Lifting cofibration and fibration structures

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1 Introduction

Derived functors of functors between non-additive categories are abundant in modern mathematics. One of the most popular frame works for constructing and working with such functors is Quillen's concept of a closed model category [9]. To verify that given classes of weak equivalence, cofibrations and fibrations constitute a closed model structure on a category one usually has to rely on small object arguments; there are only a few examples, where one can do without them.

In relative homological algebra one often has to deal with weak equivalences which are genuine homotopy equivalences after forgetting part of the structure. Genuine homotopy equivalences are usually not detected by a fixed set of maps and hence resist small object arguments. So they are rarely part of the data of a closed model structure. Fortunately, fibration or cofibration structures as studied by Baues [1] suffice to treat right respectively left derived functors in such situations.

In this paper we construct lifts of fibration and cofibration structures if we are given a pair of enriched adjoint functors

$$L : \mathcal{B} \leftrightarrows \mathcal{A} : R$$

between suitably enriched categories, the typical situation for classical relative homological algebra, where we are in a chain complex enriched setting (see Section 8). Moreover, our categories have well-behaved cylinder- and path space functors.

The original motivation for this paper comes from the attempt to interpret the "standard complex" definition of topological Hochschild homology (e.g.

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see [4, X.2.1]) as a relative Tor functor. Here we are in a topologically enriched setting, which is our main source of applications.

We start with the recollection of some basic facts of $\mathcal{V}$-enriched category theory over a ground category $\mathcal{V}$ in Section 2. In Section 3 we discuss the "absolute" homotopy theory in suitable $\mathcal{V}$-enriched categories, defined by the given cylinder and path space; we obtain compatible fibration and cofibration structures. Section 4 till 6 treat the "relative" homotopy theory: in Section 4 and 5 we lift the cofibration structure in $\mathcal{B}$ via the left adjoint L to $\mathcal{A}$. In Section 6 we shortly address the dual situation, namely lifting the fibration structure in $\mathcal{A}$ via R to $\mathcal{B}$.

As examples we study the topological enriched, the chain complex enriched, and the $\mathcal{C}at$-enriched cases with emphasis on the topological one. Explicit examples are the categories of diagrams of topological spaces, of topological operads, of algebras over a topological operad, and of module spectra. In Section 8 we link the chain complex enriched case to classical relative homological algebra. We close with a sketch of the $\mathcal{C}at$-enriched case.

The results of this paper are used in [11] and [13]. They also provide a good frame work to study homotopy invariance of algebraic structures on spaces, which will be exploited in a later paper.

2 The ground category $\mathcal{V}$

Let $\mathcal{V} = (\mathcal{V}_0, \square, \Phi)$ be a symmetric monoidal category with unit object Φ. We assume $\mathcal{V}$ to be closed, i.e. there is a bifunctor

$$\mathcal{V}(-,-) : \mathcal{V}_0^{op} \times \mathcal{V}_0 \to \mathcal{V}_0$$

satisfying the Hom-tensor adjunction

2.1 $\mathcal{V}_0(X \square Y, Z) \cong \mathcal{V}_0(X, \mathcal{V}(Y, Z))$

In this paper we consider $\mathcal{V}$-categories $\mathcal{C}$, i.e. $\mathcal{V}$-enriched categories. $\mathcal{C}_0$ will denote the underlying category. Note that $\mathcal{V}_0$ is canonically $\mathcal{V}$-enriched by the objects $\mathcal{V}(X, Y)$.

Recall that a $\mathcal{V}$-category $\mathcal{C}$ is $\mathcal{V}$-complete and $\mathcal{V}$-cocomplete if each functor $F : \mathcal{J} \to \mathcal{C}_0$, where $\mathcal{J}$ is a small indexing category, has a limit and a colimit in $\mathcal{C}_0$, and we have natural isomorphisms in $\mathcal{V}_0$

$$\mathcal{C}(X, \lim F) \cong \lim \mathcal{C}(X, F) \quad \text{and} \quad \mathcal{C}(\operatorname{colim} F, X) \cong \mathcal{C}(F, X)$$

A $\mathcal{V}$-category $\mathcal{C}$ is called tensored and cotensored if there are functors

2.2 $\mathcal{C}_0 \times \mathcal{V}_0 \to \mathcal{C}_0, \quad (X,K) \mapsto X \otimes K$
$\mathcal{C}_0 \times \mathcal{V}_0^{op} \to \mathcal{C}_0, \quad (X,K) \mapsto X^K$

and natural isomorphisms in $\mathcal{V}_0$

$$\mathcal{C}(X \otimes K, Y) \cong \mathcal{V}(K, \mathcal{C}(X,Y)) \cong \mathcal{C}(X, Y^K).$$

Note that $\mathcal{V}$ is tensored by $-\Box-$ and cotensored by $\mathcal{V}(-,-)$. A pair of $\mathcal{V}$-functors of $\mathcal{V}$-categories

$$F : \mathcal{B} \leftrightarrows A : U$$

is called a $\mathcal{V}$ adjoint pair, if there are isomorphisms in $\mathcal{V}_0$

$$\mathcal{B}(X, U(Y)) \cong \mathcal{A}(F(X), Y)$$

natural in $X \in \mathcal{B}$ and $Y \in \mathcal{A}$.

We make the following assumptions:

2.3 Assumptions:

(1) The ground category $\mathcal{V}$ is $\mathcal{V}$-complete and $\mathcal{V}$-cocomplete, tensored and cotensored.

(2) We are given a functor $\nabla : \triangle \to \mathcal{V}$ with $\nabla[0] = \Phi$. Here $\triangle$ is the category of ordered sets $[n] = \{0, 1, 2, \ldots, n\}$ and order preserving maps.

(3) The map $(i_0, i_1) : \Phi \sqcup \Phi \to \nabla[1]$, induced by the two injections $[0] \to [1]$, is a strong cofibration (see Definition 3.2).

Following Kelly [6] we call a morphism $f : \Phi \to Z$ an element of Z. For simplicity, we often denote the elements $i_0, i_1 : \Phi \to \nabla[1]$ by 0 and 1.

2.4 Examples of categories $\mathcal{V}$:

(1) $\mathcal{T}$op, the category of k-spaces in the sense of [12, 5(ii)] with $X \Box Y = X \times Y$, $\mathcal{T}\mathrm{op}(X,Y)$ the k-function space, and $\nabla : \triangle \to \mathcal{T}\mathrm{op}$ the standard simplex functor.

(2) $\mathcal{T}\mathrm{op}_*$ the based version of (1) with $X \Box Y = X \wedge Y$, $\mathcal{T}\mathrm{op}_*(X,Y)$ the based k-function space, and $\nabla[n] = \Delta^n_+$, the standard n-simplex with an extra point $+$ as base point.

(3) $\mathcal{HT}$op and $\mathcal{HT}$op$_*$, the categories of (based) compactly generated weak Hausdorff spaces with structures as in (1) and (2).

(4) $\mathcal{SM}od_R$, the category of simplicial R-modules, R a commutative ring, with $(X_\bullet \Box Y_\bullet)_n = X_n \otimes Y_n$, $\mathcal{SM}od_R(X_\bullet, Y_\bullet)_n$ the R-module of simplicial R-linear maps $X_\Box \otimes R[\triangle(-,[n])] \to Y_\bullet$, where $R[-]$ is the free R-module functor. We define $\nabla[n] = R[\triangle(-,[n])]$.

(5) $\mathcal{C}plx_R$, the category of unbounded chain complexes of R-modules, R a commutative ring. We take $X_\bullet \Box Y_\bullet$ and $\mathcal{C}plx_R(X_\bullet, Y_\bullet)$ to be the usual tensor product resp. Hom-complex of chain complexes, and define $\nabla[n]$ to be the normalized chain complex associated with $R[\triangle(-,[n])]$.

(6) $\mathcal{C}at$, the category of small categories with $X \Box Y = X \times Y$, $\mathcal{C}at(X,Y)$ the functor category, and $\nabla[n]$ the category with n objects and exactly one morphism between any two objects.

Examples of $\mathcal{V}$-categories will be discussed in the example sections. For a summary of what we need to know about enriched category theory see [10], for a detailed account we recommend [6].

3 Homotopy theory in $\mathcal{V}$-categories

3.1 General assumptions: $\mathcal{C}$ is a $\mathcal{V}$-category which is $\mathcal{V}$-complete, $\mathcal{V}$-cocomplete, tensored and cotensored. We also assume the Assumptions 2.3 for $\mathcal{V}$.

We denote the initial object of $\mathcal{C}$ by $\emptyset$ and the terminal one by $*$.

Recall that a morphism $j : A \to X$ has the left lifting property (LLP) for $p : E \to B$ and p has the right lifting property (RLP) for j, if each commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

has a diagonal filler $h : X \to E$ making the diagram commute.

3.2 Definition: A map $f : X \to Y$ in $\mathcal{C}$ is called

(i) a *fibration*, if it has the RLP for all maps $Z \otimes i_0 : Z \otimes 0 \to Z \otimes \nabla[1]$

(ii) a *cofibration*, if it has the LLP for all maps $Z^{i_0} : Z^{\nabla[1]} \to Z^{\Phi} \cong Z$

(iii) a *strong cofibration*, if it has the LLP for all maps $(p^{\nabla[1]}, E^{i_0}) : E^{\nabla[1]} \to B^{\nabla[1]} \times_B E$, where $p : E \to B$ is a fibration

(iv) a *strong fibration*, if it has the RLP for all maps $(j \otimes \nabla[1] \cup Z \otimes 0)$: $A \otimes \nabla[1] \cup_{A \otimes 0} Z \otimes 0 \to Z \otimes \nabla[1]$, where $j : A \to Z$ is a cofibration

(v) an *equivalence*, if it is a homotopy equivalence with respect to the cylinder functor $- \otimes \nabla[1]$.

An object X of $\mathcal{C}$ is called *(strongly) fibrant* resp. *(strongly) cofibrant*, if $X \to *$ is a (strong) fibration resp. $\emptyset \to X$ a (strong) cofibration.

3.3 Notation: Let $\mathbf{sfib}_{\mathcal{C}}, \mathbf{fib}_{\mathcal{C}}, \mathbf{scof}_{\mathcal{C}}, \mathbf{cof}_{\mathcal{C}}, \mathbf{eq}_{\mathcal{C}}$ denote the classes of (strong) fibrations, (strong) cofibrations, and equivalences in $\mathcal{C}$ respectively.

We have the following characterizations of the various classes (for a sample proof see [10]).

3.4 Proposition: (1) $j \in \mathbf{cof}_{\mathcal{C}} \Leftrightarrow j$ has the LLP for all $p \in \mathbf{sfib}_{\mathcal{C}} \cap \mathbf{eq}_{\mathcal{C}}$
(2) $p \in \mathbf{sfib}_{\mathcal{C}} \Leftrightarrow p$ has the RLP for all $j \in \mathbf{cof}_{\mathcal{C}} \cap \mathbf{eq}_{\mathcal{C}}$
(3) $j \in \mathbf{cof}_{\mathcal{C}} \cap \mathbf{eq}_{\mathcal{C}} \Leftrightarrow j$ has the LLP for all $p \in \mathbf{sfib}_{\mathcal{C}}$
(4) $j \in \mathbf{scof}_{\mathcal{C}} \cap \mathbf{eq}_{\mathcal{C}} \Leftrightarrow j$ has the LLP for all $p \in \mathbf{fib}_{\mathcal{C}}$
(5) $p \in \mathbf{fib}_{\mathcal{C}} \Leftrightarrow p$ has the RLP for all $j \in \mathbf{scof}_{\mathcal{C}} \cap \mathbf{eq}_{\mathcal{C}}$
(6) $j \in \mathbf{scof}_{\mathcal{C}} \Leftrightarrow j$ has the LLP for all $p \in \mathbf{fib}_{\mathcal{C}} \cap \mathbf{eq}_{\mathcal{C}}$
(7) $p \in \mathbf{fib}_{\mathcal{C}} \cap \mathbf{eq}_{\mathcal{C}} \Leftrightarrow p$ has the RLP for all $j \in \mathbf{scof}_{\mathcal{C}}$
(8) $p \in \mathbf{sfib}_{\mathcal{C}} \cap \mathbf{eq}_{\mathcal{C}} \Leftrightarrow p$ has the RLP for all $j \in \mathbf{cof}_{\mathcal{C}}$

In [10] we proved

3.5 Proposition: $(\mathcal{C}, \mathbf{cof}, \mathbf{sfib}, - \otimes \nabla[1], (-)^{\nabla[1]}, \emptyset, *)$ and $(\mathcal{C}, \mathbf{scof}, \mathbf{fib}, - \otimes \nabla[1], (-)^{\nabla[1]}, \emptyset, *)$ are IP-categories in the sense of [1, (I.4a)]. In particular, $(\mathcal{C}, \mathbf{eq}, \mathbf{cof})$ and $(\mathcal{C}, \mathbf{eq}, \mathbf{scof})$ are proper cofibration categories with all objects strongly fibrant, and $(\mathcal{C}, \mathbf{eq}, \mathbf{fib})$ and $(\mathcal{C}, \mathbf{eq}, \mathbf{sfib})$ are proper fibration categories with all objects strongly cofibrant. $\square$

For the reader's convenience we recall the definitions of the structures addressed in the second part of (3.5).

3.6 Definition: A *cofibration category* is a category $\mathcal{C}$ with an initial object $\emptyset$ and two subcategories $\mathbf{cof}_{\mathcal{C}}$ and $\mathbf{we}_{\mathcal{C}}$, whose morphisms are called *cofibrations* and *weak equivalences* respectively. Morphisms in $\mathbf{cof}_{\mathcal{C}} \cap \mathbf{we}_{\mathcal{C}}$ are called *trivial cofibrations*. An object A is called *cofibrant*, if $\emptyset \to A$ is a cofibration, and *fibrant*, if each trivial cofibration $A \to X$ has a retraction. The following axioms holds

(C1) Given $A \stackrel{f}{\to} B \stackrel{g}{\to} C$, if two of $f, g, g \circ f$ are in $\mathbf{we}_{\mathcal{C}}$, so is the third. Isomorphisms are trivial cofibrations.

(C2) Pushouts along cofibrations i exist.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "\overline{f}"] \\ X \arrow[r, "\overline{i}"] & X \cup_A B \end{tikzcd}$$

If i is a (trivial) cofibration, so is $\overline{i}$.

(C3) Every map factors into a cofibration followed by a weak equivalence.

(C4) Any object X has a fibrant resolution RX, i.e. there is a trivial cofibration $e_X : X \to RX$ with RX fibrant.

We call $\mathcal{C}$ *proper*, if the following additional axiom holds.

(P) In the pushout of (C2), if i is a cofibration and f a weak equivalence, then $\overline{f}$ is a weak equivalence.

A proper cofibration category is a cofibration category as defined by Baues [1]. Our definition is due to Majewski [7]. Many of the results of [1] also hold for this weaker version. In particular we have

3.7 Lifting lemma: Consider the commutative (solid) diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

with i a cofibration and p a weak equivalence.

(1) If X is fibrant, there is an h such that $f = h \circ i$.

(2) If X and Y are fibrant, there is h such that $h \circ i = f$ and $p \circ h \cong g \operatorname{rel} A$, and two such h are homotopic $\operatorname{rel} A$.

Here homotopy $\operatorname{rel} A$ is defined using a cylinder object $I_A B$ of $B \operatorname{rel} A$ obtained by factoring the folding map into a cofibration and a weak equivalence.

$$\begin{tikzcd} B \cup_A B \arrow[rr] \arrow[dr] & & B \\ & I_A B \arrow[ur] & \end{tikzcd}$$

4 Relative homotopy theory

4.1 General assumption: Throughout this section we assume that we are given a $\mathcal{V}$-adjoint pair

$$L : \mathcal{B} \leftrightarrows \mathcal{A} : R$$

between $\mathcal{V}$-categories satisfying (3.1).

The purpose of this section is to construct a cofibration structure on $\mathcal{A}$, whose weak equivalences are those maps f for which $R(f)$ is an equivalence.

4.2 Lemma: (1) L preserves (strong) cofibrations and equivalences.

(2) R preserves (strong) fibrations and equivalences.

Proof: L preserves tensors and R cotensors. Hence they preserve equivalences. By passing to the adjoint situation one proves that L preserves cofibrations and R fibrations. The argument for strong cofibrations and strong fibrations is similar. $\square$

4.3 Definition: A morphism $f : X \to Y$ in $\mathcal{A}$ is called an

(1) *R-fibration*, if $R(f) \in \mathbf{fib}_{\mathcal{B}}$

(2) *R-equivalence*, if $R(f) \in \mathbf{eq}_{\mathcal{B}}$

(3) *R-cofibration*, if it has the LLP for all trivial R-fibrations

(4) *trivial R-(co)fibration*, if it is an R-equivalence and an R-(co)fibration.

We denote the corresponding classes by $R\mathbf{fib}$, $R\mathbf{eq}$, and $R\mathbf{cof}$.

4.4 Lemma: (1) $R\mathbf{cof}$ is closed under composition, retracts, cobase change, arbitrary sums, and sequential colimits

(2) Given a commutative ladder

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, "f_0"] & A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & {} \\ B_0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & {} \end{tikzcd}$$

such that f_0 and the maps $B_i \cup_{A_i} A_{i+1} \to B_{i+1}$ are R-cofibrations for all $i \geq 0$. Then $\operatorname{colim} f_n : \operatorname{colim} A_n \to \operatorname{colim} B_n$ is an R-cofibration.

(3) $R\mathbf{fib}$ is closed under composition, sections, base change, arbitrary products, and sequential limits.

(4) Given a commutative ladder

$$\begin{tikzcd} A_0 \arrow[d, "f_0"] & A_1 \arrow[l] \arrow[d, "f_1"] & A_2 \arrow[l] \arrow[d, "f_2"] & {} \arrow[l] \\ B_0 & B_1 \arrow[l] & B_2 \arrow[l] & {} \arrow[l] \end{tikzcd}$$

such that f_0 and the maps $A_{i+1} \to A_i \times_{B_i} B_{i+1}$ are R-fibrations. Then $\lim A_n \to \lim B_n$ is an R-fibration.

(5) $R\mathbf{cof} \subset s\mathbf{cof}_{\mathcal{A}}$, $\mathbf{fib}_{\mathcal{A}} \subset R\mathbf{fib}$, $\mathbf{eq}_{\mathcal{A}} \subset R\mathbf{eq}$

(6) $L(B)$ is R-cofibrant for each $B \in ob\,\mathcal{B}$. Retracts of R-cofibrant objects are R-cofibrant.

(7) Every $A \in ob\,\mathcal{A}$ is R-fibrant.

Proof: (1) and (2) holds because R-cofibrations are characterized by a left lifting property. (3) and (4) hold because R preserves limits and $\mathbf{fib}_{\mathcal{B}}$ is closed under the corresponding constructions. (5) follows from (3.4) and (4.2). By passing to adjoints, one obtains (6), and (7) holds because R preserves terminal objects. □

4.5 Lemma: Let $j : A \to X$ be a trivial R-cofibration. Then there is a retraction $r : X \to A$ and a homotopy $j \circ r \simeq id_X\ rel A$.

Proof: Consider the mapping path space diagram in $\mathcal{A}$

$$\begin{tikzcd}
A \arrow[dr, "s"] \arrow[drr, bend left, "X^{\pi}\circ j"] \arrow[ddr, bend right, "id"'] & & & \\
& P_j \arrow[r, "\overline{j}"'] \arrow[d, "p"] \arrow[dr, phantom, "\text{pullback}"] & X^{\nabla[1]} \arrow[r, "X^{i_1}"] \arrow[d, "X^{i_0}"] & X \\
& A \arrow[r, "j"'] & X &
\end{tikzcd}$$

($\pi : \nabla[1] \to \nabla[0]$ is induced by $[1] \to [0]$). Since $\mathcal{A}$ is a category with a natural path object in the sense of [1] (see [10]), $q = X^{i_1} \circ \overline{j}$ is a fibration, s is an equivalence, and $q \circ s = j$. Hence q is a trivial R-fibration, and we have a map $u : X \to P_j$ making the following diagram commute

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "j"'] & P_j \arrow[d, "q"] \\
X \arrow[r, equal] \arrow[ur, dashed] & X
\end{tikzcd}$$

$r = p \circ u$ is a retraction of j. Since $u \circ j$ and $q \circ u$ are equivalences, u is an equivalence, and so are j and q. Since $j \in s\mathbf{cof}_{\mathcal{A}}$, there is an r with the required property by the lifting lemma in the cofibration category $(\mathcal{A}, \mathbf{eq}, s\mathbf{cof}_{\mathcal{A}})$. $\square$

4.6 Corollary: (1) $R\mathbf{cof} \cap R\mathbf{eq}$ is closed under cobase change.

(2) $(\mathcal{A}, R\mathbf{eq}, R\mathbf{cof})$ satisfies Axioms (C1), (C2), and (C3) of a cofibration category, in which all objects are fibrant.

The factorization axiom:

We prove the factorization axiom by constructing a suitable simplicial resolution. For this we need additional assumptions.

A morphism $f : M \to X$ is an object in the category $M/\mathcal{A}$, the category under M. The forgetful functor

$$R_M : M/\mathcal{A} \to \mathcal{A} \to \mathcal{B}$$

has a left adjoint

$$L_M : \mathcal{B} \to M/\mathcal{A}, \quad B \mapsto M \sqcup L(B).$$

The composite $F_M = R_M \circ L_M$ is a monad on $\mathcal{B}$.

4.7 The (simplicial) *Godement resolution* of $(X,f) \in M/\mathcal{A}$ is the simplicial map

$$\varepsilon : B_\bullet(L_M, F_M, R_M(X,f)) \to X_\bullet,$$

where $B_n(L_M, F_M, R_M(X,f)) = L_M \circ (F_M)^n \circ R_M(X,f)$ and $X_\bullet$ is the constant simplicial object on (X,f). The structure maps and ε are induced by the adjunction maps of L_M and R_M. Moreover,

$$R_M(\varepsilon) : R_M B_\bullet(L_M, F_M, R_M(X,f)) \to R_M X_\bullet$$

has a natural section η, and there is a simplicial homotopy

$$\eta \circ R_M(\varepsilon) \simeq id.$$

An object (X,f) is $M/\mathcal{A}$ is called *R-cofibrant*, if $f : M \to X$ is an R-cofibration. A morphism $h : (X,f) \to (Y,g)$ in $M/\mathcal{A}$ is called an *R-equivalence*, if $h : X \to Y$ is an R-equivalence.

By (4.4), each $B_n(L_M, F_M, R_M(X,f)$ is R-cofibrant in $M/\mathcal{A}$. Hence the factorization axiom (C4) holds for $(\mathcal{A}, R\mathbf{eq}, R\mathbf{cof})$, if the following assumptions are satisfied:

4.8 Assumptions: Let $\mathcal{S}(M/\mathcal{A})$ denote the category of simplicial objects in $M/\mathcal{A}$. For each $M \in \mathcal{A}$ we assume that there is a realization functor

$$\mathcal{S}(M/\mathcal{A}) \to M/\mathcal{A}, \quad X_\bullet \mapsto \varrho(X_\bullet)$$

satisfying

(1) if each X_n is R-cofibrant, so in $\varrho(X_\bullet)$

(2) the simplicial map ε of (4.7) induces an R-equivalence

$$\overline{\varepsilon}_X : \varrho(B_\bullet(L_M, F_M, R_M(X,f)) \to (X,f)$$

4.9 Proposition: If (4.8) and the general assumption (4.1) hold, then $(\mathcal{A}, R\mathbf{eq}, R\mathbf{cof})$ is a cofibration category such that all objects are fibrant.

We have the following obvious addendum:

4.10 Addendum: If R maps R-cofibrations to cofibrations and preserves pushouts along R-cofibrations, then $(\mathcal{A}, R\mathbf{eq}, R\mathbf{cof})$ is proper. This holds, in particular, if R has a right $\mathcal{V}$-adjoint.

5 Standard realizations

In this section we discuss the standard realizations. By assumption we are given a functor

$$\nabla : \triangle \to \mathcal{V}.$$

Let $\mathcal{I}nj \subset \triangle$ denote the subcategory of injective morphisms. For a simplicial object $X_\bullet$ in $\mathcal{A}$ we define the *thin realization* $|X_\bullet|_\mathcal{A}$ and the *fat realization* $\|X_\bullet\|_\mathcal{A}$ to be the coends of

$$\triangle^{op} \times \triangle \xrightarrow{X \times \nabla} \mathcal{A} \times \mathcal{V} \xrightarrow{-\otimes -} \mathcal{A}$$

respectively of its restriction to $\mathcal{I}nj^{op} \times \mathcal{I}nj$. We also use the suggestive notation $X_\bullet \otimes_\triangle \nabla$ respectively $X_\bullet \otimes_{\mathcal{I}nj} \nabla$ for $|X_\bullet|$ and $\|X_\bullet\|$.

Let $(\mathcal{I}nj \downarrow [n])^*$ be the over category with the terminal object $id_{[n]}$ deleted. Dually, let $\mathcal{S}ur \subset \triangle$ be the category of surjective morphisms and $(\mathcal{S}ur^{op} \downarrow [n])^*$ the over category obtained from $\mathcal{S}ur^{op}$, again with the terminal object $id_{[n]}$ deleted.

5.1 Definition: Let $X_\bullet$ be a simplicial object and $K^\bullet$ a cosimplicial object in $\mathcal{A}$

(1) the *n-th latching objects* of $X_\bullet$ respectively $K^\bullet$ are

$$\begin{aligned} sX_n &= \operatorname{colim}_{(\mathcal{S}ur^{op}\downarrow[n])^*} X \\ \partial K^n &= colim_{(\mathcal{I}nj\downarrow[n])^*} K \end{aligned}$$

where $X : (\mathcal{S}ur^{op} \downarrow [n])^* \to \mathcal{A}$ and $K : (\mathcal{I}nj \downarrow [n])^* \to \mathcal{A}$ are the functors determined by $X_\bullet$ and $K^\bullet$.

(2) The canonical maps $sX_n \to X_n$ and $\partial K^n \to K^n$ are called the *n-th latching maps.*

(3) $X_\bullet$ and $K^\bullet$ are called *proper* if all latching maps are strong cofibrations.

By [10], the category $\mathcal{A}$ is a proper cofibration category with strong cofibrations and equivalences as structures maps. Hence the well-known proof applies to show

5.2 Proposition: If $f : X_\bullet \to Y_\bullet$ is a map of proper simplicial objects in $\mathcal{A}$ and $g : K^\bullet \to L^\bullet$ is a map of proper cosimplicial objects in $\mathcal{V}$ such that each f_n and g^n is an equivalence, then

$$f \otimes_\triangle g : X_\bullet \otimes_\triangle K^\bullet \to Y_\bullet \otimes_\triangle L^\bullet$$

is a homotopy equivalence.

The following axioms replace the Assumptions 4.8.

5.3 Realization axioms:

(1) $\nabla : \Delta \to \mathcal{V}$ is a proper cosimplicial object in $\mathcal{V}$.

(2) R preserves the fat realization up to homotopy, i.e. the natural map

$$\|RX_\bullet\|_{\mathcal{B}} \to R(\|X_\bullet\|_{\mathcal{A}})$$

is an equivalence in $\mathcal{B}$.

(3) $\| - \|_{\mathcal{B}}$ maps the simplicial homotopy $R_M\varepsilon$ of (4.7) to an equivalence.

(4) If $X_\bullet$ is the constant simplicial object on $X \in \mathcal{A}$, then the map

$$\xi_X : \|X_\bullet\|_{\mathcal{A}} \to X,$$

induced by the maps $\nabla([n]) \to *$, is an equivalence.

5.4 Proposition: The realization axioms imply the Assumptions 4.8.

Proof: Let $X_\bullet = \{X_n, f_n\}$ be a simplicial object in $M/\mathcal{A}$, $M \in \mathcal{A}$. The maps $f_n : M \to X_n$ define a simplicial map $f : M_\bullet \to X_\bullet$ in $\mathcal{SA}$ from the constant simplicial object on M to $X_\bullet$. We define $\varrho(X_\bullet) \in M/\mathcal{A}$ to be the pushout in $\mathcal{A}$

$$\begin{tikzcd} \|M_\bullet\|_{\mathcal{A}} \arrow[r, "\|f\|"] \arrow[d, "\xi"'] & \|X_\bullet\|_{\mathcal{A}} \arrow[d] \\ M \arrow[r] & \varrho(X_\bullet) \end{tikzcd}$$

Claim: If each (X_n, f_n) is R-cofibrant in $M/\mathcal{A}$, then $\|f\|$ is an R-cofibration in $\mathcal{A}$. Hence $\varrho(X_\bullet)$ is R-cofibrant in $M/\mathcal{A}$.

Proof: Let $\|X_\bullet\|^{(n)}$ denote the "n-skeleton" of $\|X_\bullet\|_{\mathcal{A}}$. Then $\|f\|^{(0)} = f_0 : M \to X_0$ is an R-cofibration. By (4.4) it suffices to show that

$$\|M_\bullet\|^{(n)} \cup_{\|M_\bullet\|^{(n-1)}} \|X_\bullet\|^{(n-1)} \to \|X_\bullet\|^{(n)}$$

is an R-cofibration. Since

$$\|M_\bullet\|^{(n)} \cup_{\|M_\bullet\|^{(n-1)}} \|X_\bullet\|^{(n-1)} \cong M \otimes \nabla[n] \cup_{M\otimes\partial\nabla[n]} \|X_\bullet\|^{(n-1)},$$

we have to show that

$$M \otimes \nabla[n] \cup_{M\otimes\partial\nabla[n]} \|X_\bullet\|^{(n-1)} \to X_n \otimes \nabla[n] \cup_{X_n\otimes\partial\nabla[n]} \|X_\bullet\|^{(n-1)}$$

is an R-cofibration. Applying Reedy's patching lemma we need to know that

$$(M \otimes \nabla[n]) \cup_{M \otimes \partial\nabla[n]} (X_n \otimes \partial\nabla[n]) \to X_n \otimes \nabla[n]$$

is an R-cofibration. Passing to adjoints this is equivalent to showing that $E^{\nabla[n]} \to B^{\nabla[n]} \times_{\mathcal{B}^{\partial\nabla[n]}} E^{\partial\nabla[n]}$ is a trivial R-fibration, provided $E \to B$ is a trivial R-fibration. Since R is $\mathcal{V}$-right adjoint, it preserves cotensors. By definition $RE \to RB$ is a trivial fibration in $\mathcal{B}$. Hence

$$(RE)^{\nabla[n]} \to (RB)^{\nabla[n]} \times_{(RB)^{\partial\nabla[n]}} (RE)^{\partial\nabla[n]}$$

is a trivial fibration by [10, 3.6]. This proves the claim.

For $(X, f) \in M/\mathcal{A}$ let $B_\bullet(X, f) = B_\bullet(L_M, F_M, R_M(X, f))$. Let $X_\bullet$ denote the constant simplicial object on X. The simplicial map ε of (4.7) defines a map $\|\varepsilon\|$ inducing $\overline{\varepsilon}_X$

$$\begin{tikzcd}
\|M_\bullet\|_{\mathcal{A}} \arrow[r] \arrow[d, "\xi_M"'] & \|B_\bullet(X,f)\|_{\mathcal{A}} \arrow[r, "\|\varepsilon\|"] \arrow[d, "q"] & \|X_\bullet\|_{\mathcal{A}} \arrow[d, "\xi_X"] \\
M \arrow[r] & \varrho(B_\bullet(X,f)) \arrow[r, "\overline{\varepsilon}_X"] & X
\end{tikzcd}$$

By (5.3), $\|\varepsilon\|$ is an R-equivalence. Since $\|M_\bullet\|_{\mathcal{A}} \to \|B_\bullet(X, f)\|_\bullet$ is an R-cofibration, it is a strong cofibration in $\mathcal{A}$. But $(\mathcal{A}, s\mathbf{cof}_{\mathcal{A}}, \mathbf{eq}_{\mathcal{A}})$ is a proper cofibration category. Since ξ_M and ξ_X are equivalences, so is q and hence $R(\overline{\varepsilon}_X)$. □

Thin realizations: In some examples the thin realization has considerable advantages over the fat realization.

5.5 Proposition: Suppose that the assumptions (5.3) hold for the thin realization and that $B_\bullet(X, f)$ is R-proper, i.e. $sB_n(X, f) \to B_n(X, f)$ is an R-cofibration. Then

$$M \longrightarrow |B_\bullet(X, f)|_{\mathcal{A}} \xrightarrow{|\varepsilon|} |X_\bullet|_{\mathcal{A}} = X$$

is a factorization of $f : M \to X$ into an R-cofibration and an R-equivalence.

Proof: The assumptions (5.3) imply that $|\varepsilon|$ is an R-equivalence. The map

$$M \to |B_\bullet(X, f)|^{(0)}_{\mathcal{A}} = M \sqcup LR(X)$$

is an R-cofibration, since $LR(X)$ is R-cofibrant. Hence the result follows from

5.6 Lemma: If $X_\bullet$ is an R-proper simplicial object in $\mathcal{A}$, then

$$|X_\bullet|^{(n-1)} \to |X_\bullet|^{(n)}$$

is an R-cofibration.

Proof: Since

$$\begin{tikzcd}
(sX_n \otimes \nabla[n]) \cup_{sX_n \otimes \partial\nabla[n]} (X_n \otimes \partial\nabla[n]) \arrow[r, "j"] \arrow[d] & X_n \otimes \nabla[n] \arrow[d] \\
|X_\bullet|^{(n-1)} \arrow[r] & |X_\bullet|^{(n)}
\end{tikzcd}$$

is a pushout, it suffices to show that j is an R-cofibration. This follows from

5.7 Lemma: If $f : X \to Y$ is an R-cofibration in $\mathcal{A}$ and $i : K \to L$ is a strong cofibration in $\mathcal{V}$, then

$$j : (X \otimes L) \cup_{X \otimes K} (Y \otimes K) \to Y \otimes L$$

is an R-cofibration.

Proof Let $p : E \to B$ be a trivial R-fibration. We have to show that j has the LLP for p. Passing to adjoints this is equivalent to showing that

$$p^j : E^L \to B^L \times_{B^K} E^K$$

is a trivial R-fibration. Since R preserves limits and cotensors we have to show that

$$(Rp)^j : (RE)^L \to (RB)^L \times_{(RB)^K} (RE)^K$$

is a trivial fibration in $\mathcal{B}$. But this holds by [10, (2.9),(3.6)]. $\square$

6 The dual case

Again we assume (4.1). We want to lift the fibration structure in $\mathcal{A}$ to a fibration structure in $\mathcal{B}$.

6.1 Definition: A morphism $f : X \to Y$ is $\mathcal{B}$ is called an

(1) *L-cofibration*, if $L(f) \in \mathbf{cof}_{\mathcal{A}}$

(2) *L-equivalence*, if $L(f) \in \mathbf{eq}_{\mathcal{A}}$

(3) *L-fibration*, if it has the RLP for all trivial L-cofibrations

(4) *trivial L-(co)fibration*, if it is an L-equivalence and an L-(co)fibration.

The results of Sections 4 and 5 dualize verbatim. For the factorization axiom we consider a morphism $f : X \to B$ in $\mathcal{B}$ as an object in the category $\mathcal{B}/B$, the category $\mathcal{B}$ over B. The forgetful functor

$$L_{\mathcal{B}} : \mathcal{B}/B \to \mathcal{B} \to \mathcal{A}$$

has a right adjoint

$$R_B : \mathcal{A} \to \mathcal{B}/B, \quad A \mapsto R(A) \times B.$$

We obtain a cosimplicial resolution of $(X, f) \in \mathcal{B}/B$

$$\varepsilon : X^\bullet \to B^\bullet(R_B, L_B \circ R_B, L_B(X, f)).$$

The assumptions dual to (4.8) are

6.2 Assumptions: Let $\mathcal{S}^{co}(\mathcal{B}/B)$ denote the category of cosimplicial objects in $\mathcal{B}/B$. For each $B \in \mathcal{B}$ we assume that there is a corealization functor

$$\varepsilon : \mathcal{S}^{co}(\mathcal{B}/B) \to \mathcal{B}/B, \quad X^\bullet \mapsto \varrho^{co}(X^\bullet)$$

satisfying

(1) if each X^n is L-fibrant, so is $\varrho^{co}(X^\bullet)$,

(2) the cosimplicial map ε induces an L-equivalence

$$\overline{\varepsilon}_X : (X, f) \to \varrho^{co}(B^\bullet(R_B, L_B \circ R_B, L_B(X, f)))$$

6.3 Proposition: If (6.2) and assumption (4.1) hold, then $(\mathcal{B}, L\mathbf{eq}, L\mathbf{fib})$ is a fibration category such that all objects are cofibrant.

Let $X^\bullet$ be a cosimplicial object in $\mathcal{B}$. We define the *thin corealization* $|X^\bullet|^{co}$ and its *fat* version $\|X^\bullet\|^{co}$ to be the end of

$$\triangle \times \triangle^{op} \xrightarrow{X^\bullet \times \nabla^{op}} \mathcal{B} \times \mathcal{V}^{op} \xrightarrow{\text{cotensor}} \mathcal{B}$$

respectively its restriction to $\mathcal{I}nj \times \mathcal{I}nj^{op}$.

6.4 Corealization axioms:

(1) $\nabla : \triangle \to \mathcal{V}$ is a proper cosimplicial object in $\mathcal{V}$

(2) L preserves fat corealizations up to homotopy

(3) $\| - \|^{co}_{\mathcal{A}}$ maps the cosimplicial homotopy $L_B\varepsilon$ to an equivalence

(4) if $X^\bullet$ is the constant cosimplicial object on $X \in \mathcal{B}$, then the map

$$\eta : X \to \|X^\bullet\|^{co}_{\mathcal{B}},$$

induced by the maps $\nabla[n] \to *$, is a homotopy equivalence.

6.5 Proposition: The corealization axioms imply (6.2).

Finally, a word about the dual of (5.5).

6.6 Definition: Let $X^\bullet$ be a cosimplicial object in $\mathcal{B}$. The *n-th matching object* of $X^\bullet$ is

$$sX^n = \lim_{\mathcal{S}ur\downarrow[n]^*} X^\bullet$$

$X^\bullet$ is called *coproper* if the canonical map $X^n \to sX^n$ is a fibration for all n.

6.7 Proposition: Suppose the assumptions (6.4) hold for the thin corealization and that $B^\bullet(R_B, L_B \circ L_B(X,f))$ is L-coproper. Then

$$X \xrightarrow{|\varepsilon|^{co}} |B^\bullet(R_B, L_B \circ R_B, L_B(X,f)|^{co}_{\mathcal{B}} \longrightarrow B$$

is a factorization of $f : X \to B$ into an L-equivalence followed by an L-fibration. $\square$

7 Topologically enriched categories

Realization functors: Let $\mathcal{V}$ and $\nabla : \triangle \to \mathcal{V}$ be one of the examples (2.4 (1), (2), (3)). It is well-known that ∇ is a proper cosimplicial object.

Let $\mathcal{A}$ be a $\mathcal{V}$-complete and -cocomplete, tensored and cotensored category, and let $X_\bullet$ be a simplicial object in $\mathcal{A}$. Then

$$\|X_\bullet\|_{\mathcal{A}} = X_\bullet \otimes_{Inj} \nabla \cong X_\bullet \otimes_\triangle \triangle \otimes_{Inj} \nabla$$

It is well-known that $\triangle \otimes_{Inj} \nabla$ is a proper cosimplicial object and that the evaluation

$$\triangle \otimes_{Inj} \nabla \longrightarrow \nabla$$

is dimensionswise a homotopy equivalence. So we deduce from (5.2).

7.1 Proposition: If $X_\bullet$ is a proper simplicial object in $\mathcal{A}$, then the natural map $\|X_\bullet\|_{\mathcal{A}} \to |X_\bullet|_{\mathcal{A}}$ is an equivalence. $\square$

If $X_\bullet$ is the constant simplicial object on $X \in \mathcal{A}$, then $|X_\bullet|_{\mathcal{A}} \cong X$. Hence (1) and (4) of the Realization axioms hold for the fat as well as the thin realization. We will now show that Axiom (3) holds for both realizations. The following result is an immediate consequence of [10, Prop. 3.4] and the fact that $(\mathcal{A}, cof_{\mathcal{A}}, eq_{\mathcal{A}})$ is a proper cofibration category with all objects fibrant and cofibrant.

7.2 Lemma: Let $f_\bullet : X_\bullet \to Y_\bullet$ be a simplicial map of simplicial objects such that each $f_n : X_n \to Y_n$ is a homotopy equivalence. Then:

(1) $\|f_\bullet\| : \|X_\bullet\|_{\mathcal{A}} \to \|Y_\bullet\|_{\mathcal{A}}$ is a homotopy equivalence,

(2) if $X_\bullet$ and $Y_\bullet$ are proper, then $|f_\bullet| : |X_\bullet|_{\mathcal{A}} \to |Y_\bullet|_{\mathcal{A}}$ is a homotopy equivalence. $\square$

7.3 Lemma: Let $X_\bullet$ be a simplicial object in $\mathcal{A}$ and $K_\bullet$ a simplicial object in $\mathcal{V}$. Then $X_\bullet \otimes K_\bullet$ is a bisimplicial object in $\mathcal{A}$. Let $d(X_\bullet \otimes K_\bullet)$ denote its diagonal simplicial object. Then

(1) there is a natural isomorphism

$$|d(X_\bullet \otimes K_\bullet)|_{\mathcal{A}} \cong |X_\bullet|_{\mathcal{A}} \otimes |K_\bullet|_{\mathcal{V}}$$

(2) there is a sequence of natural homotopy equivalences:

$$\|d(X_\bullet \otimes K_\bullet)\|_{\mathcal{A}} \simeq \|X_\bullet\|_{\mathcal{A}} \otimes \|K_\bullet\|_{\mathcal{V}}$$

Proof:
$$\begin{aligned} |d(X_\bullet \otimes K_\bullet)|_{\mathcal{A}} &= d(X_\bullet \otimes K_\bullet) \otimes_\triangle \nabla \\ &\cong (X_\bullet \otimes K_\bullet) \otimes_{\triangle\times\triangle} (\triangle \times \triangle \otimes_\triangle \nabla) && (a) \\ &\cong (X_\bullet \otimes K_\bullet) \otimes_{\triangle\times\triangle} (\nabla \times \nabla) && (b) \\ &\cong (X_\bullet \otimes_\triangle \nabla) \otimes (K_\bullet \otimes_\triangle \nabla) && (c) \end{aligned}$$

In (a) we consider the sets $\triangle([n],[m])$ as discrete topological spaces (if $\mathcal{V}$ is a based we have to take $\triangle([n],[m])_+$ replace the product by the smash product). In (b) we use that

$$(\triangle \times \triangle) \otimes_\triangle \nabla \cong \nabla \times \nabla$$

as bicosimplicial spaces. Part (c) is straight forward.

Let $\tau X_\bullet = X_\bullet \otimes_\triangle B(\triangle,\triangle,\triangle)$ where we have the twosided topological bar-construction on the right. Then $\tau X_\bullet$ is a proper simplicial object and there is a natural homotopy equivalence

$$\tau X_\bullet \xrightarrow{\simeq} X_\bullet$$

in the weak sense, i.e. each morphism $\tau X_n \to X_n$ is a homotopy equivalence. $d(\tau X_\bullet \otimes \tau K_\bullet)$ is proper. Using (1) and (7.2) we obtain

$$\begin{tikzcd} \|d(X_\bullet \otimes K_\bullet)\|_{\mathcal{A}} & \|d(\tau X_\bullet \otimes \tau K_\bullet)\|_{\mathcal{A}} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & |d(\tau X_\bullet \otimes \tau K_\bullet)|_{\mathcal{A}} \arrow[d, "\cong"] \\ \|X_\bullet\|_{\mathcal{A}} \otimes \|K_\bullet\|_{\mathcal{V}} & \|\tau X_\bullet\|_{\mathcal{A}} \otimes \|\tau K_\bullet\|_{\mathcal{V}} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & |\tau X_\bullet|_{\mathcal{A}} \otimes |\tau K_\bullet|_{\mathcal{V}} \end{tikzcd}$$

□

7.4 Corollary: If $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ are simplicially homotopic, then $\|f_\bullet\|_{\mathcal{A}}$ and $\|g_\bullet\|_{\mathcal{A}}$ respectively $|f_\bullet|_{\mathcal{A}}$ and $|g_\bullet|_{\mathcal{A}}$ are homotopic in $\mathcal{A}$.

Proof: Consider $\triangle(-,[1])$ (respectively $\triangle(-,[1])_+$ is the based cases) as simplicial object in $\mathcal{V}$. Then a simplicial homotopy is a simplicial map $X_\bullet \otimes \triangle(-,[1]) \to Y_\bullet$. Now apply (7.3). □

7.5 Diagrams in $\mathcal{T}$op: Let $\mathcal{A}$ and $\mathcal{B}$ be topologically enriched small categories and $F : \mathcal{A} \to \mathcal{B}$ a continuous functor. Let $\mathcal{T}\mathrm{op}^{\mathcal{A}}$ denote the category of $\mathcal{A}$-diagrams, i.e. of continuous functors $X : \mathcal{A} \to \mathcal{T}\mathrm{op}$. It is $\mathcal{T}$op-complete and -cocomplete, tensored and cotensored in the obvious way. The forgetful functor $R : \mathcal{T}\mathrm{op}^{\mathcal{B}} \to \mathcal{T}\mathrm{op}^{\mathcal{A}}$ has a continuous left adjoint

$$L : \mathcal{T}\mathrm{op}^{\mathcal{A}} \longrightarrow \mathcal{T}\mathrm{op}^{\mathcal{B}}, \quad X \longmapsto \mathcal{B} \otimes_{\mathcal{A}} X$$

and a continuous right adjoint

$$Q : \mathcal{T}\mathrm{op}^{\mathcal{A}} \longrightarrow \mathcal{T}\mathrm{op}^{\mathcal{B}}, \quad X \longmapsto \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, X).$$

Hence R preserves limits and colimits, in particular fat and thin realizations, and $(\mathcal{T}\mathrm{op}^{\mathcal{B}}, R\mathbf{eq}, R\mathbf{cof})$ is a proper cofibration category.

Of special interest is the case $R : ob\mathcal{A} \to \mathcal{A}$, which leads to the usual homotopy theory of diagrams. The R-equivalences are those maps of diagrams $f : X \to Y$, for which $f(A) : X(A) \to Y(A)$ is a homotopy equivalence for each $A \in ob\mathcal{A}$. If we specialize even further by taking $\mathcal{A}$ to be a category with one object $*$ and $\mathcal{A}(*,*)$ to be a topological group G, we deal with the homotopy theory of G-spaces.

We have similar results for $\mathcal{T}\mathrm{op}_*$, $\mathcal{HT}\mathrm{op}$, and $\mathcal{HT}\mathrm{op}_*$.

7.6 Topological operads: An *operad* as defined in [2] (there called a category of operators in standard form) is a topologically enriched permutative category $(\mathcal{A}, \oplus, 0)$ with $ob\mathcal{A} = \mathbb{N}$, and $m \oplus n = m + n$, such that

$$\begin{array}{rcl} \coprod\limits_{r_1+\ldots+r_n=m} \mathcal{A}(r_1,1) \times \ldots \times \mathcal{A}(r_n,1) \times_{\Sigma_{r_1} \times \ldots \times \Sigma_{r_n}} \times \Sigma_m & \longrightarrow & \mathcal{A}(m,n) \\ (f_1, \ldots, f_n; \sigma) & \longmapsto & (f_1 \oplus \ldots \oplus f_n) \circ \sigma \end{array}$$

is a homeomorphisms. Let $\mathcal{O}p$ denote the category of operads. $\mathcal{O}p$ is $\mathcal{T}$op-enriched, and it is well-known that it satisfies the Assumptions 3.1.

Operads encode algebraic structures on topological spaces. If $\mathcal{A}$ is an operad, an $\mathcal{A}$-*algebra* is a continuous strictly permutative functor $(\mathcal{A}, \oplus, 0) \to (\mathcal{T}\mathrm{op}, \times, *)$.

The category $\mathcal{C}oll_*$ of based collections has $\mathbb{N}$-graded spaces $(X_n; n \in \mathbb{N})$ as objects, where X_n is a right Σ_n-space and X_1 is based. Morphisms are graded equivariant maps $f = (f_n)$, such that f_1 is based.

$\mathcal{C}oll_*$ is $\mathcal{T}$op-enriched in the obvious way, tensored and cotensored, where $(X_n) \otimes K = (Y_n)$ with $Y_n = X_n \times K$ for $n \neq 1$ and $Y_1 = X_1 \wedge K_+$. The forgetful functor

$$R : \mathcal{O}p \longrightarrow \mathcal{C}oll_*, \quad \mathcal{A} \longmapsto (\mathcal{A}(n,1); n \in \mathbb{N})$$

has a continuous left adjoint, constructed in [3, Chap.III]. Since the thin realization of simplicial spaces preserves products, R preserves thin realizations by the argument of [8, 4.4]. To be able to apply (5.5) we have to ensure, that $B_\bullet(\mathcal{A}, f)$ is R-proper for each map of operads $f : \mathcal{M} \to \mathcal{A}$. This is the case, provided our operads are well-pointed, i.e. the inclusion $\{id_1\} \subset \mathcal{A}(1,1)$ is a closed cofibration.

Let $\mathcal{O}p_w \subset \mathcal{O}p$ be the full subcategory of well-pointed operads. Then $(\mathcal{O}p_w, R\mathbf{eq}, R\mathbf{cof})$ is a cofibration category. This result can be used to study homotopy invariance of algebraic structures on spaces. In [13] it is applied to construct universal E_∞-operads.

7.7 $\mathcal{A}$-algebras: Let $\mathcal{A}$ be a well-pointed topological operad and let $\mathcal{T}\mathrm{op}^{\mathcal{A}}$ denote the category of $\mathcal{A}$-algebras in $\mathcal{T}\mathrm{op}$. The forgetful functor

$$R : \mathcal{T}\mathrm{op}^{\mathcal{A}} \longrightarrow \mathcal{T}\mathrm{op}, \quad X \longmapsto X(1)$$

has a continuous left adjoint

$$L : \mathcal{T}\mathrm{op} \longrightarrow \mathcal{T}\mathrm{op}^{\mathcal{A}}, \quad Y \longrightarrow \coprod_{n=0}^{\infty} \mathcal{A}(n,1) \times_{\Sigma_n} Y^n$$

As in (7.6), R preserves thin realizations. Since $\mathcal{A}$ is well-pointed $B_\bullet(X, f)$ is R-proper for each $\mathcal{A}$-algebra X. Hence $(\mathcal{T}\mathrm{op}^{\mathcal{A}}, R\mathbf{eq}, R\mathbf{cof})$ is a cofibration category.

If we consider $\mathcal{A}$-algebras in $\mathcal{T}\mathrm{op}_*$, base point relations enter the definition of L. We obtain the analogous result for well-pointed algebras.

7.8 Spectra: Let $f : A \to B$ be a map of ring spectra in the sense of [4], i.e. of monoids in the symmetric monoidal category of S-module spectra, where S is the sphere spectrum. Let ${}_A\mathcal{M}od$ denote the category of A-module spectra. It is $\mathcal{T}\mathrm{op}_*$-enriched, $\mathcal{T}\mathrm{op}_*$-complete and -cocomplete, tensored and cotensored. For details see [8] or [4].

The forgetful functor

$$R : {}_B\mathcal{M}od \longrightarrow {}_A\mathcal{M}od$$

has a left $\mathcal{T}\mathrm{op}_*$-adjoint

$$L : {}_A\mathcal{M}od \longrightarrow {}_B\mathcal{M}od, \quad X \longmapsto B \wedge_A X$$

and a right $\mathcal{T}\mathrm{op}_*$-adjoint $X \mapsto F_A(B, X)$, the function spectrum. Since $B \wedge_A$-preserves colimits, $\|X_\bullet\|_{{}_A\mathcal{M}od}$ is a B-module, if $X_\bullet$ is a simplicial B-module. By the argument of [8, 4.4], the functor R preserves fat realizations. The same applies to the thin realization. Hence $({}_B\mathcal{M}od, R\mathbf{eq}, R\mathbf{cof})$ is a proper cofibration category. We will use this structure to study the topological Hochschild homology of spectra.

8 Chain complex-enriched categories

Let R be a commutative ring and $\mathcal{C}plx_R$ the category of unbounded chain complexes of R-modules. Let $\nabla : \triangle \to \mathcal{C}plx_R$ map $[n]$ to the normalized chain complex associated with the simplicial R-module $R[\triangle(-,[n])]$.

The considerations of Section 7 concerning realization functors translate to chain complexes in the known standard fashion. This is different for the examples (7.6) and (7.7). In both cases the left adjoint is not additive and hence not a $\mathcal{C}plx_R$-enriched functor. So the only example we want to discuss is a change of rings functor. Let $F : R \to S$ be a ring homomorphisms. The forgetful functor

$$F^* : \mathcal{C}plx_S \longrightarrow \mathcal{C}plx_R$$

has a left adjoint

$$F_* : \mathcal{C}plx_R \longrightarrow \mathcal{C}plx_S$$

namely scalar extension. They form a $\mathcal{C}plx_R$-adjoint pair. As in Example 7.8, F^* preserves thin and fat realizations. Hence $(\mathcal{C}plx_S, F^*\mathbf{eq}, F^*\mathbf{cof})$ is a cofibration category. It is proper, because F^* has a right adjoint $\mathrm{Hom}_R(S,-)$. This example links our approach to classical relative homological algebra.

8.1 Proposition: A chain map $p : X \to Y$ in $\mathcal{C}plx_S$ is an F^*-fibration iff each $p_n : X_n \to Y_n$ has an R-linear section.

Proof: By definition, p is an F^*-fibration iff it is a fibration in $\mathcal{C}plx_R$. Let I denote the 1-simplex in $\mathcal{C}plx_R$, i.e. $I_0 = R \otimes R,\ I_1 = R,\ I_n = 0$ otherwise and

$$d : I_1 \longrightarrow I_0, \quad r \longmapsto (r, -r)$$

For $A \in \mathcal{C}plx_R$ we have $A \otimes \nabla[1] = A \otimes_R I$, and p is a fibration iff the commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_0"'] & X \arrow[d, "p"] \\ A \otimes_R I \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

admits a diagonal filler h. Here i_0 is induced by the map $R \to I_0$, $r \mapsto (0, -r)$. Passing to adjoints, we have to find a filler for

$$\begin{tikzcd}
R \arrow[r, "\overline{f}"] \arrow[d, "i_0"'] & \mathrm{Hom}_R(A,X) \arrow[d, "p_*"] \\
I \arrow[r, "\overline{g}"] \arrow[ur, dashed, "\overline{h}"] & \mathrm{Hom}_R(A,Y)
\end{tikzcd}$$

Given $\overline{f}$, the chain map maps $\overline{g}$ correspond bijectively to the maps

$$\overline{g}_1 : R \longrightarrow (\mathrm{Hom}_R(A,Y))_1 = \prod_{n\in\mathbb{Z}} \mathrm{Hom}_R(A_n, Y_{n+1})$$

The same holds for $\overline{h}$. Hence $\overline{h}$ always exists iff

$$(p_*)_1 : \prod_{n\in\mathbb{Z}} \mathrm{Hom}_R(A_n, X_{n+1}) \longrightarrow \prod_{n\in\mathbb{Z}} \mathrm{Hom}_R(A_n, Y_{n+1})$$

is surjective for any A. This is equivalent to the existence of sections of $p_n : X_n \to Y_n$ as R-modules. $\square$

8.2 Definition: A map $p : X \to Y$ of S-modules is called *R-split surjective* if it admits an R-linear section. An S-module A is called *R-split projective*, if for any R-split surjective map $p : X \to Y$ and any S-linear map $g : A \to Y$ there is an S-linear lift $f : A \to X$ of g.

8.3 Proposition: If $A \in \mathcal{C}plx_S$ is F^*-cofibrant, then each A_n is *R-split projective.* As a partial converse, any bounded below complex of R-split projective S-modules is F^*-cofibrant.

Proof: We follow in part the argument of [5, (2.3.6)]. Let $p : X \to Y$ be an R-split surjection of S-modules. Let D^nX denote the contractible complex with X in dimensions n and $n-1$ and 0 otherwise. By (8.1), D^np is a trivial F^*-fibration. An S-linear map $g : A_{n-1} \to Y$ defines a chain map $g : A \to D^nY$.

$$\begin{tikzcd}
{} \arrow[d, "0"'] & {} \arrow[d, "0"'] & {} \arrow[d, "d"] \\
X \arrow[r, "p"] \arrow[d, "id"'] & Y \arrow[d, "id"'] & A_n \arrow[l, "g\circ d"'] \arrow[d, "d"] \\
X \arrow[r, "p"] \arrow[d, "0"'] & Y \arrow[d, "0"'] & A_{n-1} \arrow[l, "g"'] \arrow[d, "d"] \\
{} & {} & {}
\end{tikzcd}$$

Since A is F^* cofibrant, this chain map lifts to a chain map $f : A \to D^n X$. Hence A_{n-1} is R-split projective.

Conversely, suppose we are given a bounded below chain complex A of R-split projective S-modules and a chain map $p : X \to Y$ of S-complexes, which is a trivial fibration as map of R-complexes. Then the kernel K of p is an S-complex which is contractible as R-complex. Suppose we are given $g : A \to Y$, we construct a lift $f : A \to X$ of g inductively. Since A is bounded below, we have no trouble to start the induction, and we suppose that we already have f_k for $k < n$ such that $p_k \circ f_k = g_k$ and $df_k = f_{k-1}d$. Since p_n is R-split surjective, there is a map $\overline{u} : A_n \to X_n$ such that $p_n \circ u = g_n$. For $F = d\overline{u} - f_{n-1}d : A_n \to X_{n-1}$ we have $p_{n-1} \circ F = d \circ F = 0$, so that F is in fact a map $F : A_n \to Z(K_{n-1})$ into the cycles of K_{n-1}. Since K is contractible as R-complex, the boundary map $d : K_n \to Z(K_{n-1})$ is R-split surjective. Hence there is an S-linear map $G : A_n \to K_n$ such that $d \circ G = F$. Now take $f_n = \overline{u} - G$. $\square$

9 $\mathcal{C}at$-enriched categories

Let $\mathcal{V}$ be the category $\mathcal{C}at$ of small categories and $\nabla : \triangle \to \mathcal{C}at$ the funcor of Example 2.4.6. Then ∇ is proper, and two categories are homotopy equivalent if they are equivalent in the category theoretical sense. This type of homotopy equivalence is central in the study of structured categories.

Consider the sets $\triangle([k], [m])$ as discrete categories. Then

$$\triangle(-, [m]) : \triangle^{op} \to \mathcal{C}at$$

is a simplicial object in $\mathcal{C}at$, and we have

9.1 Lemma:

$$(\triangle \times \triangle) \otimes_{\triangle} \nabla \cong \nabla \times \nabla$$

as bicosimplicial categories.

The proof is simple, because $\partial\nabla[n] = \nabla[n]$ for $n \geq 2$. The lemma implies that the thin realization preserves products (compare the proof of (7.3)). Now the results of Section 7 translate to this case. In particular, Examples 7.6 and 7.7 carry over to $\mathcal{C}at$-operads and their algebras.

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