sum_{i<j}\sum_{k=1}^{n-2}c_k^{n-1}\int_0^t e^{-\binom{n}{2}}e^{-(t-s)\binom{k}{2}}\|\phi_{ij}^{(n)}p_s^{(n)}f\|_{L^{p_k}(\varrho_{n-1})}\,ds \\
&\quad\le e^{c(n-1)-t\binom{n-1}{2}}\left(\binom{n}{2}\int_0^t e^{-s(\binom{n}{2}-\frac{p_{n-1}}{p_{n-1}-1}\binom{n-1}{2})}(t-s)^{-\frac{p_{n-1}}{4(p_{n-1}-1)}}\,ds\right)^{\frac{p_{n-1}-1}{p_{n-1}}}\cdot \\
&\qquad\quad\cdot\left(\sum_{i<j}\int_0^t\int\phi_{ij}^{(n)}p_s^{(n)}(|f|^{p_{n-1}})\,d\varrho_{n-1}\,ds\right)^{\frac{1}{p_{n-1}}} \\
&\qquad+\sum_{k=1}^{n-2}c_k^{n-1}e^{-t\binom{k}{2}}\left(\binom{n}{2}\int_0^t e^{-s(\binom{n}{2}-\frac{p_k}{p_k-1}\binom{k}{2})}\,ds\right)^{\frac{p_k-1}{p_k}}\cdot \\
&\qquad\quad\cdot\left(\sum_{i<j}\int_0^t\int\phi_{ij}^{(n)}p_s^{(n)}(|f|^{p_k})\,d\varrho_{n-1}\,ds\right)^{\frac{1}{p_k}} \\
&\quad\le e^{c(n-1)-t\binom{n-1}{2}}\left(\frac{\binom{n}{2}}{1-\frac{p_{n-1}}{4(p_{n-1}-1)}}\right)^{\frac{p_{n-1}-1}{p_{n-1}}}\|f\|_{L^{p_{n-1}}(\varrho_n)} \\
&\qquad+\sum_{k=1}^{n-2}c_k^{n-1}e^{-t\binom{k}{2}}\left(\frac{\binom{n}{2}}{\binom{n}{2}-\frac{p_k}{p_k-1}\binom{k}{2}}\right)^{\frac{p_k-1}{p_k}}\|f\|_{L^{p_k}(\varrho_n)} \\
&\quad\le\sum_{k=1}^{n-1}c_k^n e^{-t\binom{k}{2}}\|f\|_{L^{p_k}(\varrho_n)}\ ,t\in(0,1]\ ,
\end{aligned}$$

where we used the inequalities

$$\left(\frac{\binom{n}{2}}{\binom{n}{2}-\frac{p_k}{p_k-1}\binom{k}{2}}\right)^{\frac{p_k-1}{p_k}}\le\exp\left(\frac{\binom{k}{2}}{\binom{n}{2}-\frac{p_k}{p_k-1}\binom{k}{2}}\right);1\le k\le n-1,$$

and $\left(\frac{1}{1-\frac{p_{n-1}}{4(p_{n-1}-1)}}\right)^{\frac{p_{n-1}-1}{p_{n-1}}}\le\frac{4p_{n-1}-4}{3p_{n-1}-4}\le 2,\ n\ge 2.$

Consequently, if $t\in(0,1]$,

$$\begin{aligned}
\|q_t^{(n)}f\|_{L^\infty(\nu^n)}&\le e^{-t\binom{n}{2}}\|p_t^{(n)}f\|_{L^\infty(\nu^n)}+\sum_{i<j}\int_0^t e^{-s\binom{n}{2}}\|q_{t-s}^{(n-1)}\phi_{ij}^{(n)}p_s^{(n)}f\|_{L^\infty(\nu^n)}\,ds \\
&\le e^{cn-t\binom{n}{2}}t^{-\frac14}\|f\|_{L^{p_n}(\varrho_n)}+\sum_{k=1}^{n-1}c_k^n e^{-t\binom{k}{2}}\|f\|_{L^{p_k}(\varrho_n)}\ .
\end{aligned}$$

Step 3: There exists a positive constant c such that

$$\|q_t^{(n)} f\|_{L^\infty(\nu^n)} \le e^{\frac{c}{t^3}} \|e^{|\ell_f|}\|_{L^1(\Pi)} \ , t \in (0,1],$$

for all $f \in \mathcal{B}_b(S^n)$, $n \ge 1$.

Proof: First note that $\|f\|_{L^{p_k}(\varrho_n)} \le \left(p_k! \int e^{|f|}\, d\varrho_n\right)^{\frac{1}{p_k}} \le 4\bar{\alpha}k\|e^{|f|}\|_{L^1(\varrho_n)}$. Since

$$\sum_{\ell=k+2}^{n} \frac{1}{\binom{\ell}{2} - \frac{p_k}{p_k-1}\binom{k}{2}} \le 2\binom{k}{2} \int_k^\infty \frac{dx}{x^2 - \frac{p_k}{p_k-1}k(k-1)}$$
$$\le \frac{\sqrt{k(k-1)}}{2} \left(\frac{p_k-1}{p_k}\right)^{\frac{1}{2}} \log\left(\frac{k + \sqrt{\frac{p_k}{p_k-1}k(k-1)}}{k - \sqrt{\frac{p_k}{p_k-1}k(k-1)}}\right) ,$$

and $k - \sqrt{\frac{p_k}{p_k-1}k(k-1)} = \sqrt{k}(\sqrt{k} - \sqrt{\frac{p_k}{p_k-1}(k-1)}) \ge \frac{1}{2}(1 - \frac{1}{4\bar{\alpha}}) \ge \frac{1}{4}$, hence $\frac{k+\sqrt{\frac{p_k}{p_k-1}k(k-1)}}{k-\sqrt{\frac{p_k}{p_k-1}k(k-1)}} \le 12k$, it follows that $\exp\left(\sum_{\ell=k+2}^n \frac{\binom{k}{2}}{\binom{l}{2}-\frac{p_k}{p_k-1}\binom{k}{2}}\right) \le \exp((12k)\log(12k))$ and hence there exists a positive constant c_1 such that $c_k^n 4\bar{\alpha}k \le e^{c_1 k^{\frac{3}{2}}}$ for all k and n. Young's inequality $st \le \frac{1}{4}s^4 + \frac{3}{4}t^{\frac{4}{3}}$, $s,t \ge 0$, now implies that

$$\|q_t^{(n)} f\|_{L^\infty(\nu^n)} \le e^{cn - t\binom{n}{2}} t^{-\frac{1}{4}} \|f\|_{L^\infty(\nu^n)} + \sum_{k=1}^{n-1} e^{c_1 k^{\frac{3}{2}} - t\binom{k}{2}} \|e^{|f|}\|_{L^1(\varrho_n)}$$
$$\le e^{cn - t\binom{n}{2}} t^{-\frac{1}{4}} \|f\|_\infty + \left(e^{c_1} + \sum_{k=2}^{n-1} e^{\frac{c_1^4 (2k)^3}{4((k-1)t)^3} - \frac{t}{4}\binom{k}{2}}\right) \|e^{|f|}\|_{L^1(\varrho_n)}$$
$$\le e^{cn - t\binom{n}{2}} t^{-\frac{1}{4}} \|f\|_\infty + e^{\frac{c_2}{t^3}} \|e^{|f|}\|_{L^1(\varrho_n)} \ , t \in (0,1],$$

for some positive constant c_2 independent of n.

For $g \in \mathcal{B}_b(S^n)$ define $S_m g := \frac{1}{m}\sum_{k=1}^m g \circ p_k \in \mathcal{B}_b(S^{nm})$, where $p_k : (S^n)^m \to S^n$ denotes the projection onto the k–th coordinate. Fix now $f \in \mathcal{B}_b(S^n)$ and note that $\langle e^{|S_m f|}, \mu^{nm}\rangle \to \exp(|\langle f, \mu^n\rangle|)$ for all $\mu \in \mathcal{M}_1(S)$ by the law of large numbers. Since

$$\|q_t^{(n)} f\|_{L^\infty(\nu^n)} = \|S_m q_t^{(n)} f\|_{L^\infty(\nu^{nm})} = \|q_t^{(nm)} S_m f\|_{L^\infty(\nu^{nm})}$$

for all m it follows that

$$\|q_t^{(n)} f\|_{L^\infty(\nu^n)} = \lim_{m\to\infty} \|q_t^{(nm)} S_m f\|_{L^\infty(\nu^{nm})}$$
$$\le \lim_{m\to\infty} e^{cnm - t\binom{nm}{2}} t^{-\frac{1}{4}} \|f\|_\infty + e^{\frac{c_2}{t^3}} \|e^{|S_m f|}\|_{L^1(\varrho_{nm})}$$
$$= e^{\frac{c_2}{t^3}} \lim_{m\to\infty} \int \langle e^{|S_m f|}, \mu^{nm}\rangle \, \Pi(d\mu) = e^{\frac{c_2}{t^3}} \|e^{|\ell_f|}\|_{L^1(\Pi)} \ , t \in (0,1] \ ,$$

by Lesbesgue's theorem. Hence Step 3 is proved.

Fix again $f \in \mathcal{B}_b(S^n)$. Since ϱ_n and ν^n are equivalent measures by 3.4 it follows from Step 3 that for $t \ge 1$ $\|q_t^{(n)} f\|_{L^\infty(\nu^n)} = \|q_t^{(n)} f\|_{L^\infty(\varrho_n)} = \|q_1^{(n)} f\|_{L^\infty(\varrho_n)} = \|q_1^{(n)} f\|_{L^\infty(\nu^n)} \le e^c \|e^{|\ell_f|}\|_{L^1(\Pi)}$ if

$t > 1$, hence $\|q_t^{(n)} f\|_{L^\infty(\nu^n)} \le e^{c(1+\frac{1}{t^3})} \|e^{|\ell_f|}\|_{L^1(\Pi)}$ for all $t > 0$ which implies the first assertion. The second assertion follows from

$$
\begin{aligned}
\|T_t \ell_f\|_{L^\infty(\Pi)} &= \lim_{p\to\infty} \left(\int |\langle q_t^{(n)} f, \mu^n \rangle|^p \, \Pi(d\mu) \right)^{\frac{1}{p}} \le \lim_{p\to\infty} \left(\int |q_t^{(n)} f|^p \, d\varrho_n \right)^{\frac{1}{p}} \\
&= \|q_t^{(n)} f\|_{L^\infty(\varrho_n)} = \|q_t^{(n)} f\|_{L^\infty(\nu^n)}.
\end{aligned}
$$

The last assertion is easily implied by the fact that $\{\ell_f | f \in \mathcal{B}_b(S^n)\,, n \ge 1\} \subset L^\infty(\Pi)$ is dense w.r.t. pointwise, uniform bounded convergence. $\square$

As one of the main tools in the proof of 3.3 (ii) let us define for $n \in \mathbb{N}$ the following seminorm N_n on measurable functions:

$$
N_n(f) := \sup_{\substack{v \in \mathcal{B}^+(S^n) \\ \int e^v - 1 \, d\varrho_n = 1}} \left(\int f^2 v \, d\varrho_n \right)^{\frac{1}{2}} \,, f \in \mathcal{B}(S^n) \,.
$$

Clearly, $N_n(\lambda f) = |\lambda| N_n(f)$ for $\lambda \in \mathbb{R}$ and $N_n(f+g) \le N_n(f) + N_n(g)\,; f, g \in \mathcal{B}_b(S^n)$.

Lemma 3.6. *Let $f \in \mathcal{B}_b(S^n)$. Then*

$$
\left(\int f^2 \log \left(\frac{f^2}{\|f\|^2_{L^2(\varrho_n)}} + 1 \right) d\varrho_n \right)^{\frac{1}{2}} \le N_n(f) \le \left(\int f^2 \left(\log \left(\frac{f^2}{\|f\|^2_{L^2(\varrho_n)}} \right) + 1 \right) d\varrho_n \right)^{\frac{1}{2}} .
$$

Proof. For the proof of the lower bound note that $v_0 := \log \left(\frac{f^2}{\|f\|^2_{L^2(\varrho_n)}} + 1 \right) \in \mathcal{B}^+(S^n)$ satisfies $\int e^{v_0} - 1 \, d\varrho_n = \int \frac{f^2}{\|f\|^2_{L^2(\varrho_n)}} \, d\varrho_n = 1$ and thus $N_n(f) \ge \left(\int f^2 v_0 \, d\varrho_n \right)^{\frac{1}{2}}$. For the proof of the upper bound note that Young's inequality $st \le s \log s - s + e^t$, $s \ge 0$, $t \in \mathbb{R}$, implies that for f such that $\|f\|_{L^2(\varrho_n)} = 1$

$$
\int f^2 v \, d\varrho_n \le \int f^2 \log f^2 - f^2 \, d\varrho_n + \int e^v \, d\varrho_n = \int f^2 \left(\log \left(\frac{f^2}{\|f\|^2_{L^2(\varrho_n)}} \right) + 1 \right) d\varrho_n
$$

for all $v \in \mathcal{B}^+(S^n)$ with $\int e^v - 1 \, d\varrho_n = 1$, hence $N_n(f) \le \left(\int f^2 \left(\log \left(\frac{f^2}{\|f\|^2_{L^2(\varrho_n)}} \right) + 1 \right) d\varrho_n \right)^{\frac{1}{2}}$. The general case $\|f\|_{L^2(\varrho_n)} \ne 1$ now follows from homogeneity. $\square$

3.6 implies in particular that $N_n(\cdot)$ respects ϱ_n–classes, i.e., $N_n(f) = N_n(g)$ if $f = g$ ϱ_n–a.e., hence $N_n(\cdot)$ induces a norm on $L^{2+\varepsilon}(\varrho_n)$, $\varepsilon > 0$.

Lemma 3.7. *Let $f \in \mathcal{B}_b(S^n)$ and $m \ge 1$. Then $N_{n+m}(f) \le N_n(f)$.*

Proof. Let $v \in \mathcal{B}^+(S^{n+m})$ be such that $\int e^v - 1 \, d\varrho_n = 1$ and $\bar{v}(x) \in \mathcal{B}^+(S^n)$ be a ϱ_{n+m}–version version of $\log E_{\varrho_{n+m}}[e^v | \sigma(x_1, \dots x_n)]$. Then $\int e^{\bar{v}} - 1 \, d\varrho_n = \int e^{\bar{v}} - 1 \, d\varrho_{n+m}$

$= \int E_{\varrho_{n+m}}[e^v|\sigma(x_1,\ldots x_n)] - 1\, d\varrho_{n+m} = 1$ and thus

$$\left(\int f^2 v\, d\varrho_{n+m}\right)^{\frac{1}{2}} = \left(\int f^2 E_{\varrho_{n+m}}[v|\sigma(x_1,\ldots,x_n)]\, d\varrho_{n+m}\right)^{\frac{1}{2}}$$
$$\leq \left(\int f^2 \log E_{\varrho_{n+m}}[e^v|\sigma(x_1,\ldots,x_n)]\, d\varrho_{n+m}\right)^{\frac{1}{2}}$$
$$= \left(\int f^2 \bar{v}\, d\varrho_n\right)^{\frac{1}{2}} \leq N_n(f)\ ,$$

which implies the assertion. □

Lemma 3.8. *There exists a positive constant c such that*

$$N_n(p_t^{(n)} f) \leq (|\log t| + 1)^{\frac{1}{2}}\, e^{cn+\frac{1}{2}t\binom{n}{2}} \|f\|_{L^2(\varrho_n)}, t \in (0,1],$$

for all $f \in \mathcal{B}_b(S^n)$, $n \geq 1$.

Proof. It suffices to consider the case $\int f^2\, d\varrho_n = 1$. Since $\|p_t^{(n)} f\|_\infty \leq (Mt^{-\alpha})^n\, \|f\|_{L^1(\nu^n)}$ (cf. (3.3)) and thus

$$\|p_t^{(n)} f\|_\infty \leq \left(Mt^{-\alpha}\right)^n c^{1-n} e^{(\delta+\frac{2\beta}{\delta})\binom{n+1}{2}}, t \in (0,1]\ ,$$

for some constant $c > 0$ by 3.4 (ii), it follows that

$$N_n(p_t^{(n)} f) \leq \left(\int (p_t^{(n)} f)^2 \left(\log\left(\frac{(p_t^{(n)} f)^2}{\|p_t f\|^2_{L^2(\varrho_n)}}\right) + 1\right) d\varrho_n\right)^{\frac{1}{2}}$$
$$\leq \left(\int (p_t^{(n)} f)^2 \left(\log\left(\frac{(Mt^{-\alpha})^{2n} c^{2(1-n)} e^{(\delta+\frac{2\beta}{\delta})n(n+1)}}{\|p_t^{(n)} f\|^2_{L^2(\varrho_n)}}\right) + 1\right) d\varrho_n\right)^{\frac{1}{2}}$$
$$\leq \left(\left(c_1 n^2 + 2\alpha n|\log t|\right) \|p_t^{(n)} f\|^2_{L^2(\varrho_n)} - \|p_t^{(n)} f\|^2_{L^2(\varrho_n)} \log \|p_t^{(n)} f\|^2_{L^2(\varrho_n)}\right)^{\frac{1}{2}}$$

for some positive constant c_1. Since $-x\log x \leq e^{-1}$, hence $-\|p_t^{(n)} f\|^2_{L^2(\varrho_n)} \log \|p_t^{(n)} f\|^2_{L^2(\varrho_n)} \leq e^{-1}$, and $\int (p_t^{(n)} f)^2\, d\varrho_n \leq \int p_t^{(n)}(f^2)\, d\varrho_n \leq e^{t\binom{n}{2}} \int q_t^{(n)}(f^2)\, d\varrho_n = e^{t\binom{n}{2}} \int f^2\, d\varrho_n$, we obtain that

$$N_n(p_t^{(n)} f) \leq \left(\left(c_1 n^2 + 2\alpha n|\log t|\right) e^{t\binom{n}{2}} + e^{-1}\right)^{\frac{1}{2}}$$
$$= \left(\left(c_1 n^2 + 2\alpha n|\log t|\right) e^{t\binom{n}{2}} + e^{-1}\right)^{\frac{1}{2}} \|f\|_{L^2(\varrho_n)}\ ,$$

which implies the assertion. □

Proposition 3.9. *Let c be the constant specified in 3.8. Let*

$$c_k^n := 3(1+k^2)\binom{k+1}{2} \prod_{l=k+1}^{n}\left(1+\frac{1}{l^2}\right) \prod_{l=k+2}^{n}\left(\frac{\binom{l}{2}}{\binom{l}{2}-\binom{k}{2}}\right)\ , n \geq 1,\ 1 \leq k \leq n-1\ ,$$

and $c_n(t) := \left(\sum_{k=1}^{n-1} c_k^n e^{2ck-t\binom{k}{2}} + (1+n^2)(1+|\log t|)e^{2cn-t\binom{n}{2}}\right)^{\frac{1}{2}}$. *Then* $N_n(q_t^{(n)} f) \le c_n(t)\|f\|_{L^2(\varrho_n)}$, $t \in (0,1]$, *for all* $f \in \mathcal{B}_b(S^n)$.

Proof. First consider the case $n = 1$. Since $q_t^{(1)} f = p_t f$ 3.8 implies that

$$N_n(p_t^{(n)} f) \le e^c \left(|\log t| + 1\right)^{\frac{1}{2}} \|f\|_{L^2(\varrho_n)} = c_1(t)\|f\|_{L^2(\varrho_n)} .$$

Hence the assertion is proved in this case.

Next suppose that the assertion is proved for $n-1$. Let (X_n, N_n) be the abstract completion of $L^\infty(\varrho_n)$ w.r.t. N_n. 3.6. implies that $L^{2+\varepsilon}(\varrho_n) \hookrightarrow X_n$ continuously for all $\varepsilon > 0$. In particular, if $f \in C(S^n)$ then $[0,t] \to X_n$, $s \mapsto q_{t-s}^{(n-1)} \phi_{ij}^{(n)} p_s^{(n)} f$, is continuous and thus Bochner's inequality implies that

$$N_n\left(\int_0^t e^{-s\binom{n}{2}} q_{t-s}^{(n-1)} \phi_{ij}^{(n)} p_s^{(n)} f\, ds\right) \le \int_0^t e^{-s\binom{n}{2}} N_n(q_{t-s}^{(n-1)} \phi_{ij}^{(n)} p_s^{(n)} f)\, ds$$

holds for $1 \le i < j \le n$. For general $f \in \mathcal{B}_b(S^n)$ we can find a uniformly bounded sequence $(f_n) \subset C(S^n)$ converging to f pointwise everywhere (cf. [EK1, 4.2]). Consequently, $\lim_{n\to\infty} p_s^{(n)} f_n = p_s^{(n)} f$ uniformly bounded and pointwise everywhere for all s and thus $\lim_{n\to\infty} q_{t-s}^{(n-1)} \phi_{ij}^{(n)} p_s^{(n)} f_n = q_{t-s}^{(n-1)} \phi_{ij}^{(n)} p_s^{(n)} f$ uniformly bounded and pointwise everywhere and thus in X_n too by Lebesgue's theorem. It follows that the last inequality extends also to the limit $f \in \mathcal{B}_b(S^n)$. Consequently, by the triangle inequality,

$$\begin{aligned} N_n(q_t^{(n)} f) &\le e^{-t\binom{n}{2}} N_n(p_t^{(n)} f) + \sum_{1\le i<j\le n} N_n\Big(\int_0^t e^{-s\binom{n}{2}} q_{t-s}^{(n-1)}(\phi_{ij}^{(n)} p_s^{(n)} f)\, ds\Big) \\ &\le e^{-t\binom{n}{2}} N_n(p_t^{(n)} f) + \sum_{1\le i<j\le n} \int_0^t e^{-s\binom{n}{2}} N_n(q_{t-s}^{(n-1)}(\phi_{ij}^{(n)} p_s^{(n)} f))\, ds . \end{aligned}$$

Since $N_n(q_{t-s}^{(n-1)}(\phi_{ij}^{(n)} p_s^{(n)} f)) \le N_{n-1}(q_{t-s}^{(n-1)}(\phi_{ij}^{(n)} p_s^{(n)} f))$ by 3.7 and $N_{n-1}(q_{t-s}^{(n-1)}(\phi_{ij}^{(n)} p_s^{(n)} f)) \le c_{n-1}(t-s)\|\phi_{ij}^{(n-1)} p_s^{(n)} f\|_{L^2(\varrho_{n-1})}$ by assumption, it follows that

$$\begin{aligned} &\sum_{1\le i<j\le n} \int_0^t e^{-s\binom{n}{2}} N_n(q_{t-s}^{(n-1)}(\phi_{ij}^{(n)} p_s^{(n)} f))\, ds \\ &\quad\le \sum_{1\le i<j\le n} \int_0^t e^{-s\binom{n}{2}} c_{n-1}(t-s)\|\phi_{ij}^{(n-1)} p_s^{(n)} f\|_{L^2(\varrho_{n-1})}\, ds \\ &\quad\le \left(\binom{n}{2}\int_0^t e^{-s\binom{n}{2}} c_{n-1}^2(t-s)\, ds\right)^{\frac{1}{2}} \left(\sum_{1\le i<j\le n} \int_0^t e^{-s\binom{n}{2}} \int \phi_{ij}^{(n-1)} (p_s^{(n)} f)^2\, d\varrho_{n-1}\, ds\right)^{\frac{1}{2}} \\ &\quad= \Big(\sum_{k=1}^{n-2} c_k^{n-1} \binom{n}{2} \int_0^t e^{-s\binom{n}{2}} e^{2ck-(t-s)\binom{k}{2}}\, ds \\ &\qquad + (1+(n-1)^2)\binom{n}{2}\int_0^t e^{-s\binom{n}{2}} e^{2c(n-1)-(t-s)\binom{n-1}{2}} (1+|\log(t-s)|)\, ds\Big)^{\frac{1}{2}} \|f\|_{L^2(\varrho_n)} \\ &\quad\le \Big(\sum_{k=1}^{n-2} c_k^{n-1} \frac{\binom{n}{2}}{\binom{n}{2}-\binom{k}{2}} e^{2ck-t\binom{k}{2}} + 3(1+(n-1)^2)\binom{n}{2} e^{2c(n-1)-t\binom{n-1}{2}}\Big)^{\frac{1}{2}} \|f\|_{L^2(\varrho_n)} . \end{aligned}$$

Consequently,

$$
\begin{aligned}
N_n(q_t^{(n)}f) &\le e^{cn-\frac{1}{2}t\binom{n}{2}}(1+|\log t|)^{\frac{1}{2}}\|f\|_{L^2(\varrho_n)} + \left(\sum_{k=1}^{n-1} c_k^{n-1}\frac{\binom{n}{2}}{\binom{n}{2}-\binom{k}{2}}e^{2ck-t\binom{k}{2}}\right)^{\frac{1}{2}}\|f\|_{L^2(\varrho_n)}\\
&\le \left((1+n^2)e^{-t\binom{n}{2}}(1+|\log t|) + (1+\frac{1}{n^2})\sum_{k=1}^{n-1} c_k^{n-1}\frac{\binom{n}{2}}{\binom{n}{2}-\binom{k}{2}}e^{2ck-t\binom{k}{2}}\right)^{\frac{1}{2}}\|f\|_{L^2(\varrho_n)}\\
&= c_n(t)\|f\|_{L^2(\varrho_n)}\ ,
\end{aligned}
$$

where we used the inequality $\sqrt{a}+\sqrt{b} \le \left((1+n^2)a + (1+\frac{1}{n^2})b\right)^{\frac{1}{2}}$, $a, b \ge 0$, in the last but one step. □

Proof of 3.3. As already mentioned, 3.3 (i) follows from 3.5.

(ii) Let $c_n(t)$ be as in 3.9. First note that there exists a positive constant c_2 such that $\limsup_{n\to\infty} c_n^2(t) \le e^{\frac{c_2}{t^3}}$, $t \in (0,1]$.

Indeed, since

$$
\prod_{\ell=k+1}^{n}(1+\frac{1}{\ell^2}) \le \exp\left(\sum_{\ell=k+1}^{n}\frac{1}{\ell^2}\right) \le \exp\left(\frac{1}{k}\right)\ ,
$$

and

$$
\begin{aligned}
\prod_{\ell=k+2}^{n}\frac{\binom{\ell}{2}}{\binom{\ell}{2}-\binom{k}{2}} &\le \exp\left(\sum_{\ell=k+2}^{n}\frac{\binom{k}{2}}{\binom{\ell}{2}-\binom{k}{2}}\right) \le \exp\left(\binom{k}{2}2\int_k^{\infty}\frac{dx}{x^2-k(k-1)}\right)\\
&= \exp\left(\frac{\sqrt{k(k-1)}}{2}\log\left(\frac{k+\sqrt{k(k-1)}}{k-\sqrt{k(k-1)}}\right)\right) \qquad (3.5)\\
&\le \exp\left(\frac{k}{2}\log(4k)\right) \le \exp\left(3k^{\frac{3}{2}}\right)\ ,
\end{aligned}
$$

therefore

$$
\begin{aligned}
c_n^2(t) &\le \sum_{k=1}^{n-1}3(1+k^2)\binom{k+1}{2}e^{\frac{1}{k}+3k^{\frac{3}{2}}+2ck-t\binom{k}{2}} + (1+n^2)(1+|\log t|)e^{2cn-t\binom{n}{2}}\\
&\le 144\sum_{k=1}^{n-1}e^{(5+2c)k+3k^{\frac{3}{2}}-t\binom{k}{2}} + 4(1+|\log t|)e^{(1+2c)n-t\binom{n}{2}}\\
&\le 144\left(e^{8+2c}+\sum_{k=2}^{n-1}e^{\frac{(8+2c)^4(2k)^3}{4(t(k-1))^3}-\frac{t}{4}\binom{k}{2}}\right) + 4(1+|\log t|)e^{(1+2c)n-t\binom{n}{2}}\ ,
\end{aligned}
$$

where we have applied Young's inequality $st \le \frac{1}{4}s^4 + \frac{3}{4}t^{\frac{4}{3}}$, $s, t \ge 0$, in the last step, it follows that

$$
\limsup_{n\to\infty} c_n^2(t) \le 144\left(e^{8+2c} + e^{\frac{(2(8+2c))^4}{t^3}}\sum_{k=2}^{\infty}e^{-\frac{t}{4}\binom{k}{2}}\right) \le e^{\frac{c_2}{t^3}}\ , t \in (0,1],
$$

for some positive constant c_2.

Fix now $f \in \mathcal{B}_b(S^n)$ and let $S_m f := \frac{1}{m}\sum_{k=1}^m f \circ p_k \in \mathcal{B}_b(S^{nm})$, where $p_k : (S^n)^m \to S^n$ denotes the projection onto the k–th coordinate. 3.6 and 3.9 now imply that

$$\int \left(q_t^{(nm)} S_m f\right)^2 \left(\log\left(\frac{\left(q_t^{(nm)} S_m f\right)^2}{\|q_t^{(nm)} S_m f\|^2_{L^2(\varrho_{nm})}}\right) + 1\right) d\varrho_{nm}$$

$$\leq N_{nm}(q_t^{(nm)} S_m f)^2 \leq c^2_{nm}(t) \int (S_m f)^2 \, d\varrho_{nm} \; .$$

Note that $\lim_{m\to\infty}\langle (S_m f)^2, \mu^{nm}\rangle = \langle f, \mu^n\rangle^2$ for all $\mu \in \mathcal{M}_1(S)$ and thus $\lim_{n\to\infty}\langle (S_m f)^2, \mu^{nm}\rangle = \langle f, \mu^n\rangle^2$ in $L^1(\Pi)$ by Lebesgue's theorem. Hence, Jensen's inequality implies that

$$\begin{aligned}
\int (T_t\ell_f)^2 \log(T_t\ell_f)^2 \, d\Pi &= \lim_{m\to\infty} \int \langle q_t^{(nm)} S_m f, \mu^{nm}\rangle^2 \log\langle q_t^{(nm)} S_m f, \mu^{nm}\rangle^2 \, \Pi(d\mu) \\
&\leq \limsup_{m\to\infty} \int \langle (q_t^{(nm)} S_m f)^2 \log(q_t^{(nm)} S_m f)^2, \mu^{nm}\rangle \, \Pi(d\mu) \\
&= \limsup_{m\to\infty} \int (q_t^{(nm)} S_m f)^2 \log(q_t^{(nm)} S_m f)^2 \, d\varrho_{nm} \\
&\leq \limsup_{m\to\infty} \int (q_t^{(nm)} S_m f)^2 \log\left(\frac{(q_t^{(nm)} S_m f)^2}{\|q_t^{(nm)} S_m f\|_{L^2(\varrho_{nm})}} + 1\right) d\varrho_{nm} \\
&\qquad + \|q_t^{(nm)} S_m f\|^2_{L^2(\varrho_{nm})} \log \|q_t^{(nm)} S_m f\|^2_{L^2(\varrho_{nm})} \\
&\leq \limsup_{m\to\infty} N_{nm}(q_t^{(nm)} S_m f)^2 + \|q_t^{(nm)} S_m f\|^2_{L^2(\varrho_{nm})} \log \|q_t^{(nm)} S_m f\|^2_{L^2(\varrho_{nm})} \\
&\leq \limsup_{m\to\infty} c^2_{nm}(t) \int (S_m f)^2 \, d\varrho_{nm} + \|q_t^{(nm)} S_m f\|^2_{L^2(\varrho_{nm})} \log \|q_t^{(nm)} S_m f\|^2_{L^2(\varrho_{nm})} \\
&\leq e^{\frac{c_2}{t^3}} \|\ell_f^2\|^2_{L^2(\Pi)} + \|T_t\ell_f\|^2_{L^2(\Pi)} \log \|T_t\ell_f\|^2_{L^2(\Pi)} \, , t \in (0,1],
\end{aligned}$$

where we used 3.6 in the last but one inequality. If $t \geq 1$ then clearly

$$\int (T_t\ell_f)^2 \log\left(\frac{(T_t\ell_f)^2}{\|T_t\ell_f\|^2_{L^2(\Pi)}}\right) d\Pi \leq e^{c_2} \|T_{t-1}\ell_f\|^2_{L^2(\Pi)} \leq e^{c_2} \|\ell_f\|^2_{L^2(\Pi)} \; ,$$

which implies 3.3 (ii) for $F = \ell_f$, $f \in \mathcal{B}_b(S^n)$, $n \geq 1$. Since $\{\ell_f | f \in \mathcal{B}_b(S^n), n \geq 1\} \subset L^2(\Pi)$ dense, the assertion now follows for general $F \in L^2(\Pi)$ by taking limits. $\square$

4. Strong Feller property

Throughout this section we suppose that S is a compact subset of $\mathbb{R}^d$. Let A be a linear operator generating a Feller semigroup (p_t) with unique invariant probability measure ν such that $\mathrm{supp}(\nu) \equiv S$. We assume in addition to (U) and (L) (cf. Section 3) that

(F) $p_t f \in C^1(S)$ if $f \in \mathcal{B}_b(S)$, $t > 0$, and there exist $c_0 > 0$ and $\alpha \in (0,1)$ such that $\sum_{k=1}^d \|\partial_k p_t f\|_\infty \leq c_0 t^{-\alpha} \|f\|_{L^\infty(\nu)}$ for all $f \in \mathcal{B}_b(S)$, $t > 0$.

Example 4.1. Let $S = [0,1]^d$ and $A = \frac{1}{2}\Delta$ with periodic boundary conditions. If $d = 1$ it is well–known that $p_t f(x) = \sum_{k\in\mathbb{Z}} \frac{1}{(2\pi t)^{\frac{1}{2}}} \int_0^1 f(y) \exp(-\frac{(x-y-k)^2}{2t})\, dy \in C^1([0,1])$ and $\frac{d}{dx} p_t f(x) \leq t^{-\frac{1}{2}} \|f\|_{L^\infty(dx)}$ for all $f \in \mathcal{B}_b([0,1])$, $t > 0$. Hence $p_t^{(d)} f \in C^1(S)$ and

$$\sum_{k=1}^{d} \|\partial_k p_t^{(d)} f\|_\infty \leq d t^{-\frac{1}{2}} \|f\|_{L^\infty(\nu)} \ , t > 0,$$

for all $f \in \mathcal{B}_b(S)$, hence (F) is satisfied. (U) and (L) are obvious. For more general diffusion operators consider Example 6.c) below.

Recall that $\mathcal{M}_1(S)$ endowed with the weak topology is a metrizable space. A compatible metric is given by

$$d_w(\mu, \bar\mu) := \sup_{\substack{f: S\to\mathbb{R} \\ \frac{|f(x)-f(y)|}{|x-y|}\leq 1}} \left| \int f\, d\mu - \int f\, d\bar\mu \right| \ ; \mu, \bar\mu \in \mathcal{M}_1(S) \ .$$

The main result of this section will be the following

Theorem 4.2. *Assume that the semigroup (p_t) generated by A satisfies (U), (L) and (F). Let (T_t) be the semigroup generated by the FV-operator with mutation A and denote by Π its unique invariant measure. Then:*

(i) There exists a positive constant c such that

$$(4.1) \quad |T_t F(\mu) - T_t F(\bar\mu)| \leq (1 + t^{1-\alpha}) e^{c(1+\frac{1}{t^3})} \|e^{|F|}\|_{L^1(\Pi)} \cdot d_w(\mu, \bar\mu) \,, t > 0; \mu, \bar\mu \in \mathcal{M}_1(S),$$

for all $F \in C(\mathcal{M}_1(S))$. In particular, (T_t) is strong Feller, i.e., $T_t(\mathcal{B}_b(\mathcal{M}_1(S))) \subset C(\mathcal{M}_1(S))$, $t > 0$, and (4.1) extends to all $F \in \mathcal{B}_b(\mathcal{M}_1(S))$.

((ii) If $\varepsilon > 0$ then $T_t : L^{p+\varepsilon}(\Pi) \to L^p(\Pi)$ is compact for all $p \in [1, \infty]$.

For the proof of 4.2 let us state a series of Lemmas first:

Lemma 4.3. *Let $c_n(t) = c_0 n t^{-\alpha} e^{-t\binom{n}{2}} + c_0 \sum_{k=1}^{n-1} k\binom{k+1}{2} e^{\sum_{l=k+2}^{n} \frac{\binom{k}{2}}{\binom{l}{2}-\binom{k}{2}}} \frac{1}{1-\alpha} t^{1-\alpha} e^{-t\binom{n}{2}}$, $n \geq 1$. Then $q_t^{(n)} f \in C^1(S^n)$ and*

$$\sum_{k=1}^{dn} \|\partial_k q_t^{(n)} f\|_\infty \leq c_n(t) \|f\|_\infty \ , t > 0 \ ,$$

for all $f \in \mathcal{B}_b(S^n)$.

Proof. We will prove the assertion using induction. If $n = 1$ then the assertion is true by assumption on (p_t).

Suppose now that the assertion is proved for $n-1$ and define $c_k := k\binom{k+1}{2} e^{\sum_{l=k+2}^{n-1} \frac{\binom{k}{2}}{\binom{l}{2}-\binom{k}{2}}}$, $1 \leq k \leq n-1$. Then

$$q_t^{(n)} f = e^{-t\binom{n}{2}} p_t^{(n)} f + \sum_{1\leq i<j\leq n} \int_0^t e^{-s\binom{n}{2}} q_{t-s}^{(n-1)} \phi_{ij}^{(n)} p_s^{(n)} f\, ds \in C^1(S^n) \ ,$$

since $p_t^{(n)} f \in C^1(S^n)$ and $q_{t-s}^{(n-1)} \phi_{ij}^{(n)} p_s^{(n)} f \in C^1(S^{n-1}) \in C^1(S^n)$ by assumption.

Note that by assumption $\sum_{k=1}^{d(n-1)} \|\partial_k q_{t-s}^{(n-1)} \phi_{ij}^{(n)} p_s^{(n)} f\|_\infty \le c_{n-1}(t-s)\|\phi_{ij}^{(n)} p_s^{(n)} f\|_\infty$ $\le c_{n-1}(t-s)\|f\|_\infty$ and therefore

$$
\begin{aligned}
\sum_{k=1}^{dn} \|\partial_k q_t^{(n)} f\|_\infty &\le c_0 n t^{-\alpha} e^{-t\binom{n}{2}} \|f\|_\infty + \sum_{1\le i<j\le n} \int_0^t e^{-s\binom{n}{2}} \sum_{k=1}^{d(n-1)} \|\partial_k q_{t-s}^{(n-1)} \phi_{ij}^{(n)} p_s^{(n)} f \, ds\|_\infty \, ds \\
&\le c_0 n t^{-\alpha} e^{-t\binom{n}{2}} \|f\|_\infty + c_0 \sum_{1\le i<j\le n} \int_0^t e^{-s\binom{n}{2}} (n-1)(t-s)^{-\alpha} e^{-(t-s)\binom{n-1}{2}} \, ds \, \|f\|_\infty \\
&\quad + c_0 \sum_{1\le i<j\le n} \int_0^t e^{-s\binom{n}{2}} \left(\sum_{k=1}^{n-2} c_k e^{-(t-s)\binom{k}{2}} \frac{1}{1-\alpha} (t-s)^{1-\alpha} \right) ds \, \|f\|_\infty \\
&\le c_0 n t^{-\alpha} e^{-t\binom{n}{2}} \|f\|_\infty + c_0 \binom{n}{2} (n-1) e^{-t\binom{n-1}{2}} \frac{1}{1-\alpha} t^{1-\alpha} \|f\|_\infty \\
&\quad + c_0 \sum_{k=1}^{n-2} c_k \frac{\binom{n}{2}}{\binom{n}{2} - \binom{k}{2}} \frac{1}{1-\alpha} t^{1-\alpha} e^{-t\binom{k}{2}} \, \|f\|_\infty \\
&\le \left(c_0 n t^{-\alpha} e^{-t\binom{n}{2}} + c_0 \sum_{k=1}^{n-1} k \binom{k+1}{2} e^{\sum_{l=k+2}^{n} \frac{\binom{k}{2}}{\binom{l}{2} - \binom{k}{2}}} \frac{1}{1-\alpha} t^{1-\alpha} \right) \|f\|_\infty \, . \quad \square
\end{aligned}
$$

Lemma 4.4. *Let $f \in C^1(S^n)$. Then*

$$
|\langle f, \mu^n \rangle - \langle f, \bar\mu^n \rangle| \le \sum_{k=1}^{dn} \|\partial_k f\|_\infty \, d_w(\mu, \bar\mu) \, ; \mu, \bar\mu \in \mathcal{M}_1(S) \ .
$$

Proof. Let $\gamma(s) := s \cdot \mu + (1-s) \cdot \bar\mu$, $s \in [0,1]$. Then

$$
\begin{aligned}
|\langle f, \mu^n \rangle - \langle f, \bar\mu^n \rangle| &\le \left| \int_0^1 \sum_{k=1}^{n} \int_S \int_{S^{n-1}} f(x) \gamma(s)^{n-1} (dx_1 \ldots \hat{dx_k} \ldots dx_n) \mu(dx_k) - \bar\mu(dx_k) \, ds \right| \\
&= | \int_0^1 \int_S \sum_{k=1}^{n} \int_{S^{n-1}} f(x_2, \ldots, x_{k-1}, x_1, x_k, \ldots, x_n) \gamma(s)^{n-1} (dx_2 \ldots dx_n) \mu(dx_1) - \\
&\qquad - \int_S \sum_{k=1}^{n} \int_{S^{n-1}} f(x_2, \ldots, x_{k-1}, x_1, x_k, \ldots, x_n) \gamma(s)^{n-1} (dx_2 \ldots dx_n) \bar\mu(dx_1) \, ds| \\
&\le L \cdot d_w(\mu, \bar\mu) \ ,
\end{aligned}
$$

where the ˆ in the first line indicates that the term below it is absent, and where L denotes the Lipschitz constant of

$$
x_1 \mapsto \sum_{k=1}^{n} \int_0^1 \int_{S^{n-1}} f(x_2, \ldots, x_{k-1}, x_1, x_k, \ldots, x_n) \gamma(s)^{n-1} (dx_2 \ldots dx_n) \, ds \ .
$$

Clearly, $L \le \sum_{k=1}^{dn} \|\partial_k f\|_\infty$. Combining the two inequalities we obtain the assertion. $\square$

Lemma 4.5. *There exists a positive constant c such that*

$$\left|\langle q_t^{(n)} f, \mu^n\rangle - \langle q_t^{(n)} f, \bar{\mu}^n\rangle\right| \le (1+t^{1-\alpha})e^{c(1+\frac{1}{t^3})}\|e^{|\ell_f|}\|_{L^1(\Pi)} d_w(\mu,\bar{\mu}) \ ; \mu, \bar{\mu} \in \mathcal{M}_1(S) \ , t > 0,$$

for all $f \in \mathcal{B}_b(S^n)$.

Proof. Let $c_n(t)$ be as in 4.3. Since $\exp\left(\sum_{l=k+2}^n \frac{\binom{k}{2}}{\binom{l}{2}-\binom{k}{2}}\right) \le \exp(3k^{\frac{3}{2}})$ (cf. (3.5)) it follows from Young's inequality $st \le \frac{1}{4}s^4 + \frac{3}{4}t^{\frac{4}{3}}$, $s,t \ge 0$, again that

$$\begin{aligned} c_n(t) &\le c_0 n t^{-\alpha} e^{-t\binom{n}{2}} + c_0 \frac{6}{1-\alpha} t^{1-\alpha} \sum_{k=1}^{n-1} e^{4k^{\frac{3}{2}} - t\binom{k}{2}} \\ &\le c_0 n t^{-\alpha} e^{-t\binom{n}{2}} + c_0 \frac{6}{1-\alpha} t^{1-\alpha} \left(e^4 + \sum_{k=2}^{n-1} e^{\frac{4^4(2k)^3}{4(t(k-1))^3} - \frac{t}{4}\binom{k}{2}}\right) \\ &\le c_0 n t^{-\alpha} e^{-t\binom{n}{2}} + c_0 \frac{6}{1-\alpha} t^{1-\alpha} \left(e^4 + e^{\frac{4^6}{t^3}} \sum_{k=2}^{n-1} e^{-\frac{t}{4}\binom{k}{2}}\right) . \end{aligned}$$

Thus we conclude that there exists a positive constant c_1 such that $c_n(t) \le (1+t^{1-\alpha})e^{c_1(1+\frac{1}{t^3})}$, $t > 0$. Since ν has full support, hence ν^n too, and $q_t^{(n)} f \in C(S^n)$, $t > 0$, we obtain from 3.5 that $\|q_t^{(n)} f\|_\infty = \|q_t^{(n)} f\|_{L^\infty(\nu^n)} \le e^{c_2(1+\frac{1}{t^3})}\|e^{|\ell_f|}\|_{L^1(\Pi)}$, $t > 0$, for some positive constant c_2. Combining 4.3 and 4.4 we conclude that for $\mu, \bar{\mu} \in \mathcal{M}_1(S)$

$$\begin{aligned} \left|\langle q_t^{(n)} f, \mu^n\rangle - \langle q_t^{(n)} f, \bar{\mu}^n\rangle\right| &\le \sum_{k=1}^{dn} \|\partial_k q_t^{(n)} f\|_\infty \, d_w(\mu,\bar{\mu}) \\ &\le c_n(\frac{t}{2})\|q_{t/2}^{(n)} f\|_\infty \cdot d_w(\mu,\bar{\mu}) \\ &\le c_n(\frac{t}{2}) e^{c_2(1+\frac{8}{t^3})}\|e^{|\ell_f|}\|_{L^1(\Pi)} \cdot d_w(\mu,\bar{\mu}) \\ &\le (1+t^{1-\alpha})e^{(c_1+c_2)(1+\frac{8}{t^3})}\|e^{|\ell_f|}\|_{L^1(\Pi)} \cdot d_w(\mu,\bar{\mu}) \ , t > 0 \ . \quad \square \end{aligned}$$

Proof of Theorem 4.2. (i) Let $C := \{\langle f, \mu^n\rangle | f \in C_b(S^n)\, n \ge 1\}$. Then 4.5 implies that for some positive constant

$$(4.2) \qquad |T_t F(\mu) - T_t F(\bar{\mu})| \le (1+t^{1-\alpha})e^{c(1+\frac{1}{t^3})}\|e^{|F|}\|_{L^1(\Pi)} \cdot d_w(\mu,\bar{\mu}) \ , t > 0 \ .$$

Since $C \subset C(\mathcal{M}_1(S))$ dense w.r.t. $\|\cdot\|_\infty$ (4.2) extends to all $F \in C(\mathcal{M}_1(S))$, which proves the first assertion. For general $F \in \mathcal{B}_b(\mathcal{M}_1(S))$ we can find a uniformly bounded sequence $(F_n) \subset C(\mathcal{M}_1(S))$ converging to F pointwise everywhere (cf. [EK1, 4.2]). In particular, $\sup_{n\ge1}\|e^{|F_n|}\|_{L^1(\Pi)} < +\infty$, which implies that $(T_t F_n)_{n\ge1} \subset C(\mathcal{M}_1(S))$ precompact by Arzela-Ascoli. Hence there exists $G \in C(\mathcal{M}_1(S)$ such that $\lim_{k\to\infty} T_t F_{n_k} = G$ uniformly for some subsequence $(T_t F_{n_k})$. Since on the other hand $T_t F_n \to T_t F$ pointwise everywhere we conclude that $T_t F \in C(\mathcal{M}_1(S))$ and (4.2) extends to all F.

(ii) For the proof of (ii) let us consider the two cases $p = \infty$ and $p < \infty$ separately. If $p = \infty$ and $(F_n) \subset L^\infty(\Pi)$ bounded (i) implies that $(T_t F_n) \subset C(\mathcal{M}_1(S))$ is bounded and uniformly continuous, hence precompact by Arzela-Ascoli, which implies the assertion. To prove compactness in the case $p < +\infty$ fix a sequence $(F_n) \subset L^{p+\varepsilon}(\Pi)$ such that $sup_{n\geq 1}\|F_n\|_{L^{p+\varepsilon}(\Pi)} \leq M$. Let $F_n^c := F_n 1_{\{|F_n|\leq c\}}$, $c > 0$. Then $(T_t(F_n^c)) \subset C(\mathcal{M}_1(S))$ precompact for all $c > 0$. Consequently, we can find a subsequence (n_k) such that $(T_t F_{n_k}^m)_{k\geq 1}$ is L^p–Cauchy for all m. Note that

$$\int |T_t(F_n - F_n^m)|^p \, d\Pi \leq \int |F_n - F_n^m|^p \, d\Pi \leq \left(\frac{1}{m}\right)^\varepsilon \int |F_n|^{p+\varepsilon} \, d\Pi \leq \left(\frac{1}{m}\right)^\varepsilon M$$

and thus $\lim_{m\to\infty} \sup_{n\geq 1} \int |T_t(F_n - F_n^m)|^p \, d\Pi = 0$. hence, if $\varepsilon_0 > 0$, we can find m_{ε_0} such that $\sup_{n\geq 1} \int |T_t(F_n - F_n^{m_{\varepsilon_0}})|^p \, d\Pi \leq \frac{\varepsilon_0}{3}$ and k_{ε_0} such that $\|T_t(F_{n_k}^{m_{\varepsilon_0}} - F_{n_l}^{m_{\varepsilon_0}})\|_{L^p(\Pi)} \leq \frac{\varepsilon_0}{3}$ for all $k, l \geq k_{\varepsilon_0}$. Consequently, $\|T_t(F_{n_k} - F_{n_l})\|_{L^p(\Pi)} \leq \varepsilon_0$ for all $k, l \geq k_{\varepsilon_0}$ which implies the assertion. $\square$

5. Compactness

Let S be a compact subset of $\mathbb{R}^d$ and A be a Feller generator with unique invariant probability measure ν satisfying $\text{supp}(\nu) \equiv S$.

Theorem 5.1. *Suppose that the semigroup (p_t) generated by A satisfies (U), (L) and (F). Let (T_t) be the semigroup of the associated FV-operator and Π be its unique invariant measure. Then:*

(i) $T_t : L^2(\Pi) \to L^2(\Pi)$ is compact for all $t > 0$.

(ii) The embedding $D(L_2) \hookrightarrow L^2(\Pi)$ is compact.

Proof of 5.1. (i) Fix $t > 0$ and let $(F_n) \subset L^2(\Pi)$ be bounded. We have to show that $(T_t F_n)$ contains a strongly convergent subsequence. We may suppose that $\|F_n\|_{L^2(\Pi)} \leq 1$. Let $G_n^m := (T_{t/2}F_n)1_{\{|T_{t/2}F_n|\leq m\}}$. Then $T_{t/2}G_n^m \in C(\mathcal{M}_1(S))$ by 4.2 (i) and $|T_{t/2}G_n^m(\mu) - T_{t/2}G_n^m(\bar\mu)| \leq (1 + t^{1-\alpha})e^{c(1+\frac{1}{t^3})+m} d_w(\mu, \bar\mu)$. By Arzela-Ascoli $(T_{t/2}G_n^m)_{n\geq 1}$ contains a uniformly convergent subsequence for all m. Hence, passing to a subsequence, again denoted by (F_n), we may suppose that $(T_{t/2}G_n^m)_{n\geq 1}$ is $L^2(\Pi)$–convergent for all m.

Note that 3.3 (ii) implies that $((T_{t/2}F_n)^2)$ is uniformly integrable. Consequently,

$$\begin{aligned}\lim_{m\to\infty} \sup_{n\geq 1} \int (T_{t/2}G_n^m - T_t F_n)^2 \, d\Pi &\leq \lim_{m\to\infty} \sup_{n\geq 1} \int (G_n^m - T_{t/2}F_n)^2 \, d\Pi \\ &\leq \lim_{m\to\infty} \sup_{n\geq 1} \int_{\{|T_{t/2}F_n|>m\}} (T_{t/2}F_n)^2 \, d\Pi = 0 \ .\end{aligned}$$

If $\varepsilon > 0$ is arbitrary we can therefore find m_ε such that $\sup_{n\geq 1} \|T_{t/2}G_n^{m_\varepsilon} - T_t F_n\|_{L^2(\Pi)} \leq \frac{\varepsilon}{3}$ and for m_ε there exist n_ε such that $\|T_{t/2}G_k^{m_\varepsilon} - T_{t/2}G_l^{m_\varepsilon}\|_{L^2(\Pi)} \leq \frac{\varepsilon}{3}$ for all $k, l \geq n_\varepsilon$. Consequently, $\|T_t F_k - T_t F_l\|_{L^2(\Pi)} \leq \|T_t F_k - T_{t/2}G_k^{m_\varepsilon}\|_{L^2(\Pi)} + \|T_{t/2}G_k^{m_\varepsilon} - T_{t/2}G_l^{m_\varepsilon}\|_{L^2(\Pi)} + \|T_{t/2}G_l^{m_\varepsilon} - T_t F_l\|_{L^2(\Pi)} \leq \varepsilon$ for all $k, l \geq n_\varepsilon$. Hence, (i) is proved.

(ii) Let (G_α) be the resolvent of L, i.e., $G_\alpha F = \int_0^\infty e^{\alpha t} T_t F \, dt$, $\alpha > 0$. Since (T_t) is compact for all $t > 0$ we conclude that G_α is compact too. Indeed, if $(F_n) \subset L^2(\Pi)$ is weakly convergent to F (i) implies that (T_tF_n) is strongly L^2–convergent to T_tF for all $t > 0$. Consequently, by Lebesgue's theorem,

$$\lim_{n\to\infty} \|G_\alpha F_n - G_\alpha F\|_{L^2(\Pi)} \le \lim_{n\to\infty} \int_0^\infty e^{-\alpha t} \|T_tF_n - T_tF\|_{L^2(\Pi)}\, dt = 0 \ .$$

Recall that $(L_2, D(L_2))$ denotes the L^2–generator of (T_t) (cf. Section 2). If $(F_n) \subset D(L_2)$ is bounded, it follows that $((1-L_2)F_n) \subset L^2(\Pi)$ is bounded and thus contains a subsequence $((1-L_2)F_{n_k})$ weakly convergent to some element $H \in L^2(\Pi)$. But then $F_{n_k} = G_1(1-L_2)F_{n_k} \to G_1H$ strongly in $L^2(\Pi)$, hence the assertion follows. □

Application to the existence of absolutely continuous measures for Girsanov-type perturbations.

In this subsection we are going to model FV-operators with interactive selection. Since the selective model is a first-order perturbation of the neutral model we will first consider an abstract setting as follows: Let S be a compact metric space, A be a Feller generator with unique invariant measure ν, and L be the corresponding FV-operator (with no selection). Consider a measurable function $B : S \times \mathcal{M}_1(S) \to \mathbb{R}$ such that

$$\alpha_0 := \sup_\mu \operatorname{var}_\mu(B(\cdot, \mu)) < \infty \ . \tag{5.1}$$

Denote by Π the unique invariant measure of L and let $D(L_2)$ be the domain of the corresponding L^2–realization. Note that by the invariance of Π

$$\int \operatorname{var}(\nabla F)\, d\Pi = \int L(F^2)\, d\Pi - 2\int LF\cdot F\, d\Pi = -2\int LF\, F\, d\Pi \le 2\|LF\|_{L^2(\Pi)}\|F\|_{L^2(\Pi)}$$

for all $F \in \mathcal{F}C_b^\infty(D(A))$. Since $\mathcal{F}C_b^\infty(D(A)) \subset D(L_2)$ dense, it follows that $F \mapsto \nabla.F(\mu) - \langle \nabla.F, \mu\rangle$ can be uniquely extended to all $F \in D(L_2)$. Consequently, the first-order perturbation

$$L_2^B F(\mu) := L_2F(\mu) + \frac{1}{2}\operatorname{cov}_\mu(B(\cdot,\mu), \nabla.F(\mu)) \ , F \in D(L_2) \ ,$$

is well–defined.

Proposition 5.2. *$(L_2^B, D(L_2))$ generates a Markovian C_0–semigroup (T_t^B).*

Remark 5.3. Let $G = \ell_g$, $g \in D(A^{(n)})$, and $B(x,\mu) = \nabla_x G(\mu)$. Then (5.1) is clearly satisfied. If P is a solution for the $C_{\mathcal{M}_1(S)}([0,\infty))$–martingale problem for L, (X_t) is the canonical coordinate process, and $\mathcal{F}_t := \sigma\{X_s | s \le t\}$, $t \ge 0$, it follows that

$$R_t := \exp(G(X_t) - G(X_0) - 2\int_0^t e^{-\frac{G}{2}} L e^{\frac{G}{2}}(X_s)\, ds) \, , t \ge 0 \ ,$$

is an $(P, (\mathcal{F}_t))$–martingale with $E[R_t] = 1$. If we define Q by $\frac{dQ}{dP}|_{\mathcal{F}_t} = R_t$ it follows that Q is a solution of the martingale problem for L_2^B (cf. [EK2, Th. 3.3]). In particular, the semigroup (U_t) defined by $U_tF(\mu) = E_\mu[F(X_t)R_t]$, $t \ge 0$, $F \in \mathcal{B}_b(\mathcal{M}_1(S))$, uniquely determines a C_0–semigroup on $L^2(\Pi)$ whose generator extends $(L_2^B, D(L_2))$. Since $(L_2^B, D(L_2))$ is maximal, i.e., there exists only one extension of $(L_2^B, D(L_2))$ generating a C_0–semigroup on $L^2(\Pi)$ ($(L_2^B, D(L_2))$ itself) it follows that (U_t) and (T_t^B) coincide. Equivalently, $(L_2^B, D(L_2))$ is the L^2–realization of the Girsanov transform Q of P.

Lemma 5.4.

(i) Let $\varphi \in C^\infty(\mathbb{R})$, $\|\dot\varphi\|_\infty < \infty$. Then

$$\int L_2 F\, \varphi(F)\, d\Pi = -\frac{1}{2}\int var_\mu(\nabla F(\mu)\dot\varphi(F(\mu))\, \Pi(d\mu)$$

for all $F \in D(L_2)$.

(ii) $\int L_2^B F\, F^+\, d\Pi \le \frac{\alpha_0^2}{8}\int (F^+)^2\, d\Pi$ for all $F \in D(L_2)$.

Proof. (i) Let $\psi \in C^\infty(\mathbb{R})$ be such that $\dot\psi = \varphi$. Since $\mathcal{F}C_b^\infty(D(A)) \subset D(L_2)$ dense it suffices to consider $F \in \mathcal{F}C_b^\infty(D(A))$. But then $\psi(F) \in \mathcal{F}C_b^\infty(D(A))$ and $L\psi(F(\mu)) = \varphi(F(\mu))LF(\mu) + \frac{1}{2}\mathrm{var}_\mu(\nabla F(\mu))\dot\varphi(F(\mu))$. Consequently,

$$0 = \int L\psi(F(\mu))\, \Pi(d\mu) = \int LF(\mu)\varphi(F(\mu))\, \Pi(d\mu) + \frac{1}{2}\int \mathrm{var}_\mu(\nabla F(\mu))\dot\varphi(F(\mu))\Pi(d\mu)\ ,$$

which implies the assertion.

(ii) Again it suffices to consider $F \in \mathcal{F}C_b^\infty(D(A))$. Let $\varphi_\varepsilon \in C^\infty(\mathbb{R})$ be such that $\varphi_\varepsilon \ge -\varepsilon$, $0 \le \dot\varphi_\varepsilon$, $\varphi_\varepsilon(t) = t$ if $t \ge 0$. Then (i) implies

$$\begin{aligned}
\int L_2^B F\, F^+\, d\Pi &= \lim_{\varepsilon\to 0}\int L_2^B F\varphi_\varepsilon(F)\, d\Pi \\
&= \lim_{\varepsilon\to 0} -\frac{1}{2}\int \mathrm{var}(\nabla F)\dot\varphi_\varepsilon(F)\, d\Pi + \frac{1}{2}\int \mathrm{cov}(B, \nabla F)F^+\, d\Pi \\
&\le -\frac{1}{2}\int \mathrm{var}(F)1_{\{F\ge 0\}}\, d\Pi + \frac{1}{2}\int \mathrm{var}(F)1_{\{F\ge 0\}}\, d\Pi + \frac{\alpha_0^2}{8}\int (F^+)^2\, d\Pi\ ,
\end{aligned}$$

which implies the assertion. $\square$

Proof of 5.2. Since

$$\begin{aligned}
\int L_2^B F\, F\, d\Pi &= \int L_2 F\, F\, d\Pi + \frac{1}{2}\int \mathrm{cov}(B, \nabla F)\, F\, d\Pi \\
&\le -\frac{1}{2}\int \mathrm{var}(\nabla F)\, d\Pi + \frac{1}{8}\int \mathrm{var}(B)F^2\, d\Pi + \frac{1}{2}\int \mathrm{var}(\nabla F)\, d\Pi \\
&\le \frac{\alpha_0}{8}\int F^2\, d\Pi
\end{aligned}$$

for all $F \in D(L_2)$ it follows that $(L_2^B - \frac{\alpha_0}{8}, D(L_2))$ is negative definite. To prove that $L_2^B - \frac{\alpha_0}{8}$ (hence L_2^B too) generates a C_0–semigroup, it is therefore enough to show that $(\alpha + \frac{\alpha_0}{8} - L_2^B)(D(L_2)) = L^2(\Pi)$ for one (hence all) $\alpha > 0$. To this end let (G_α) be the resolvent of L and note that $F \mapsto \mathrm{cov}(B, \nabla F)$ is a strict contraction if $\alpha > 2\alpha_0^2$, since $\frac{1}{2}\int \mathrm{cov}(B, \nabla G_\alpha F)^2\, d\Pi \le \frac{\alpha_0^2}{2}\int \mathrm{var}(\nabla G_\alpha F)\, d\Pi = -\alpha_0^2\int L_2 G_\alpha F\, G_\alpha F\, d\Pi \le \alpha_0^2\|L_2 G_\alpha F\|_{L^2(\Pi)}\|G_\alpha F\|_{L^2(\Pi)} \le \frac{2\alpha_0^2}{\alpha}\|F\|^2_{L^2(\Pi)}$. Consequently, $T : F \mapsto F - \mathrm{cov}(B, \nabla G_\alpha F)$ is invertible if $\alpha > 2\alpha_0^2$. Since $(\alpha - L_2^B)G_\alpha \circ T^{-1}F = T^{-1}F - \frac{1}{2}\mathrm{cov}(B, \nabla G_\alpha T^{-1}F) = T \circ T^{-1}F = F$, the assertion follows. Let (V_α) be the resolvent generated by $L_2^B - \frac{\alpha_0}{8}$. Then (V_α) is positivity preserving, i.e., $V_\alpha F \ge 0$ if $F \ge 0$. To see this let $G := V_\alpha F$. Since $G^- = (-G)^+$ it follows from 5.3 (ii) that $\alpha \int (G^-)^2 \, d\Pi \leq \int (\alpha + \frac{\alpha_0}{8} - L_2^B)(-G)(-G)^+ \, d\Pi = -\int F(-G)^+ \, d\Pi \leq 0$, i.e., $G \geq 0$. Hence the semigroup (U_t) generated by $L_2^B - \frac{\alpha_0}{8}$ is positivity preserving too, since $U_t F = \lim_{\alpha \to \infty} e^{t(\alpha(\alpha V_\alpha - 1))} F \geq 0$ for $F \geq 0$, and so is the semigroup (T_t^B) generated by L_2^B, since $T_t^B = e^{\frac{\alpha_0}{8} t} U_t$, $t \geq 0$. Since $L_2^B 1 = 0$, i.e., $T_t^B 1 = 1$, it follows that (T_t^B) is Markovian. $\square$

Let us now assume again that S is a compact subset of $\mathbb{R}^d$.

Theorem 5.5. *Suppose that the semigroup (p_t) generated by A satisfies (U), (L) and (F). Let (T_t) be the semigroup generated by the corresponding FV-operator and denote by Π its unique invariant probability measure. Let $(L_2, D(L_2))$ be the corresponding L_2–realization and $B : S \times \mathcal{M}_1(S) \to \mathbb{R}$ satisfying (5.1). Then there exists an invariant measure Π^B for the semigroup (T_t^B) generated by $L_2^B F = L_2 F + \frac{1}{2} cov(B, \nabla F)$, $F \in D(L_2)$, which is absolutely continuous w.r.t. Π and has a square–integrable density.*

Proof. Since $D(L_2) \hookrightarrow L^2(\Pi)$ is compact by 5.1 and the topologies induced by the graph norm of L_2 and L_2^B respectively are equivalent it follows from [Ka, IV.5.26] that $\operatorname{ind}(L_2^B) = \operatorname{ind}(\frac{\alpha_0+1}{8} - L_2^B) = 0$. Since on the other hand $\dim \ker(L_2^B) > 0$ (since $L_2^B 1 = 0$) we conclude that $\dim \operatorname{koker}(L_2^B) > 0$. Hence there exists $H \in L^2(\Pi) \setminus \{0\}$ such that $\int L_2^B F H \, d\Pi = 0$ for all $F \in D(L_2)$. If we denote the adjoint semigroup of (T_t^B) by $(\hat{T}_t^B)$ the last equality implies in particular $\hat{T}_t^B H = H$, $t \geq 0$. We will show in the following that we may suppose that $H \geq 0$. Since T_t^B is positivity preserving we conclude that its adjoint $\hat{T}_t^B$ is positivity preserving too, because $F \geq 0$ implies that $\int \left((\hat{T}_t^B F)^- \right)^2 \, d\Pi = -\int \hat{T}_t^B F (\hat{T}_t^B F)^- \, d\Pi = -\int F \, T_t^B (\hat{T}_t^B F)^- \, d\Pi \leq 0$, i.e., $\hat{T}_t^B F \geq 0$. Consequently, $\hat{T}_t^B (H^+) \geq (\hat{T}_t^B H)^+ = H^+$, and on the other hand $\int \hat{T}_t^B (H^+) - H^+ \, d\Pi = \int H^+ T_t^B 1 - H^+ \, d\Pi = 0$, i.e., $\hat{T}_t^B (H^+) = H^+$ and thus, $\hat{T}_t^B (H^-) = H^-$ too. Since at least one, H^+ or H^-, $\neq 0$ it follows that there exists $H \geq 0$, $\int H \, d\Pi = 1$, such that $\int L_2^B F \, H d\Pi = 0$ for all $F \in D(L_2)$ and hence $H \, d\Pi$ is an invariant probability measure for the semigroup (T_t^B) generated by $(L_2^B, D(L_2))$. $\square$

Example 5.6. (population homozygosity, cf. [EK2, p. 357]) Let $D := \{(x, x) | x \in S\}$ be the diagonal in S^2 and $\lambda > 0$. Let $G(\mu) := \lambda \langle 1_D, \mu^2 \rangle$ and $B(x, \mu) = \nabla_x G(\mu)$. Then B satisfies (5.1).

6. Examples

a) Fleming-Viot operators with derivations as mutation operators.

This class of examples is the simplest class of Fleming–Viot operators. Let A be the generator of a Feller semigroup (p_t) with unique invariant probability measure ν and let L be the corresponding Fleming-Viot operator with no selection and no recombination. The corresponding invariant measure Π can be identified explicitely in the special case where the semigroup (p_t) commutes with the nonlinear operators $\phi_{ij}^{(n)}$. In this case we obtain in particular that

$$(p_t f)^2(x) = \phi_{12}^{(2)} p_t^{(2)} (f \otimes f)(x) = p_t^{(1)} \phi_{12}^{(2)} (f \otimes f)(x) = p_t(f^2)(x) \; .$$

It follows that the measure $p_t(x, \cdot)$ is equal to a Dirac measure, i.e., there exists $\xi_x(t) \in S$ such that $p_t f(x) = f(\xi_x(t))$. By the Markov-property, $t \mapsto \xi_x(t)$ induces a deterministic flow on the space S and the corresponding generator A is a differential operator "of first order". Moreover, the domain $D(A)$ of the generator of (p_t) on $C(S)$ is an algebra and the derivation property $A(fg) = Af\,g + f\,Ag$ holds, since $f, g \in D(A)$ implies that

$$\|\frac{1}{t}(p_t(fg) - fg) - (Af\,g + f\,Ag)\|_\infty \leq \|\frac{1}{t}(p_t f - f) - Af\|_\infty \|g\|_\infty + \|\frac{1}{t}(p_t g - g) - Ag\|_\infty \|g\|_\infty \to 0\,,$$

if $t \to 0$. This implies in particular that $-A$ is the dual operator of A (w.r.t. the measure ν).

Example 6.1. Let $S = [0, 1]$ and A be the time-derivative with periodic boundary conditions. Then $p_t f(x) = f((x + t) \text{mod } 1)$. Since (p_t) is a Feller semigroup with unique invariant measure dt, the invariant measure Π for the corresponding FV-transition semigroup is unique too.

The following Proposition identifies the measure Π.

Proposition 6.2. *Let A be a derivation with unique invariant measure ν. Let Π be the uniquely determined invariant measure for the corresponding FV-operator. Then*

$$\Pi(B) = \int 1_B(\delta_x)\nu(dx)\ , B \in \mathcal{B}(\mathcal{M}_1(S))\,.$$

Proof. It suffices to prove that for $f \in \mathcal{B}_b(S^n)$

$$\int \langle f, \mu^n\rangle\, \Pi(d\mu) = \int f(x, \dots, x)\nu(dx)\ .$$

We proceed by induction. If $n = 1$ the assertion is obvious. Suppose that the statement is proved for all $f \in \mathcal{B}_b(S^{n-1})$. Since

$$\int \langle f, \mu^n\rangle\, \Pi(d\mu) = \sum_{1\leq i<j\leq n} \int_0^\infty e^{-t\binom{n}{2}} \int \langle \phi_{ij}^{(n)} p_t^{(n)} f, \mu^{n-1}\rangle\, \Pi(d\mu)\, dt$$

(cf. the proof of 3.4), it follows that

$$\begin{aligned}\int \langle f,\mu^n\rangle\, \Pi(d\mu) &= \sum_{1\leq i<j\leq n} \int_0^\infty e^{-t\binom{n}{2}} \int \phi_{ij}^{(n)} p_t^{(n)} f(x, \dots, x)\,(dx)\, dt \\ &= \sum_{1\leq i<j\leq n} \int_0^\infty e^{-t\binom{n}{2}} \int p_t^{(n-1)} \phi_{ij}^{(n)} f(x, \dots, x)\,(dx)\, dt = \int f(x, \dots, x)\,\nu(dx)\ . \quad \square\end{aligned}$$

Remark 6.3. The last Proposition implies in particular that the measure Π has no full support. Moreover, if $f \in D(A^{(n)})$,

$$L\ell_f(\mu) = \sum_{i<j} \langle \phi_{ij}^{(n)} f, \mu^{n-1}\rangle - \langle f, \mu^n\rangle + \langle A^{(n)} f, \mu^n\rangle = \langle A^{(n)} f, \mu^n\rangle\ \Pi\text{-a.s.}$$

Consequently, the diffusion part of the Fleming-Viot operator L corresponding to A vanishes and L is a differential operator of first order w.r.t. its unique invariant measure Π.

The explicit expression of Π also allows one to identify the dual operator $\hat{L}$ of L:

Proposition 6.4. *The dual operator $\hat{L}$ of L (w.r.t. the measure Π) is the FV–operator with mutation operator $-A$ (the dual operator $\hat{A}$ of A (w.r.t. the measure ν)).*

Proof. Let $f, g \in D(A^{(n)})$. First note that

$$\int A^{(n)} f(x, \dots, x) g(x, \dots, x)\, \nu(dx) = -\int f(x, \dots, x) A^{(n)} g(x, \dots, x)\, \nu(dx)\ .$$

Since, by 6.2,

$$\sum_{i<j} \int \left(\langle \phi_{ij}^{(n)} f, \mu^{n-1}\rangle - \langle f, \mu^n\rangle \right) \langle g, \mu^n\rangle\, \Pi(d\mu)$$
$$= \sum_{i<j} \int \left(\phi_{ij}^{(n)} f(x, \dots, x) - f(x, \dots, x) \right) g(x, \dots, x) \nu(dx) = 0\ ,$$

and similarly,

$$\sum_{i<j} \int \left(\langle \phi_{ij}^{(n)} g, \mu^{n-1}\rangle - \langle g, \mu^n\rangle \right) \langle f, \mu^n\rangle\, \Pi(d\mu) = 0\ ,$$

it follows that

$$\begin{aligned}\int L\langle f, \mu^n\rangle \langle g, \mu^n\rangle\, \Pi(d\mu) &= \int \langle A^{(n)} f, \mu^n\rangle \langle g, \mu^n\rangle\, \Pi_\nu(d\mu)\\ &= \int A^{(n)} f(x, \dots, x) g(x, \dots, x)\, \nu(dx)\\ &= -\int f(x, \dots, x) A^{(n)} g(x, \dots, x)\, \nu(dx)\\ &= \int \langle f, \mu^n\rangle \left(\sum_{i<j} \left(\langle \phi_{ij}^{(n)} g, \mu^{n-1}\rangle - \langle g, \mu^n\rangle \right) - \langle A^{(n)} g, \mu^n\rangle \right) \Pi(d\mu)\ . \quad \square\end{aligned}$$

The derivation property of A implies in particular that A has no mass gap and 6.2 (cf. also 2.8) implies that the corresponding FV-operator has no mass gap either. Clearly, if $|S| = +\infty$, (p_t) neither satisfies (U), (L) nor (F), and the semigroup generated by the corresponding FV-operator is neither ultracontractive nor hypercontractive. However, this is not true in general for the associated resolvent as follows from the next example:

Example 6.5. Consider 6.1 again. If $D(L_2)$ denotes the domain of the L^2-generator of the associated FV-semigroup it follows quite surprisingly that $D(L_2) \subset L^\infty(\Pi)$ continuously. In particular, the resolvent generated by L is ultracontractive, i.e., $(\alpha - L)^{-1} : L^2(\Pi) \to L^\infty(\Pi)$ is bounded for all $\alpha > 0$.

Proof: Fix $f \in D(\frac{d^n}{dt})$ and let $\tilde{f}(x) := f(x, \dots, x)$. Note that $L\ell_f(\delta_x) = \sum_{k=1}^n \partial_k f(x, \dots, x) = \frac{d\tilde{f}}{dx}(x)$. Since $|\tilde{f} - \langle \tilde{f}\rangle|_\infty^2 \le \int \frac{d\tilde{f}}{dx}^2(x)dx$ it follows from 6.2 that

$$\|\ell_f - \langle \ell_f\rangle\|_\infty^2 = \|\tilde{f} - \langle \tilde{f}\rangle\|_\infty^2 \le \int \frac{d\tilde{f}}{dx}^2(x)dx = \int (L\ell_f)^2\, d\Pi\ .$$

Since $\{\ell_f | n \ge 1\ , f \in D(\frac{d^n}{dt})\}$ is a dense subset of $D(L)$, the last inequality now implies that $F \in L^\infty(\Pi)$ for all $F \in D(L)$ and $\|F - \langle F\rangle\|_\infty \le \|LF\|_2$. The ultracontractivity of $(\alpha - L)^{-1}$, $\alpha > 0$, is now an easy consequence.

b) Fleming-Viot operators with parent independent mutations.

In this case the mutation operator A is given by

$$Af(x) = \frac{\theta}{2} \int f(y) - f(x) \, \nu_0(dy) \, , f \in C(S) \ ,$$

for some probability measure ν_0 on the space of types S and some $\theta > 0$. Fleming-Viot operators with parent–independent mutation are the best studied class of Fleming-Viot operators. This is so because in this case the invariant measure, in this case denoted by m_{θ,ν_0}, can be described explicitely. In fact, let $(\rho_1, \rho_2, \dots)$ have a Poisson–Dirichlet distribution with parameter θ and let $(\xi_n)_{n\in\mathbb{N}}$ be i.i.d. with distribution ν_0 and independent of $(\rho_1, \rho_2, \dots)$. Then

$$m_{\theta,\nu_0}[A] = P\left[\sum_{n=1}^{\infty} \rho_n \delta_{\xi_n} \in A\right] \ .$$

Moreover, m_{θ,ν_0} is a symmetrizing measure for the corresponding Fleming-Viot operator L_{θ,ν_0} (cf. [EK2]). Let (p_t) (resp. (T_t)) be the semigroup generated by A (resp. L_{θ,ν_0}). Clearly, $p_t f(x) = e^{-\frac{\theta}{2}t} f(x) + (1 - e^{-\frac{\theta}{2}t})\langle f\rangle$, so that $\|p_t f - \langle f\rangle\|_\infty \le 2e^{-\frac{\theta}{2}t}\|f\|_\infty$. 2.7 now implies that $\|T_t F - \langle F\rangle\|_{L^\infty(m_{\theta,\nu_0})} \le ce^{-\frac{\theta}{2}t}\|F\|_{L^\infty(m_{\theta,\nu_0})}$, $t \ge 0$, for some constant c. Since (T_t) is symmetric 2.10 (ii) now implies that $\|T_t F - \langle F\rangle\|_{L^2(m_{\theta,\nu_0})} \le e^{-\frac{\theta}{2}t}\|F\|_{L^2(m_{\theta,\nu_0})}$, $t \ge 0$, i.e., L_{θ,ν_0} has a mass gap of size $\frac{\theta}{2}$. It follows from [St2, 2.2 (i) and 3.4] that this result is optimal. Moreover, it has been shown in [St2, 3.6] that (T_t) is hypercontractive if and only if $|\text{supp }(\nu)| < \infty$. Clearly, (T_t) is neither strong Feller nor compact if $|\text{supp }(\nu)| = \infty$, since $T_t \ell_f(\mu) = e^{-\frac{\theta}{2}t}\langle f, \mu\rangle + (1 - e^{-\frac{\theta}{2}t})\langle f, \nu_0\rangle$, $f \in \mathcal{B}_b(S)$.

c) Mutations induced by diffusions on $\mathbb{T}^d$.

In this example we will generalize Example 4.1 to a general diffusion operator on the d–dimensional torus $\mathbb{T}^d$. We consider $\mathbb{T}^d$ as the set $[0,1]^d$ by identifying opposite boundaries. Let

$$Af(x) = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij} f(x) + \sum_{i=1}^{d} \partial_i f(x) \ , f \in C^2(\mathbb{T}^d),$$

be a differential operator satisfying

$$\lambda_0 |h|^2 \le \sum_{i,j=1}^{n} a_{ij}(x) h_i h_j \le \lambda_0^{-1}|h|^2 \, , h \in \mathbb{R}^d \ ,$$

for some strictly positive constant λ_0 and

$$a_{ij}, b_i \in C^3(\mathbb{T}^d) \, , 1 \le i,j \le d \ .$$

In the following, we will construct an associated Feller semigroup (p_t) on $\mathbb{T}^d$ having a unique invariant probability measure with full support and satisfying (U), (L) and (F).

To this end let us introduce the following notation: if $g \in \mathcal{B}(\mathbb{T}^d)$, let $\bar{g} \in \mathcal{B}(\mathbb{R}^d)$ be its unique periodic extension to $\mathbb{R}^d$, i.e., $\bar{g}(x) = g(x)$ if $x \in \mathbb{T}^d$ and $\bar{g}(x+k) = \bar{g}(x)$ for all $k \in \mathbb{Z}^d$.

Consider the differential operator

$$\bar{A}f(x) \sum_{i,j=1}^{n} \bar{a}_{ij}(x)\partial_{ij}f(x) + \sum_{i=1}^{n} \bar{b}_i(x)\partial_i f(x) \ , f \in C_b^2(\mathbb{R}^d) \ .$$

Clearly, $(\bar{a}_{ij})$ is strictly uniformly elliptic again and $\bar{a}_{ij}$, $\bar{b}_i \in C_b^3(\mathbb{R}^d)$, $1 \le i, j \le d$. By [Fr, Th. 1.10] there exists a fundamental solution $\bar{p}_t(x,y)$ of $\bar{A}u = \partial_t u$ satisfying the following estimates:

$$(6.1) \qquad |\bar{p}_t(x,y)| \le c_1 t^{-\frac{d}{2}} \exp(-\lambda \frac{|x-y|^2}{2t}) \ , x, y \in \mathbb{R}^d, t > 0 \ ,$$

$$(6.2) \qquad |\frac{\partial}{\partial x_k} \bar{p}_t(x,y)| \le c_2 t^{-\frac{d+1}{2}} \exp(-\lambda \frac{|x-y|^2}{2t}) \ , x, y \in \mathbb{R}^d, t > 0 \ ,$$

for some positive constants c_1, c_2 and $\lambda > 0$. By [Fr, Th 1.12 and 2.10] $\bar{p}_t f(x) = \int f(y) p_t(x,y)\, dy$ is the unique solution of the Cauchy problem $\partial_t u = \bar{A}u$, $u(0,\cdot) = f$, $f \in C_b(\mathbb{R}^d)$. Moreover, $\bar{p}_t f \ge 0$ if $f \ge 0$ by [Fr, Th. 2.11] and $\bar{p}_t 1 = 1$ by uniqueness.

Let us prove a lower bound on $\bar{p}_t(x,y)$. First note that there exists $\delta_1, m, r > 0$ such that

$$(6.3) \qquad \sup_{y \in B_{2d}(0)} \bar{p}_t(x,y) \ge m \ \forall x \in \mathbb{T}^d \ , t \in (0, \delta_1).$$

Indeed, fix $\varphi \in C_0(\mathbb{R}^d)$, $1_S \le \varphi \le 1_{B_{2d}(0)}$. Let $\delta_1 > 0$ be such that $\|\bar{p}_t\varphi - \varphi\|_{\infty,S} \le \frac{1}{2}$, $t \in (0, \delta_1)$ and $m := \frac{1}{2\langle\varphi\rangle}$. Then $\frac{1}{2} \le \bar{p}_t\varphi(x) = \int_{\mathbb{T}^d} \varphi(x)\bar{p}_t(x,y)\, dy \le \sup_{y \in B_{2d}(0)} \bar{p}_t(x,y)\langle\varphi\rangle$, $t \in (0, \delta_1)$, which proves (6.3). Let

$$\bar{A}^* f(x) := \sum_{i,j=1}^{d} \bar{a}_{ij}(x)\partial_{ij}f(x) + \sum_{i=1}^{d}(2\sum_{j=1}^{d} \partial_j \bar{a}_{ij}(x) - \bar{b}_i(x))\partial_i f(x)$$

be the formal adjoint of $\bar{A}$. Then by [Fr, Appendix, Th. 5] (cf. also the proof of [Fr, Appendix, Cor. 1]) (Harnack's inequality) it follows that there exists a positive constant c, depending only on λ_0, d and the coefficients of $\bar{A}^*$, such that for $t \in (0, \delta_2)$, $\delta_2 > 0$ sufficiently small,

$$\bar{p}_t(x,y) \ge \bar{p}_{\frac{t}{2}}(x,\bar{y}) \exp(-c\frac{|y-\bar{y}|^2}{t} - 2) \ , x, y, \bar{y} \in \mathbb{R}^d \ ,$$

and now (6.3) implies that

$$(6.4) \qquad \bar{p}_t(x,y) \ge m \exp(-c\frac{|y-\bar{y}|^2}{t} - 2) \ge m \exp(-\frac{c3d}{t} - 2) \ , x, y \in \mathbb{T}^d \ , t \in (0, \delta_1 \wedge \delta_2).$$

Let us now define $p_t f(x) := \bar{p}_t \bar{f}(x) = \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{T}^d} f(y)\bar{p}_t(x, y+k)\, dy$, $f \in C(\mathbb{T}^d)$, $t \ge 0$. Clearly, (p_t) is a Feller semigroup on $\mathbb{T}^d$ whose generator extends $(A, C^2(\mathbb{T}^d))$.

Since $p_t(x,y) \ge \bar{p}_t(x,y)$ (6.4) implies that (p_t) satisfies (L). In particular, (p_t) is strictly positive and thus by [Ku, Th. 1.3.6] (p_t) has a unique invariant probability measure ν with full support. Moreover, ν is absolutely continuous w.r.t. dx (cf. [Ku, 1.3.5]). The proof of [Ku, 1.3.5] shows in addition that the density of ν is bounded from below by some strictly positive constant α_0. Consequently, by (6.1)

$$|p_t f(x)| \leq c_1 t^{-\frac{d}{2}} \sum_{k \in Z^d} \int_{\mathbb{T}^d} |f(y)| \exp(-\lambda \frac{|x-y-k|^2}{2t}) \, dy \leq M t^{-\frac{d}{2}} \|f\|_{L^1(\nu)} \ , f \in \mathcal{B}_b(\mathbb{T}^d) \, , t > 0,$$

hence (U) is satisfied.

Finally, note that (6.2) implies that

$$\begin{aligned} |\partial_k p_t f(x)| &\leq c_2 t^{-\frac{d+1}{2}} \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{T}^d} |f(y)| \exp(-\lambda \frac{|x-y-k|^2}{2t}) \, dy \\ &\leq c_0 t^{-\frac{1}{2}} \|f\|_{L^\infty(\nu)} \ , t > 0, f \in \mathcal{B}_b(\mathbb{T}^d). \end{aligned}$$

Consequently, (F) is satisfied.

Let (T_t) be the semigroup generated by the FV-operator with mutation A and denote by Π its unique invariant probability measure. We may now apply Theorems 4.2 and 5.1 to conclude that (i) (T_t) is strong Feller and (ii) compact on $L^2(\Pi)$.

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References

[Da] Davies, E.B., *Heat kernels and spectral theory*, Cambridge University Press, Cambrigde, 1989.

[DawM] Dawson, D., March, P., *Resolvent Estimates for Fleming-Viot Operators and Uniqueness of Solutions to Related Martingale Problems*, Journal of Funct. Anal. **132** (1995), 417–472.

[DK] Donnelly, P., Kurtz, T.G., *A countable representation of the Fleming-Viot measure-valued diffusion*, Ann. Probab. **24** (1999), 698–742.

[EG] Ethier, S.N., Griffiths, R.C., *The transition function of a Fleming–Viot process*, Ann. Probab. **21** (1993), 1571–1590.

[EK1] Ethier, S.N., Kurtz, T.G., *Markov Processes, Characterization and Convergence*, John Wiley & Sons, New York, 1986.

[EK2] Ethier, S.N., Kurtz, T.G., *Fleming–Viot processes in population genetics*, SIAM J. Control And Optimization **31** (1993), 345–386.

[EK3] Ethier, S.N., Kurtz, T.G., *Convergence to Fleming-Viot processes in the weak-atomic topology*, Stochastic Process. Appl. **54** (1994), 1–27.

[EK4] Ethier, S.N., Kurtz, T.G., *Coupling and ergodic theorems for Fleming-Viot processes*, Ann. Probab. **26** (1998), 533–561.

[Fr] Friedman, A. ,, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[Ka] Kato, T., *Perturbation Theory of Linear Operators*, Springer, Berlin, 1980.

[KS] Konno, N. Shiga, T., *Stochastic Partial Differential Equations for some Measure-Valued Diffusions*, Probab. Th. Rel. Fields **79** (1988), 201–225.

[Ku] Kunita, H., *Stochastic flows and stochastic differential equations*, Cambridge studies in advanced mathematics, vol. 24, Cambridge University Press, Cambridge, 1990.

[ORS] Overbeck, L., Röckner, M., Schmuland, B., *An analytic approach to Fleming-Viot processes with interactive selection*, Ann. Probab. **23** (1995), 1–36.

[ReSiI] Reed, M., Simon, B., *Methods of modern mathematical physics I. Functional Analysis*, Academic Press, New York, 1980.

[ReSiII] Reed, M., Simon, B., *Methods of modern mathematical physics II. Fourier Analysis*, Academic Press, New York, 1975.

[Sh] Shimakura, N., *Equations différentielles provenant de la génetique des populations*, Tôhoka Math. J. **29** (1977), 287–318.

[St1] Stannat, W., *The Theory of Generalized Dirichlet Forms and Its Applications in Analysis and Stochastics*, Mem. of the AMS, vol. 142 (678), 1999.

[St2] Stannat, W., *On the Validity of the Log–Sobolev Inequality for symmetric Fleming–Viot operators*, to appear in: Ann. Probab.