Some associative algebras related to $U(\mathfrak{g})$ and twisted generalized Weyl algebras

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Abstract

We prove that both Mickelsson step algebras and Orthogonal Gelfand-Zetlin algebras are twisted generalized Weyl algebras. Using an analogue of the Shapovalov form we construct all weight simple graded modules and some classes of simple weight modules over a twisted generalized Weyl algebra, improving the results from [MT1], where a particular class of algebras was considered and only special modules were classified.

1 Introduction

In the representation theory of infinite-dimensional associative algebras the description of all representations is usually rather difficult and therefore the investigations are naturally restricted to some special classes, for example, the so-called weight modules (with respect to a fixed subalgebra). A naive visualization of such module is usually the lattice of its weights together with the action of the generators of our algebra on weight components (subspaces). This inspired two of us to introduce in [MT2] a construction of associative algebras, called *twisted generalized Weyl construction* (TGWC in the sequel), which "agrees" with the picture described above. The construction generalizes the notion of twisted generalized Weyl algebra (TGWA) from [MT1] and the earlier notion of generalized Weyl algebra, originally defined by Bavula (see [B] and references therein). As it was shown in [MT1, MT2], many known algebras like certain (quantized) universal enveloping algebras, (quantized) Weyl algebras, (quantized) CCR-algebra and others can be realized via TGWC.

Another motivation for TGWC was a question of Yu.Drozd to find a natural generalization of the Bavula's construction, which covers, in particular, the universal enveloping algebras of semi-simple complex Lie algebras. An evidence that some TGWC-obtained algebras are close to the enveloping algebras was established in [MT1, Example 2], where certain similarity between the supports of weight modules was obtained.

The aim of this paper is to deeper this connection. There are two classes of associative algebras, known to be closely related to $U(\mathfrak{gl}(n,\mathbb{C}))$. The first one is the class of *Mickelsson* *step algebras* ([Mi] or [Z, Chapter 4]), connected with highest weight $U(\mathfrak{gl}(n,\mathbb{C}))$-modules. The second one is the class of *orthogonal Gelfand-Zetlin algebras* (OGZ-algebras), defined in [Ma] using the formulae from the famous Gelfand-Zetlin construction of simple finite-dimensional $U(\mathfrak{gl}(n,\mathbb{C}))$-modules. We prove that Mickelsson algebras as well as a certain extension of OGZ algebras are TGWAs. Hence, we give a partial answer to the mentioned question of Drozd using the fact that $U(\mathfrak{gl}(n,\mathbb{C}))$ is an OGZ algebra.

The paper is organized as follows: In Section 2 we define all main objects of our interest, namely TGWC and TGWA in Subsection 2.1, Mickelsson algebras in Subsection 2.2 and extended OGZ algebras in Subsection 2.3. In Sections 3 and 4 we show how to obtain respectively extended OGZ algebras and Mickelsson algebras via the twisted generalized Weyl construction. In Section 5 we prove that these algebras are in fact twisted generalized Weyl algebras using an analogue of the Shapovalov form on TGWC. In Section 6 we apply the Shapovalov form to construct weight simple graded modules over a TGWA in an abstract situation, extending the results from [MT1]. These results can be easily used to construct certain simple weight modules over Mickelsson step algebras and extended OGZ algebras. Finally, in Section 7 we reduce the classification of simple weight modules over a TGWA to the classification of simple modules over a certain subalgebra and invistigate the structure of the last one in several cases. In particular, we give some sufficient condition for this subalgebra to be commutative and show that its graded elements always commute or anticommute (in the case when the basic ring is a domain).

2 Preliminaries

2.1 TGWC and TGWA

Fix a positive integer, k, and set $\mathbb{N}_k = \{1, 2, \dots, k\}$. Let R be a ring with a unit element, $\{\sigma_i \,|\, 1 \le i \le n\}$ a set of pairwise commuting automorphisms of R and M a matrix, $(\mu_{i,j})_{i,j\in\mathbb{N}_k}$, whose entries are invertible elements from the center of R, stable under all σ_i (e.g. $\mu_{i,j} = 1$ for all i, j). Fix central elements $0 \ne t_i \in R$, $i \in \mathbb{N}_k$, satisfying the following relations:

$$t_i t_j = \mu_{i,j}\mu_{j,i}\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i), \quad i, j \in \mathbb{N}_n, i \ne j.$$

Define $\mathcal{A}$ to be a unital R-algebra generated over R by indetermines X_i, Y_i, $i \in \mathbb{N}_k$, subject to the relations

- $X_i r = \sigma_i(r) X_i$ for any $r \in R$, $i \in \mathbb{N}_k$;
- $Y_i r = \sigma_i^{-1}(r) Y_i$ for any $r \in R$, $i \in \mathbb{N}_k$;
- $X_i Y_j = \mu_{i,j} Y_j X_i$ for any $i, j \in \mathbb{N}_k$, $i \ne j$;
- $Y_i X_i = t_i$, $i \in \mathbb{N}_k$;
- $X_i Y_i = \sigma_i(t_i)$, $i \in \mathbb{N}_k$.

We will say that $\mathcal{A}$ is obtained from R, M, $\{\sigma_i\}$ and $\{t_i\}$ by *twisted generalized Weyl construction.*

Algebra $\mathcal{A}$ possesses a natural structure of $\mathbb{Z}^k$-graded algebra by setting $\deg R = 0$, $\deg X_i = g_i$, $\deg Y_i = -g_i$, $i \in \mathbb{N}_k$, where g_i, $i \in \mathbb{N}_k$ are the standard generators of $\mathbb{Z}^k$. For a graded $\mathcal{A}$-module M we set $\mathrm{grsupp}\, M = \{g \in \mathbb{Z}^k \mid M_g \neq 0\}$.

Let now R be commutative. The *twisted generalized Weyl algebra* $\hat{\mathcal{A}} = \mathcal{A}(R, \sigma_1, \dots, \sigma_k, t_1, \dots, t_k)$ of rank k is defined as the quotient ring $\mathcal{A}/I$, where I is the (unique) maximal graded two-sided ideal of $\mathcal{A}$ intersecting R trivially.

Denote by $\mathfrak{M}$ the set of maximal ideals $\mathfrak{m} \subset R$. For $\mathfrak{m} \in \mathfrak{M}$ and an $\mathcal{A}$-module ($\hat{A}$-module) V we set $V_{\mathfrak{m}} = \{v \in V \mid \mathfrak{m}v = 0\}$. An $\mathcal{A}$-module ($\hat{A}$-module), M, will be called *weight* provided $M = \sum_{\mathfrak{m}\in\mathfrak{M}} M_{\mathfrak{m}}$. For a weight module M we set $\mathrm{supp}\, M = \{\mathfrak{m} \in \mathfrak{M} \mid M_{\mathfrak{m}} \neq 0\}$. We will also denote by W the group, generated by all σ_i. By definition, W is a commutative group of finite rank.

Remark 1. *We note that the definition above is more general than one used in [MT1]. In that paper there were some additional assumptions on $\{\sigma_i\}$ and $\{t_i\}$ associated with a biserial graph and all $\mu_{i,j}$ were supposed to equal 1. It was already noticed in [MT2] that these assumptions are superfluous for $*$-representations. However, the constructions of simple weight modules over TGWAs from [MT1] heavily depend on these assumptions. In Section 6 we present a construction of simple weight modules for TGWA in the present setup, which covers all the results from [MT1].*

We refer the reader to [MT1, MT2] for further properties of TGWA and TGWC.

2.2 Mickelsson (step) algebras

In this Subsection we follow [Z, Chapter 4] and mostly use the same notation. With each reductive pair $(\mathfrak{g}, \mathfrak{k})$ we are going to associate an associative algebra, operating on the set of $\mathfrak{k}$-highest weight vectors of any $\mathfrak{g}$-module. For the further properties of these algebras and their applications we refer to [vH, Mi, Z].

Let $(\mathfrak{g}, \mathfrak{k})$ be a reductive pair of complex finite-dimensional Lie algebras and $\Delta_{\mathfrak{k}} = \Delta_{\mathfrak{k}}^+ \cup \Delta_{\mathfrak{k}}^-$ be the root system of $\mathfrak{k}$ with respect to the Cartan subalgebra $\mathfrak{h}$, decomposed into positive and negative roots. For a root, α, we will denote by X_α the corresponding element from a fixed Weyl-Chevalley basis. For any $\mathfrak{g}$-module V we will denote by V^+ the set $\{v \in V \mid X_\alpha v = 0 \text{ for all } \alpha \in \Delta_{\mathfrak{k}}^+\}$. For the algebra $\mathfrak{n}_+$, generated by all X_α, $\alpha \in \Delta_{\mathfrak{k}}^+$, we denote by I_+ the left ideal $U(\mathfrak{g})\mathfrak{n}_+$ of $U(\mathfrak{g})$ and set $V(\mathfrak{g}, \mathfrak{k}) = U(\mathfrak{g})/I_+$. Then the *Mickelsson step algebra* $S(\mathfrak{g}, \mathfrak{k})$, associated with $(\mathfrak{g}, \mathfrak{k})$, is defined as $V(\mathfrak{g}, \mathfrak{k})^+$. A slightly more convenient algebra appears if we invert $U(\mathfrak{h})$. Let $D(\mathfrak{h})$ denote the fraction field of $U(\mathfrak{h})$. Set $U'(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} D(\mathfrak{h})$, $I'_+ = U'(\mathfrak{g})\mathfrak{n}_+$, $V'(\mathfrak{g}, \mathfrak{k}) = U'(\mathfrak{g})/I'_+$ and $Z(\mathfrak{g}, \mathfrak{k}) = V'(\mathfrak{g}, \mathfrak{k})^+$.

Let $\mathfrak{g}_n = \mathfrak{gl}(\mathfrak{n}, \mathbb{C})$, $\mathfrak{h}_n$ be the Cartan subalgebra of diagonal matrices. In this paper we will be interested in the algebra $AZ_n = Z(\mathfrak{gl}_{n+1}, \mathfrak{gl}_n \oplus \mathbb{C}e_{n+1,n+1})$. According to [Z, Section 4.5] this algebra has the following presentation. It is generated (over the field $D_{n+1} = D(\mathfrak{h}_{n+1})$) by elements z_i, $i \in \{\pm1, \pm2, \dots, \pm n\}$ subject to the following relations:

- $z_i z_j = \alpha_{i,j} z_j z_i$, $i + j \neq 0$;

- $z_i z_{-i} = \sum_{j=1}^{n} \beta_{i,j} z_{-j} z_j + \gamma_i$, $i = 1, 2, \ldots, n$;

- $[h_j, z_i] = (\varepsilon_i - \varepsilon_{n+1})(h_j) z_i$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n+1$;

- $[h_j, z_{-i}] = (\varepsilon_{n+1} - \varepsilon_i)(h_j) z_{-i}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n+1$;

where

$$\alpha_{i,j} = \alpha_{-j,-i} = \frac{\varphi^{+}_{i,j}}{\varphi_{i,j}}, 1 \leq i < j \leq n; \quad \alpha_{i,j} = 1, \operatorname{sign}(i) \neq \operatorname{sign}(j);$$

$$\beta_{i,j} = \delta^{-}_{i} \gamma_{i,j} \delta^{+}_{j}; \quad \gamma_i = \delta^{-}_{i} \varphi^{-}_{i,n+1}; \quad \varphi_{i,j} = h_i - h_j + j - i, \quad \varphi^{\pm}_{i,j} = \varphi_{i,j} \pm 1;$$

$$\gamma_{i,j} = (1 - \varphi_{i,j})^{-1}; \quad \delta^{\pm}_{i} = \prod_{k=i+1}^{n} \frac{\varphi^{\pm}_{i,k}}{\varphi_{i,k}}; \quad \varepsilon_i(h_j) = \delta_{i,j}, i, j = 1, 2, \ldots, n+1.$$

2.3 (Extended) OGZ-algebras

Let $\mathbb{F}$ be an arbitrary field of characteristic zero. Fix $n \in \mathbb{N}$ and $r = (r_1, r_2, \ldots, r_n) \in \mathbb{N}^n$ and set $|r| = \sum_{i=1}^{n} r_i$. Consider a vector space, $\mathcal{L} = \mathcal{L}(\mathbb{F}, r)$, of dimension k. We will call the elements of $\mathcal{L}$ *tableaux* and consider them as double indexed families

$$[l] = \{l_{i,j} \,|\, i = 1, \ldots, n; j = 1, \ldots, r_i\}.$$

The element r will be called the *signature* of $[l]$. We will denote by $\delta^{i,j} = [\delta^{i,j}]$, $1 \leq i \leq n$, $1 \leq j \leq r_i$, the Kronecker tableau, i.e. $\delta^{i,j}_{i,j} = 1$ and $\delta^{i,j}_{p,q} = 0$ for $p \neq i$ or $q \neq j$. Denote by $\mathcal{L}_0$ the subset of $\mathcal{L}$ that consists of all $[l]$ satisfying the following conditions:

1. $l_{1,j} = 0$, $j = 1, \ldots, r_1$;

2. $l_{n,j} = 0$, $j = 1, \ldots, r_n$;

3. $l_{i,j} \in \mathbb{Z}$, $2 \leq i \leq n-1$, $1 \leq j \leq r_i$.

Fix some $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{N}^n$. Consider a field, Λ, of rational functions in $|r|$ variables $\lambda_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq r_i$. Let $[\mathfrak{l}] \in \mathcal{L}(\Lambda, r)$ be the tableau defined by $\mathfrak{l}_{i,j} = \lambda_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq r_i$. Consider a vector space, $M = M([\mathfrak{l}])$, over Λ with the base $v_{[t]}$, $[t] \in [\mathfrak{l}] + \mathcal{L}_0$ (here $[t]$ is a formal index and thus M is infinite-dimensional over Λ). For $[t] \in [\mathfrak{l}] + \mathcal{L}_0$, $2 \leq i \leq n-1$ and $1 \leq j \leq r_i$ denote

$$a^{\pm}_{i,j}([t]) = \mp \frac{\prod_{m} (t_{i\pm 1,m} - t_{i,j})}{\prod_{m \neq j} (t_{i,m} - t_{i,j})}.$$

For $2 \le i \le n-1$, $1 \le j \le r_i$ we define Λ-linear operators $X^{\pm}_{i,j} : M \to M$ by $X^{\pm}_{i,j} v_{[t]} = a^{\pm}_{i,j}([t]) v_{[t]+[\delta^{i,j}]}$ and $H_{i,j} : M \to M$ by $H_{i,j} v_{[t]} = t_{i,j} v_{[t]}$. It follows immediately from the definition, that all polynomials in $H_{i,j}$ are invertible, so we can consider the localization ring $\mathcal{Q} = \mathcal{Q}(r)$ of $\mathbb{C}[H_{i,j}, 1 \le i \le n, 1 \le j \le r_i]$ with respect to the multiplicative set, generated by $H_{i,j} - H_{i,l} + m$ for all $i = 2, \dots, n-1$, $j = 1, \dots, r_i$, $m \in \mathbb{Z}$. We define the *extended orthogonal Gelfand-Zetlin algebra* $\mathcal{U} = \mathcal{U}(r)$ of signature r, as the $\mathbb{F}$-algebra, generated over $\mathbb{F}$ by $\mathcal{Q}$ and $X^{\pm}_{i,j}$, $2 \le i \le n-1$, $1 \le j \le r_i$. To obtain the original *orthogonal Gelfand-Zetlin algebra* $\hat{\mathcal{U}}$ of signature $r-1$, one has to take $r_1 = 0$ (repeating the above definition) and to consider a subalgebra of $\mathcal{U}$, generated by $X^{\pm}_i = \sum_{j=1}^{r_i} X^{\pm}_{i,j}$ and symmetric polynomials in $H_{i,j}$, $1 \le j \le r_i$ for all i. In particular, it is known ([Ma, Section 4]) that $U(\mathfrak{gl}(n, \mathbb{C}))$ is isomorphic to some OGZ algebra. The definition and properties of (extended) OGZ algebras are closely related to generic Gelfand-Zetlin $\mathfrak{gl}(n, \mathbb{C})$-modules ([DFO]).

3 Extended OGZ-algebras via TGWC

Fix $k \in \mathbb{N}$. Let $r = (k-1, k, k+1)$ and consider the corresponding extended OGZ-algebra $\mathcal{U} = \mathcal{U}(r)$. The aim of this Section is to show that $\mathcal{U}$ can be obtained using the twisted generalized Weyl construction.

For $i = 1, 2, \dots, k$ set $A_i = X^{+}_{2,i}$ and $B_i = X^{-}_{2,i}$. For $i = 1, 2, \dots, k$ we define the following elements of $\mathcal{Q}$:

$$T_i = -\frac{\prod_{j=1}^{k+1}(H_{3,j} - H_{2,i}) \prod_{j=1}^{k-1}(H_{1,j} - H_{2,i} - 1)}{\prod_{j \ne i}(H_{2,j} - H_{2,i}) \prod_{j \ne i}(H_{2,j} - H_{2,i} - 1)}.$$

For $i = 1, 2, \dots, k$ we also define the endomorphisms σ_i of $\mathcal{Q}$ as follows:

$$\sigma_i(H_{2,i}) = H_{2,i} - 1; \quad \sigma_i(H_{k,l}) = H_{k,l}, \quad k \ne 2 \text{ or } l \ne i.$$

Lemma 3.1. *1. $\{\sigma_i | i = 1, 2, \dots, k\}$ are pairwise commuting automorphisms of $\mathcal{Q}$.*

2. $T_i T_j = \sigma_i^{-1}(T_j)\sigma_j^{-1}(T_i)$ holds for any $i \ne j \in \{1, 2, \dots, k\}$.

Proof. One sees that the endomorphism of $\mathcal{Q}$ defined by setting $\sigma_i^{-1}(H_{2,i}) = H_{2,i} + 1$ and $\sigma_i^{-1}(H_{k,l}) = H_{k,l}$, $k \ne 2$ or $l \ne i$ is inverse to σ_i, hence σ_i is an automorphism. The commutativity of $\{\sigma_i\}$ is obvious. One gets the second statement by direct calculation. □

Lemma 3.2. *$B_i A_i = T_i$ and $A_i B_i = \sigma_i(T_i)$ holds for any $i = 1, 2, \dots, k$.*

Proof. Straightforward calculation. □

Lemma 3.3. *$A_i q = \sigma_i(q) A_i$ and $B_i q = \sigma_i^{-1}(q) B_i$, $q \in \mathcal{Q}$, $i = 1, 2, \dots, k$.*

Proof. It is sufficient to check that the equalities hold on the basis of $M([\mathfrak{l}])$. Let $v_{[t]}$ be a basis element. We have

$$A_i q v_{[t]} = q(t_{1,1},\ldots,t_{3,k+1})A_i v_{[t]} = -q(t_{1,1},\ldots,t_{3,k+1})\frac{\prod_{j=1}^{k+1}(t_{3,j}-t_{2,i})}{\prod_{j\neq i}(t_{2,j}-t_{2,i})}v_{[t]+[\delta^{2,i}]}$$

and

$$\sigma_i(q)A_i v_{[t]} = -\sigma_i(q)\left(\frac{\prod_{j=1}^{k+1}(t_{3,j}-t_{2,i})}{\prod_{j\neq i}(t_{2,j}-t_{2,i})}v_{[t]+[\delta^{2,i}]}\right) = -\frac{\prod_{j=1}^{k+1}(t_{3,j}-t_{2,i})}{\prod_{j\neq i}(t_{2,j}-t_{2,i})}\sigma_i(q)\left(v_{[t]+[\delta^{2,i}]}\right) =$$

$$= -\frac{\prod_{j=1}^{k+1}(t_{3,j}-t_{2,i})}{\prod_{j\neq i}(t_{2,j}-t_{2,i})}q(t_{1,1},\ldots,t_{2,i-1},t_{2,i}-1+1,t_{2,i+1},\ldots,t_{3,k+1})v_{[t]+[\delta^{2,i}]},$$

as desired. The second equality follows by similar arguments. □

Lemma 3.4. *$A_iB_j = B_jA_i$ for any $i,j = 1,2,\ldots,k$, $i\neq j$.*

Proof. As above, we check the equalities on the basis. We have

$$A_iB_jv_{[t]} = \frac{\prod_{l=1}^{k-1}(t_{1,l}-t_{2,j})}{\prod_{l\neq j}(t_{2,l}-t_{2,j})}A_i(v_{[t]-[\delta^{2,j}]}) =$$

$$= -\frac{\prod_{l=1}^{k+1}(t_{3,l}-t_{2,i})}{(t_{2,j}-t_{2,i}-1)\prod_{l\neq i,j}(t_{2,l}-t_{2,i})}\cdot\frac{\prod_{l=1}^{k-1}(t_{1,l}-t_{2,j})}{\prod_{l\neq j}(t_{2,l}-t_{2,j})}v_{[t]-[\delta^{2,j}]+[\delta^{2,i}]}$$

and

$$B_jA_iv_{[t]}=-\frac{\prod\limits_{l=1}^{k+1}(t_{3,l}-t_{2,i})}{\prod\limits_{l\neq i}(t_{2,l}-t_{2,i})}B_j(v_{[t]+[\delta^{2,i}]})=$$

$$=-\frac{\prod\limits_{l=1}^{k-1}(t_{1,l}-t_{2,j})}{(t_{2,i}-t_{2,j}+1)\prod\limits_{l\neq i,j}(t_{2,l}-t_{2,j})}\cdot\frac{\prod\limits_{l=1}^{k+1}(t_{3,l}-t_{2,i})}{\prod\limits_{l\neq i}(t_{2,l}-t_{2,i})}v_{[t]-[\delta^{2,j}]+[\delta^{2,i}]}.$$

Clearly, the results are the same. □

We define the elements $s_{i,j}\in\mathcal{Q}$, $i,j=1,2,\dots,k$ as follows:

$$s_{i,j}=\frac{H_{2,j}-H_{2,i}-1}{H_{2,j}-H_{2,i}+1}.$$

Lemma 3.5. *$A_iA_j=s_{i,j}A_jA_i$ and $B_iB_j=s_{i,j}^{-1}B_jB_i$ for all $i,j=1,2,\dots,k$, $i\neq j$.*

Proof. We again will check only the first equality, applying it to the basis elements.

$$A_iA_jv_{[t]}=-\frac{\prod\limits_{l=1}^{k+1}(t_{3,l}-t_{2,j})}{\prod\limits_{l\neq j}(t_{2,l}-t_{2,j})}A_i(v_{[t]+[\delta^{2,j}]})=$$

$$=\frac{\prod\limits_{l=1}^{k+1}(t_{3,l}-t_{2,i})}{(t_{2,j}-t_{2,i}+1)\prod\limits_{l\neq i,j}(t_{2,l}-t_{2,i})}\cdot\frac{\prod\limits_{l=1}^{k+1}(t_{3,l}-t_{2,j})}{\prod\limits_{l\neq j}(t_{2,l}-t_{2,j})}v_{[t]+[\delta^{2,j}]+[\delta^{2,i}]}$$

and analogously

$$s_{i,j}A_jA_iv_{[t]}=s_{i,j}\frac{\prod\limits_{l=1}^{k+1}(t_{3,l}-t_{2,j})}{(t_{2,i}-t_{2,j}+1)\prod\limits_{l\neq i,j}(t_{2,l}-t_{2,j})}\cdot\frac{\prod\limits_{l=1}^{k+1}(t_{3,l}-t_{2,i})}{\prod\limits_{l\neq i}(t_{2,l}-t_{2,i})}v_{[t]+[\delta^{2,j}]+[\delta^{2,i}]}.$$

One has that the results are the same, completing the proof. □

Let $\mathcal{A}$ be the TGWC, associated with $\mathcal{Q}$, $\{\mu_{i,j}=1\}$, $\{\sigma_i\}$, $\{T_i\}$.

Theorem 3.1. *$\mathcal{U}$ is isomorphic to the quotient of $\mathcal{A}$ modulo the ideal I, generated by all $X_iX_j - s_{i,j}X_jX_i$ and all $Y_iY_j - s_{i,j}^{-1}Y_jY_i$.*

Proof. From Lemmas 3.3–3.5 it follows that there is a natural epimorphism $\varphi : \mathcal{A} \to \mathcal{U}$ such that $\varphi(q) = q$, $q \in \mathcal{Q}$; $\varphi(X_i) = A_i$ and $\varphi(Y_i) = B_i$ for all $i = 1, 2, \ldots, k$. We have only to prove that the kernel of φ coincides with I. Clearly $\varphi(I) = 0$. Set

$$Z_i^l = \begin{cases} X_i^l, & l \geq 0 \\ Y_i^{-l}, & l < 0 \end{cases}, \quad C_i^l = \begin{cases} A_i^l, & l \geq 0 \\ B_i^{-l}, & l < 0 \end{cases}.$$

Let $x \in \mathcal{A}$ be such that $\varphi(x) = 0$. Because of the relations in $\mathcal{A}$ we can write:

$$x + I = \sum_{l_1,\ldots,l_k \in \mathbb{Z}_+} q_{l_1,\ldots,l_k} Z_1^{l_1} \ldots Z_k^{l_k} + I.$$

Applying φ, we get

$$\varphi(x) = \sum_{l_1,\ldots,l_k \in \mathbb{Z}_+} q_{l_1,\ldots,l_k} C_1^{l_1} \ldots C_k^{l_k} = 0.$$

Applying this equality to $v_{[t]}$ we see that the later holds if and only if all $q_{l_1,\ldots,l_k} = 0$, which forces $x \in I$. This completes the proof. $\square$

Remark 2. *All the arguments and results of this Section remain valid for (extended) OGZ-algebras, associated with the quantum algebra $U_q(gl_n)$, see [MT3, Section 5].*

4 Mickelsson algebras via TGWC

The aim of this Section is to show how to construct Mickelsson algebras using the twisted generalized Weyl construction. We will use the presentation of AZ_n given in Subsection 2.2. It will be more transparent to rewrite the weight conditions in the following detailed form:

$$\begin{array}{ll} z_ih_j = h_jz_i, j \neq i, n+1; & z_{-i}h_j = h_jz_{-i}, j \neq i, n+1; \\ z_ih_i = (h_i - 1)z_i; & z_{-i}h_i = (h_i + 1)z_{-i}; \\ z_ih_{n+1} = (h_{n+1} + 1)z_i; & z_{-i}h_{n+1} = (h_{n+1} - 1)z_{-i}. \end{array} \quad (1)$$

We set $t_i = z_{-i}z_i$ and denote by R the algebra, generated by $t_1, \ldots, t_n$ over the field D_{n+1}.

Lemma 4.1. *The algebra R is commutative.*

Proof. First we will show that the elements t_i commute pairwise. As z_i and z_j commute if i and j have different signs, we have $t_it_j = z_{-i}z_iz_{-j}z_j = z_{-i}z_{-j}z_iz_j = \alpha_{-i,-j}z_{-j}z_{-i}\alpha_{i,j}z_jz_i$. From the definition we get $\alpha_{i,j} = \alpha_{j,i}^{-1} = \alpha_{-i,-j}^{-1}$ and hence

$$t_it_j = \frac{\varphi_{i,j}}{\varphi_{i,j}^+} z_{-j}z_{-i}\frac{\varphi_{i,j}^+}{\varphi_{i,j}} z_jz_i.$$

From (1) it follows that $z_{-i}\varphi_{i,j} = \varphi^+_{i,j} z_{-i}$ and thus $z_{-i}\varphi^+_{i,j}\varphi^{-1}_{i,j} = (\varphi^+_{i,j}+1)(\varphi^+_{i,j})^{-1} z_{-i}$. We get

$$t_i t_j = \frac{\varphi_{i,j}}{\varphi^+_{i,j}} z_{-j} z_{-i} \frac{\varphi^+_{i,j}}{\varphi_{i,j}} z_j z_i = \frac{\varphi_{i,j}}{\varphi^+_{i,j}} z_{-j} \frac{\varphi^+_{i,j}+1}{\varphi^+_{i,j}} z_{-i} z_j z_i =$$
$$\frac{\varphi_{i,j}}{\varphi^+_{i,j}} \frac{\varphi^+_{i,j}}{\varphi_{i,j}} z_{-j} z_{-i} z_j z_i = z_{-j} z_{-i} z_j z_i = t_j t_i.$$

To complete the proof, it is sufficient to check that $h_j t_i = t_i h_j$ for all $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n+1$. By (1), we have $h_j t_i = h_j z_{-i} z_i = z_{-i} z_i h_j = t_i h_j$ if $j \neq i, n+1$; $h_i t_i = h_i z_{-i} z_i = z_{-i}(h_i - 1) z_i = z_{-i} z_i h_i = t_i h_i$ and $h_{n+1} t_i = h_{n+1} z_{-i} z_i = z_{-i}(h_{n+1}+1) z_i = z_{-i} z_i h_{n+1} = t_i h_{n+1}$. □

Define the endomorphisms σ_i, $i = 1, 2, \ldots, n$ of R as follows:

$$\sigma_i(h_k) = h_k, \quad k \neq i, n+1; \quad \sigma_i(h_i) = h_i - 1; \quad \sigma_i(h_{n+1}) = h_{n+1} + 1;$$
$$\sigma_i(t_j) = \frac{\varphi^-_{i,j}}{\varphi^-_{i,j} - 1} t_j, \quad j < i; \quad \sigma_i(t_j) = \frac{\varphi_{i,j}}{\varphi^-_{i,j}} t_j, \quad j > i; \quad \sigma_i(t_i) = \sum_{k=1}^{n} \beta_{i,k} t_k + \gamma_i.$$

We remark that $\sigma_i(t_i) = z_i z_{-i}$ directly by the definition.

Lemma 4.2. *Each endomorphism σ_i is in fact an automorphism of R.*

Proof. Define the endomorphism σ_i^{-1} as follows:

$$\sigma_i^{-1}(h_k) = h_k, \quad k \neq i, n+1; \quad \sigma_i^{-1}(h_i) = h_i + 1; \quad \sigma_i^{-1}(h_{n+1}) = h_{n+1} - 1;$$
$$\sigma_i^{-1}(t_j) = \frac{\varphi^-_{i,j}}{\varphi_{i,j}} t_j, \quad j < i; \quad \sigma_i^{-1}(t_j) = \frac{\varphi_{i,j}}{\varphi^+_{i,j}} t_j, \quad j > i;$$
$$\sigma_i^{-1}(t_i) = \frac{1}{\sigma_i^{-1}(\beta_{i,i})} \left(t_i - \sigma_i^{-1} \left(\sum_{k \neq i} \beta_{i,k} t_k - \gamma_i \right) \right).$$

One can easily check that $\sigma_i \circ \sigma_i^{-1} = \sigma_i^{-1} \circ \sigma_i$ is the identity, completing the proof. □

Lemma 4.3. *1. $z_i r = \sigma_i(r) z_i$ for all $r \in R$, $i = 1, 2, \ldots, n$.*

2. $z_{-i} r = \sigma_i^{-1}(r) z_{-i}$ for all $r \in R$, $i = 1, 2, \ldots, n$.

3. $t_i t_j = \sigma_i^{-1}(t_j) \sigma_j^{-1}(t_i)$, $i \neq j$.

Proof. From (1) it follows that the first two statements hold for $r \in D_{n+1}$. So, it is enough to check it for all t_j. Let $j \neq i$. We have $z_i t_j = z_i z_{-j} z_j = z_{-j} z_i z_j = z_{-j} \alpha_{i,j} z_j z_i = \sigma_j^{-1}(\alpha_{i,j}) t_j z_i$ and $z_{-i} t_j = \alpha_{-i,-j} t_j z_{-i}$. For $j < i$ we have $\alpha_{i,j} = \varphi_{i,j}/\varphi^-_{i,j}$ and hence

$$\sigma_j^{-1}(\alpha_{i,j}) = \sigma_j^{-1}\left(\frac{h_i - h_j + j - i}{h_i - h_j + j - i - 1} \right) = \frac{\varphi^-_{i,j}}{\varphi^-_{i,j} - 1}.$$

For $j > i$ we have $\alpha_{i,j} = \varphi^+_{i,j}/\varphi_{i,j}$ and hence

$$\sigma_j^{-1}(\alpha_{i,j}) = \sigma_j^{-1}\left(\frac{h_i - h_j + j - i + 1}{h_i - h_j + j - i}\right) = \frac{\varphi_{i,j}}{\varphi^-_{i,j}}.$$

Moreover, $\alpha_{-i,-j} = \alpha_{j,i} = \varphi^-_{i,j}/\varphi_{i,j}$ for $j < i$ and $\alpha_{-i,-j} = \alpha_{i,j}^{-1} = \varphi_{i,j}/\varphi^+_{i,j}$ for $j > i$. Finally, for $r = t_i$ we have $z_i t_i = z_i z_{-i} z_i = \sigma_i(t_i) z_i$ and, expressing t_i from $z_i z_{-i} = \sum_{k=1}^{n} \beta_{i,k} t_k + \gamma_i$ and using the definition of σ_i^{-1} (see the proof of Lemma 4.2), we get

$$\begin{aligned} z_{-i} t_i = z_{-i} \left(\frac{1}{\beta_{i,i}}\left(z_i z_{-i} - \sum_{k\neq i} \beta_{i,k} t_k - \gamma_i\right)\right) = \\ = \sigma_i^{-1}\left(\frac{1}{\beta_{i,i}}\right)\left(t_i - \sigma_i^{-1}\left(\sum_{k\neq i}\beta_{i,k} t_k - \gamma_i\right)\right) z_{-i} = \sigma_i^{-1}(t_i) z_{-i}. \end{aligned}$$

The first and the second statements are proved.

The last statement follows immediately from the definition of σ_i^{-1} and Lemma 4.1. □

Proposition 4.1. *The automorphisms $\{\sigma_i, i = 1, 2, \ldots, n\}$ are pairwise commuting.*

Proof. We have to prove that $\sigma_i(\sigma_j(r)) = \sigma_j(\sigma_i(r))$ holds for all $r \in R$ and all $i, j = 1, 2, \ldots, n$. It is easy to see that the equality holds for $r \in D_{n+1}$. Let $r = t_k$, $k \neq i, j$. Set $r_{j,k} = \varphi^-_{j,k}/(\varphi^-_{j,k} - 1)$ for $k < j$ and $r_{j,k} = \varphi_{j,k}/\varphi^-_{j,k}$ for $k > j$. Then $\sigma_j(t_k) = r_{j,k} t_k$, $j \neq k$ and we have

$$\begin{aligned} \sigma_i(\sigma_j(t_k)) = \sigma_i(r_{j,k} t_k) = r_{j,k}\sigma_i(t_k) = r_{j,k} r_{i,k} t_k = \\ = r_{i,k} r_{j,k} t_k = r_{i,k}\sigma_j(t_k) = \sigma_j(r_{i,k} t_k) = \sigma_j(\sigma_i(t_k)). \end{aligned}$$

To complete the proof we have only to consider the (most non-trivial) case $r = t_i$. First we assume $i < j$. By the definition of σ's we have

$$\sigma_i(\sigma_j(t_i)) = \sigma_i\left(\frac{\varphi^+_{i,j}}{\varphi^+_{i,j} + 1} t_i\right) = \frac{\varphi_{i,j}}{\varphi^+_{i,j}}\left(\sum_{k=1}^{n} \beta_{i,k} t_k + \gamma_i\right), \tag{2}$$

$$\sigma_j(\sigma_i(t_i)) = \sum_{k=1}^{n} \sigma_j(\beta_{i,k})\sigma_j(t_k) + \sigma_j(\gamma_i) = S_1 + S_2 + \sigma_j(\beta_{i,j})\sigma_j(t_j) + \sigma_j(\gamma_i), \tag{3}$$

where $S_1 = \sum_{k=1}^{j-1} \sigma_j(\beta_{i,k})\sigma_j(t_k)$ and $S_2 = \sum_{k=j+1}^{n} \sigma_j(\beta_{i,k})\sigma_j(t_k)$. We want to rewrite our expressions for S_1 and S_2.

Recall that $\beta_{i,k} = \delta_i^- \gamma_{i,k} \delta_k^+$. By the definition of δ's and γ's we have $\sigma_j(\gamma_{i,k}) = \gamma_{i,k}$ and

$$\sigma_j(\delta_i^-) = \sigma_j \left(\prod_{l=i+1}^{n} \frac{\varphi_{i,l}^-}{\varphi_{i,l}} \right) = \prod_{\substack{l=i+1 \\ l \neq i}}^{n} \frac{\varphi_{i,l}^-}{\varphi_{i,l}} \cdot \sigma_j \left(\frac{\varphi_{i,j}^-}{\varphi_{i,j}} \right) = \prod_{\substack{l=i+1 \\ l \neq i}}^{n} \frac{\varphi_{i,l}^-}{\varphi_{i,l}} \cdot \frac{\varphi_{i,j}}{\varphi_{i,j}^+} = \frac{(\varphi_{i,j})^2}{\varphi_{i,j}^- \varphi_{i,j}^+} \delta_i^-;$$

$$\sigma_j(\delta_k^+) = \sigma_j \left(\prod_{l=k+1}^{n} \frac{\varphi_{k,l}^+}{\varphi_{k,l}} \right) = \prod_{\substack{l=k+1 \\ l \neq j}}^{n} \frac{\varphi_{k,l}^+}{\varphi_{k,l}} \sigma_j \left(\frac{\varphi_{k,j}^+}{\varphi_{k,j}} \right) = \prod_{\substack{l=k+1 \\ l \neq j}}^{n} \frac{\varphi_{k,l}^+}{\varphi_{k,l}} \cdot \frac{\varphi_{k,j}^+ + 1}{\varphi_{k,j}^+} = \frac{\varphi_{k,j}(\varphi_{k,j}^+ + 1)}{(\varphi_{k,j}^+)^2} \delta_k^+.$$

And therefore

$$\sigma_j(\beta_{i,k}) = \frac{(\varphi_{i,j})^2 \varphi_{k,j} (\varphi_{k,j}^+ + 1)}{\varphi_{i,j}^- \varphi_{i,j}^+ (\varphi_{k,j}^+)^2} \beta_{i,k}.$$

Since $\sigma_j(t_k) = \dfrac{\varphi_{j,k}^-}{\varphi_{j,k}^- - 1} t_k = \dfrac{\varphi_{k,j}^+}{\varphi_{k,j}^+ + 1} t_k$, we obtain

$$S_1 = \frac{\varphi_{i,j}}{\varphi_{i,j}^+} \sum_{k=1}^{j-1} \frac{\varphi_{i,j} \varphi_{k,j}}{\varphi_{i,j}^- \varphi_{k,j}^+} \beta_{i,k} t_k.$$

Similarly, we get the following new expression for S_2:

$$S_2 = \frac{\varphi_{i,j}}{\varphi_{i,j}^+} \sum_{k=j+1}^{n} \frac{\varphi_{i,j} \varphi_{k,j}}{\varphi_{i,j}^- \varphi_{k,j}^+} \beta_{i,k} t_k.$$

Further, as $\sigma_j(\delta_j^+) = \displaystyle\prod_{l=j+1}^{n} \frac{\varphi_{j,l}}{\varphi_{j,l}^-} = \frac{1}{\delta_j^-}$, we have

$$\sigma_j(\beta_{i,j}) = \sigma_j(\delta_i^-) \sigma_j(\gamma_{i,j}) \sigma_j(\delta_j^+) = \frac{(\varphi_{i,j})^2}{\varphi_{i,j}^- \varphi_{i,j}^+} \delta_i^- \cdot \frac{(-1)}{\varphi_{i,j}} \cdot \frac{1}{\delta_j^-} = - \frac{\varphi_{i,j}}{\varphi_{i,j}^- \varphi_{i,j}^+} \cdot \frac{\delta_i^-}{\delta_j^-}.$$

Finally,

$$\sigma_j(\gamma_i) = \sigma_j(\delta_i^- \varphi_{i,n+1}^-) = \sigma_j(\delta_i^-) \sigma_j(\varphi_{i,n+1}^-) = \frac{(\varphi_{i,j})^2}{\varphi_{i,j}^- \varphi_{i,j}^+} \delta_i^- (\varphi_{i,n+1}^- - 1).$$

Inserting the obtained expressions to (3), and using $\varphi_{j,j} = 0$ we get

$$\begin{aligned} \sigma_j(\sigma_i(t_i)) &= \frac{\varphi_{i,j}}{\varphi_{i,j}^+} \sum_{k=1}^{j-1} \frac{\varphi_{i,j} \varphi_{k,j}}{\varphi_{i,j}^- \varphi_{k,j}^+} \beta_{i,k} t_k + \frac{\varphi_{i,j}}{\varphi_{i,j}^+} \sum_{k=j+1}^{n} \frac{\varphi_{i,j} \varphi_{k,j}}{\varphi_{i,j}^- \varphi_{k,j}^+} \beta_{i,k} t_k - \\ &\quad - \frac{\varphi_{i,j}}{\varphi_{i,j}^- \varphi_{i,j}^+} \frac{\delta_i^-}{\delta_j^-} \left(\sum_{k=1}^{n} \beta_{j,k} t_k + \gamma_j \right) + \frac{(\varphi_{i,j})^2}{\varphi_{i,j}^- \varphi_{i,j}^+} \delta_i^- (\varphi_{i,n+1}^- - 1) = \\ &= \frac{\varphi_{i,j}}{\varphi_{i,j}^+} \left(\sum_{k=1}^{n} \frac{\varphi_{i,j} \varphi_{k,j}}{\varphi_{i,j}^- \varphi_{k,j}^+} \beta_{i,k} t_k - \frac{\delta_i^-}{\varphi_{i,j}^- \delta_j^-} \sum_{k=1}^{n} \beta_{j,k} t_k - \frac{\delta_i^-}{\varphi_{i,j}^- \delta_j^-} \gamma_j + \frac{\varphi_{i,j}}{\varphi_{i,j}^-} \delta_i^- (\varphi_{i,n+1}^- - 1) \right). \end{aligned} \tag{4}$$

Now we use

$$\frac{\varphi_{i,j}\varphi_{k,j}}{\varphi_{i,j}^-\varphi_{k,j}^+}\beta_{i,k} - \frac{\delta_i^-}{\varphi_{i,j}^-\delta_j^-}\beta_{j,k} = \frac{1}{\varphi_{i,j}^-}\left(\frac{\varphi_{i,j}\varphi_{k,j}}{\varphi_{k,j}^+}\delta_i^- \frac{1}{\varphi_{k,i}^+}\delta_k^+ - \frac{\delta_i^-}{\delta_j^-}\delta_j^- \frac{1}{\varphi_{k,j}^+}\delta_k^+\right) =$$

$$= \frac{1}{\varphi_{i,j}^-}\frac{\delta_i^-\delta_k^+}{\varphi_{k,i}^+}\frac{\varphi_{i,j}\varphi_{k,j}-\varphi_{k,i}^+}{\varphi_{k,j}^+} = \beta_{i,k}\frac{\varphi_{i,j}\varphi_{k,j}-\varphi_{k,i}^+}{\varphi_{i,j}^-\varphi_{k,j}^+} = \beta_{i,k}\frac{\varphi_{i,j}\varphi_{k,j}-\varphi_{k,i}^+}{(\varphi_{i,j}-1)(\varphi_{k,j}+1)} =$$

$$= \beta_{i,k}\frac{\varphi_{i,j}\varphi_{k,j}-\varphi_{k,i}^+}{\varphi_{i,j}\varphi_{k,j}-(\varphi_{k,j}-\varphi_{i,j}+1)} = \beta_{i,k}$$

and

$$-\frac{\delta_i^-}{\varphi_{i,j}^-\delta_j^-}\gamma_j + \frac{\varphi_{i,j}}{\varphi_{i,j}^-}\delta_i^-(\varphi_{i,n+1}^- - 1) = -\frac{\delta_i^-}{\varphi_{i,j}^-\delta_j^-}\delta_j^-\varphi_{j,n+1}^- + \frac{\varphi_{i,j}}{\varphi_{i,j}^-}\delta_i^-(\varphi_{i,n+1}^- - 1) =$$

$$= \delta_i^-\frac{\varphi_{i,j}(\varphi_{i,n+1}^- - 1) - \varphi_{j,n+1}^-}{\varphi_{i,j}^-} = \delta_i^-\frac{\varphi_{i,j}\varphi_{i,n+1}^- - (\varphi_{i,n+1}^- - \varphi_{j,n+1}^-) - \varphi_{j,n+1}^-}{\varphi_{i,j}^-} =$$

$$= \delta_i^-\frac{\varphi_{i,n+1}^-(\varphi_{i,j}-1)}{\varphi_{i,j}^-} = \delta_i^-\varphi_{i,n+1}^- = \gamma_i$$

to get from (4) the following:

$$\sigma_j(\sigma_i(t_i)) = \frac{\varphi_{i,j}}{\varphi_{i,j}^+}\left(\sum_{k=1}^n \beta_{i,k}t_k + \gamma_i\right).$$

Now from (2) we have $\sigma_i(\sigma_j(t_i)) = \sigma_j(\sigma_i(t_i))$ for $i < j$. The case $j < i$ can be treated using the same arguments, completing the proof. $\square$

Let $\mathcal{B}$ denote the TGWC, associated with R, $\{\mu_{i,j} = 1\}$, $\{\sigma_i\}$, $\{t_i\}$.

Theorem 4.1. *AZ_n is isomorphic to the quotient of $\mathcal{B}$ modulo the ideal J, generated by $X_iX_j - \frac{\varphi_{i,j}^+}{\varphi_{i,j}}X_jX_i$, $1 \le i < j \le n$ and $Y_iY_j - \frac{\varphi_{i,j}}{\varphi_{i,j}^+}Y_jY_i$, $1 \le i < j \le n$.*

Proof. From Lemmas 4.1–4.3 and Proposition 4.1 it follows that there is a natural epimorphism $\varphi : \mathcal{B} \to AZ_n$ such that $\varphi(r) = r$, $r \in R$; $\varphi(X_i) = z_i$ and $\varphi(Y_i) = z_{-i}$ for all $i = 1, 2, \dots, n$. We have only to prove that the kernel of φ coincides with J, but this follows immediately from the presentation of AZ_n. $\square$

The discussion above motivates the study of the following natural question: is it true that $R \simeq D_{n+1}[t_1, \dots, t_n]$? It turns out that it is.

Proposition 4.2. *The monomials $t_1^{k_1}t_2^{k_2}\dots t_n^{k_n}Z_1^{k_1}Z_2^{k_2}\dots Z_n^{k_n}$, where $Z_i = z_i$ or $Z_i = z_{-i}$, form a basis of AZ_n over D_{n+1}, in particular, $R \simeq D_{n+1}[t_1, \dots, t_n]$.*

Proof. Follows from the Diamond Lemma ([Be, Theorem 1.2]) by standard arguments. $\square$

5 Shapovalov forms on TGWC with applications to $\mathcal{A}$ and $\mathcal{B}$

Let A be the TGWC, associated with some R, M, $\{\sigma_i\}$ and $\{t_i\}$, $i = 1, 2, \dots, n$. Assume that $\mu_{i,j} = \mu_{j,i}$ for all i, j. Till the end of the paper we assume that R is a domain.

Lemma 5.1. *There is a unique antiinvolution, $*$, on A such that $(X_i)^* = Y_i$ for any $i = 1, 2, \dots, n$ and $(r)^* = r$ for any $r \in R$.*

Proof. The uniqueness is trivial as A is generated by R, all X_i and all Y_i. To prove the existence we realize A as the quotient of the free associative R-algebra over $\{X_i\} \cup \{Y_i\}$ modulo the ideal I, generated by defining relations (see Subsection 2.1). Clearly, if $\mu_{i,j} = \mu_{j,i}$, the ideal I is stable under the corresponding antiinvolution on the free algebra, which induces the necessary antiinvolution on A. $\square$

Regard A as a $\mathbb{Z}^n$-graded algebra in a natural way and denote by $\mathfrak{p} : A \to A_0$ the graded projection on the zero component. For $u, v \in A$ put $F^l(u, v) = \mathfrak{p}(u^* v) \in A_0 = R$ and $F^r(u, v) = \mathfrak{p}(uv^*) \in A_0 = R$. As we will see these forms are quite analogous to the Shapovalov form ([S]), so we will call F^l the *left Shapovalov form* on A and F^r the *right Shapovalov form* on A.

Lemma 5.2. *1. $F^l : A \times A \to R$ and $F^r : A \times A \to R$ are R-bilinear form.*

2. $F^l(xu, v) = F^l(u, x^* v)$ and $F^r(u, vx) = F^r(ux^*, v)$ for all $u, x, v \in A$.

3. $F^l(u, v) = F^l(v, u)$ and $F^r(u, v) = F^r(v, u)$ for all $u, v \in A$.

4. $F^l(A_g, A_h) = 0$ and $F^r(A_g, A_h) = 0$ for any $g \neq h \in \mathbb{Z}^n$.

5. The ideal, generated by the intersection of the kernels of F^l and F^r coincides with the maximal graded ideal of A intersecting R trivially.

Proof. The first and the forth statements are obvious. The second one follows from $F^l(xu, v) = \mathfrak{p}((xu)^* v) = \mathfrak{p}(u^* x^* v) = F^l(u, x^* v)$ and analogous arguments work for F^r. To prove the third one we note that $r = r^*$ for any $r \in R = A_0$. Hence $\mathfrak{p}(x) = \mathfrak{p}(x^*)$ for any $x \in A$ and therefore $F^l(u, v) = \mathfrak{p}(u^* v) = \mathfrak{p}(v^* u) = F^l(v, u)$. For F^r one has $F^r(u, v) = \mathfrak{p}(uv^*) = \mathfrak{p}(vu^*) = F^r(v, u)$.

To prove the last statement we denote by I the maximal graded ideal of A intersecting R trivially and by J the ideal, generated by the intersection of the kernels of F^l and F^r. Let $v \in I$, $\deg(v) = g \in \mathbb{Z}^n$. Then $A_{-g} v = 0$ and $v A_{-g} = 0$ as $I \cap R = 0$ and hence $F^l(u, v) = \mathfrak{p}(u^* v) = 0$ and $F^r(v, u) = \mathfrak{p}(vu^*) = 0$ for any $u \in A_g$. This shows that $I \subset J$. By the definition, J is a two-sided graded ideal of A. To prove $J \subset I$ we need only to show that $J \cap R = 0$. Assume that $x_1, x_2 \in A$ are graded elements and u is a graded element from the intersection of the kernels of F^l and F^r. If $\deg(u) = g$ then we have $uA_{-g} = A_{-g} u = 0$, as $F^l(A_{-g}, u) = F^r(u, A_{-g}) = 0$. Suppose that $0 \neq r = x_1 u x_2 \in R$

and consider the element $t = x_1^* x_1 u x_2 x_1 \in R$. We have $t = w(r) x_1^* x_1$ for some $w \in W$. As w is an automorphism, all $t_i \neq 0$ and R is a domain, we get $t \neq 0$. From the other hand, $t = x_1^* x_1 (u x_2 x_1)$, where $x_2 x_1 \in A_{-g}$. This implies $t = 0$, a contradiction. Thus $J = I$ completing the proof. □

From the proof above we immediately get the following.

Corollary 5.1. *The intersection of the kernels of F^l and F^r coincides with I.*

Corollary 5.2. *The kernel of F^l coincides with I (and coincides with the kernel of F^r).*

Proof. For this it is enough to show that the kernel of F^l is an ideal. It is a graded left ideal, intersecting R trivially, because of the second statement of Lemma 5.2. To get that it is a right ideal we use the last arguments from the proof of Lemma 5.2. Let u be a graded element from the kernel of F^l and assume that ux does not belong to this kernel for some graded element $x \in A$. Then there exists a graded element $v \in A$ such that $0 \neq vux = F^l(v^*, ux) = r \in R$. Now we consider the element $v^* vuxv \in R$ and get the same contradiction as in the proof of Lemma 5.2. □

In the sequel we will use only the left Shapovalov form F^l, which we will denote simply by F and will call it the *Shapovalov form* on A. Corollary 5.2 allows us to formulate the following criterion for distinguishing a TGWA.

Corollary 5.3. *Let A be as above and J be a graded two-sided ideal of A, stable under $*$. Denote by $\tilde{F}$ the form induced by F on the quotient $\tilde{A} = A/J$. Then $\tilde{A}$ is isomorphic to the TGWA $\hat{A}$ if and only if $\tilde{F}$ is non-degenerate on $\tilde{A}$.*

Proof. Follows immediately from Corollary 5.2 and the definition of a TGWA. □

Corollary 5.4. *Let A be as above and J be a graded two-sided ideal of A, stable under $*$ and $\tilde{A} = A/J$. Assume that and each A_g, $g \in \mathbb{Z}^n$ is a cyclic left R-module. Then $\tilde{A}$ is a TGWA.*

Proof. Fix $0 \neq g = (g_1, g_2, \dots, g_n) \in \mathbb{Z}^n$. For $i = 1, 2, \dots, n$ set $Z_i = X_i$ and $Z_i^{-1} = Y_i$. Then $v_g = Z_1^{g^1} \dots Z_n^{g_n}$ generates A_g as a left R-module. For any $r, r' \in R$ the value $F^l(rv_g, r'v_g)$ is a product of non-zero elements from R, hence non-zero. Now the statement follows from Corollary 5.3. □

Corollary 5.5. *The algebras $\mathcal{A}$ and $\mathcal{B}$ are TGWA.*

Proof. First we remark that $\mu_{i,j} = \mu_{j,i} = 1$ in all cases. In both cases the zero component is a polynomial ring over a field, hence is a domain. From Theorems 3.1 and 4.1 it also follows that all graded components are cyclic left modules over the zero component and the statement follows from Corollary 5.4. □

6 Application of the Shapovalov form to construction of simple weight modules over a TGWA

Assume that we are in the situation of the previous Section with A and $\hat{A}$ being a TGWC and a TGWA respectively. Let F (resp. $\hat{F}$) denote the Shapovalov form on A (resp. $\hat{A}$). Our aim here is to use $\hat{F}$ for construction of simple weight $\hat{A}$-modules.

Consider $\hat{A}$ as a regular left $\hat{A}$-module and fix an ideal, $\mathfrak{m}$, in R. Set $N(\mathfrak{m}) = \{x \in \hat{A} \,|\, \hat{F}(x,y) \in \mathfrak{m} \text{ for any } y \in \hat{A}\}$. Recall that $\mathfrak{M}$ denotes the set of maximal ideals of R.

Lemma 6.1. *1. $N(\mathfrak{m})$ is a graded submodule of $\hat{A}$;*

2. $N(\mathfrak{m})_0 = \mathfrak{m}$;

3. If $\mathfrak{m} \in \mathfrak{M}$ then $M(\mathfrak{m}) = \hat{A}/N(\mathfrak{m})$ is a simple graded $\hat{A}$-module.

Proof. Let $x \in N(\mathfrak{m})$ and $a \in \hat{A}$. Then $\hat{F}(ax,y) = \hat{F}(x,a^*y) \in \mathfrak{m}$ for any $y \in \hat{A}$ and hence $ax \in N(\mathfrak{m})$. Therefore $N(\mathfrak{m})$ is a submodule of $\hat{A}$. As $\hat{F}$ separates the graded components of $\hat{A}$, $N(\mathfrak{m})$ is automatically graded. This proves the first statement. As A is unital, the second statement is obvious.

Finally, assume that $\mathfrak{m} \in \mathfrak{M}$. Let $v \in A_g$ for some $g \in \mathbb{Z}^n$ such that its image in $M(\mathfrak{m})_g$. Then there exists a graded element, $y \in \hat{A}$, such that $\hat{F}(y,v) \notin \mathfrak{m}$ and hence the image of y^*v in $R/\mathfrak{m}$ is also non-zero. As $\mathfrak{m}$ is maximal, $R/\mathfrak{m}$ is a field hence $\hat{A}v$ contains $M(\mathfrak{m})_0$, which clearly generates $M(\mathfrak{m})$. This completes the proof. □

From Lemma 6.1 it also follows that $N(\mathfrak{m})$ is the maximal graded submodule of $M(\mathfrak{m})$ whose intersection with $M(\mathfrak{m})_0$ equals $\mathfrak{m}$. From this it follows that, up to a shift of grading, all weight simple ($\mathbb{Z}^n$-) graded $\hat{A}$-modules are exhausted by $\{M(\mathfrak{m})\}$.

Corollary 6.1. *Let $\mathfrak{m} \in \mathfrak{M}$ such that $g(\mathfrak{m}) \neq \mathfrak{m}$ for any $0 \neq g \in \mathrm{grsupp}(M(\mathfrak{m}))$. Then the module $M(\mathfrak{m})$ is a simple weight $\hat{A}$-module such that $M(\mathfrak{m})_\mathfrak{m} \neq 0$. Moreover, if $g(\mathfrak{m}) \neq \mathfrak{m}$ for any $0 \neq g \in \mathbb{Z}^n$, then $M(\mathfrak{m})$ is the unique simple weight $\hat{A}$-module such that $M(\mathfrak{m})_\mathfrak{m} = M(\mathfrak{m})_0 \neq 0$.*

Proof. As $M(\mathfrak{m})$ is generated by a weight element (any non-zero element from $M(\mathfrak{m})_0$), it is a weight module. Under our assumptions, R separates the graded components of $M(\mathfrak{m})$, i.e. the graded decomposition of $M(\mathfrak{m})$ coincides with its weight decomposition. This implies that $M(\mathfrak{m})$ is a simple $\hat{A}$-module. In the case $g(\mathfrak{m}) \neq \mathfrak{m}$ for any $0 \neq g \in \mathbb{Z}^n$ it's uniqueness follows easily from general nonsense (see [DFO, Theorem 18] or Proposition 7.1 below). □

Remark 3. *We have to note that Corollary 6.1 is an extension of [MT1, Theorems 1] and a partial extension of [MT1, Theorem 2] to a wider class of TGWA. The direct construction of modules used in [MT1, Theorem 1] can be applied only to TGWA considered in [MT1], the later being associated with a biserial graph. In the present paper and in Corollary 6.1 we have removed this restriction. This is important, as the natural presentation of both $\mathcal{A}$*

and $\mathcal{B}$ obtained above does not fit in the framework of [MT1]. At the same time, the results of this Section can be applied to both $\mathcal{A}$ and $\mathcal{B}$. In the next Section we will also present a construction of a new class of modules for TGWA.

7 Towards the classification of simple weight modules over a TGWA

We assume that we are in the situation of Section 6 and retain the notation from it. Fix $\mathfrak{m} \in \mathfrak{M}$ and set $W(\mathfrak{m}) = \{w \in \mathbb{Z}^n \,|\, w(\mathfrak{m}) = \mathfrak{m}\}$. Then $W(\mathfrak{m})$ is an abelian group of finite rank and we can fix a set of independent generators, $\{s_1, \dots, s_k\}$ of $W(\mathfrak{m})$. Denote by $B = B(\mathfrak{m})$ the graded subalgebra $\oplus_{g \in W(\mathfrak{m})} A_g$ of A. Clearly $R \subset B$.

Lemma 7.1. *A is a $\mathbb{Z}^n/W(\mathfrak{m})$-graded left (right) B-module and $A_0 \simeq B$ with respect to this gradation.*

Proof. Obvious. □

Proposition 7.1. *The functor $X \mapsto X_\mathfrak{m}$ induces a natural bijection between simple weight A-modules M such that $M_\mathfrak{m} \neq 0$ and simple B-modules N such that $\mathfrak{m}N = 0$ (or, in other words, with simple $B/(\mathfrak{m})$-modules).*

Proof. Let M be a simple weight A-module such that $M_\mathfrak{m} \neq 0$. As $w(\mathfrak{m}) = \mathfrak{m}$ for any $w \in W(\mathfrak{m})$, we get $BM_\mathfrak{m} \subset M_\mathfrak{m}$, hence our functor is well-defined. Assume that $N \subset M_\mathfrak{m}$ is a non-zero B-submodule. Then AN is an A-submodule of M and $(AN)_0 = N$ with respect to $\mathbb{Z}^n/W(\mathfrak{m})$-grading by Lemma 7.1 (this is equivalent to $AN \cap M_\mathfrak{m} = N$). The last contradicts the simplicity of M.

Conversely, let N be a simple B-modules such that $\mathfrak{m}N = 0$. Then the A-module $M = A \otimes_B N$ is clearly a weight A-module with $\mathrm{supp}(M) \subset W \cdot \mathfrak{m}$ and it surjects on any weight A-module V, generated by $V_\mathfrak{m}$ such that $V_\mathfrak{m} \simeq N$ as a B-module. From Lemma 7.1 it also follows that $M_\mathfrak{m} = N$, in particular, $M \neq 0$. Denote by M' the sum of all submodules of M, whose intersection with $M_\mathfrak{m}$ is zero. Then M/M' is a simple weight A-module and $(M/M')_\mathfrak{m} \neq 0$. Now one sees that the constructed maps are inverse to each other, completing the proof. □

By Proposition 7.1, the classification of simple weight A-modules is reduced to the classification of modules over certain subalgebras of A. The simplest case, namely $B = R$, can be treated by methods presented in Section 6. Below we will present some results on the structure of B in some cases, when $W(\mathfrak{m})$ is non-trivial. In particular, we show that in many cases this algebra is commutative. Denote by T the set of all l such that g_l occurs in the reduced decomposition of some s_i and by W_T the group, generated by all σ_l, $l \in T$. In the sequel we assume that for any $r \in R$ and any $w \in W(\mathfrak{m})$ the actions of r and $w(r)$ on $M(\mathfrak{m})$ coincide (this is true, for example, in the case $W \simeq \mathbb{Z}^n/W(\mathfrak{m})$).

Lemma 7.2. *Assume that $\mu_{i,j}=1$ for all i,j and for any i,j there is no l such that the generators s_i and s_j both contain $g_l^{\pm 1}$ in their reduced decomposition. Also assume that for any $l\in T$ the element t_l is invertible on $M(\mathfrak{m})$. Then $B/(\mathfrak{m})$ is commutative.*

Proof. First we note that, under the conditions of the lemma, all $w(t_i)$, $w\in W_T$ are invertible in $B/(\mathfrak{m})$.

It is sufficient to prove that A_{s_i} commutes with A_{s_j} for any i, j, which we fix throughout the proof. Let $s_i=g_{i_1}^{m_1}\dots g_{i_a}^{m_a}g_{j_1}^{-l_1}\dots g_{j_u}^{-l_u}$ and $s_j=g_{b_1}^{c_1}\dots g_{b_t}^{c_t}g_{r_1}^{-q_1}\dots g_{r_p}^{-q_p}$. Denote $X(s_i)=X_{i_1}^{m_1}\dots X_{i_a}^{m_a}$, $X(s_j)=X_{b_1}^{c_1}\dots X_{b_t}^{c_t}$, $Y(s_i)=Y_{j_1}^{l_1}\dots Y_{j_u}^{l_u}$, $Y(s_j)=Y_{r_1}^{q_1}\dots X_{r_p}^{q_p}$. Under the conditions of lemma we have that all lower indices of X's and Y's are different. Set $\sigma_{X(s_i)}=\sigma_{i_1}^{m_1}\dots\sigma_{i_a}^{m_a}$, $\sigma_{X(s_j)}=\sigma_{b_1}^{c_1}\dots\sigma_{b_t}^{c_t}$, $\sigma_{Y(s_i)}=\sigma_{j_1}^{l_1}\dots\sigma_{j_u}^{l_u}$ and $\sigma_{Y(s_j)}=\sigma_{r_1}^{q_1}\dots\sigma_{r_p}^{q_p}$. We divide the proof into a sequence of steps.
Step 1. First we prove the equalities

$$\begin{aligned}
\sigma_{Y(s_j)}^{-1}(X(s_i)^*X(s_i))\sigma_{X(s_i)}^{-1}(Y(s_j)Y(s_j)^*)&=X(s_i)^*X(s_i)Y(s_j)Y(s_j)^*;\\
\sigma_{X(s_j)}^{-1}(X(s_i)^*X(s_i))\sigma_{X(s_i)}^{-1}(X(s_j)^*X(s_j))&=X(s_i)^*X(s_i)X(s_j)^*X(s_j);\\
\sigma_{Y(s_j)}^{-1}(Y(s_i)Y(s_i)^*)\sigma_{Y(s_i)}^{-1}(Y(s_j)Y(s_j)^*)&=Y(s_i)Y(s_i)^*Y(s_j)Y(s_j)^*;\\
\sigma_{Y(s_i)}^{-1}(X(s_j)^*X(s_j))\sigma_{X(s_j)}^{-1}(Y(s_i)Y(s_i)^*)&=X(s_j)^*X(s_j)Y(s_i)Y(s_i)^*.
\end{aligned}$$

We will prove only the first one and all other can be done analogously. We start with $Y(s_j)X(s_i)=X(s_i)Y(s_j)$. Multiplying with $Y(s_j)^*$ from both sides and by $X(s_i)^*$ from the left we get

$$X(s_i)^*Y(s_j)^*Y(s_j)X(s_i)Y(s_j)^*=X(s_i)^*Y(s_j)^*X(s_i)Y(s_j)Y(s_j)^*.$$

Hence

$$X(s_i)^*X(s_i)Y(s_j)^*\sigma_{X(s_i)}^{-1}(\sigma_{Y(s_j)}^{-1}(Y(s_j)^*Y(s_j)))=Y(s_j)^*X(s_i)^*X(s_i)Y(s_j)Y(s_j)^*.$$

Therefore

$$Y(s_j)^*\sigma_{Y(s_j)}^{-1}(X(s_i)^*X(s_i))\sigma_{X(s_i)}^{-1}(\sigma_{Y(s_j)}^{-1}(Y(s_j)^*Y(s_j)))=Y(s_j)^*X(s_i)^*X(s_i)Y(s_j)Y(s_j)^*.$$

Now we can multiply the last equality with $Y(s_j)$ from the left. The element $Y(s_j)Y(s_j)^*$ decomposes into a product of $w(t_l)$, $l\in T$, $w\in W_T$. As these elements are invertible in $B/(\mathfrak{m})$, we get the desired equality as soon as $\sigma_{Y(s_j)}^{-1}(Y(s_j)^*Y(s_j))=Y(s_j)Y(s_j)^*$. The last follows from the equalities

$$\begin{aligned}
Y(s_j)Y(s_j)^*&=\prod_{d=1}^{u}\sigma_{j_{d-1}}^{-l_{d-1}}\circ\dots\circ\sigma_{j_1}^{-l_1}\left(t_{j_d}\sigma_{j_d}^{-1}(t_{j_d})\dots\sigma_{j_d}^{-(l_d-1)}(t_{j_d})\right),\\
Y(s_j)^*Y(s_j)&=\prod_{d=1}^{u}\sigma_{j_{d+1}}^{l_{d+1}}\circ\dots\circ\sigma_{j_u}^{l_u}\left(\sigma_{j_d}(t_{j_d})\sigma_{j_d}^2(t_{j_d})\dots\sigma_{j_d}^{l_d}(t_{j_d})\right)
\end{aligned}$$

checked by direct calculation.
Step 2. We proceed with the following equalities:

$$X(s_i)X(s_j)X(s_i)^*X(s_i) = X(s_j)X(s_i)\sigma^{-1}_{X(s_j)}(X(s_i)^*X(s_i));$$
$$X(s_i)X(s_j)\sigma^{-1}_{X(s_i)}(X(s_j)^*X(s_j)) = X(s_j)X(s_i)X(s_j)^*X(s_j);$$
$$Y(s_i)Y(s_j)\sigma_{Y(s_i)}(Y(s_i)Y(s_i)^*) = Y(s_j)Y(s_i)\sigma_{Y(s_j)}(\sigma_{Y(s_i)}(Y(s_i)Y(s_i)^*));$$
$$Y(s_i)Y(s_j)\sigma_{Y(s_j)}(\sigma_{Y(s_i)}(Y(s_j)Y(s_j)^*)) = Y(s_j)Y(s_i)\sigma_{Y(s_j)}(Y(s_j)Y(s_j)^*)$$

and again will prove only the first one. We have $X(s_i)^*X(s_j) = X(s_j)X(s_i)^*$. Multiplying with $X(s_i)$ from both sides and moving $X(s_i)^*X(s_i)$ we get the necessary equality.
Step 3. Finally, we have:

$$Y(s_i)X(s_i)Y(s_j)X(s_j) = Y(s_i)Y(s_j)X(s_i)X(s_j) = Y(s_j)Y(s_i)\cdot$$
$$\cdot\,\sigma_{Y(s_j)}(\sigma_{Y(s_i)}(Y(s_i)Y(s_i)^*))(\sigma_{Y(s_i)}(Y(s_i)Y(s_i)^*))^{-1}X(s_j)X(s_i)X(s_j)^*X(s_j)\cdot$$
$$\cdot\,(\sigma^{-1}_{X(s_i)}(X(s_j)^*X(s_j)))^{-1} = Y(s_j)Y(s_i)X(s_j)X(s_i) = Y(s_j)X(s_j)Y(s_i)X(s_i).$$

□

Lemma 7.3. *Assume that $\sigma_i^k(\mathfrak{m}) = \mathfrak{m}$, $k \in \mathbb{N}$, all t_j are invertible on $M(\mathfrak{m})$ and $\mu_{i,j} = 1$ for all i, j. Then for any $a \in \hat{A}$ and any $v \in M(\mathfrak{m})$ the equality $a(X_i^k(v)) = X_i^k(a(v))$ holds. In particular, the map $v \mapsto X_i^k(v)$ is an automorphism of $M(\mathfrak{m})$.*

Proof. We need to check the necessary equality on generators. For $a \in R$ this follows from $\sigma_i^k(\mathfrak{m}) = \mathfrak{m}$ and our assumption on the action of R on $M(\mathfrak{m})$, for $a = Y_j$, $j \neq i$, this follows from $\mu_{i,j} = 1$. For $a = X_i$ this is obvious. Let $a = Y_i$. We have

$$Y_iX_i^k(v) = t_iX_i^{k-1}(v) = t_iX_i^{k-1}X_iY_i(\sigma_i(t_i))^{-1}(v) =$$
$$= X_i^kY_i\sigma_i^{-k+1}(t_i)(\sigma_i(t_i))^{-1}(v) = X_i^kY_i(v).$$

Now let $a = X_j$, $j \neq i$. We have

$$X_jX_i^k(v) = X_i^kX_j\sigma_i^{-k}(t_j)t_j^{-1}(v) = X_i^kX_j(v).$$

□

Remark 4. *We note that in general the algebra $B/(\mathfrak{m})$ is not commutative. Indeed, take $n = 3$, $R = \mathbb{C}[t]$, $\sigma_1 = \sigma_2 = \sigma_3 : t \mapsto -t$, $t_1 = t_2 = t_3 = t$, $\mu_{i,j} = 1$ for all i, j. Clearly $t^2 = t_it_j = \sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i) = (-t)(-t) = t^2$ and $\sigma_1^{-1}\sigma_3 = \sigma_2^{-1}\sigma_3 = e$. But $Y_1X_3Y_2X_3 = -Y_2X_3Y_1X_3$.*

Now we want to show that this is a common feature, in fact, we will show that the graded elements in B either commute or anticommute in their action on $M(\mathfrak{m})$.

Lemma 7.4. *Assume that $W(\mathfrak{m}) \subset \{g \in \mathbb{Z}^n \mid g(\mathfrak{m}) \in \mathrm{grsupp}(M(\mathfrak{m}))\}$. For any $i \in T$ and any $\mathfrak{n} \in \mathrm{supp}(M(\mathfrak{m}))$ the actions $t_i : M(\mathfrak{m})_\mathfrak{n} \to M(\mathfrak{m})_\mathfrak{n}$ and $\sigma_i(t_i) : M(\mathfrak{m})_\mathfrak{n} \to M(\mathfrak{m})_\mathfrak{n}$ are bijective.*

Proof. Assume that g_i occurs with a non-zero coefficient in the decomposition of s_j. Consider a generator, Z, of $\hat{A}_j$, having the form $Z = Z'Z_i$, where $Z_i = X_i$ or $Z_i = Y_i$. As $s_j(\mathfrak{n}) \in \mathrm{grsupp}(M(\mathfrak{m}))$, we have $Z^*Z : M(\mathfrak{m})_\mathfrak{n} \to M(\mathfrak{m})_\mathfrak{n}$ is a bijection. Taking into account that $W(\mathfrak{m})$ is a group, we get $\sigma_i^{\pm 1} : \mathrm{supp}(M(\mathfrak{m})) \to \mathrm{supp}(M(\mathfrak{m}))$. If $Z_i = X_i$ we have $t_i \notin \mathfrak{n}$ and if $Z_i = Y_i$ we have $\sigma_i(t_i) \notin \mathfrak{n}$. As $\mathfrak{n}$ is arbitrary and $\mathfrak{m}$ is a maximal ideal, the proof is finished. □

From now on we will work under the assumption on the group $W(\mathfrak{m})$, which appeared in the previous Lemma: $W(\mathfrak{m}) \subset \{g \in \mathbb{Z}^n \mid g(\mathfrak{m}) \in \mathrm{grsupp}(M(\mathfrak{m}))\}$. We recall that we also assume R to be a domain. Now we want to extend the ring R in order to be able to take square roots of the elements $t_i \in T$. Let T' be the subset of T consisting of all those i such that there does not exist $\hat{t}_i \in R$ with $\hat{t}_i^2 = t_i$ (for $i \in T \setminus T'$ we fix such $\hat{t}_i$). Let $P = R[x_w^i \mid w \in W, i \in T']$. Setting $\sigma_j(x_w^i) = x_{\sigma_j \cdot w}^i$ we extend σ_j to an automorphism of P. These extensions commute for different j and we get an action of W on P by automorphisms. Let I be the minimal ideal of P, containing all $(x_0^i)^2 - t_i$ and stable under W-action. Set $R' = P/I$, clearly R can be naturally identified with polynomials of zero degree in R'. Let $\hat{A}'$ denote the TGWA associated with R', M, $\{\sigma_i\}$, t_i.

Lemma 7.5. *The ideal $\mathfrak{m}'$, generated by all $ax_w^i + b$, $a, b \in \mathfrak{m}$ is a maximal ideal of R', whose intersection with R equals $\mathfrak{m}$.*

Proof. Follows by direct calculation. □

The inclusion $R \subset R'$ induces a natural inclusion $R/\mathfrak{m} \subset R'/\mathfrak{m}'$, which induces an inclusion of $M(\mathfrak{m})$ to $M(\mathfrak{m}')$ as $\hat{A}$-modules.

Lemma 7.6. *Let $g, h \in W(\mathfrak{m})$, $a \in \hat{A}'_g$, $b \in \hat{A}'_h$, $v \in M(\mathfrak{m}')$. Then $abv = \pm bav$. In particular, $abv = \pm bav$ for any $a \in \hat{A}_g$, $b \in \hat{A}_h$, $v \in M(\mathfrak{m})$.*

Proof. First we note that from Lemma 7.4 it follows that all x_0^i, $\sigma_i(x_0^i)$, $i \in T'$ and all $\hat{t}_i$, $\sigma_i(\hat{t}_i)$, $i \in T' \setminus T'$ are invertible on $M(\mathfrak{m}')$. We will write $\sqrt{t_i} = \hat{t}_i$, $i \in T \setminus T'$ and $\sqrt{t_i} = x_0^i$, $i \in T'$. By definition we have $\sqrt{\sigma_i(t_j)} = \sigma_i(\sqrt{t_j})$. Consider the elements $X_i' = X_i(\sqrt{t_i})^{-1}$ and $Y_i' = Y_i(\sqrt{\sigma_i(t_i)})^{-1}$, $i \in T$ acting on $M(\mathfrak{m}')$. We claim that all these elements (anti)commute (till the end of this proof all the equalities will be operator equalities on $M(\mathfrak{m})$). Indeed, the only non-trivial relations to be checked are $X_j'Y_i' = \pm Y_i'X_j'$, $X_j'X_i' = \pm X_i'X_j'$ and $Y_j'Y_i' = \pm Y_i'Y_j'$ ($i, j \in T$). We will do the first and the second ones, the third follows by applying $*$ to the second. For the first one we start with the case $i = j$ and have

$$X_i'Y_i' = X_i(\sqrt{t_i})^{-1}Y_i\left(\sqrt{\sigma_i(t_i)}\right)^{-1} = X_iY_i(\sigma_i(t_i))^{-1} = 1 =$$

$$= Y_iX_it_i^{-1} = Y_i\left(\sqrt{\sigma_i(t_i)}\right)^{-1}X_i\left(\sqrt{t_i}\right)^{-1} = Y_i'X_i'.$$

For $i \neq j$ we have

$$X_i'Y_j' = X_i(\sqrt{t_i})^{-1}Y_j\left(\sqrt{\sigma_j(t_j)}\right)^{-1} = X_iY_j\left(\sqrt{\sigma_j(t_i)}\right)^{-1}\left(\sqrt{\sigma_j(t_j)}\right)^{-1} =$$
$$= \pm\mu_{i,j}Y_jX_i\left(\sqrt{\sigma_i^{-1}(\sigma_j(t_j))}\right)^{-1}(\sqrt{t_i})^{-1} = \pm Y_j'X_i',$$

where we have used the equality $t_it_j = \mu_{i,j}^2\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i)$.

For the second one we start with $X_iX_jY_iX_i = \mu_{j,i}X_iY_iX_jX_i$, which is equivalent to $X_iX_jt_i = \mu_{j,i}\sigma_i(t_i)X_jX_i$. We proceed as follows:

$$X_iX_jt_i = \mu_{j,i}\sigma_i(t_i)X_jX_i \quad \Leftrightarrow \quad X_i'\sqrt{t_i}X_j'\sqrt{t_j}t_i = \mu_{j,i}\sigma_i(t_i)X_j'\sqrt{t_j}X_i'\sqrt{t_i} \quad \Leftrightarrow$$
$$\Leftrightarrow \quad X_i'X_j'\sqrt{\sigma_j^{-1}(t_i)}\sqrt{t_j}t_i = \mu_{j,i}X_j'X_i'\sigma_j^{-1}(t_i)\sqrt{\sigma_i^{-1}(t_j)}\sqrt{t_i}.$$

As $t_it_j = \mu_{j,i}^2\sigma_j^{-1}(t_i)\sigma_i^{-1}(t_j)$ and R is a domain, we have the equality $\sqrt{\sigma_j^{-1}(t_i)}\sqrt{t_j}t_i = \pm\mu_{j,i}\sigma_j^{-1}(t_i)\sqrt{\sigma_i^{-1}(t_j)}\sqrt{t_i}$. The necessary statement now follows from the invertability of the later elements.

Now write $a = r_1a'$ and $b = r_2b'$, where $r, i \in R'$ and both a' and b' are products of X_i' and Y_i'. As $g, h \in W(\mathfrak{m})$, we have $ra' = a'r$ and $rb' = b'r$ for any $r \in R'$. Finally, on $M(\mathfrak{m})$ we have $ab = r_1a'r_2b' = r_1r_2a'b' = \pm r_1r_2b'a' = \pm r_2b'r_1a' = \pm ba$, as desired. □

Corollary 7.1. *Under the conditions of Lemma 7.6 the equality $xy = \pm yx$ holds for any graded elements $x, y \in B/(\mathfrak{m})$.*

Proof. Follows from the fact that the representation of $B/(\mathfrak{m})$ in $M(\mathfrak{m})$ is faithful. □

Remark 5. *Representations of the algebras, whose generators (anti)commute, were intensively studied. In particular, in [Sa] one can find a complete classification of unitarizable simple modules over the complex skew-polynomial ring of this form.*

Lemma 7.7. *1. Let N be a simple graded $\hat{A}$-module, generated by a graded element, v, such that $\mathfrak{m}v = 0$. Then the map $\bar{1} \mapsto v$ extends to an isomoprphism from $M(\mathfrak{m})$ onto N.*

2. For any $j = 1, \dots, k$ there exists a graded automorphism, φ_j, of the module $M(\mathfrak{m})$ such that $\varphi_j(M(\mathfrak{m})_g) = M(\mathfrak{m})_{gs_j}$.

Proof. The first statement follows from the construction of $M(\mathfrak{m})$. The second statement follows from Lemma 6.1 and the first one, if one remarks that $\mathfrak{m}M(\mathfrak{m})_{s_i} = 0$. □

Assume that the automorphisms φ_j, given by Lemma 7.7, commute (e.g this is the case of Lemma 7.3 if the group $W(\mathfrak{m})$ is generated by $g_i^{k_i}$, $i \in T$). Let G be the group, generated by all φ_j, given by Lemma 7.7. Clearly, $G \simeq W(\mathfrak{m})$, as G is commutative and we can fix the isomorphism $\psi : W(\mathfrak{m}) \to G$, sending s_j to φ_j, $j = 1, 2, \dots, k$. Recall that we have assumed $W(\mathfrak{m}) \operatorname{grsupp}(M(\mathfrak{m})) \subset \operatorname{grsupp}(M(\mathfrak{m}))$. Let K be a set of representatives of all orbits of $W(\mathfrak{m})$ acting on $\operatorname{grsupp}(M(\mathfrak{m}))$. As $\operatorname{grsupp}(M(\mathfrak{m})) \subset \mathbb{Z}^n$, the elements of a fixed orbit bijectively correspond to elements of $W(\mathfrak{m})$. Consider the R-module $M(\mathfrak{m}, K) = \oplus_{g \in K} M(\mathfrak{m})_g$. We define on $M(\mathfrak{m}, K)$ an action of X_i and Y_i as follows: let $v \in M(\mathfrak{m})_g$, $g \in K$, then there exist a unique element, $h \in W(\mathfrak{m})$ (it depends on X_i, Y_i, g and K), such that $h(\sigma_i(g)) \in K$ (resp. $h(\sigma_i^{-1}(g)) \in K$), define $X_i \cdot v = \psi(h)(X_i(v))$ (resp. $Y_i \cdot v = \psi(h)(Y_i(v))$).

Theorem 7.1. *$M(\mathfrak{m}, K)$ is a simple weight $\hat{A}$-module.*

Proof. The statement about simplicity is obvious, since R will separate the components of $M(\mathfrak{m}, K)$ graded by the quotient of W modulo the image of $W(\mathfrak{m})$ in it. So, we have to show that $M(\mathfrak{m}, K)$ is a $\hat{A}$-module, for which it is necessary to check all the relations from the definition of $\hat{A}$. Let $u = \sum_i u_i = 0$ be a graded relation with monomial u_i. Then on $M(\mathfrak{m}, K)$ our relation will take the form $\sum_i \gamma_i(u_i(v)) = 0$, $v \in M(\mathfrak{m}, K)$, for some $\gamma_i \in G$. The commutativity of $W(\mathfrak{m})$ implies that all γ_i do not depend on i (denote this common element by γ), which forces $\gamma(\sum_i u_i(v)) = 0$. The last is true since $M(\mathfrak{m})$ is a $\hat{A}$-module. This completes the proof. □

Finally, we want to remark that Corollary 6.1 and Theorem 7.1 can be used to construct simple weight modules over $\mathcal{A}$ and $\mathcal{B}$. In the case of algebra $\mathcal{A}$ the ring $\mathcal{Q}$ is quite simple and it follows directly from the definition of σ_i that $W \simeq \mathbb{Z}^n$ and $w(\mathfrak{m}) \neq \mathfrak{m}$ for any $\mathfrak{m} \in \mathfrak{M}(\mathcal{Q})$ and any $0 \neq w$. In the second case the situation is worse since R is a polynomial ring over the field D_{n+1}, which is not algebraically closed. This requires more technical efforts (as description of W-orbits on R and maximal ideals in R) in each concrete case.

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