GENERATORS OF MEHLER-TYPE SEMIGROUPS AS PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. We study semigroups $(P_t)_{t\geq 0}$ on a Hilbert space E, given by a Mehler-type formula:

$$P_t f(x) = \int_E f\big(T_t x + y\big)\, \mu_t(dy).$$

Under reasonable assumptions, the $L^p(E,\mu)$–generator $\mathcal{A}_0$ of $(P_t)_{t\geq 0}$ turns out to be expressible as a pseudo-differential operator, provided μ is an invariant measure for $(P_t)_{t\geq 0}$. The question of L^p–uniqueness is also answered positively.

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Contents

1. Introduction and main results

One motivation to study Ornstein-Uhlenbeck processes on infinite dimensional (e.g. separable Hilbert) spaces E, i.e. solutions to stochastic differential equations of type

$$dX_t = AX_t\, dt + dW_t$$

(where $A : E \to E$ is linear, generating a C_0–semigroup on E and $(W_t)_{t\geq 0}$ is an E–valued Brownian motion), is that in contrast to an infinite dimensional Brownian motion, they can have an invariant (probability) measure (cf. e.g. [DPZ96]). Furthermore, the presence of a linear drift in the stochastic equation can have smoothing effects via the semigroup generated by it. This is extensively exploited in the so-called "mild solution approach" to stochastic partial differential equations (cf. e.g. [DPZ92]). So, in this sense infinite dimensional Ornstein-Uhlenbeck processes are better reference processes than infinite dimensional Brownian motions as far as one studies diffusions, i.e. processes with continuous sample paths, whose generators are differential operators.

In the study of processes with jumps, more precisely, processes with càdlàg paths, whose generators are pseudo-differential operators, usually in finite dimensions, so-called Levy-processes are considered as reference processes (cf. e.g. [Jak96, JS00, Hoh95, Hoh98]). In infinite dimensions, analogously one could take processes with stationary independent increments $(Y_t)_{t\geq 0}$ as reference processes on E. But, like infinite dimensional Brownian motions, such processes do not have an invariant measure. Therefore, one would like to have an analogue of Ornstein-Uhlenbeck processes in this case, i.e. solutions to

$$dX_t = AX_t\, dt + dY_t$$

with $A : E \to E$ linear and generating a C_0–semigroup on E. Such processes can be constructed quite easily (cf. [FR00] and the references therein, in particular [CM87]).

The corresponding transition semigroups are called generalized Mehler semigroups (as generalizations of the classical Mehler formula for Ornstein-Uhlenbeck processes) and have been studied in [BRS96] and, particularly, in [FR00]. They are given by the formula

$$P_t f(x) = \int_E f\big(T_t x + y\big)\, \mu_t(dy),$$

where $(T_t)_{t\geq 0}$ is the strongly continuous semigroup of linear operators on E, generated by A, and $(\mu_t)_{t\geq 0}$ is a family of probability measures on $\mathcal{B}(E)$. However, although P_t maps $C_b(E)$, i.e. the set of all bounded continuous functions on E, into $C_b(E)$, there is no complete norm $\|\cdot\|$ on $C_b(E)$ known which makes $(P_t)_{t\geq 0}$ strongly continuous.

So, the idea is to study $(P_t)_{t\geq 0}$ as a semigroup of operators on (real) $L^p(E,\mu)$, $p \geq 1$, where μ is its invariant measure, provided the latter exists (see [FR00] for easy to check conditions).

The purpose of this paper is to calculate the generator of $(P_t)_{t\geq 0}$ on $L^p(E,\mu)$ explicitly. Heuristically, this generator is easily seen to be a pseudo-differential operator of a very simple type, see Eq. (1.3) below. To explain this more precisely, we need to recall a few facts on generalized Mehler semigroups. Below (for simplicity) we assume that E is a separable real Hilbert space.

We fix a strongly continuous semigroup $(T_t)_{t\geq 0}$ with generator A on E. For a family of probability measures $(\mu_t)_{t\geq 0}$ the semigroup property for the corresponding $(P_t)_{t\geq 0}$ is equivalent to the formula

$$\mu_{t+s} = (\mu_t \circ T_s^{-1}) * \mu_s,$$

or in terms of Fourier transforms, for all $l \in E'$

$$\hat{\mu}_{t+s}(l) = \hat{\mu}_s(l)\hat{\mu}_t(T_s^* l),$$

$s, t > 0$ (cf. [BRS96, p.203]). Let us remind the reader that, by definition,

$$\mathcal{F}(\mu_t)(l) := \hat{\mu}_t(l) = \int_E e^{i\langle y,l\rangle}\, \mu_t(dy).$$

Under mild additional hypotheses on $\hat{\mu}_t$ [BRS96, p.205, Lemma 2.6] one necessarily has for all $\xi \in E'$

$$\hat{\mu}_t(\xi) = \exp\Big\{ - \int_0^t \lambda(T_s^*\xi)\, ds \Big\} \tag{1.1}$$

for some negative-definite function $\lambda : E' \to \mathbb{C}$, such that $\lambda(0) = 0$.

Conversely, given $\lambda : E' \to \mathbb{C}$ negative-definite with $\lambda(0) = 0$ and continuous w.r.t. the Sazonov topology, one may construct (given T_t)

the μ_t's, and therefore P_t. Therefore, we shall make the following hypothesis throughout the paper:

(H1) $\lambda : E' \to \mathbb{C}$ is a Sazonov-continuous negative definite function with $\lambda(0) = 0$ so that (1.1) holds.

If λ is merely continuous it is possible [FR00, p.11, Theorem 2.8] to embed E into a bigger Hilbert space, to which T_t, $t \geq 0$, naturally extend, in such a way that the hypothesis above is satisfied.

A part of our main results will be proved under the following hypothesis:

(H2) There exists a probability measure μ on $\mathcal{B}(E')$ which is invariant under P_t, i.e. such that for all $t \geq 0$ and all bounded, $\mathcal{B}(E)$–measurable functions $f : E \to \mathbb{R}$ one has

$$\int_E P_t f(x)\, d\mu(x) = \int_E f(x)\, d\mu(x).$$

We refer to [FR00, Lemma 6.2 and Theorem 3.1] for conditions implying existence or uniqueness of μ.

Let $\nu \in M = \mathcal{M}_b^{\mathbb{C}}(E')$ (i.e. the space of complex-valued measures with finite total variation) and let $\varphi = \mathcal{F}(\nu)$ (we use the notations and work in the general context of [BLR99] here). Then one has for all $x \in E$

$$\begin{aligned}
P_t\varphi(x) &= \int_E \varphi\big(T_t x + y\big)\, \mu_t(dy) \\
&= \int_E \left(\int_{E'} e^{i\langle T_t x + y, \xi\rangle}\, \nu(d\xi) \right) \mu_t(dy) \\
&= \int_{E'} \left(\int_E e^{i\langle T_t x + y, \xi\rangle}\, \mu_t(dy) \right) \nu(d\xi) \qquad (1.2) \\
&= \int_{E'} e^{i\langle T_t x, \xi\rangle}\, \hat{\mu}_t(\xi)\nu(d\xi) \\
&= \int_{E'} \exp\left\{ i\langle x, T_t^*\xi\rangle - \int_0^t \lambda\big(T_s^*\xi\big)\, ds \right\} \nu(d\xi),
\end{aligned}$$

the application of Fubini's Theorem being legitimate here because μ_t and ν are both *finite* measures.

Clearly, for all $\varphi \in \mathcal{F}(M)$ as above

$$\lim_{t \to 0} P_t\varphi(x) = \varphi(x) \qquad \text{for all } x \in E.$$

Consequently, e.g. by [MR92, Proposition II.4.3], $(P_t)_{t\geq 0}$ extends to a strongly continuous semigroup on $L^2(E, \mu)$, for any invariant measure μ as in (H2). Since $(P_t)_{t\geq 0}$ is Markovian, this then also holds on $L^p(E, \mu)$, for all $p \geq 1$.

Differentiating that expression *formally* w.r.t. t and setting $t = 0$, one is led (as pointed out in [FR00, Remark 4.4.,p.20]) to conjecture that when (H2) holds, the $L^p(E,\mu)$–generator $\mathcal{A}_p$ of $(P_t)_{t\geq 0}$ should have the following form:

$$\mathcal{A}_p\varphi(x) = \int_{E'} \Big(i\langle x, A^*\xi\rangle - \lambda(\xi)\Big)e^{i\langle x,\xi\rangle}\,\nu(d\xi). \tag{1.3}$$

We are, in fact, going to establish this expression rigorously for all functions φ belonging to a certain dense subspace W of $L^p(E,\mu)$. For this, we need to (and shall) assume in the whole paper the following:

(H3) There exists an orthonormal basis $(\xi_n)_{n\in\mathbb{N}}$ of E', consisting of eigenvectors of A^* (i.e. the dual of the generator A on E').

Remark. Condition (H3) is satisfied whenever A is self-adjoint with compact resolvent. We therefore have numerous easy examples.

Now, let W_0 be the space of functions φ that have a representation of the form

$$\varphi(x) = f\big(\langle x,\xi_1\rangle,\ldots,\langle x,\xi_m\rangle\big),$$

for all $x \in E$ and for $m \geq 1$ an integer, $f \in \mathcal{S}(\mathbb{R}^m,\mathbb{C})$ (i.e. the Schwartz space of complex-valued functions, "rapidly decreasing" at infinity as well as all their derivatives). With the notations above, let $g_0 : \mathbb{R}^m \to \mathbb{C}$ denote the inverse Fourier transform of f, i.e. the function g_0, such that for all $y \in \mathbb{R}^m$

$$f(y) = \int_{\mathbb{R}^m} e^{i\langle y,v\rangle} g_0(v)\,dv,$$

and let $\nu_0(dv) := g_0(v)\,dv$, where dv denotes the Lebesgue measure on $\mathbb{R}^m$. Let $\Pi_m : \mathbb{R}^m \to E'$ be defined by

$$\Pi_m(v_1,\ldots,v_m) = v_1\xi_1 + \cdots + v_m\xi_m,$$

and let $\nu = \nu_0 \circ \Pi_m^{-1}$. Then a very classical computation [BLR99, Lemma 1.3, p. 103] yields that $\varphi = \mathcal{F}(\nu)$. As, obviously, $\nu \in M$, $\varphi \in \mathcal{F}(M)$, whence $W_0 \subset \mathcal{F}(M)$. It is clear that W_0 is a ($\mathbb{C}$–) vector subspace of $C_b(E,\mathbb{C})$. Let W be the ($\mathbb{R}$–) vector space of $\mathbb{R}$–valued elements of W_0. With the notations above and $\varphi \in W_0$, it will be that $\varphi \in W$ as soon as for all $\beta \in \mathbb{R}$

$$g_0(-\beta) = \overline{g_0(\beta)}. \tag{1.4}$$

From this and the hypothesis made on A^*, it is easy to see that W separates points of E and is dense in (real) $L^p(E,\mu)$.

Let $\mathcal{A}$ be defined by:

(1.5)
$$\mathcal{A}\varphi(x) := \int_{E'} \Big(i\langle x, A^*\xi\rangle - \lambda(\xi)\Big) e^{i\langle x,\xi\rangle}\, \mathcal{F}^{-1}(\varphi)(d\xi), \quad \varphi \in W,\ x \in E.$$

We intend to prove the following results:

Theorem 1.1. *Suppose* (H1) *and* (H3) *hold. Then*

(i) $\mathcal{A}$ *maps* W *into* $C_b(E)$.

(ii) $P_t\varphi(x) - \varphi(x) = \int_0^t P_s\mathcal{A}\varphi(x)\, ds$ *for all* $x \in E$.

Theorem 1.2 (computation of the generator)**.** *Suppose* (H1), (H2) *and* (H3) *hold. Then for each* $\varphi \in W$

$$\lim_{t\to 0} \frac{1}{t}\big(P_t\varphi - \varphi\big)$$

exists in $L^p(E,\mu)$, *for all* $p \geq 1$, *and equals* $\mathcal{A}\varphi$. *In other words,* $W \subset D(\mathcal{A}_p)$ *and* $\mathcal{A}_p\big|_W = \mathcal{A}$.

Theorem 1.3 (L^p–uniqueness)**.** *Suppose* (H1), (H2) *and* (H3) *hold. Then*

(i) If for all $n \in \mathbb{N}$ *and* $F_n := \mathrm{span}\{\xi_1, \dots, \xi_n\}$, $\lambda\big|_{F_n}$ *is infinitely differentiable, then* $P_t(W) \subseteq W$.

(ii) Let $p \geq 1$. *Then* $(W, \mathcal{A})$ *is a* core *for* $\mathcal{A}_p$, *i.e.* W *is dense in* $D(\mathcal{A}_p)$ *for the* graph norm $\|\cdot\|_{\mathrm{gr}}$, *defined for* $f \in D(\mathcal{A}_p)$ *by*

$$\|f\|_{\mathrm{gr}} := \|f\|_{L^p(E,\mu)} + \|\mathcal{A}_p f\|_{L^p(E,\mu)}.$$

2. Proof of a special case

In this section, we prove Part (i) of Theorem 1.1 in full generality, but Part (ii) only under the following additional assumption.

(2.1) For all $n \in \mathbb{N}$ and $F_n := \mathrm{span}\{\xi_1, \dots, \xi_n\}$, $\lambda\big|_{F_n}$ is continuously differentiable.

(i) Let $\varphi \in W$. By definition, there exists $m \geq 1$, $(\xi_1, \dots, \xi_m) \in (E')^m$, $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ and $g \in \mathcal{S}(\mathbb{R}^m, \mathbb{C})$, such that—denoting by f the Fourier transform of g_0, i.e. for all $y \in \mathbb{R}^m$

$$f(y) = \int_{\mathbb{R}^m} e^{i\langle y,v\rangle} g_0(v)\, dv,$$

—one has

(2.2)
$$A^*\xi_j = \alpha_j\xi_j, \quad j \in \{1, \dots, m\}$$
$$\varphi(x) = f\big(\langle x, \xi_1\rangle, \dots, \langle x, \xi_m\rangle\big), \quad x \in E.$$

Setting, as above, $\nu_0 = g_0(v)\,dv$,

$$\Pi_m : \mathbb{R}^m \to E'$$

$$\Pi_m : (v_1, \dots, v_m) \mapsto \sum_{i=1}^m v_i \xi_i,$$

and $\nu = \nu_0 \circ \Pi_m^{-1} = \mathcal{F}^{-1}(\varphi)$, one has

$$\begin{aligned}
\mathcal{A}\varphi(x) &= \int_{E'} \Big(i\langle x, A^*\xi\rangle - \lambda(\xi) \Big) e^{i\langle x,\xi\rangle}\,\nu(d\xi) \\
&= \int_{\mathbb{R}^m} \bigg(i\Big\langle x, A^*\Big(\sum_{j=1}^m v_j\xi_j\Big)\Big\rangle - \lambda\Big(\sum_{j=1}^m v_j\xi_j\Big)\bigg) \times \\
&\qquad\qquad \times \exp\Big\{ i\Big\langle x, \sum_{j=1}^m v_j\xi_j \Big\rangle\Big\} g_0(v)\,dv \\
&= \sum_{j=1}^m A_j(x) + B(x),
\end{aligned}$$

where we have set

$$B(x) = -\int_{\mathbb{R}^m} \lambda\Big(\sum_{j=1}^m v_j\xi_j\Big) \exp\Big\{ i\Big\langle x, \sum_{j=1}^m v_j\xi_j \Big\rangle\Big\} g_0(v)\,dv$$

and

$$A_j(x) = i\big\langle x, A^*\xi_j\big\rangle \int_{\mathbb{R}^m} v_j \exp\Big\{ i\Big\langle x, \sum_{k=1}^m v_k\xi_k \Big\rangle\Big\} g_0(v)\,dv.$$

We are going to establish that $(A_j)_{1\le j\le m}$ and B belong to $C_b(E)$. Here, we have to notice that B, as well as each A_j, is real-valued because of (1.4) and the fact that for all $\xi \in E'$ $\lambda(-\xi) = \overline{\lambda(\xi)}$. Then

$$\mathcal{A}\varphi = \sum_{j=1}^m A_j + B \in C_b(E),$$

hence $\mathcal{A}$ maps W into $C_b(E)$.

For A_j, the statement is obvious. Concerning B, we recall that by a well-known property of negative-definite functions on finite-dimensional spaces (actually re-proved in Lemma 3.2), there exists a constant $C = C(\xi_1, \dots, \xi_m)$, such that for all $(v_1, \dots, v_m) \in \mathbb{R}^m$ one has

$$0 \le \Big| \lambda\Big(\sum_{i=1}^m v_i\xi_i\Big) \Big| \le C\Big(1 + \sum_{i=1}^m v_i^2\Big).$$

Then for all $x \in E$

$$\big|B(x)\big| \leq C \int_{\mathbb{R}^m} \Big(1 + \sum_{i=1}^m v_i^2\Big)\big|g_0(v)\big|\,dv,$$

and the expression on the right-hand side is finite as $g_0 \in \mathcal{S}$. The continuity of B follows by Lebesgue's Dominated Convergence Theorem.

(ii) Let us keep the previous notations and assume (2.1) to hold. Since $\mathcal{A}\varphi \in C_b(E)$, $P_s\mathcal{A}\varphi$, $s \geq 0$ is well-defined and we have by definition of P_s and $\mathcal{A}$ that

$$\begin{aligned}
P_s\mathcal{A}\varphi(x) &= \int_E \mathcal{A}\varphi\big(T_s x + y\big)\,\mu_s(dy) \\
&= \int_E \Big(\int_{E'} \big(i\big\langle T_s x + y, A^*\xi\big\rangle - \lambda(\xi)\big)e^{i\langle T_s x + y, \xi\rangle}\,\nu(d\xi)\Big)\,\mu_s(dy) \\
&= \int_E \Big(\int_{E'} i\langle y, A^*\xi\rangle e^{i\langle T_s x + y, \xi\rangle}\,\nu(d\xi)\Big)\,\mu_s(dy) + \\
&\qquad + \int_E \big(i\langle T_s x, A^*\xi\rangle - \lambda(\xi)\big)e^{i\langle T_s x, \xi\rangle}\Big(\int_{E'} e^{i\langle y, \xi\rangle}\,\mu_s(dy)\Big)\,\nu(d\xi) \\
&= B_1(s,x) + B_2(s,x) \quad \text{(say)}.
\end{aligned}$$

But then

$$\begin{aligned}
B_1(s,x) &= \int_E \Big(\int_{\mathbb{R}^m} i\Big\langle y, A^*\Big(\sum_{k=1}^m v_k\xi_k\Big)\Big\rangle \times \\
&\qquad\qquad \times \exp\Big\{i\Big\langle T_s x + y, \sum_{k=1}^m v_k\xi_k\Big\rangle\Big\}g_0(v)\,dv\Big)\,\mu_s(dy) \\
&= \int_E \Big(\int_{\mathbb{R}^m} i\Big\langle y, \sum_{k=1}^m \alpha_k v_k \xi_k\Big\rangle e^{i\langle T_s x + y, \Pi_m(v)\rangle} g_0(v)\,dv\Big)\,\mu_s(dy) \\
&= \sum_{k=1}^m \alpha_k \int_E \Big(\int_{\mathbb{R}^m} i\langle y, \xi_k\rangle v_k e^{i\langle T_s x + y, \Pi_m(v)\rangle} g_0(v)\,dv\Big)\,\mu_s(dy) \\
&= \sum_{k=1}^m \alpha_k \int_E \Big(\int_{\mathbb{R}^m} i\langle y, \xi_k\rangle e^{i\langle y, \Pi_m(v)\rangle} h_k(v)\,dv\Big)\,\mu_s(dy),
\end{aligned}$$

where we have set

$$h_k(v) := e^{i\langle T_s x, \Pi_m(v)\rangle} v_k g_0(v).$$

Clearly, for fixed x, $h_k \in \mathcal{S}(\mathbb{R}^m, \mathbb{C})$, which will be enough to legitimate all of the integrations by parts carried out later.

From what we have seen we may write

$$\begin{aligned}B_1(s,x) &= \sum_{k=1}^{m} \alpha_k \int_E \left(\int_{\mathbb{R}^m} \frac{\partial}{\partial v_k} \Big[e^{i\langle y, \Pi_m(v)\rangle} \Big] h_k(v)\, dv \right) \mu_s(dy) \\ &= \sum_{k=1}^{m} \alpha_k \int_E \left(- \int_{\mathbb{R}^m} e^{i\langle y, \Pi_m(v)\rangle} \frac{\partial h_k}{\partial v_k}(v)\, dv \right) \mu_s(dy),\end{aligned}$$

integration by parts being permitted as

$$\Big| e^{i\langle y, \Pi_m(v)\rangle} h_k(v) \Big| = \big| h_k(v) \big| = \big| v_k g_0(v) \big| \xrightarrow[|v|\to+\infty]{} 0.$$

Therefore, one has

$$\begin{aligned}B_1(s,x) &= \sum_{k=1}^{m} \alpha_k \int_E \left(- \int_{\mathbb{R}^m} e^{i\langle y, \Pi_m(v)\rangle} \frac{\partial h_k}{\partial v_k}(v)\, dv \right) \mu_s(dy) \\ &= - \sum_{k=1}^{m} \alpha_k \int_{\mathbb{R}^m} \frac{\partial h_k}{\partial v_k}(v) \left(\int_E e^{i\langle y, \Pi_m(v)\rangle}\, \mu_s(dy) \right) dv \\ &= - \sum_{k=1}^{m} \alpha_k \int_{\mathbb{R}^m} \frac{\partial h_k}{\partial v_k}(v)\, \hat{\mu}_s\big(\Pi_m(v)\big)\, dv,\end{aligned}$$

the use of Fubini's Theorem being legitimate as $\frac{\partial h_k}{\partial v_k} \in \mathcal{S}$. If we now proceed to a *second* integration by parts, we get

$$\begin{aligned}B_1(s,x) &= \sum_{k=1}^{m} \alpha_k \int_{\mathbb{R}^m} \frac{\partial}{\partial v_k} \Big(\hat{\mu}_s\big(\Pi_m(v)\big) \Big) h_k(v)\, dv \\ &= \sum_{k=1}^{m} \alpha_k \int_{\mathbb{R}^m} \frac{\partial}{\partial v_k} \left(\exp\left\{ - \int_0^s \lambda\big(T_r^*\big(\Pi_m(v)\big)\big)\, dr \right\} \right) h_k(v)\, dv \\ &= \sum_{k=1}^{m} \alpha_k \int_{\mathbb{R}^m} \exp\left\{ - \int_0^s \lambda\big(T_r^*\big(\Pi_m(v)\big)\big)\, dr \right\} \times \\ &\qquad \times \left(- \int_0^s D\lambda\Big(T_r^*\big(\Pi_m(v)\big)\Big)\big(T_r^*(\xi_k)\big)\, dr \right) e^{i\langle T_s x, \Pi_m(v)\rangle} v_k g_0(v)\, dv.\end{aligned}$$

Applying now the Fundamental Theorem of Calculus to the function

$$u \mapsto \lambda\big(T_u^*\big(\Pi_m(v)\big)\big)$$

on the interval $[0, s]$, one obtains

$$\begin{aligned}
&\lambda\big(T_s^*\left(\Pi_m(v)\right)\big) - \lambda\big(\Pi_m(v)\big) \\
&\qquad = \int_0^s D\lambda\big(T_r^*\left(\Pi_m(v)\right)\big)\left(T_r^* A^*\left(\Pi_m(v)\right)\right) dr \\
&\qquad\qquad = \sum_{j=1}^m \alpha_j v_j \int_0^s D\lambda\big(T_r^*\left(\Pi_m(v)\right)\big)\left(T_r^*(\xi_j)\right) dr.
\end{aligned}$$

It now appears that

$$\begin{aligned}
B_1(s,x) &= \int_{\mathbb{R}^m} \exp\left\{ i\big\langle T_s x, \Pi_m(v)\big\rangle - \int_0^s \lambda\big(T_r^*\left(\Pi_m(v)\right)\big)\, dr \right\} \times \\
&\qquad \times \Big(\lambda\big(\Pi_m(v)\big) - \lambda\big(T_s^*\left(\Pi_m(v)\right)\big)\Big) g_0(v)\, dv \\
&= \int_{E'} \exp\left\{ i\langle T_s x, \xi\rangle - \int_0^s \lambda\big(T_r^*\xi\big)\, dr \right\} \Big(\lambda(\xi) - \lambda\big(T_s^*\xi\big)\Big)\, \nu(d\xi) \\
&= \int_{E'} e^{i\langle x, T_s^*\xi\rangle} \Big(\lambda(\xi) - \lambda\big(T_s^*\xi\big)\Big) \hat{\mu}_s(\xi)\, \nu(d\xi).
\end{aligned}$$

But one obviously has

$$B_2(s,x) = \int_E \Big(i\big\langle x, T_s^* A^* \xi\big\rangle - \lambda(\xi)\Big) e^{i\langle x, T_s^*\xi\rangle} \hat{\mu}_s(\xi)\, \nu(d\xi).$$

Therefore

$$\begin{aligned}
P_s \mathcal{A}\varphi(x) &= B_1(s,x) + B_2(s,x) \\
&= \int_{E'} \Big(i\big\langle x, T_s^* A^* \xi\big\rangle - \lambda\big(T_s^*\xi\big)\Big) \times \\
&\qquad \times \exp\left\{ i\langle x, T_s^*\xi\rangle - \int_0^s \lambda\big(T_r^*\xi\big)\, dr \right\} \nu(d\xi),
\end{aligned} \tag{2.3}$$

where, once more, we have replaced $\hat{\mu}_s(\xi)$ by its Fourier expression (1.1).

For fixed x one may write, using Fubini's Theorem

$$\begin{aligned}
\int_0^t P_s\mathcal{A}\varphi(x)\,ds &= \int_{E'}\biggl(\int_0^t \Bigl(i\bigl\langle x, T_s^*A^*\xi\bigr\rangle - \lambda\bigl(T_s^*\xi\bigr)\Bigr) \times \\
&\qquad\qquad \times \exp\biggl\{i\langle x, T_s^*\xi\rangle - \int_0^s \lambda\bigl(T_r^*\xi\bigr)\,dr\biggr\}\,ds\biggr)\,\nu(d\xi) \\
&= \int_{E'}\biggl(\int_0^t \frac{d}{ds}\biggl(\exp\biggl\{i\langle x, T_s^*\xi\rangle - \int_0^s \lambda\bigl(T_r^*\xi\bigr)\,dr\biggr\}\biggr)\,ds\biggr)\,\nu(d\xi) \\
&= \int_{E'}\biggl[\exp\biggl\{i\langle x, T_s^*\xi\rangle - \int_0^s \lambda\bigl(T_r^*\xi\bigr)\,dr\biggr\}\biggr]_0^t\,\nu(d\xi) \\
&= \int_{E'}\biggl(\exp\biggl\{i\langle x, T_t^*\xi\rangle - \int_0^t \lambda\bigl(T_r^*\xi\bigr)\,dr\biggr\} - e^{i\langle x,\xi\rangle}\biggr)\,\nu(d\xi) \\
&= \int_{E'} e^{i\langle x, T_t^*\xi\rangle}\hat{\mu}_t(\xi)\,\nu(d\xi) - \int_{E'} e^{i\langle x,\xi\rangle}\,\nu(d\xi) \\
&= P_t\varphi(x) - \varphi(x),
\end{aligned}$$

and assertion (ii) is proved under the additional assumtion (2.1). □

3. Proofs of Theorems 1.1 and 1.2

Part (i) of Theorem 1.1 was proved in the previous section, so it remains to prove Part (ii). To this end let M be the Levy measure [FR00, p.6] associated with λ, i.e. M is a Borel measure on E with $M\bigl(\{0\}\bigr) = 0$ and

$$\int_E \bigl(1 \wedge \|x\|^2\bigr)\,M(dx) < +\infty, \tag{3.1}$$

and for all $\xi \in E'$

(3.2)

$$\lambda(\xi) := -i\langle\xi, b\rangle + \frac{1}{2}\langle\xi, R\xi\rangle - \int_E \Bigl(e^{i\langle\xi,x\rangle} - 1 - \frac{i\langle\xi, x\rangle}{1+\|x\|^2}\Bigr)\,M(dx),$$

where $b \in E$ and $R : E' \to E$ is such that $R \circ \iota$ is a non-negative symmetric trace class operator, ι denoting the canonical isomorphism $\iota : E \to E'$.

For $0 \le \varepsilon \le 1$, let M_ε be the "doubly truncated" measure on E, defined by

$$M_\varepsilon(dx) = \mathbb{1}_{\{a|\varepsilon\le\|a\|\le 1/\varepsilon\}}(x)\,M(dx),$$

where we use the convention $\frac{1}{0} := +\infty$. It is obvious that M_ε is a Borel measure with $M_\varepsilon\bigl(\{0\}\bigr) = 0$ and that is has the property (3.1). Replacing M by M_ε in (3.2) therefore yields a Sazonov-continuous, negative-definite $\lambda_\varepsilon : E' \to \mathbb{C}$ with $\lambda_\varepsilon(0) = 0$ [FR00, p.5]. Note that, since $M_0 = M$, we have $\lambda_0 = \lambda$. By the Bochner-Minlos Theorem, we see that there is, for each $t \geq 0$, a probability measure $\mu_{t,\varepsilon}$ on E, satisfying

$$\hat{\mu}_{t,\varepsilon}(\xi) = \exp\left\{ -\int_0^t \lambda_\varepsilon\big(T_s^* \xi\big)\, ds \right\} \qquad \forall \xi \in E'.$$

Let $P_{t,\varepsilon}$ be the semigroup on $C_b(E)$, associated with $(T_t)_{t\geq 0}$ and λ_ε, as P_t is with $(T_t)_{t\geq 0}$ and λ, i.e.

$$P_{t,\varepsilon} f(x) = \int_E f\big(T_t x + y\big)\, \mu_{t,\varepsilon}(dy).$$

Lemma 3.1. *As $\varepsilon \to 0$, $\lambda_\varepsilon \to \lambda$ uniformly on bounded subsets of E'.*

Proof. Let $r > 0$. We are going to study the difference $\lambda(\xi) - \lambda_\varepsilon(\xi)$ for $\xi \in \overline{B_r^{E'}(0)}$. By definition, one has for all $\xi \in E'$

$$\begin{aligned}\lambda(\xi) - \lambda_\varepsilon(\xi) &= \int_E \Big(e^{i\langle \xi, x\rangle} - 1 - \frac{i\langle \xi, x\rangle}{1 + \|x\|^2}\Big)\big(\mathbb{1}_{\{a|\varepsilon \leq \|a\| \leq 1/\varepsilon\}}(x) - 1\big)\, M(dx)\\ &= I_{1,\varepsilon}(\xi) + I_{2,\varepsilon}(\xi),\end{aligned}$$

where we have set

$$I_{1,\varepsilon}(\xi) := -\int_{B_\varepsilon^E(0)} \Big(e^{i\langle \xi, x\rangle} - 1 - \frac{i\langle \xi, x\rangle}{1 + \|x\|^2}\Big)\, M(dx)$$

and

$$I_{2,\varepsilon}(\xi) := -\int_{E \setminus \overline{B_{1/\varepsilon}^E(0)}} \Big(e^{i\langle \xi, x\rangle} - 1 - \frac{i\langle \xi, x\rangle}{1 + \|x\|^2}\Big)\, M(dx).$$

Since for all $\theta \in \mathbb{R}$

$$\big| e^{i\theta} - 1 - i\theta \big| \leq \frac{\theta^2}{2},$$

we have

$$\begin{aligned}\Big| e^{i\langle \xi, x\rangle} - 1 - \frac{i\langle \xi, x\rangle}{1 + \|x\|^2}\Big| &\leq \Big| e^{i\langle \xi, x\rangle} - 1 - i\langle \xi, x\rangle \Big| + \Big| i\langle \xi, x\rangle - \frac{i\langle \xi, x\rangle}{1 + \|x\|^2}\Big|\\ &\leq \frac{1}{2}\langle \xi, x\rangle^2 + \Big| \frac{i\langle \xi, x\rangle \|x\|^2}{1 + \|x\|^2}\Big|\\ &\leq \frac{1}{2}\|\xi\|^2\|x\|^2 + \|\xi\|\|x\|(\|x\|^2 \wedge 1)\\ &\leq \Big(\frac{r^2}{2} + r\varepsilon\Big)(\|x\|^2 \wedge 1),\end{aligned}$$

whenever $x \in \overline{B_\varepsilon^E(0)}$, $\xi \in \overline{B_r^{E'}(0)}$. Hence

$$\left|I_{1,\varepsilon}(\xi)\right| \leq \left(\frac{r^2}{2} + r\varepsilon\right) \int_{B_\varepsilon^E(0)} \left(1 \wedge \|x\|^2\right) M(dx),$$

which tends to 0 with ε, by the definition of a Levy measure (see (3.1)). Therefore, $I_{1,\varepsilon}(\xi)$ tends to 0 as $\varepsilon \to 0$, uniformly in ξ on $\overline{B_r^{E'}(0)}$.

For $I_{2,\varepsilon}(\xi)$, things are even easier, since for $\xi \in \overline{B_r^{E'}(0)}$

$$\begin{aligned}\left|I_{2,\varepsilon}(\xi)\right| &\leq \int_{E\setminus\overline{B_{1/\varepsilon}^E(0)}} \left| e^{i\langle \xi, x\rangle} - 1 - \frac{i\langle \xi, x\rangle}{1+\|x\|^2}\right| M(dx) \\ &\leq \int_{E\setminus\overline{B_{1/\varepsilon}^E(0)}} \left(2 + \frac{\|\xi\|\|x\|}{\|x\|^2}\right) M(dx) \\ &\leq \left(2 + r\varepsilon\right) \int_{E\setminus\overline{B_{1/\varepsilon}^E(0)}} \left(1 \wedge \|x\|^2\right) M(dx),\end{aligned}$$

which tends to 0 with ε. Hence, uniform convergence to 0 on $\overline{B_r^{E'}(0)}$ is also valid for $I_{2,\varepsilon}$. Thus, the result follows. □

Lemma 3.2. *There exists a constant $D \geq 0$ such that for all $\varepsilon \in [0,1]$ and all $\xi \in E'$*

$$\left|\lambda_\varepsilon(\xi)\right| \leq D\left(1 + \|\xi\|_{E'}^2\right).$$

Proof. Let us use the Levy formula and the definition of M_ε. We get

$$\begin{aligned}\lambda_\varepsilon(\xi) = &- i\langle \xi, b\rangle + \frac{1}{2}\langle \xi, R\xi\rangle \\ &- \int_{\overline{B_1^E(0)}} \left(e^{i\langle \xi, x\rangle} - 1 - i\langle \xi, x\rangle\right) \mathbb{1}_{\{a|\varepsilon \leq \|a\| \leq 1/\varepsilon\}}(x)\, M(dx) \\ &- \int_{\overline{B_1^E(0)}} \frac{i\langle \xi, x\rangle \|x\|^2}{1+\|x\|^2} \mathbb{1}_{\{a|\varepsilon \leq \|a\| \leq 1/\varepsilon\}}(x)\, M(dx) \\ &- \int_{E\setminus\overline{B_1^E(0)}} \left(e^{i\langle \xi, x\rangle} - 1 - \frac{i\langle \xi, x\rangle}{1+\|x\|^2}\right) \mathbb{1}_{\{a|\varepsilon \leq \|a\| \leq 1/\varepsilon\}}(x)\, M(dx),\end{aligned}$$

whence for all $\xi \in E'$

$$
\begin{aligned}
\big|\lambda_\varepsilon(\xi)\big| &\le \|b\|\|\xi\| + \frac{1}{2}|||R|||\|\xi\|^2 + \int_{\overline{B_1^E(0)}} \frac{1}{2}\langle \xi, x\rangle^2 \, M(dx) \\
&\quad + \int_{\overline{B_1^E(0)}} \big|\langle \xi, x\rangle\big| \|x\|^2 \, M(dx) + \int_{E\setminus\overline{B_1^E(0)}} \Big(2 + \frac{|\langle \xi, x\rangle|}{2\|x\|}\Big) \, M(dx) \\
&\le \|b\|\|\xi\| + \frac{1}{2}|||R|||\|\xi\|^2 + \frac{1}{2}\|\xi\|^2 \int_{\overline{B_1^E(0)}} \|x\|^2 \, M(dx) \\
&\qquad + \|\xi\| \int_{\overline{B_1^E(0)}} \|x\|^2 \, M(dx) + \Big(2 + \frac{\|\xi\|}{2}\Big) \int_{E\setminus\overline{B_1^E(0)}} M(dx).
\end{aligned}
$$

Setting

$$
K := \int_E \big(1 \wedge \|x\|^2\big) \, M(dx),
$$

we see that for all $\varepsilon \in \,]0,1]$ and for all $\xi \in E'$

$$
\begin{aligned}
\big|\lambda_\varepsilon(\xi)\big| &\le \|b\|\|\xi\| + \frac{1}{2}|||R|||\|\xi\|^2 + \frac{K}{2}\|\xi\|^2 + K\|\xi\| + \Big(2 + \frac{\|\xi\|}{2}\Big)K \\
&\le \frac{\|b\|}{2}\big(1 + \|\xi\|^2\big) + \frac{1}{2}|||R|||\|\xi\|^2 + \frac{K}{2}\|\xi\|^2 + \frac{3K}{4}\big(1 + \|\xi\|^2\big) + 2K.
\end{aligned}
$$

Hence we obtain the desired result. □

Proposition 3.3. *For each finite-dimensional subspace F of E', $\lambda_\varepsilon\big|_F$ is infinitely differentiable.*

Proof. Let $\big(\eta_1, \ldots, \eta_m\big)$ denote a basis of F and for $\varepsilon \in \,]0,1[$ define $f_\varepsilon : \mathbb{R}^m \to \mathbb{C}$ as

$$
f_\varepsilon\big(u_1, \ldots, u_m\big) := \lambda_\varepsilon\Big(\sum_{j=1}^m u_j\eta_j\Big) + i\sum_{j=1}^m u_j\langle \eta_j, b\rangle - \frac{1}{2}\sum_{j,k=1}^m u_j u_k \langle \eta_j, R\eta_k\rangle.
$$

Obviously, it suffices to check that $f_\varepsilon \in C^1(\mathbb{R}^m, \mathbb{C})$, but

$$
\begin{aligned}
&f_\varepsilon\big(u_1, \ldots, u_m\big) \\
&\quad = -\int_E \left(\exp\Big\{i\Big\langle \sum_{j=1}^m u_j\eta_j, x\Big\rangle\Big\} - 1 - \frac{i\Big\langle \sum_{j=1}^m u_j\eta_j, x\Big\rangle}{1 + \|x\|^2}\right) M_\varepsilon(dx),
\end{aligned}
$$

and the integral

$$
-\int_E \left(\exp\Big\{i\Big\langle \sum_{j=1}^m u_j\eta_j, x\Big\rangle\Big\} \, i\langle \eta_k, x\rangle - \frac{i\langle \eta_k, x\rangle}{1 + \|x\|^2}\right) M_\varepsilon(dx) \tag{3.3}
$$

converges uniformly in u, since the integrand is bounded in absolute value by $2\big|\langle \eta_k, x\rangle\big|$, and M_ε is a finite measure with support in $\overline{B^E_{1/\varepsilon}(0)}$. Therefore, $\frac{\partial f_\varepsilon}{\partial u_k}$ exists and equals expression (3.3). Proceeding similarly with the function in (3.3) and with its derivatives, we obtain the result. $\square$

Let us now fix $\varphi \in W$ ($\subseteq \mathcal{F}(M)$) and set, as before, $\nu := \mathcal{F}^{-1}(\varphi)$, $\nu = \nu_0 \circ \Pi_m^{-1}$, $\nu_0 = g_0(v)\, dv$ ($g_0 \in \mathcal{S}\big(\mathbb{R}^m; \mathbb{C}\big)$), and

$$\Pi_m : \mathbb{R}^m \to E',$$
$$\big(v_1, \dots, v_m\big) \mapsto \sum_{j=1}^m v_j \xi_j.$$

Proposition 3.4. *For each $\xi \in E'$, $\hat{\mu}_{t,\varepsilon}(\xi) \to \hat{\mu}_t(\xi)$ as $\varepsilon \to 0$.*

Proof. Since

$$\hat{\mu}_{t,\varepsilon}(\xi) = \exp\bigg\{ -\int_0^t \lambda_\varepsilon\big(T_s^*\xi\big)\, ds\bigg\}$$

and

$$\hat{\mu}_t(\xi) = \exp\bigg\{ -\int_0^t \lambda\big(T_s^*\xi\big)\, ds\bigg\},$$

it suffices to prove

$$\int_0^t \lambda_\varepsilon\big(T_s^*\xi\big)\, ds \xrightarrow[\varepsilon\to 0]{} \int_0^t \lambda\big(T_s^*\xi\big)\, ds.$$

But this follows from Lemma 3.1 and the fact that $\big\{T_s^*\xi \;\big|\; 0 \le s \le t\big\}$ is bounded, which is obvious since $s \mapsto T_s^*\xi$ is continuous. $\square$

Corollary 3.5. *$P_{t,\varepsilon}\varphi \to P_t\varphi$ uniformly as $\varepsilon \to 0$.*

Proof. One has for all $x \in E$ by the usual calculations

$$\begin{aligned} P_t\varphi(x) - P_{t,\varepsilon}\varphi(x) &= \int_{E'} e^{i\langle T_t x, \xi\rangle}\, \hat{\mu}_t(\xi)\, \nu(d\xi) - \int_{E'} e^{i\langle T_t x, \xi\rangle}\, \hat{\mu}_{t,\varepsilon}(\xi)\, \nu(d\xi) \\ &= \int_{E'} e^{i\langle T_t x, \xi\rangle} \big(\hat{\mu}_t(\xi) - \hat{\mu}_{t,\varepsilon}(\xi)\big)\, \nu(d\xi). \end{aligned}$$

Therefore,

$$\begin{aligned} \big\|P_t\varphi - P_{t,\varepsilon}\varphi\big\|_\infty &\le \int_{E'} \big|\hat{\mu}_t(\xi) - \hat{\mu}_{t,\varepsilon}(\xi)\big|\, \nu(d\xi) \\ &= \int_F \big|\hat{\mu}_t(\xi) - \hat{\mu}_{t,\varepsilon}(\xi)\big|\, \nu(d\xi), \end{aligned}$$

since ν is carried by F. But, by Proposition 3.4, $\hat{\mu}_{t,\varepsilon}(\xi) \to \hat{\mu}_t(\xi)$ as $\varepsilon \to 0$, for any given $\xi \in F$. Since $\big|\hat{\mu}_t(\xi) - \hat{\mu}_{t,\varepsilon}(\xi)\big| \leq 2$ uniformly in ξ, Lebesgue's Dominated Convergence Theorem now finishes the proof. □

Proposition 3.6. *Let $g : E \to \mathbb{R}$ be given by $g = h \circ \Pi'_m$, where $\Pi'_m(x) := \big(\langle x, \xi_1\rangle, \ldots, \langle x, \xi_m\rangle\big)$ and $h \in C_b(\mathbb{R}^m)$. Then, as $\varepsilon \to 0$, $P_{t,\varepsilon} g \to P_t g$ in the sense of* pointwise *convergence.*

Proof. Clearly,

$$
\begin{aligned}
P_t g(x) &= \int_E g\big(T_t x + y\big)\, \mu_t(dy) \\
&= \int_E h\big(\Pi'_m(T_t x + y)\big)\, \mu_t(dy) \\
&= \int_{\mathbb{R}^m} h\big(\Pi'_m(T_t x) + u\big)\, \sigma_t(du),
\end{aligned}
$$

where $\sigma_t := \mu_t \circ (\Pi'_m)^{-1}$, and similarly

$$
P_{t,\varepsilon} g(x) = \int_{\mathbb{R}^m} h\big(\Pi'_m(T_t x) + u\big)\, \sigma_{t,\varepsilon}(du),
$$

where $\sigma_{t,\varepsilon} = \mu_{t,\varepsilon} \circ (\Pi'_m)^{-1}$. For fixed x, $u \mapsto h\big(\Pi'_m(T_t x) + u\big)$ is bounded and continuous on $\mathbb{R}^m$, so it suffices to prove the *weak convergence* of $\sigma_{t,\varepsilon}$ to σ_t for $\varepsilon \to 0$. But this follows from P. Levy's Theorem, since for given $v \in (\mathbb{R}^m)'$ $(\simeq \mathbb{R}^m)$

$$
\begin{aligned}
\hat{\sigma}_{t,\varepsilon}(v) &= \int_{\mathbb{R}^m} e^{i\langle u, v\rangle}\, \sigma_{t,\varepsilon}(du) \\
&= \int_E e^{i\langle \Pi'_m(x), v\rangle}\, \mu_{t,\varepsilon}(dx) \\
&= \int_E e^{i\langle x, \Pi_m(v)\rangle}\, \mu_{t,\varepsilon}(dx) \\
&= \hat{\mu}_{t,\varepsilon}\big(\Pi_m(v)\big) = \hat{\mu}_{t,\varepsilon}\big(v_1\xi_1 + \cdots + v_m\xi_m\big),
\end{aligned}
$$

which tends to $\hat{\mu}_t\big(v_1\xi_1 + \cdots + v_m\xi_m\big) = \hat{\sigma}_t(v)$ as $\varepsilon \to 0$, by Prop. 3.4. □

Let $x \in E$. By the result of Section 2, applied to $P_{t,\varepsilon}$ (which is justified because, according to Proposition 3.3, $\lambda_\varepsilon\big|_F$ is C^1), one may write

$$
P_{t,\varepsilon}\varphi(x) - \varphi(x) = \int_0^t P_{s,\varepsilon}\mathcal{A}_\varepsilon\varphi(x)\, ds,
$$

where $\mathcal{A}_\epsilon$ is defined as in (1.5) with λ_ε replacing λ. Hence (for $\mathcal{A}$ as in (1.5))

$$\begin{aligned}
&P_t\varphi(x) - \varphi(x) - \int_0^t P_s\mathcal{A}\varphi(x)\,ds \\
&= \big(P_t\varphi(x) - P_{t,\varepsilon}\varphi(x)\big) + \int_0^t P_{s,\varepsilon}\big(\mathcal{A}_\varepsilon - \mathcal{A}\big)\varphi(x)\,ds + \int_0^t \big(P_{s,\varepsilon} - P_s\big)\mathcal{A}\varphi(x)\,ds \\
&\qquad = C_1(t,x,\varepsilon) + C_2(t,x,\varepsilon) + C_3(t,x,\varepsilon) \qquad \text{(say)}.
\end{aligned}$$

We are going to show that for fixed t and x each of these three terms tends to 0 as $\varepsilon \to 0$. This is obvious for C_1, by Corollary 3.5. For C_2 we have

$$\big\|P_{s,\varepsilon}\big(\mathcal{A}_\varepsilon - \mathcal{A}\big)\varphi\big\|_\infty \le \big\|\big(\mathcal{A}_\varepsilon - \mathcal{A}\big)\varphi\big\|_\infty.$$

But from the definition of $\mathcal{A}$ and $\mathcal{A}_\varepsilon$ it follows that for all $x \in E$

$$\big(\mathcal{A}_\varepsilon - \mathcal{A}\big)\varphi(x) = \int_{E'} e^{i\langle x,\xi\rangle}\big(\lambda(\xi) - \lambda_\varepsilon(\xi)\big)\,\nu(d\xi).$$

Therefore,

$$\begin{aligned}
\big|C_2(t,x,\varepsilon)\big| &\le t\int_{E'}\big|\lambda(\xi) - \lambda_\varepsilon(\xi)\big|\,\nu(d\xi) \\
&= t\int_F \big|\lambda(\xi) - \lambda_\varepsilon(\xi)\big|\,\nu(d\xi) \\
&= t\int_{\mathbb{R}^m}\big|\lambda\big(\Pi_m(v)\big) - \lambda_\varepsilon\big(\Pi_m(v)\big)\big|\,g_0(v)\,dv,
\end{aligned}$$

which tends to 0 as $\varepsilon \to 0$ by Lemmas 3.1 and 3.2, since $g \in \mathcal{S}(\mathbb{R}^m, \mathbb{C})$.

For C_3, since $\mathcal{A}\varphi \in C_b(E)$, by Theorem 1.1(i), for fixed s one has $P_{s,\varepsilon}\mathcal{A}\varphi(x) \to P_s\mathcal{A}\varphi(x)$ as $\varepsilon \to 0$ by Proposition 3.6, which applies by (1.5). Since

$$\big|P_{s,\varepsilon}\mathcal{A}\varphi(x) - P_s\mathcal{A}\varphi(x)\big| \le 2\big\|\mathcal{A}\varphi\big\|_\infty$$

uniformly in s, the Dominated Convergence Theorem yields $C_3(t,x,\varepsilon) \to 0$ as $\varepsilon \to 0$.

Hence, the equality

$$P_t\varphi(x) - \varphi(x) = \int_0^t P_s\mathcal{A}\varphi(x)\,ds$$

holds for all $x \in E$, which finishes the proof of Theorem 1.1(ii). ☐

Proof of Theorem 1.2. Let $\varphi \in W$ and let $p \ge 1$. As $\mathcal{A}\varphi \in C_b(E)$ by Theorem 1.1(i), hence in particular $\mathcal{A}\varphi \in L^p(E,\mu)$, it follows that $s \mapsto P_s\mathcal{A}\varphi$ is continuous from $\mathbb{R}_+$ to $L^p(E,\mu)$. Hence the integral in Theorem 1.1(ii) is a μ–version of the corresponding $L^p(E,\mu)$–valued Bochner integral and the assertion follows immediately from Theorem 1.1(ii) by the Fundamental Theorem of Calculus (applied to $L^p(E,\mu)$–valued functions). □

4. Proof of Theorem 1.3

(i) Let $\varphi \in W$ be given, with ν, f and g_0 having their usual meaning, and $t \geq 0$ be fixed. We have seen that for all $x \in E$

$$P_t\varphi(x) = \int_{E'} \exp\left\{ i\langle x, T_t^*\xi\rangle - \int_0^t \lambda\big(T_s^*\xi\big)\,ds \right\} \nu(d\xi) \\ = h\big(\langle x,\xi_1\rangle, \ldots, \langle x,\xi_m\rangle\big),$$

where

$$\text{(4.1)} \quad h(u_1,\ldots,u_m) \\ = \int_{\mathbb{R}^m} \exp\left\{ i\sum_{j=1}^m e^{t\alpha_j} u_j v_j \right\} \exp\left\{ - \int_0^t \lambda\Big(\sum_{j=1}^m e^{s\alpha_j} v_j \xi_j\Big)\,ds \right\} g_0(v)\,dv.$$

But, as $g_0 \in \mathcal{S}(\mathbb{R}^m,\mathbb{C})$, λ is C^∞ on $F_m := \operatorname{span}\{\xi_1,\ldots,\xi_m\}$, and since the real part of a negative definite function is always non-negative, it is not difficult to see that $h \in \mathcal{S}(\mathbb{R}^m,\mathbb{C})$ (and is real-valued).

(ii) Fix $\varphi = f\big(\langle\,\cdot\,,\xi_1\rangle,\ldots,\langle\,\cdot\,,\xi_m\rangle\big) = \mathcal{F}(\nu) \in W$ and let g_0 be as above. Then, according to the proof of (i), $P_t\varphi \in W$, hence by (1.2) we see that $P_t f = \mathcal{F}(\nu_t)$ for some $\nu_t \in M$. In fact, ν_t is just the image measure under T_t^* of the measure

$$\tilde{\nu}_t(d\xi) := \exp\left\{ - \int_0^t \lambda\big(T_s^*\xi\big)\,ds \right\} \nu(d\xi) = \hat{\mu}_t(\xi)\,\nu(d\xi),$$

i.e.

$$\nu_t = \tilde{\nu}_t \circ (T_t^*)^{-1}. \tag{4.2}$$

Now, by a well-known result [Are86], it is enough to show that any strongly continuous semigroup $\overline{P_t}$ on $L^p(E,\mu)$, whose generator $\overline{\mathcal{A}}$ has the property that $\overline{\mathcal{A}}\big|_W = \mathcal{A}\big|_W$, coincides with P_t. This will be proved to follow from an application of Duhamel's Formula.

In order to justify the following computations, we need to know that

$$\frac{d}{ds} P_{s,\varepsilon}\varphi = \mathcal{A}_\varepsilon P_{s,\varepsilon}\varphi \qquad \text{in the } L^p(E,\mu) \text{ sense} \tag{4.3}$$

for $\varphi \in W$, fixed $\varepsilon > 0$, and $s \geq 0$, where $P_{s,\varepsilon}$ is as in Section 3.[1] But this is easy to see: By Theorem 1.1(ii), the equality

$$P_{t,\varepsilon}\varphi - \varphi = \int_0^t P_{s,\varepsilon}\mathcal{A}_\varepsilon\varphi\, ds \tag{4.4}$$

holds *pointwise*. Furthermore, $\mathcal{A}_\varepsilon\varphi \in C_b(E)$ by Theorem 1.1(i) and

$$\left\|P_{s,\varepsilon}\mathcal{A}_\varepsilon\varphi\right\|_\infty \leq \left\|\mathcal{A}_\varepsilon\varphi\right\|_\infty$$

for all s. But (2.3) (applied to $P_{s,\varepsilon}$, $\mathcal{A}_\varepsilon$, λ_ε) and Lemma 3.2 imply that $s \mapsto P_{s,\varepsilon}\mathcal{A}_\varepsilon\varphi(x)$ is continuous on $\mathbb{R}_+$ for all $x \in E$. Both facts together imply that $s \mapsto P_{s,\varepsilon}\mathcal{A}_\varepsilon\varphi$ is continuous from $\mathbb{R}_+$ to $L^p(E,\mu)$, and hence (4.3) follows from (4.4) by the Fundamental Theorem of Calculus.

Hence, by Duhamel's Formula

$$\begin{aligned}
\overline{P_t}\varphi - P_{t,\varepsilon}\varphi &= \int_0^t \frac{d}{ds}\left[\overline{P_s}P_{t-s,\varepsilon}\varphi\right] ds \\
&= \int_0^t \left[\overline{P_s}\,\overline{\mathcal{A}}P_{t-s,\varepsilon}\varphi - \overline{P_s}\mathcal{A}_\varepsilon P_{t-s,\varepsilon}\varphi\right] ds
\end{aligned}$$

(because $P_{t-s,\varepsilon}\varphi \in P_{t-s,\varepsilon}(W) \subseteq W \subseteq D(\overline{\mathcal{A}})$ by (i) and because of (4.3))

$$= \int_0^t \overline{P_s}\big(\overline{\mathcal{A}} - \mathcal{A}_\varepsilon\big)\, P_{t-s,\varepsilon}\varphi\, ds.$$

Hence, for $c,\omega > 0$ such that $|||\overline{P_t}|||_{L^p\to L^p} \leq ce^{\omega t}$, $t \geq 0$,

$$\begin{aligned}
\left\|\overline{P_t}\varphi - P_{t,\varepsilon}\varphi\right\|_{L^p(E,\mu)} &\leq \int_0^t \left\|\overline{P_s}\big(\overline{\mathcal{A}} - \mathcal{A}_\varepsilon\big)\, P_{t-s,\varepsilon}\varphi\right\|_{L^p(E,\mu)} ds \\
&\leq ce^{\omega t}\int_0^t \left\|\big(\overline{\mathcal{A}} - \mathcal{A}_\varepsilon\big)\, P_{t-s,\varepsilon}\varphi\right\|_{L^p(E,\mu)} ds \\
&\leq ce^{\omega t}\int_0^t \left\|\big(\mathcal{A} - \mathcal{A}_\varepsilon\big)\, P_{t-s,\varepsilon}\varphi\right\|_\infty ds.
\end{aligned}$$

But, setting $P_{t-s,\varepsilon}\varphi = \mathcal{F}(\nu_{t-s,\varepsilon})$, by (1.5) we have for all $x \in E$,

$$\big(\mathcal{A} - \mathcal{A}_\varepsilon\big)\, P_{t-s,\varepsilon}\varphi(x) = \int_{E'} e^{i\langle x,\xi\rangle}\big(\lambda_\varepsilon(\xi) - \lambda(\xi)\big)\, \nu_{t-s,\varepsilon}(d\xi).$$

Therefore,

$$\left\|\big(\mathcal{A} - \mathcal{A}_\varepsilon\big)\, P_{t-s,\varepsilon}\varphi\right\|_\infty \leq \sup_{x\in E}\left|\int_{E'} e^{i\langle x,\xi\rangle}\big(\lambda_\varepsilon(\xi) - \lambda(\xi)\big)\, \nu_{t-s,\varepsilon}(d\xi)\right|.$$

[1] The point is that μ need not be an invariant measure for $P_{t,\varepsilon}$!

and it follows that

$$\left\|\overline{P_t}\varphi - P_{t,\varepsilon}\varphi\right\|_{L^p(E,\mu)} \leq tce^{\omega t} \sup_{0\leq u\leq t}\sup_{x\in E}\left|\int_{E'} e^{i\langle x,\xi\rangle}\big(\lambda_\varepsilon(\xi)-\lambda(\xi)\big)\,\nu_{u,\varepsilon}(d\xi)\right|.$$

Let us assume for a moment that, for fixed t,

$$\sup_{0\leq u\leq t}\sup_{x\in E}\left|\int_{E'} e^{i\langle x,\xi\rangle}\big(\lambda_\varepsilon(\xi)-\lambda(\xi)\big)\,\nu_{u,\varepsilon}(d\xi)\right| \xrightarrow[\varepsilon\to 0]{} 0. \tag{4.5}$$

Then it follows from the above that

$$\left\|\overline{P_t}\varphi - P_{t,\varepsilon}\varphi\right\|_{L^p(E,\mu)} \xrightarrow[\varepsilon\to 0]{} 0,$$

i.e. $P_{t,\varepsilon}\varphi \to \overline{P_t}\varphi$ in $L^p(E,\mu)$ as $\varepsilon\to 0$. But $P_{t,\varepsilon}\varphi\to P_t\varphi$ in $L^p(E,\mu)$ as $\varepsilon\to 0$ by Prop. 3.6 and Lebesgue's Dominated Convergence Theorem (as $\left|P_{t,\varepsilon}\varphi(x)-P_t\varphi(x)\right|\leq 2\|\varphi\|_\infty$ for all x). Since W is dense in $L^p(E,\mu)$ it follows that $\overline{P_t}=P_t$ for all $t>0$, which completes the proof, provided we can show (4.5).

By (4.2) (applied to λ_ϵ) we have for all $u\in[0,t]$, $x\in E$, that (since $\left|\hat\mu_{t,\epsilon}(\xi)\right|\leq 1$)

$$\begin{aligned}
&\left|\int_{E'} e^{i\langle x,\xi\rangle}\big(\lambda_\varepsilon(\xi)-\lambda(\xi)\big)\,\nu_{u,\varepsilon}(d\xi)\right| \\
&\quad = \left|\int_{E'} e^{i\langle x,T_u^*\xi\rangle}\big(\lambda_\varepsilon(T_u^*\xi)-\lambda(T_u^*\xi)\big)\,\hat\mu_{t,\varepsilon}(\xi)\,\nu(d\xi)\right| \\
&\quad \leq \int_{\mathbb{R}^m}\Big|\lambda_\varepsilon\big(T_u^*\big(\Pi_m(v)\big)\big)-\lambda\big(T_u^*\big(\Pi_m(v)\big)\big)\Big|\,|g_0|(v)\,dv \\
&\quad = \int_{\mathbb{R}^m}\Big|\lambda_\varepsilon\Big(\sum_{j=1}^m e^{\alpha_j u}v_j\xi_j\Big)-\lambda\Big(\sum_{j=1}^m e^{\alpha_j u}v_j\xi_j\Big)\Big|\,|g_0|(v)\,dv \\
&\quad \leq \sup_{\xi\in\overline{B_{cr}^{E'}(0)}}\big|\lambda_\varepsilon(\xi)-\lambda(\xi)\big|\int_{\{|v|\leq r\}}|g_0|(v)\,dv+ \\
&\qquad\qquad + 2D\int_{\{|v|\geq r\}}\big(1+c^2\|v\|^2\big)\,|g_0|(v)\,dv,
\end{aligned}$$

where $r>0$, $c:=\exp\big(\max_{1\leq j\leq m}|\alpha_j|\,t\big)$ and where we used Lemma 3.2. Letting first $\varepsilon\to 0$ and then $r\to\infty$, (4.5) follows from Lemma 3.1 and the fact that $g_0\in\mathcal{S}$. □

Remark 4.1. The result of this section applies in particular to the example considered in [FR00, pp. 45–46]:

θ is a finite positive symmetric measure on E, concentrated on $\partial B_1^E(0)$, $0 < p \leq 2$, λ is defined by

$$\lambda(\xi) = \int_E \left|\langle \xi, x\rangle\right|^p \theta(dx), \qquad \xi \in E',$$

and $T_t = e^{-t}I$. In this case

$$P_t\varphi(x) = \int_E \varphi\big(e^{-t}x + (1-e^{-pt})^{\frac{1}{p}}y\big)\,\mu(dy),$$

where $\hat{\mu}(\xi) = e^{-\frac{1}{p}\lambda(\xi)}$, $\xi \in E'$.

Remark 4.2. Here is an example where λ satisfies (H1) and is C^∞:

$$\lambda(\xi) = \frac{\alpha\left\|C\xi\right\|^2}{\alpha + \left\|C\xi\right\|^2},$$

where $\alpha > 0$ and $C : E' \to E'$ a symmetric, positive, nuclear operator. The negative-definiteness of λ here follows from the Bernstein representation

$$\frac{\alpha y}{\alpha + y} = \int_0^{+\infty} \big(1 - e^{-sy}\big)\alpha^2 e^{-\alpha s}\,ds,$$

already used in [BLR99, p.8].

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