CONTROLLED WILD ALGEBRAS AND τ-WILD ALGEBRAS

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Dedicated to Professor Yukio Tsushima on the occasion of his 60-th birthday

Introduction

Finite dimensional algebras over an algebraically closed field are devided into two disjoint classes by Drozd's Tame and Wild dichotomy (see [4] and [2]). In [2], Crawley-Boevey gave a conjecture about a characterization of tame algebras. He showed that all but finite n-dimensional indecomposable modules over tame algebra are τ-invariant for all natural number n, and conjectured that the converse is true, where we call an indecomposable module X *τ-invariant* if $X \cong \tau X$, where τ is Auslander-Reiten translation.

The contraposition of the conjecture says that any wild algebra has infinitely many non-isomorphic indecomposable τ-variant modules of a same dimension, where we call an indecomposable module X *τ-variant* if $X \ncong \tau X$. Therefore we are interested to find τ-variant modules over wild algebras. We call an algebra A *τ-wild* if A has infinitely many non-isomorphic indecomposable τ-variant modules of a same dimension. Therefore the conjecture is equivalent that any wild algebra is τ-wild.

On the other hand, Ringel proposed a notion of wild algebra with a nice functor which defines the wildness. We call it *controlled wild* (see section 2). It is conjectured that any wild algebra is controlled wild, and Han gave many examples of controlled wild algebras by using covering theory in [5].

Thus we are interested in relations between τ-wild algebras and controlled wild algebras, especialy τ-variant modules over controlled wild algebras. One of our results, Theorem 4.1, gives a sufficient condition when controlled wild algebras are τ-wild, and all examples in [5] satisfy the sufficient condition.

In sention 3, we give a way to find controlled wild algebras. It says that for any controlled wild algebras A, any subalgebra B of A with rad B=rad A is also controlled wild. It seems different from the ways in [3] and [5].

1. Preliminaries

Throughout this paper, all algebras are basic over an algebraically closed field k. We assume that all algebras are finite-dimensional unless otherwise stated and all modules are finite-dimensional. We denote $\operatorname{mod} A$ the category of finite-dimesional left A-modules. For any full subcategory $\mathcal{C}$ of $\operatorname{mod} A$, we set that, for any A-modules V and W, $\operatorname{Hom}_A(V,W)_{\mathcal{C}} = \{g \in \operatorname{Hom}_A(V,W) | g$ factors through a module in $\mathcal{C}\}$. If $\mathcal{C}$ is closed under direct summs, then $\operatorname{Hom}_A(V,W)_{\mathcal{C}}$ is subspace of $\operatorname{Hom}_A(V,W)$.

Let A and B be algebras which dimensions are not necessarily finite, and $\mathcal{C}$ a full subcategory of $\operatorname{mod} B$. We call a faithful exact functor $F : \operatorname{mod} A \to \operatorname{mod} B$ *controlled by* $\mathcal{C}$ if (1) $\mathcal{C}$ is closed under direct sums and direct summands and (2) F preserves indecomposability and isomorphism calsses of modules and (3) $\operatorname{Hom}_B(FX, FY) = F\operatorname{Hom}_A(X,Y) \oplus \operatorname{Hom}_B(FX,FY)_{\mathcal{C}}$ for any A-modules X and Y.

We denote $k\langle x,y\rangle$ the free associative algebra with two indeterminates. An algebra A is called *wild* if there exists a A-$k\langle x,y\rangle$-bimodule M which is a free right $k\langle x,y\rangle$-module of finite rank such that the functor $M \otimes_{k\langle x,y\rangle} - : \operatorname{mod} k\langle x,y\rangle \to \operatorname{mod} A$ preservs indecomposability and isomorphism calasses of modules. We set that $F{:=}M \otimes_{k\langle x,y\rangle} -$ and call simply the pair (F, A) a *wild pair*.

2. The definition of controlled wild algebras

An algebra A is called *controlled wild* if there exists a wild pair (F, A) such that F is controlled by a full subcategory $\mathcal{C}$ of $\operatorname{mod} A$. We call simply the algebra A *controlled by* $\mathcal{C}$ and define the *controlling index* of A, $c(A) = inf_{\mathcal{C}}\ \{card(\operatorname{ind}\mathcal{C}) | A$ is controlled by $\mathcal{C}\}$.

Obviously, strictly wild algebras are controlled wild algebras with $c(A)$=0, where an algebra A is called *strictly wild* if there exists a wild pair (F, A) such that F is full.

Remark 1. For any wild pair (F, A) with F controlled by $\mathcal{C}$, any homomorphism in $F\operatorname{Hom}_{k\langle x,y\rangle}(V,W)$ does not factor through semisimple modules. Therefore we can assume that C includes semisimple modules, when we do not strictly mention the controlling index.

Remark 2. The definition of controlled wild algebras above is equivalent to the original one below (see [5]). In this paper, we use the definition above for convenience.

An algebra A is called *controlled wild* if there exists a faithful exact functor $F : \operatorname{mod} k\langle x,y\rangle \to \operatorname{mod} A$ and a full subcategory $\mathcal{C}$ of $\operatorname{mod} A$

closed under direct sums and direct summands such that for any A-modules X and Y,

$\mathrm{Hom}_A(FX, FY)_{\mathcal{C}} \leq \mathrm{rad}\,\mathrm{Hom}_A(FX, FY)$ and
$\mathrm{Hom}_A(FX, FY) = F\,\mathrm{Hom}_{k\langle x,y\rangle}(X, Y) \oplus \mathrm{Hom}_A(FX, FY)_{\mathcal{C}}$.

3. Examples of controlled wild algebras

In [5], Han has given a lot of examples of controlled wild algebras by using covering theory. For examples, algebras with the radical square zero, local algebras, many monomial algebras and so on. Moreover he has shown that these examples have finite controlling index.

In [3], Dräxler has shown, by using a cleaving functor, that for any algebra A with the radical J, if there exist two prmitive idempotents e and f, and natural number n such that $\dim eJ^n f/eJ^{n+1}f \geq 3$, then A is controlled by $\mathrm{mod}\,A/J^n$. In this case, A/J^n can be of infinite representation type, however we do not have any examples of controlled wild algebras with the infinite controlling index now.

The following proposition gives us a way to find controlled wild algebras. It seems different from the ways in [3] and [5].

Proposition 3.1. *Let B be a subalgebra of A such that* $\mathrm{rad}\,A$=$\mathrm{rad}\,B$ *and $\mathcal{C}$ a full subcategory of* $\mathrm{mod}\,A$ *including semisimple modules (see Remark 1). If A is controlled by $\mathcal{C}$, then B is controlled by $\mathcal{C}$, where we take $\mathcal{C}$ as the full subcategory of* $\mathrm{mod}\,B$ *by the assumption that B is the subalgebra of A.*

Example 1. For an algebra $A = \begin{bmatrix} e_1Ae_1 & e_1Ae_2 \\ e_2Ae_1 & e_2Ae_2 \end{bmatrix}$ with local rings e_1Ae_1 and e_2Ae_2, we have decomposition $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \oplus \begin{bmatrix} e_1\,\mathrm{rad}\,Ae_1 & e_1Ae_2 \\ e_2Ae_1 & e_2\,\mathrm{rad}\,Ae_2 \end{bmatrix}$, where e_1 and e_2 are primitive orthogonal idempotents with $e_1 + e_2 = 1$. Let B be the subalgebra of A which has the following form $k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} e_1\,\mathrm{rad}\,Ae_1 & e_1Ae_2 \\ e_2Ae_1 & e_2\,\mathrm{rad}\,Ae_2 \end{bmatrix}$. Then we have that $\mathrm{rad}\,B$=$\mathrm{rad}\,A$. In general, for any algebra A and any subalgebra B of A with the quiver Q_B and Q_A respectivily, if Q_B has the form identifying some vertices of Q_A like the example above, then we have that $\mathrm{rad}\,B$=$\mathrm{rad}\,A$.

Remark 3. In [5], Han has shown that any controlled functor preserves the controlled wildness. The functor $\mathrm{mod}\,A \to \mathrm{mod}\,B$ induced from $B \hookrightarrow A$ above is not controlled, however Proposition 3.1 says that the functor preserves the controlled wildness.

We prepare the following Lemma to show Proposition 3.1.

Lemma 3.1. *Let B be a subalgebra of A such that* $\mathrm{rad}\,A$=$\mathrm{rad}\,B$*, $\mathcal{S}$ the full subcategory of* $\mathrm{mod}\,A$ *consisting of semisimple modules and $\mathcal{W}$*

a full subcategory of $\operatorname{mod} A$ *including* $\mathcal{S}$ *and closed under direct sums. Then*

1. *The functor* $\operatorname{mod} A \to \operatorname{mod} B$ *induced from* $B \hookrightarrow A$ *preserves indecomposability and isomorphism classes of indecomposable modules except simple modules;*
2. *For any* A*-modules* X *and* Y*, we have the short exact sequence*
   $0 \to \operatorname{Hom}_A(X,Y)_{\mathcal{S}} \xrightarrow{\alpha} \operatorname{Hom}_A(X,Y) \oplus \operatorname{Hom}_B(X,Y)_{\mathcal{S}} \xrightarrow{\gamma} \operatorname{Hom}_B(X,Y) \to 0,$
   where $\alpha(f){:=}(f,f)$*,* $\gamma(g,h){:=}g-h$ *and we take* $\mathcal{S}$ *for a subcategory of* $\operatorname{mod} B$ *by* $B \hookrightarrow A$*;*
3. *For any* A*-modules* X *and* Y*, we have the short exact sequence*
   $0 \to \operatorname{Hom}_A(X,Y)_{\mathcal{S}} \xrightarrow{\alpha} \operatorname{Hom}_A(X,Y)_{\mathcal{W}} \oplus \operatorname{Hom}_B(X,Y)_{\mathcal{S}} \xrightarrow{\beta} \operatorname{Hom}_B(X,Y)_{\mathcal{W}} \to 0$
   where $\alpha(f){:=}(f,f)$*,* $\beta(g,h){:=}g-h$ *and we take* $\mathcal{S}$ *and* $\mathcal{W}$ *for subcategories of* $\operatorname{mod} B$ *by* $B \hookrightarrow A$*.*

Proof. (1)Let $e_1, \ldots, e_n$ be primitive orthogonal idempotents of A with $e_1 + \cdots + e_n = 1$. For any homomorphism f in $\operatorname{Hom}_B(X,Y)$, we set that $f_{i,j}(x) = e_i f(e_j x)$ for all x in X and so we have that $f = \sum f_{i,j}$. Since $\operatorname{rad} B = \operatorname{rad} A$ and simple B-modules are induced from simple A-modules, we have that $\sum f_{i,i}$ is in $\operatorname{Hom}_A(X,Y)$ and $\sum_{i \neq j} f_{i,j}$ is in $\operatorname{Hom}_B(X,Y)_{\mathcal{S}}$. Therefore we get the epimorphism $\operatorname{Hom}_A(X,Y) \oplus \operatorname{Hom}_B(X,Y)_{\mathcal{S}} \xrightarrow{\gamma} \operatorname{Hom}_B(X,Y)$, where $\gamma(g,h) = g - h$. Since any homomorphism in $\operatorname{Hom}_B(X,Y)_{\mathcal{S}}$ is not an isomorphism for any indecomposable modules X, Y except simple modules, the assertion holds.

(2)Since $\operatorname{rad} B{=}\operatorname{rad} A$, we have that for any homomorphism f in $\operatorname{Hom}_A(X,Y)$, if the image of f is a semisimple module in $\operatorname{mod} B$, then the image of f is semisimple module in $\operatorname{mod} A$. Hence we get the exactness of the middle term. γ is an epimorphism by the proof of (1) above.

(3)We show that β is an epimorphism. For any homomorphism pq in $\operatorname{Hom}_B(X,Y)_{\mathcal{W}}$, where p is in $\operatorname{Hom}_B(X,W)$ and q is in $\operatorname{Hom}_B(W,Y)$ for a module W in $\mathcal{W}$, by the proof of (1) above, we have that $p = \sum p_{i,i} + \sum_{i \neq j} p_{i,j}$ and $q = \sum q_{i,i} + \sum_{i \neq j} q_{i,j}$ and so β is an epimorphism. □

We show the Proposition 3.1.

Proof. By Lemma 3.1(1), $\mathcal{C}$ is closed under direct summands and direct sums as a full subcategory of $\operatorname{mod} B$. Since A is controlled by $\mathcal{C}$, we have that $\operatorname{Hom}_A(FV, FW) = F \operatorname{Hom}_{k\langle x,y\rangle}(V,W) \oplus \operatorname{Hom}_A(FV,FW)_{\mathcal{C}}$ for the functor $F : \operatorname{mod} k\langle x,y\rangle \to \operatorname{mod} A$ controlled by $\mathcal{C}$. Therefore, by the Lemma 3.1(2), we have the following short exact sequence

$0 \to \operatorname{Hom}_A(FV, FW)_{\mathcal{S}}$
$\xrightarrow{\alpha} F\operatorname{Hom}_{k\langle x,y\rangle}(V, W) \oplus \operatorname{Hom}_A(FV, FW)_{\mathcal{C}} \oplus \operatorname{Hom}_B(FV, FW)_{\mathcal{S}}$
$\xrightarrow{\beta} \operatorname{Hom}_B(FV, FW) \to 0,$

where $\alpha(f) = (0, f, f)$, because $\mathcal{C}$ includes semisimple modules, and $\beta(f, g, h) = f + g - h$. Moreover, by Lemma 3.1(3), we have the following short exact sequence

$0 \to \operatorname{Hom}_A(FV, FW)_{\mathcal{S}} \xrightarrow{\alpha} \operatorname{Hom}_A(FV, FW)_{\mathcal{C}} \oplus \operatorname{Hom}_B(FV, FW)_{\mathcal{S}}$
$\xrightarrow{\beta} \operatorname{Hom}_B(FV, FW)_{\mathcal{C}} \to 0.$

Hence, by the two short exact sequences above, we have the following equality $\operatorname{Hom}_B(FV, FW) = F\operatorname{Hom}_{k\langle x,y\rangle}(V, W) \oplus \operatorname{Hom}_B(FV, FW)_{\mathcal{C}}$. □

4. τ-WILD ALGEBRAS

In this section, we consider a relation between controlled wild algebras and τ-wild algebras.

We call an indecomposable module X *τ-variant* if $\tau X \ncong X$. An algebra A is called *τ-wild* if $\operatorname{mod} A$ has infinitely many non-isomorphic indecomposable τ-variant modules of a same dimension. In [2] Crawley-Boevey conjectured that wild algebras are τ-wild. Therefore we are interested in relations between controlled wild algebras and τ-wild algebras. Our result, Theorem 4.1, gives the sufficient condition when controlled wild algebras are τ-wild.

We prepare the following two lemmas and propositon to show our main theorem.

Lemma 4.1. *Let F be a right exact functor* $\operatorname{mod} A \to \operatorname{mod} B$*, R the right adjoint functor to F and f a left almost split map $X \to Y$ in* $\operatorname{mod} A$*. For any B-module W, X is a direct summand of RW if and only if there exists a homomorphism $FX \to W$ which does not factor through Ff.*

Proof. By the adjointness, we have the following commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}_B(FY, W) & \xrightarrow{\operatorname{Hom}_B(Ff, W)} & \operatorname{Hom}_B(FX, W) \\
{\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\
\operatorname{Hom}_A(Y, RW) & \xrightarrow{\operatorname{Hom}_A(f, RW)} & \operatorname{Hom}_A(X, RW),
\end{array}$$

where g and h are isomorphisms. Since X is direct summand of RW if and only if $\operatorname{Hom}_A(f, RW)$ is not an epimorhism, the assertion holds. □

Lemma 4.2. *Let F be a left exact functor* $\operatorname{mod} A \to \operatorname{mod} B$*, L the left adjoint functor to F and g a right almost split map $Y \to X$ in* $\operatorname{mod} A$*. For any B-module W, X is a direct summand of LW if and only if there exists a homomorphism $W \to FX$ which does not factor through Fg.*

Proof. It is similar as Lemma 4.1. □

Proposition 4.1. *Let F be a faithful exact fnctor* $\operatorname{mod} A \to \operatorname{mod} B$ *controlled by $\mathcal{C}$ and R (resp. L) the right (resp. left) adjoint functor to F. For any indecomposable A-module X, if X is τ-variant and FX is τ-invariant, then τX is in* $\operatorname{add} R(\mathcal{C})$ *and $\tau^{-1}X$ is in $addL(\mathcal{C})$.*

Proof. For any non-projective indecomposable A-module X, we put ε the AR-sequence $0 \to \tau X \xrightarrow{f} M \to X \to 0$ in $\operatorname{mod} A$ and δ the AR-sequence $0 \to FX \to N \to FX \to 0$ in $\operatorname{mod} B$. Then we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
F(\varepsilon): 0 \arrow[r] & F\tau X \arrow[r, "Ff"] \arrow[d, "g"'] & FM \arrow[r] \arrow[d] & FX \arrow[r] \arrow[d, equal] & 0 \\
\delta: 0 \arrow[r] & FX \arrow[r] & N \arrow[r] & FX \arrow[r] & 0.
\end{tikzcd}$$

Since $\operatorname{Hom}_B(F\tau X, FX) = F\operatorname{Hom}_A(\tau X, X) \oplus \operatorname{Hom}_B(F\tau X, FX)_{\mathcal{C}}$, we have that $g = Fg_1 + g_3g_2$, where $g_1 \in \operatorname{Hom}_A(\tau X, X)$, $g_2 \in \operatorname{Hom}_B(F\tau X, W)$ and $g_3 \in \operatorname{Hom}_B(W, FX)$, where W is a module in $\mathcal{C}$. Since X is τ-variant, g_1 is radical map. Therefore g_1 factors through f. If g_2 factors through Ff, then g also factors through Ff, namely δ splits, a contradiction. Thus g_2 does not factor through Ff. Hence τX is a direct summand of RW, by Lemma 4.1.

The rest part is showed similarly by using Lemma 4.2. □

Theorem 4.1. *Any controlled wild algebra with the finite controlling index is τ-wild.*

Proof. Let (F, A) be wild pair such that F is controlled by $\mathcal{C}$. Since there exsists a fully faithful exact functor $G : \operatorname{mod} H \to \operatorname{mod} k\langle x, y\rangle$, where H denotes the wild hereditary algebra which quiver has two vertices $1, 2$ and three arrows form 1 to 2. Thus we have the faithful exact functor $FG : \operatorname{mod} H \to \operatorname{mod} A$ controlled by $\mathcal{C}$. Since any indecomposable H-module is τ-variant (see [6]) and $\operatorname{mod} H$ has infinitely many non-isomorphic indecomposable modules of a same dimension, the assertion holds by the Proposition 4.1 and the assumption that the controlling index is finite. □

Remark 4. According to the proof of Theorem 4.1, the controlling index in the condition of Theorem above is not necessarily finite. However we do not have any examples of controlled wild algebras with the infinite controlling index now.

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