On the absence of spectral gaps on certain loop spaces

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Abstract. Let M be a compact simply connected Riemannian manifold which contains a non–trivial closed geodesic γ such that the curvature is constant and strictly negative in a neighbourhood of γ. We show that in this case a Poincaré inequality for the H^1 gradient on the free loop space $C(S^1, M)$ endowed with Bismut measure does not hold. Similar results hold w.r.t. more general metrics, and also on based loop spaces with base point close to γ. A key ingredient in the proofs is a result which shows that in a certain sense, the concentration of a Brownian bridge on hyperbolic space near the geodesic joining the end points x and y increases rapidly as $d(x, y) \to \infty$.

1 Introduction and main results

1.1 General introduction

In [F], S. Fang proved a Poincaré inequality w.r.t. Wiener measure on the based path space over a compact Riemannian manifold. The validity of this inequality is equivalent to the existence of a spectral gap above 0 for the Ornstein Uhlenbeck operator on the path space, which is the generator of the corresponding Dirichlet form. In spite of this positive result, it is still not known if, and under which conditions, a corresponding result holds on loop spaces.

That the loop space case can not be handled as easily as the path space case is already indicated by the fact that the kernel of the Ornstein Uhlenbeck type operator on the loop space is infinite dimensional if the underlying manifold M is not simply connected. Nevertheless, one might still hope for the existence of a spectral gap above 0 in general, and for a Poincaré inequality in the ordinary form if M is simply connected. The aim of this article is to demonstrate that such results can not be expected without further geometric restrictions. More precisely, we will show (in a particular case) that a non–trivial closed geodesic which is a local minimum for the energy functional on $H^1(S^1, M)$ can provoque accumulation at 0 of the spectrum of the Ornstein Uhlenbeck operators on pinned and free loop spaces over M. A corresponding result holds for a broad class of other diffusion operators on these loop spaces as well.

We now introduce the framework needed to state our results in detail. Up to slight modifications, this framework has been used in many publications during the last years, cf. e.g. [D], [DR], [L], [M], [H1], [H2], [ES], [A1], [A2].

1.2 Measures on loop spaces, stochastic horizontal lifts, and integration by parts

Let M be a compact connected Riemannian manifold, and let $d = \dim(M)$. Let $LM = C(S^1, M)$ denote the space of continuous loops over M. In the sequel we will identify $S^1 = [0,1]/\sim$, so

$$LM = \{\omega \in C([0,1], M)\,;\, \omega(0) = \omega(1)\}.$$

For $x \in M$ let $L_xM = \{\omega \in LM;\, \omega(0) = x\}$ $(= \{\omega \in C([0,1], M);\, \omega(0) = \omega(1) = x\}$) be the pinned loop space at x. We endow the spaces LM and L_xM, $x \in M$, with their Borel σ–algebras $\mathcal{B}(LM)$ and $\mathcal{B}(L_xM)$, which are generated by the M–valued evaluation maps Π_s, $0 \le s \le 1$, $\Pi_s(\omega) = \omega(s)$. The distribution P_x of the *Brownian bridge* from x to x in time 1 is the unique probability measure on L_xM such that

$$\begin{aligned}
&\int_{L_xM} f(\omega(s_1), \omega(s_2), \ldots, \omega(s_n))\;\; P_x(d\omega) \\
= &\int_{M^n} f(x_1, x_2, \ldots, x_n)\, p_{s_1}(x, x_1) p_{s_2 - s_1}(x_1, x_2) \cdots \\
&\qquad \cdots p_{s_n - s_{n-1}}(x_{n-1}, x_n) p_{1-s_n}(x_n, x)\, \prod_{i=1}^n V(dx_i) \,/\, p_1(x, x)
\end{aligned}$$

holds for all $n \in \mathbf{N}$, $f \in C^\infty(M^n)$, and $0 < s_1 < s_2 < \ldots < s_n < 1$. Here $p_t(x, y)$ denotes the heat kernel of $\Delta/2$ on M. The S^1 invariant *normalized*

Bismut measure P on LM is determined by

$$\begin{aligned}&\int_{LM} f(\omega(s_1),\omega(s_2),\dots,\omega(s_n))\ P(d\omega)\\ =\ &\int_{M^n} f(x_1,x_2,\dots,x_n)\, p_{s_1}(x,x_1)p_{s_2-s_1}(x_1,x_2)\cdots\\ &\qquad\cdots p_{s_n-s_{n-1}}(x_{n-1},x_n)p_{1+s_1-s_n}(x_n,x)\ \prod_{i=1}^{n} V(dx_i)\ /\ \int_M p_1(x,x)\ V(dx)\end{aligned}$$

for all $n\in\mathbf{N}$, $f\in C^\infty(M^n)$, and $0<s_1<s_2<\dots<s_n\le 1$. Hence

$$P\ =\ \int P_x\, p_1(x,x)\ V(dx)\ /\ \int p_1(x,x)\ V(dx)$$

where the measures P_x, $x\in M$, have been extended trivially to LM. For the key assertions below it does not make a difference whether we are using the normalized or unnormalized Bismut measure. In particular, the operators on LM defined below are the same in both cases.

The M–valued process $(\Pi_s)_{0\le s\le 1}$ is a semimartingale both on $(L_xM,\ \mathcal{B}(L_xM)^{P_x},\ P_x)$ for every $x\in M$ and on $(LM,\ \mathcal{B}(LM)^P,\ P)$ w.r.t. the corresponding augmentations $(\mathcal{F}^x_s)_{0\le s\le 1}$, $(\mathcal{F}_s)_{0\le s\le 1}$ of the filtrations generated by the process, cf. e.g. [D]. Here $\mathcal{B}(L_xM)^{P_x}$, $\mathcal{B}(LM)^P$ denote the completions of the Borel σ–algebras. Let $\pi: O(M)\to M$ be the orthonormal frame bundle over M. We will view a frame $u\in O(M)$ as an isometry from $\mathbf{R}^d$ to $T_{\pi(u)}M$. We fix a Borel measurable map $U^{(0)}: LM\to O(M)$ such that $\pi\circ U^{(0)}=Y_0$. The corresponding *horizontal lift* of the semimartingale $(\Pi_s)_{0\le s\le 1}$ is the (up to equivalence unique) $O(M)$ valued horizontal semimartingale $(U_s)_{0\le s\le 1}$ on $(L_xM,\ \mathcal{B}(L_xM)^{P_x},\ P_x,\ (\mathcal{F}^x_s)\,)$, $(LM,\ \mathcal{B}(LM)^P,\ P,\ (\mathcal{F}_s)\,)$ respectively such that $U_0=U^{(0)}$ and $\pi\circ U_s=\Pi_s$ for all $0\le s\le 1$ hold P_x–a.s., P–a.s. respectively, cf. e.g. [HT, Satz 7.141].

For $x\in M$ and $\omega\in L_xM$, the *tangent space* $T_\omega L_xM$ consists of all continuous vector fields $X:[0,1]\to TM$ along ω that vanish at 0 and 1. Let $\mathcal{FC}^\infty$ denote the space of all smooth cylinder functions $F(\omega)=f(\omega(s_1),\dots,\omega(s_n))$, $n\in\mathbf{N}$, $f\in C^\infty(M^n)$, $s_1,\dots,s_n\in[0,1]$ on LM. For such a function F and $X\in T_\omega L_xM$, the *directional derivative* XF is given by

$$XF\ =\ \sum_{i=1}^{n}(X^{(i)}_{s_i}f)\,(\omega(s_1),\dots,\omega(s_n)),$$

where $X_s^{(i)}$ denotes the appliaction of the derivative X_s to the i–th component on M^n. Let $(U_s)_{0\le s\le 1}$ be a version of the stochastic horizontal lift w.r.t. P_x. For $h \in C([0,1],\mathbf{R}^d)$ with $h(0) = h(1) = 0$ let X^h denote the measurable vector field on L_xM given by

$$(1.1) \qquad X_s^h(\omega) \;=\; U_s(\omega)\,h(s)\;, \quad 0 \le s \le 1\,,\ \omega \in L_xM.$$

If h is in $H_0^{1,2}([0,1],\mathbf{R}^d)$, then the following crucial *integration by parts identity* holds :

$$(1.2) \qquad \int_{L_xM} X^h F\;G\;dP_x \;=\; -\int_{L_xM} F\;X_h G\;dP_x \;+\; \int_{L_xM} \beta_h\,F\,G\;dP_x$$

for all $F,\,G \in \mathcal{FC}^\infty$, where $\beta_h = \int_0^1 (h'(s) + \mathrm{Ric}_{U_s}(h(s)))\cdot db_s$. Here X_hF denotes the directional derivative of F in direction X_h, $\mathrm{Ric}_{U_s} = U_s^{-1}\mathrm{Ric}\,U_s$, and $(b_s)_{0\le s\le 1}$ is the $\mathbf{R}^d$–valued *stochastic development* of the Brownian bridge $s \mapsto \omega(s)$, cf. e.g. [HT]. The function β_h is contained in $L^p(L_xM;\,P_x)$ for every $p \in [1,\infty)$. Notice that (1.2) means that the smooth cylinder functions are contained in the domain of the adjoint X_h^* of the operator $(X_h,\,\mathcal{FC}^\infty)$ on $L^2(L_xM;\,P_x)$. For the proof of (1.2) see [D], [H2] or [ES].

1.3 H^1 metrics and gradients on loop spaces

From now on we fix a version $(U_s)_{s\ge 0}$ of the stochastic horizontal lift w.r.t. P or P_x, $x \in M$, respectively. The tangent space $T_\omega LM$ at a loop $\omega \in LM$ consists of all continuous vector fields $X : S^1 \to TM$ along ω. Since a typical loop in LM is not absolutely continuous, we have to use the stochastic horizontal lift to define an H^1 metric on TLM. For $\omega \in LM$ let $T_\omega^1 LM$ denote the space consisting of all $X \in T_\omega LM$ such that $s \mapsto U_s(\omega)^{-1}X_s$, $0 \le s \le 1$, is an absolutely continuous curve in $\mathbf{R}^d$ with square–integrable derivative. For $X \in T_\omega^1 LM$, we define the covariant derivative

$$(1.3) \qquad \frac{\nabla X}{ds}(s) \;=\; U_s(\omega)\,\frac{d}{dt}(U_t(\omega)^{-1}X_t)|_{t=s}$$

of X along ω. Note that if ω would be smooth, and $U(\omega)$ would be the usual horizontal lift, then (1.3) would yield the usual covariant derivative along ω. The H^1 metric on $T_\omega^1 LM$ is defined by

(1.4)

$$\langle X,\,Y\rangle_{T_\omega^1 LM} \;=\; \int_0^1 \langle \frac{\nabla X}{ds}(s),\,\frac{\nabla Y}{ds}(s)\rangle_{T_{\omega(s)}M}\,ds \;+\; \int_0^1 \langle X_s,\,Y_s\rangle_{T_{\omega(s)}M}\,ds\,,$$

$X, Y \in T^1_\omega LM$. W.r.t. this metric, $T^1_\omega LM$ is a Hilbert space, and the map $h \mapsto X^h(\omega)$, defined by (1.1) is an isometry between $\{h \in H^{1,2}([0,1], \mathbf{R}^d);\ U_0(\omega)h(0) = U_1(\omega)h(1)\}$ and $T^1_\omega LM$.

Now fix $x \in M$, and let $(U_s)_{0 \le s \le 1}$ be the fixed version of the stochastic horizontal lift w.r.t. P_x. Then for $\omega \in L_xM$, the H^1 tangent space $T^1_\omega L_xM$ and the covariant derivative of a vector field $X \in T^1_\omega L_xM$ along ω can be defined similarly as above. Note that a priori the stochastic horizontal lifts w.r.t. P_x and P are not related, because the measures are singular. However, it is possible to construct a joint version of the stochastic horizontal lift w.r.t. both measures, cf. [D]. By using such a version for the definitions, we have

$$T^1_\omega L_xM \;=\; \{\, X \in T^1_\omega LM\,;\ X_0 = 0\}\,.$$

The H^1 metric on T^1L_xM could now be defined as the restriction of the H^1 metric on T^1LM, cf. (1.4). In stochastic analysis, however, one usually uses the eqivalent metric

$$\langle X, Y\rangle_{T^1_\omega L_xM} \;=\; \int_0^1 \langle \frac{\nabla X}{ds}(s),\ \frac{\nabla Y}{ds}(s)\rangle_{T_{\omega(s)}M}\, ds\,, \tag{1.5}$$

$X, Y \in T^1_\omega L_xM$. The map $(h, \omega) \mapsto X^h(\omega)$ defined by (1.1) is an isometry between the trivial bundle $L_xM \times H^{1,2}_0([0,1], \mathbf{R}^d)$ and the measurable field of Hilbert spaces T^1L_xM provided $H^{1,2}_0([0,1], \mathbf{R}^d)$ is equipped with the Cameron Martin metric

$$\langle h, g\rangle_{\mathrm{CM}} \;=\; \int_0^1 h'(s) \cdot g'(s)\, ds.$$

For a smooth cylinder funtion $F(\omega) = f(\omega(s_1), \ldots, \omega(s_n))$, the *gradients* D^0F and DF are the "sections" of T^1L_xM, T^1LM respectively defined by

$$\begin{aligned} \langle (D^0F)(\omega), X\rangle_{T^1_\omega L_xM} &= XF \quad \text{for all } \omega \in L_xM \text{ and } X \in T^1_\omega L_xM, \\ \langle (DF)(\omega), X\rangle_{T^1_\omega LM} &= XF \quad \text{for all } \omega \in LM \text{ and } X \in T^1_\omega LM. \end{aligned}$$

One easily calculates that explicitly,

(1.6)

$$U_s(\omega)^{-1}(D^0F)(\omega)(s) \;=\; \sum_{i=1}^n G^0(s, s_i)\, U_{s_i}(\omega)^{-1} \mathrm{grad}^{(i)} f\, (\omega(s_1), \ldots, \omega(s_n))$$

where $G^0(s,t) = s\wedge t - s\cdot t$, $s,\, t \in [0,1]$, is the Green's function of the operator $-d^2/ds^2$ with Dirichlet boundary conditions on $(0,1)$. In particular,

$$
\begin{aligned}
|D^0F(\omega)|^2_{T^1_\omega L_xM} &= \sum_{i,j=1}^{n} \Big(G^0(s_i,s_j) U_{s_i}(\omega)^{-1} \mathrm{grad}^{(i)} f(\omega(s_1),\dots,\omega(s_n))\ , \\
&\qquad\qquad U_{s_j}^{-1}(\omega) \mathrm{grad}^{(j)} f(\omega(s_1),\dots,\omega(s_n)) \Big)_{\mathbf{R}^d} . \qquad (1.7)
\end{aligned}
$$

Corresponding representations hold for DF as well, but $G^0(s,t)$ has to be replaced by the matrix–valued Green's function $G(\omega,s,t)$ of the operator $1-d^2/ds^2$ acting on functions $h : [0,1] \to \mathbf{R}^d$ with stochastic boundary conditions $h(1) = U_1(\omega)^{-1}U_0(\omega)h(0)$ and $h'(1-) = U_1(\omega)^{-1}U_0(\omega)h'(0+)$.

A section X of the bundle T^1LM is called *measurable* if the function $U^{-1}X : LM \to H^{1,2}([0,1],\, \mathbf{R}^d)$, $(U^{-1}X)(\omega)(s) = U_s(\omega)^{-1}X_s(\omega)$, is measurable. It is called *square–integrable w.r.t. P* if $\int \langle X(\omega),\, X(\omega)\rangle_{T^1_\omega LM}\, P(d\omega) < \infty$. The space $L^2(T^1LM;\, P)$ consisting of all equivalence classes of square integrable sections is a Hilbert space. In fact, it is the *direct integral* of the measurable field of Hilbert spaces T^1LM, cf. [Di]. The Hilbert space $L^2(T^1L_xM;\, P_x)$ is defined similarly.

D and D^0 are densely defined linear operators from $L^2(LM;\, P)$ to $L^2(T^1LM;\, P)$, and from $L^2(L_xM;\, P_x)$ to $L^2(T^1L_xM;\, P_x)$ respectively. It is a consequence of the integration by parts identity (1.2) that these operators are *closable.* For the reader's convenience, a simple proof of this fact is given in the appendix. Notice that the closability of D^0 and D is equivalent to the closability of the corresponding symmetric bilinear forms

$$\mathcal{E}^0_x(F,\, G) = \int \langle D^0F(\omega),\, D^0G(\omega)\rangle_{T^1_\omega L_xM}\, P_x(d\omega), \qquad \text{and} \qquad (1.8)$$

$$\mathcal{E}(F,\, G) = \int \langle DF(\omega),\, DG(\omega)\rangle_{T^1_\omega LM}\, P(d\omega), \qquad (1.9)$$

$F,\, G \in \mathcal{F}C^\infty$, on $L^2(L_xM;\, P_x)$, $L^2(LM;\, P)$ respectively. We remark that these bilinear forms are independent of the choice of the initial frame $U^{(0)}$ and a corresponding version $(U_t)_{t\geq 0}$ of the stochastic horizontal lift made above.

We denote the domains of the closures of the operators D^0 and D by $H^{1,2}(L_xM;\, P_x)$, $H^{1,2}(LM;\, P)$ respectively. The closures of the gradients and the forms (1.8) and (1.9) themselves will again be denoted by D^0, D, $\mathcal{E}^0_x$, and

$\mathcal{E}$ respectively. D^0 and D are non–flat analogues of the Malliavin gradient which is defined on the path space over $\mathbf{R}^n$, cf. [M]. We are interested in the spectrum of the non–negative definite self–adjoint operators

$$\mathcal{L}_x^0 \;=\; (D^0)^* D^0 \qquad \text{and} \qquad \mathcal{L} \;=\; D^* D$$

which are associated to the quadratic forms $(\mathcal{E}_x^0,\, H^{1,2}(L_x M;\, P_x))$ on $L^2(L_x M;\, P_x)$ and $(\mathcal{E},\, H^{1,2}(LM;\, P))$ on $L^2(LM;\, P)$ respectively. Since the forms do not depend on the choice of $U^{(0)}$ and $(U_t)_{t\geq 0}$, the operators $\mathcal{L}_x^0$ and $\mathcal{L}$ do neither. Because of the analogy of the definition of these operators to that of the Ornstein–Uhlenbeck operator on the based path space over $\mathbf{R}^n$, one might call them the *Ornstein–Uhlenbeck operators on $L_x M$, LM* respectively. We point out, however, that the spectral properties of these operators can be very different from those of classical Ornstein–Uhlenbeck operators, cf. the results below.

1.4 Poincaré inequalities and spectral gaps

We say that a *Poincaré inequality* holds w.r.t. the H^1 metric on $L_x M$, LM respectively, if there exists a finite constant c such that

$$\text{(1.10)} \quad \mathrm{Var}_{P_x}(F) \;\leq\; c \cdot \mathcal{E}_x^0\,(F,\, F) \qquad \text{for all } F \in H^{1,2}(L_x M;\, P_x),$$

$$\text{(1.11)} \quad \mathrm{Var}_{P}(F) \;\leq\; c \cdot \mathcal{E}\,(F,\, F) \qquad \text{for all } F \in H^{1,2}(LM;\, P)$$

respectively. Here Var_μ denotes the variance w.r.t. a probability measure μ. Notice that it is enough to verify (1.10) and (1.11) for F in $\mathcal{F}C^\infty$.

We first point out that the Poincaré inequalities do not hold if M is not simply connected. In fact, in this case the loop spaces $L_x M$ and LM are the disjoint unions of their connected components (i.e., the corresponding homotopy classes of M), and it can be shown easily that each indicator function of a component Λ is contained in the kernel of D^0, D respectively, cf. the remark below Lemma 5.1. In 1994, S. Fang proved a Poincaré inequality similarly to the one above on the based path space $\mathcal{P}_x M = \{\omega \in C([0,1], M);\, \omega(0) = x\}$, cf. also [H1], [AE], [CHL], and [H2] for extensions. Notice that $L_x M$ is a submanifold of $\mathcal{P}_x M$ with finite codimension. The measure P_x can be obtained by conditioning Wiener measure on $\mathcal{P}_x M$ to $L_x M$, and the gradient D^0 is precisely the projection of the Malliavin type gradient on $\mathcal{P}_x M$ to the H^1 tangent bundle of the submanifold $L_x M$. There have been attempts to extend Fang's method of proof, which relies on a Clark-Ocone formula on $\mathcal{P}_x M$, to the loop space case, cf. e.g. [GM] and [A2]. However, so far the validity of a Poincaré inequality in the sense above could not be shown on any loop space over a non–flat simply connected Riemannian manifold that is not diffeomorphic to $\mathbf{R}^n$.

We now state our *main result* which shows that in fact, Poincaré inequalities w.r.t. the H^1 metric on loop spaces over compact simply connected Riemannian manifolds can not be expected to hold without further geometric restrictions on the base manifold. We make the following assumption on the Riemannian manifold M :

(A 1) There exists a non–trivial closed geodesic $\gamma : S^1 \to M$ such that the curvature is constant and strictly negative on a neighbourhood of $\gamma(S^1)$.

EXAMPLE. Suppose that $\dim(M) = 2$, and M contains an open subset U that is isometric to the surface of revolution in $\mathbf{R}^3$ given as the image of the map $f : (-A, A) \times \mathbf{R} \to \mathbf{R}^3$,

$$f(s, \varphi) \;=\; (\, R \cosh s \cos\varphi,\; R \cosh s \sin\varphi,\; \int_0^s (1 - R^2 \sinh^2 t)^{1/2}\, dt\,)$$

for some $R,\ A > 0$ with $\sinh A < 1/R$. Then (A 1) holds.

Note that there exists a constant $\varepsilon > 0$ such that the exponential map is a diffeomorphism from the set of all vectors of length $< \varepsilon$ in the normal bundle along $\gamma(S^1)$ to the set

$$U_\varepsilon \;=\; \{\, x \in M \,;\; \mathrm{dist}\,(x, \gamma(S^1)) < \varepsilon \,\},$$

cf. e.g. [C], Sect. 3.6. From now on we fix such an ε for which moreover the curvature is constant on U_ε.

Theorem 1.1 *If (A 1) holds then*

$$\begin{aligned} \inf \{\, \mathcal{E}(F,F)\,;\; F \in \mathcal{F}C^\infty,\, \mathrm{Var}_P(F) = 1 \,\} \;&=\; 0 \qquad \textit{and} \\ \inf \{\, \mathcal{E}^0_x(F,F)\,;\; F \in \mathcal{F}C^\infty,\, \mathrm{Var}_{P_x}(F) = 1 \,\} \;&=\; 0 \qquad \textit{for every } x \in U_\varepsilon. \end{aligned}$$

REMARKS. (i) The strict negativity of the curvature along γ implies that γ is a local minimum for the energy functional $E(\omega) = \int |d\omega/ds|^2_{\omega(s)}\, ds$ on $H^1(S^1,\, M)$, cf. e.g. [J], Thm. 4.1.1.

(ii) I strongly suspect that the assertion of the theorem holds as well under the weaker assumption that the curvature is strictly negative on $\gamma(S^1)$. In fact, in this case the proof given below can be carried out in a similar way except for the proofs of the estimates in Section 3, where we use the explicit representation of the heat kernel on the hyperbolic space H^d.

Notice that the constant functions are contained in the kernel of the operators $\mathcal{L}_x^0$, $x \in M$, and $\mathcal{L}$. We say that a non–negative self–adjoint operator with non–trivial kernel has a *spectral gap above* 0 if its spectrum is contained in $\{0\} \cup [\lambda, \infty)$ for some $\lambda > 0$. As a consequence of the proof of Theorem 1.1 that will be given below, we obtain the following corollary which is slightly stronger than Theorem 1.1 itself :

Corollary 1.2 *If (A 1) holds then the kernel of the operator $\mathcal{L}$ is infinite dimensional, or $\mathcal{L}$ does not have a spectral gap above* 0. *The same holds for the operators $\mathcal{L}_x^0$, $x \in U_\varepsilon$.*

In fact, S. Aida [A1] has shown that the kernels of the operators $\mathcal{L}_x^0$, $x \in M$, contain only the constant functions if M is simply connected. Hence the operators $\mathcal{L}_x^0$, $x \in U_\varepsilon$, do not have a spectral gap above 0 in this case.

REMARKS. (i) The absence of spectral gaps on loop spaces over not simply connected manifolds can be proven similarly under the additional assumption that the closed geodesic γ in (A 1) is homotopic to a constant loop.

(ii) The assertion of Theorem 1.1 means that for $x \in U_\varepsilon$ the strongly continuous semigroups $(\exp(t\mathcal{L}))_{t \geq 0}$ on $L^2(LM;\, P)$ and $(\exp(t\mathcal{L}_x^0))_{t \geq 0}$ on $L^2(L_xM;\, P_x)$ do not decay exponentially fast to equlibrium.

1.5 Generalizations

We now state a generalization of Theorem 1.1 which shows that a Poincaré inequality on the loop spaces considered is not only violated w.r.t. the H^1 metric, but also w.r.t. a broad class of other metrics. Let M be again a compact connected Riemannian manifold satisfying (A 1). Suppose that we are given a symmetric bilinear operator $\Gamma : \mathcal{F}C^\infty \times \mathcal{F}C^\infty \to L^1(LM;\, P)$ such that $\Gamma(F, F) \geq 0$ P–a.s. for all $F \in \mathcal{F}C^\infty$, and

$$(1.12) \qquad \Gamma(\phi(F_1, \ldots, F_n), G) \;=\; \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(F_1, \ldots, F_n)\, \Gamma(F_i, G) \qquad P\text{–a.s.}$$

for all $n \in \mathbf{N}$, $\phi \in C_b^\infty(\mathbf{R}^n)$ and $F_1, \ldots, F_n, G \in \mathcal{F}C^\infty$. Typically,

$$\Gamma(F,G) = \langle \tilde{D}F , \tilde{D}G \rangle$$

for some gradient $\tilde{D}$ and some metric $\langle \cdot, \cdot \rangle$ on LM. We assume that the non–negative definite symmetric bilinear form

$$\mathcal{E}^\Gamma(F, G) = \int_{LM} \Gamma(F, G)\, dP , \qquad F, G \in \mathcal{F}C^\infty, \tag{1.13}$$

is closable on $L^2(LM;\, P)$. We denote the closure again by $\mathcal{E}^\Gamma$ and its domain by $H^{1,2}_\Gamma(LM;\, P)$. We assume moreover that there exists a function $\alpha \in L^1(LM;\, P)$ such that

$$\Gamma(\, f \circ \Pi_s \,,\, f \circ \Pi_s \,)\, (\omega) \;\le\; \alpha(\omega) \cdot |\mathrm{grad}\, f|^2_{\omega(s)} \qquad P\text{–a.s.} \tag{1.14}$$

holds for all $f \in C^\infty(M)$ and $s \in S^1$.

For $n \in \mathbf{N}$ let $\gamma_n : S^1 \to M$ be the closed geodesic obtained by winding around γ n times, i.e., $\gamma_n((k+s)/n) = \gamma(s)$ for all $k \in \{0, 1, \ldots, n-1\}$ and $s \in [0,1]$. We choose $\varepsilon > 0$ as above. Let $\Omega_n \subset LU_\varepsilon$ denote the free homotopy class of U_ε that contains γ_n. For $\delta > 0$ let

$$\Omega_n^\delta = \{\omega \in \Omega_n \,;\, \sup_{s \in S^1} \mathrm{dist}\,(\omega(s), \gamma(S^1)) < \delta\}.$$

The conditional expectation w.r.t. the probability measure P is denoted by $E_P[\,\cdot\,|\,\cdot\,]$.

Theorem 1.3 *Consider the situation just described, and suppose that (1.14) holds for some function $\alpha \in L^1(LM;\, P)$ such that*

$$\liminf_{n \to \infty}\, \exp(-n^\beta)\, E_P[\, \alpha \,|\, \Omega_n \setminus \Omega_n^{\varepsilon/3}] = 0 \quad \textit{for some } \beta < 1/4. \tag{1.15}$$

Then

$$\inf\, \{\, \mathcal{E}^\Gamma(F, F)\, ;\, F \in H^{1,2}_\Gamma(LM;\, P),\, \mathrm{Var}_P(F) = 1\,\} = 0.$$

Corollary 1.4 *Let A be a non–negative function in $L^1(LM;\, P)$. If*

$$\liminf_{n \to \infty}\, \exp(-n^\beta)\, E_P[\, A \,|\, \Omega_n \setminus \Omega_n^{\varepsilon/3}] = 0 \quad \textit{for some } \beta < 1/4,$$

then a Poincaré inequality of type

$$\mathrm{Var}_P(F) \ \leq \ c \cdot \int_{LM} A(\omega)\, \langle DF,\, DF \rangle_{T^1_\omega LM}\, P(d\omega), \qquad F \in \mathcal{F}C^\infty,$$

does not hold for any $c > 0$*.*

REMARKS. (i) Again, corresponding results hold on based loop spaces.

(ii) Theorem 1.3 can also be used to show the non–existence of Poincaré inequalities w.r.t. H^α metrics for $0 < \alpha < 1$.

The organization of this article is as follows : In Section 2 we give a simple criterion to disprove the existence of a spectral gap. To apply this criterion on loop spaces, concentration results for Brownian bridges are crucial. In Section 3 we prove a result of this type that might also be of independent interest : A Brownian bridge from x to y on hyperbolic space concentrates in a certain sense more strongly near the minimal geodesic joining x and y if $d(x, y)$ gets large. In Section 4 we show how this result implies concentration properties for pinned Wiener and Bismut measures on manifolds satisfying (A 1). The proofs of the main results on free loop spaces are given in Section 5, and those on based loop spaces in Section 6.

2 A general anti–spectral gap result

In this section we give a simple criterion for the non–existence of a Poincaré inequality that holds in a more general framework.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $\mathcal{E}$ be a closed non–negative definite quadratic form on $L^2(\Omega;\mu)$ such that the constant functions are in the form domain $\mathrm{Dom}\,(\mathcal{E})$. We assume that $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is a *strongly local Dirichlet form that admits a carré du champ (energy density)*, i.e., there exists a symmetric bilinear map $\Gamma : \mathrm{Dom}\,(\mathcal{E}) \times \mathrm{Dom}\,(\mathcal{E}) \to L^1(\Omega;\mu)$ with the following properties :

$$\Gamma(F,F) \geq 0 \quad \mu\text{–a.e. for all } F \in \mathrm{Dom}\,(\mathcal{E}), \tag{2.1}$$

$$\mathcal{E}(F,G) = \int \Gamma(F,G)\,d\mu \quad \text{for all } F,G \in \mathrm{Dom}\,(\mathcal{E}), \tag{2.2}$$

(2.3) The composition $\phi(F_1,\dots,F_n)$ is in $\mathrm{Dom}\,(\mathcal{E})$ for all $n \in \mathbf{N}$, $F_1,\dots,F_n \in \mathrm{Dom}\,(\mathcal{E}) \cap L^\infty(\Omega;\mu)$ and $\phi \in C_b^1(\mathbf{R}^n)$, and

$$\Gamma(\phi(F_1,\dots,F_n),\,G) \;=\; \sum_{i=1}^n \frac{\partial\phi}{\partial x_i}(F_1,\dots,F_n)\,\Gamma(F_i,G) \quad \mu\text{–a.e.}$$

for all $G \in \mathrm{Dom}\,(\mathcal{E})$.

It is a consequence of (2.3), that for F, $G \in \mathrm{Dom}\,(\mathcal{E})$ and $c \in \mathbf{R}$,

$$\Gamma(F,\,G) = 0 \quad \mu\text{–a.e. on } \{\omega\in\Omega;\ F(\omega)=c\,\}, \tag{2.4}$$

(*Strong locality*, cf. [BH]). We set $\mathcal{E}(F) = \mathcal{E}(F,F)$ and $\Gamma(F) = \Gamma(F,F)$.

REMARK. All the closed quadratic forms on loop spaces introduced in Section 1 are strongly local Dirichlet forms with Carré du champ.

The following observation is almost trivial, but important :

Lemma 2.1 *Suppose that there exist disjoint sets $B_n \in \mathcal{F}$, subsets $A_n \subseteq B_n$, $A_n \in \mathcal{F}$, and functions $F_n \in \mathrm{Dom}\,(\mathcal{E})$, $n \in \mathbf{N}$, such that $F_n = 1$ μ–a.e. on A_n, $F_n = 0$ μ–a.e. on $\Omega\setminus B_n$, and*

$$\liminf_{n\to\infty}\; E_\mu[\Gamma(F_n)\,|\,B_n\setminus A_n]\cdot\mu(B_n\setminus A_n)/\mu(A_n) \;=\; 0. \tag{2.5}$$

Then

$$\inf\{\,\mathcal{E}(F)\,;\,F\in\mathrm{Dom}\,(\mathcal{E}),\ \mathrm{Var}_\mu F = 1\,\} \;=\; 0. \tag{2.6}$$

PROOF. By (A.5) in the appendix, $\Gamma(F_n)$ vanishes μ–a.e. both on $\Omega\setminus B_n$ and on A_n, whence by (2.2),

$$\frac{\mathcal{E}(F_n)}{\int F_n^2\,d\mu} \;\leq\; \frac{\int_{B_n\setminus A_n}\Gamma(F_n)\,d\mu}{\int_{A_n} 1\,d\mu} \;=\; E[\Gamma(F_n)|B_n\setminus A_n]\cdot\frac{\mu(B_n\setminus A_n)}{\mu(A_n)} \tag{2.7}$$

Since the sets B_n are disjoint, the functions F_n are orthogonal in $L^2(\Omega,\mu)$. Hence

$$\frac{\mathrm{Var}_\mu F_n}{\int F_n^2\,d\mu} \;=\; 1-\frac{(\int F_n\,d\mu)^2}{\int F_n^2\,d\mu} \;=\; 1-\left(\frac{F_n}{(\int F_n^2\,d\mu)^{1/2}}\,,\,1\right)_{L^2(\Omega;\mu)} \longrightarrow 1 \tag{2.8}$$

as $n \to \infty$ by Parseval's identity. By (2.7), (2.5) and (2.8),

$$\liminf_{n\to\infty} \mathcal{E}(F_n)/\mathrm{Var}_\mu(F_n) = 0.$$

This implies the assertion after normalizing. □

Remark. More generally, the arguments in the proof imply that

$$\inf \{ \mathcal{E}(F) \,;\, F \in \mathrm{Dom}\,(\mathcal{E}),\ \int (F - PF)^2 \, d\mu = 1 \} \ = \ 0$$

holds for every finite dimensional projection $P : L^2(\Omega; \mu) \to L^2(\Omega; \mu)$.

Example. Suppose that Ω is a complete finite dimensional Riemannian manifold, and μ is a probability measure on Ω with C^1 density w.r.t. the volume element. If there exist disjoint open sets $B_n \subset \Omega$, $n \in \mathbf{N}$, and open subsets $A_n \subset B_n$ such that

$$\inf_{n\in\mathbf{N}} \left(\frac{\mu(B_n \setminus A_n)}{\mu(A_n)} \cdot \frac{1}{\mathrm{dist}\,(\Omega \setminus B_n,\ A_n)^2} \right) \ = \ 0\ ,$$

then

$$\inf_{f\in C_0^\infty(M)} \frac{\int_M |df|^2_{T^*_\omega M}\, \mu(d\omega)}{Var_\mu\, f} \ = \ 0.$$

This follows by applying Lemma 2.1 with $F_n = 1 - \mathrm{dist}\,(\,\cdot\,, A_n)/\mathrm{dist}\,(\Omega \setminus B_n,\ A_n)$ to the closure $(\mathcal{E}, \mathrm{Dom}\,(\mathcal{E}))$ of the symmetric bilinear form $\mathcal{E}(f,g) = \int (df, dg)\, d\mu$. Arguments of this type are used in many different contexts, cf. e.g. [CGP].

In Section 5, we apply Lemma 2.1 to the situation on loop spaces described in Section 1. The sets A_n and B_n will then be chosen as appropriate neighbourhoods of geodesic loops that wind around the set U_ε n times.

3 Concentration of Brownian bridges on hyperbolic spaces

Let H^d_κ, $d \geq 2$, $\kappa < 0$, denote the hyperbolic space of dimension d and curvature κ. For x, $y \in H^d_\kappa$ let $Q_{x,y}$ denote the distribution on $C([0,1], H^d_\kappa)$ of the Brownian bridge from x to y in time 1. Let $\gamma_{x,y} : [0,1] \to H^d_\kappa$ be the unique geodesic from x to y, and let $r_{x,y} = \mathrm{dist}\,(\,\cdot\,, \gamma_{x,y}([0,1])\,)$. The aim of this section is to prove the following proposition :

Proposition 3.1 *For every $a > 0$ and $\beta < 1/4$, there exist constants K_1, K_2, $c_1 \in (0,\infty)$ such that the estimates*

$$Q_{x,y}\,[\sup_{t\in[0,1]} d(\omega(t),\gamma_{x,y}(t)) \geq u\,] \;\leq\; K_1\cdot\exp(-c_1 u^2) \tag{3.1}$$

$$Q_{x,y}\,[\sup_{t\in[0,1]} r_{x,y}(\omega(t)) \geq a\,] \;\leq\; K_2\cdot\exp(-d(x,y)^\beta) \tag{3.2}$$

hold for all $x,y \in H^d_\kappa$ and $u \geq 0$.

Remarks. (i) The crucial statement in Proposition 3.1 is that the constants can be chosen indepently of x and y. It seems that the second estimate does not hold with sup $r_{x,y}(\omega(t))$ replaced by the supremum distance of ω and $\gamma_{x,y}$.

(ii) Estimate (3.2) can be improved considerably. For our purposes, however, the form stated is sufficient.

Let $q_t(x,y)$ denote the heat kernel on H^d_κ. We first recall two well–known facts about q_t and about hyperbolic triangles, which we will frequently use in the sequel :

Lemma 3.2 *There exists a constant $K \in (1,\infty)$ such that*

$$K^{-1}\cdot\tilde{q}_t(x,y) \;\leq\; q_t(x,y) \;\leq\; K\cdot\tilde{q}_t(x,y) \qquad \textit{for all } t\in(0,1] \textit{ and } x,y\in H^d_\kappa,$$

where

$$\tilde{q}_t(x,y) \;=\; (2\pi t)^{-d/2}\, e^{-\frac{d(x,y)^2}{2t}}\; e^{-\frac{d-1}{2}\sqrt{-\kappa}\cdot d(x,y)}\cdot(1+d(x,y))^{(d-1)/2}.$$

Proof. See e.g. [Da], Thm. 5.7.2. □

Lemma 3.3 (Hyperbolic theorem of Pythagoras) *If a, b, c are the lengths of the sides of a right–angled geodesic triangle in H^d_κ with right angle opposite to the side with length c, then*

$$\cosh(\sqrt{-\kappa}\cdot c) \;=\; \cosh(\sqrt{-\kappa}\cdot a)\;\cdot\;\cosh(\sqrt{-\kappa}\cdot b). \tag{3.3}$$

In particular,

$$c \;\geq\; a \;+\; (-\kappa)^{-1/2}\log\cosh(\sqrt{-\kappa}\cdot b). \tag{3.4}$$

PROOF. (3.3) is standard, and can be easily verified in one of the explicit models of H^d_κ, cf. e.g. [Be], 19.3. In particular, $c \geq a$. Since $\cosh u = e^u(1+e^{-2u})/2$, (3.3) implies

$$e^{\sqrt{-\kappa}c} \geq e^{\sqrt{-\kappa}a}\cosh(\sqrt{-\kappa}b)\cdot(1+e^{-2\sqrt{-\kappa}a})/(1+e^{-2\sqrt{-\kappa}c}) \geq e^{\sqrt{-\kappa}a}\cosh(\sqrt{-\kappa}b),$$

and thus (3.4). □

REMARK. If the angle opposite to the side with length c is greater than $\pi/2$, then (3.3) holds with "=" replaced by ">". In particular, (3.4) is still true.

For $x, y \in H^d_\kappa$ and $t > 0$ let $\mu^{x,y}_t$ denote the distribution of $\omega \mapsto \omega(t/2)$ w.r.t. the Brownian bridge from x to y on H^d_κ in time t, i.e.,

$$\mu^{x,y}_t(dz) = \frac{q_{t/2}(x,z)\, q_{t/2}(z,y)}{q_t(x,y)}\, V(dz).$$

The next lemma is the key step in the proof of Proposition 3.1 :

Lemma 3.4 *There exist constants* $A_1,\, A_2,\, \tilde{\delta} \in (0,\infty)$ *such that*

$$\mu^{x,y}_t[\{z \in H^d_\kappa;\, r_{x,y}(z) \geq \alpha\}] \leq A_1 \cdot t^{-(d-1)/4} \cdot e^{-\tilde{\delta}\alpha^4\cdot(1+d(x,y))/t} \quad \forall\, x,y \in H^d_\kappa,\ \alpha \in [0,1],\ t \in (0,1], \tag{3.5}$$

and

$$\mu^{x,y}_t[\{z \in H^d_\kappa;\, d(z,\gamma_{x,y}(\frac{1}{2})) \geq \alpha\}] \leq A_2 \cdot t^{-(d-1)/4} \cdot e^{-\tilde{\delta}\cdot\alpha^2\cdot(\alpha^2\wedge 1)/t} \quad \forall\, x,y \in H^d_\kappa,\ \alpha \in [0,\infty),\ t \in (0,1]. \tag{3.6}$$

PROOF. For better transparency we will restrict ourselves to the case $\kappa = -1$. Using Lemma 3.2 and 3.3, the proof can be carried out similarly for other negative values of κ. Let $H^d = H^d_1$, and let $\nu = (d-1)/2$. By Lemma 3.2, it suffices to prove the assertion with $\mu^{x,y}_t$ replaced by $\tilde{\mu}^{x,y}_t = \tilde{\rho}^{x,y}_t\, dV$, where

$$\tilde{\rho}^{x,y}_t(z) = \frac{\tilde{q}_{t/2}(x,z)\,\tilde{q}_{t/2}(z,y)}{\tilde{q}_t(x,y)} \qquad \text{for all } x,\, y,\, z \in H^d \text{ and } t \in (0,1].$$

Fix $x, y \in H^d$ and $t \in (0,1]$, and let $\ell = d(x,y)$. Then

$$
\begin{aligned}
\tilde{\rho}_t^{x,y}(z) &= \left(\frac{2}{\pi t}\right)^{-d/2} e^{-(d(x,z)^2+d(z,y)^2-\ell^2/2)/t} e^{-\nu(d(x,z)+d(z,y)-\ell)} \\
&\qquad\qquad \times \left(\frac{(1+d(x,z))(1+d(z,y))}{1+\ell}\right)^{\nu} \\
&= \left(\frac{2}{\pi t}\right)^{-d/2} e^{-((d(x,z)-\ell/2)^2+(d(z,y)-\ell/2)^2)/t} e^{-(\nu+\ell/t)\cdot(d(x,z)+d(z,y)-\ell)} \\
&\qquad\qquad \times \left(\frac{(1+d(x,z))(1+d(z,y))}{1+\ell}\right)^{\nu}.
\end{aligned}
\tag{3.7}
$$

We remark first that

(3.8)
$$
e^{-\nu\cdot(d(x,z)+d(z,y)-\ell)} \cdot \left(\frac{(1+d(x,z))(1+d(z,y))}{1+\ell}\right)^{\nu} \leq K_0 \cdot (1+\ell)^{\nu} \quad \forall\, z \in H^d
$$

holds with some finite constant K_0 that does not depend on x and y. This is obvious for $z \in H^d$ with $d(x,z) + d(z,y) \leq 2\ell$. For $z \in H^d$ with $d(x,z) + d(z,y) \geq 2\ell$, the left hand side of (3.8) is dominated from above by $\exp(-\nu \cdot (d(x,z)+d(z,y))/2) \cdot (1+d(x,z))^{\nu}(1+d(z,y))^{\nu}$, which is a uniformly bounded function in x, y and z.

Now let $\tilde{\gamma}_{x,y} : \mathbf{R} \to H^d$ be the unique geodesic parametrized by arc length such that $\tilde{\gamma}_{x,y}(0) = x$ and $\tilde{\gamma}_{x,y}(\ell) = y$. In particular, $\gamma_{x,y}(t) = \tilde{\gamma}_{x,y}(\ell \cdot t)$ for $t \in [0,1]$. We complete $\tilde{E}_0(s) = \tilde{\gamma}'_{x,y}(s)$ to a parallel orthonormal frame $\{\tilde{E}_0(s), \tilde{E}_1(s), \ldots, \tilde{E}_{d-1}(s)\}$, $s \in \mathbf{R}$, along $\tilde{\gamma}_{x,y}$. Let $H^d_* = H^d \setminus \tilde{\gamma}_{x,y}(\mathbf{R})$, and let S^{d-2} denote the unit sphere in $\mathbf{R}^{d-1}$. In particular, $S^0 = \{+1, -1\}$. We introduce Fermi coordinates $u : H^d_* \to \mathbf{R}$, $\rho : H^d_* \to (0,\infty)$ and $N : H^d_* \to S^{d-2}$ by requiring that

$$
z \;=\; \exp\left(\rho(z) \sum_{i=1}^{d-1} \tilde{E}_i(u(z)) N^i(z)\right) \qquad \text{for all } z \in H^d,
\tag{3.9}
$$

where $N^i(z)$ is the i–th component of $N(z)$ in $\mathbf{R}^{d-1} \supset S^{d-2}$. Since the curvature is negative, (u, ρ, N) is a global diffeomorphism between H^d_* and $\mathbf{R} \times (0,\infty) \times S^{d-2}$, cf. e.g. [C], Sect. 3.6. We extend the functions u and ρ continuously to H^d.

For $z \in H^d$ let $\tilde{z} = \tilde{\gamma}_{x,y}(u(z))$ be the orthogonal projection of z onto $\tilde{\gamma}_{x,y}(\mathbf{R})$. Then both $(x, \tilde{z}, z)$ and $(y, \tilde{z}, z)$ are triangles with right angle at $\tilde{z}$. Since $d(z, \tilde{z}) = \rho(z)$, $d(x, \tilde{z}) = |u(\tilde{z}) - u(x)| = |u(z)|$, and $d(\tilde{z}, y) = |u(y) - u(\tilde{z})| = |\ell - u(z)|$, Lemma 3.3 implies

$$\begin{aligned} d(x,z) &\geq |u(z)| + \log\cosh \rho(z), \qquad \text{and} && (3.10) \\ d(z,y) &\geq |\ell - u(z)| + \log\cosh \rho(z). && (3.11) \end{aligned}$$

Thus

$$\begin{aligned} d(x,z) + d(z,y) &\geq |u(z)| + |\ell - u(z)| + 2\log\cosh \rho(z) \\ &\geq \ell + 2\log\cosh \rho(z) \end{aligned} \tag{3.12}$$

for all $z \in H^d$. Moreover,

$$\max\left(d(x,z), d(z,y)\right) - \ell/2 \geq |u(z) - \ell/2| + \log\cosh \rho(z)\,. \tag{3.13}$$

In fact, (3.13) is an immediate consequence of (3.10) if $u(z) \geq \ell/2$, and of (3.11) if $u(z) \leq \ell/2$. By (3.7), (3.8), (3.12) and (3.13),

(3.14)

$$\tilde{\rho}_t^{x,y}(z) \leq K_0 \cdot (\tfrac{2}{\pi t})^{d/2} e^{-\frac{1}{t}(|u(z)-\ell/2|+\log\cosh \rho(z))^2} e^{-2\frac{\ell}{t}\cdot\log\cosh \rho(z)} \cdot (1+\ell)^\nu$$

for $z \in H^d$ and $t \in (0,1]$. Integrating over z, and using that the volume element can be expressed in Fermi coordinates as $\sinh^{d-2}\rho \cosh\rho \, d\rho \, du \, dV_{S^{d-2}}(N)$, we obtain

(3.15) $\tilde{\mu}_t^{x,y}[\{z \in H^d;\ \rho(z) \geq \alpha \text{ and } u(z) \geq \beta + \ell/2\}]$

$$\begin{aligned} &\leq K_0 \cdot (\tfrac{2}{\pi t})^{d/2} \cdot V(S^{d-2}) \cdot \int_{\beta+\ell/2}^{\infty} e^{-(u-\ell/2)^2/t} du \\ &\qquad \times \int_{\alpha}^{\infty} e^{-(\log\cosh \rho)^2/t} e^{-\frac{2\ell}{t}\log\cosh\rho} \cdot (1+\ell)^\nu \sinh^{d-2}\rho \cosh\rho \, d\rho \\ &\leq K_1 \cdot t^{(1-d)/2} e^{-\beta^2/t} \int_{\alpha}^{\infty} (1+\ell)^\nu e^{(d-1)\rho} e^{-(2\ell\log\cosh\rho + (\log\cosh\rho)^2)/t} \rho^{d-2} d\rho \end{aligned}$$

for all α, $\beta \in [0,\infty)$ and $t \in (0,1]$, where K_1 is a finite constant that does not depend on t, x, y, α and β. Here we have used that

$$\int_{\beta+\ell/2}^{\infty} e^{-(u-\ell/2)^2/t} du \leq e^{-\beta^2/t} \int_0^{\infty} e^{-v^2/t} dv = \sqrt{\pi t}\, e^{-\beta^2/t}/2,$$

because $(u-\ell/2)^2 \geq \beta^2 + (u-\beta-\ell/2)^2$ if $u \geq \beta + \ell/2$.

To estimate the integral on the right hand side of (3.15) note that

$$\log\cosh r \;=\; \log(e^r \cdot (1+e^{-2r})/2) \;=\; r - \log 2 + \log(1-e^{-2r}) \sim r - \log 2$$

as $r \to \infty$, whereas $\log\cosh r \sim r^2/2$ as $r \to 0$. Since the function log cosh is continuous and strictly positive on $(0,\infty)$, there exists $\lambda > 0$ such that $\log\cosh r \geq \lambda r^2$ if $0 \leq r \leq 1$ and $\log\cosh r \geq \lambda r$ if $r \geq 1$. Let $I(\rho,t,\ell)$ denote the integrand in the integral on the right hand side of (3.15). For $\alpha_0 \geq 1$ we obtain

$$\begin{aligned} \int_{\alpha_0}^{\infty} I(\rho,t,\ell)\,d\rho \;&\leq\; (1+\ell)^{\nu} \cdot e^{-2\lambda\alpha_0\ell/t} \cdot \int_{\alpha_0}^{\infty} e^{(d-1)\rho} e^{-\lambda^2\rho^2/t}\,\rho^{d-2}\,d\rho \\ &\leq\; (1+\ell)^{\nu} \cdot e^{-2\lambda\alpha_0\ell/t} \cdot e^{(d-1)\alpha_0} \cdot e^{-\lambda^2\alpha_0^2/t}(1+\alpha_0)^{d-2} \\ &\qquad \times \int_0^{\infty} e^{(d-1)v} e^{-\lambda^2 v^2}\,(1+v)^{d-2}\,dv \\ &\leq\; K_2 \cdot e^{-\lambda\alpha_0\ell/t} \cdot e^{-\lambda^2\alpha_0^2/(2t)} \qquad \text{for all } t \in (0,1] \end{aligned} \tag{3.16}$$

with some finite constant K_2 that does not depend on t, ℓ and α_0.

Moreover, if $\ell \geq 1$ then for $\alpha_1 \in [0,1]$ we have

$$\begin{aligned} \int_{\alpha_1}^{1} I(\rho,t,\ell)\,d\rho \;&\leq\; (1+\ell)^{\nu} e^{d-1} \int_{\alpha_1}^{\infty} e^{-2\ell\lambda\rho^2/t}\,\rho^{d-2}\,d\rho \\ &=\; (1+\ell)^{\nu} e^{d-1} (2\ell\lambda/t)^{-\nu} \int_{\alpha_1\cdot\sqrt{2\ell\lambda/t}}^{\infty} e^{-r^2}\,r^{d-2}\,dr \\ &\leq\; e^{d-1}\lambda^{-\nu}\,t^{\nu}\,e^{-2\ell\lambda\alpha_1^2/t} \cdot (1+\alpha_1\cdot\sqrt{2\ell\lambda/t})^{d-2} \int_0^{\infty} e^{-v^2}\,v^{d-2}\,dv \\ &\leq\; K_3 \cdot t^{\nu} \cdot e^{-(1+\ell)\lambda\alpha_1^2/(2t)} \qquad \forall\, t \in (0,1], \end{aligned} \tag{3.17}$$

and if $\ell \leq 1$ then for $\alpha_1 \in [0,1]$,

$$\begin{aligned} \int_{\alpha_1}^{1} I(\rho,t,\ell)\,d\rho \;&\leq\; 2^{\nu} e^{d-1} \cdot \int_{\alpha_1}^{\infty} e^{-\lambda^2\rho^4/t}\,\rho^{d-2}\,d\rho \\ &\leq\; 2^{\nu} e^{d-1} \lambda^{-\nu} t^{\nu/2} \cdot \int_{\lambda^{1/2}\alpha_1 t^{-1/4}}^{\infty} e^{-r^4}\,r^{d-2}\,dr \\ &\leq\; (2/\lambda)^{\nu} e^{d-1}\,t^{\nu/2}\,e^{-\lambda^2\alpha_1^4/t}\,(1+\lambda^{1/2}\alpha_1 t^{-1/4})^{d-2} \cdot \int_0^{\infty} e^{-r^4}\,r^{d-2}\,dr \\ &\leq\; K_4 \cdot t^{\nu/2} \cdot e^{(1+\ell)\lambda^2\alpha_1^4/(4t)} \end{aligned} \tag{3.18}$$

where K_3 and K_4 are finite constants that do not depend on t, α_1 and ℓ.

By (3.15), (3.16), (3.17) and (3.18), we see that there exist constants $K_5,\ \delta_0 \in (0,\infty)$ independent of x and y such that for all $t \in (0,1]$,

$$\begin{aligned}
\tilde{\mu}_t^{x,y}[\,\rho \geq \alpha \text{ and } u \geq \ell/2+\beta\,] &\leq K_5\, t^{-\nu/2} e^{-\beta^2/t} e^{-\delta_0 \alpha^4 (1+\ell)/t} && \forall \alpha \in [0,1],\\
\tilde{\mu}_t^{x,y}[\,\rho \geq \alpha \text{ and } u \geq \ell/2+\beta\,] &\leq K_5\, e^{-\beta^2/t} e^{-\delta_0 \cdot (\alpha^2+\ell\alpha)/t} && \forall \alpha \in [1,\infty).
\end{aligned}$$

By the invariance of $\tilde{\mu}_t^{x,y}$ under the map $z \mapsto \exp(\rho(z) \sum \tilde{E}_i(\ell - u(z)) N^i(z))$ (reflection at the middle plane between x and y), the same estimates hold for $\tilde{\mu}_t^{x,y}[\rho \geq \alpha \text{ and } u \leq \ell/2 - \beta]$. Now notice that for $z \in H^d$ we have $r_{x,y}(z) = \operatorname{dist}(z, \tilde{\gamma}_{x,y}([0,\ell])) = d(z,\tilde{z}) = \rho(z)$ if $0 \leq u(z) \leq \ell$, $r_{x,y}(z) = d(z,y) \leq \rho(z) + u(z) - \ell$ if $u(z) \geq \ell$, and $r_{x,y}(z) = d(z,x) \leq \rho(z) - u(z)$ if $u(z) \leq 0$. In particular, $\rho(z) \geq r_{x,y}(z)/2$ or $|u(z) - \ell/2| \geq \ell/2 + r_{x,y}(z)/2$ for each $z \in H^d$. Hence

$$\begin{aligned}
\tilde{\mu}_t^{x,y}[r_{x,y} \geq \alpha] &\leq \tilde{\mu}_t^{x,y}[\rho \geq \alpha/2] + \tilde{\mu}_t^{x,y}[|u-\ell/2| \geq (\ell+\alpha)/2]\\
&= 2\,\tilde{\mu}_t^{x,y}[\,\rho \geq \alpha/2 \text{ and } u \geq \ell/2\,] + 2\,\tilde{\mu}_t^{x,y}[\,u \geq \ell + \alpha/2\,]\\
&\leq 2\,K_5\, t^{-\nu/2} \cdot (\, e^{-\delta_0 \alpha^4 \cdot (1+\ell)/(16t)} + e^{-(2\ell\alpha+\alpha^2)/(4t)}\,)
\end{aligned} \tag{3.19}$$

for all $t \in (0,1]$ and $\alpha \in [0,1]$. Since K_5 and δ_0 do not depend on x, y and t, (3.19) implies Estimate (3.5). Furthermore,

$$d(z,\gamma_{x,y}(1/2)) \leq d(z,\tilde{z}) + d(\tilde{z},\gamma_{x,y}(1/2)) = \rho(z) + |u(z) - \ell/2| \quad \forall z \in H^d,$$

whence

$$\begin{aligned}
&\tilde{\mu}_t^{x,y}[\{z \in H^d;\ d(z,\gamma_{x,y}(1/2)) \geq \alpha\}]\\
&\quad\leq \tilde{\mu}_t^{x,y}[\rho \geq \alpha/2] + \tilde{\mu}_t^{x,y}[|u-\ell/2| \geq \alpha/2]\\
&\quad= 2\,\tilde{\mu}_t^{x,y}[\,\rho \geq \alpha/2 \text{ and } u \geq \ell/2\,] + 2\,\tilde{\mu}_t^{x,y}[\,\rho \geq 0 \text{ and } u \geq \ell/2 + \alpha/2\,]\\
&\quad\leq 2\,K_5\, t^{-\nu/2} \cdot (\, e^{-\delta_0 \alpha^4 \cdot (1+\ell)/(16t)} + e^{-\alpha^2/(4t)}\,) \quad \forall\, t \in (0,1],\ \alpha \in [0,2],\\
\text{resp.}&\quad\leq 2\,K_5 \cdot (\, e^{-\delta_0 \cdot (\alpha^2+2\ell\alpha)/(4t)} + e^{-\alpha^2/(4t)}\,) \quad \forall\, t \in (0,1],\ \alpha \in [2,\infty).
\end{aligned}$$

This proves Estimate (3.6). □

We finally show that Proposition 3.1 follows from Lemma 3.4 :

Proof of Proposition 3.1. We first show that (3.6) implies (3.1). Afterwards, we use (3.1) and (3.5) to prove (3.2).

(i) Proof of (3.1) : Fix $x, y \in H_\kappa^d$, and let $\mathcal{P}_{x,y}(H_\kappa^d)$ denote the space of all continuous paths $\omega : [0,1] \to H_\kappa^d$ with $\omega(0) = x$ and $\omega(1) = y$. For $k \in \mathbf{N} \cup \{0\}$ and $\omega \in \mathcal{P}_{x,y}(H_\kappa^d)$ let

$$\begin{aligned} M_k(\omega) &= \max_{0 \le i \le 2^k} d(\omega(i \cdot 2^{-k}), \gamma_{x,y}(i \cdot 2^{-k})), \qquad \text{and} \\ N_k(\omega) &= \max_{0 \le i < 2^k} d(\omega((i+1/2) \cdot 2^{-k}), \gamma_{\omega(i \cdot 2^{-k}), \omega((i+1) \cdot 2^{-k})}(1/2)\,). \end{aligned}$$

Recall that since H_κ^d is simply connected and has negative curvature, the function $t \mapsto d(\gamma_1(t), \gamma_2(t))^2$ is convex for any two geodesics $\gamma_1,\ \gamma_2 : [0,1] \to H_\kappa^d$ parametrized proportional to arc–length, cf. e.g. [J], Lem. 6.5.2. In particular,

$$d(\gamma_1(t), \gamma_2(t)) \ \le\ \max\left(d(\gamma_1(0), \gamma_2(0)),\ d(\gamma_1(1), \gamma_2(1))\,\right)$$

for all $t \in [0,1]$. Hence

$$\begin{aligned} & d(\gamma_{\omega(i \cdot 2^{-k}), \omega((i+1) \cdot 2^{-k})}(1/2),\ \gamma_{x,y}((i+1/2) \cdot 2^{-k})\,) \\ & \quad \le\ \max\left(\, d(\omega(i \cdot 2^{-k}), \gamma_{x,y}(i \cdot 2^{-k}))\,,\ d(\omega((i+1) \cdot 2^{-k}), \gamma_{x,y}((i+1) \cdot 2^{-k}))\,\right) \end{aligned}$$

for all $k \ge 0$ and $0 \le i < 2^k$, and therefore

$$M_{k+1}(\omega) \ \le\ M_k(\omega) \ +\ N_k(\omega) \qquad \text{for all } \omega \in \mathcal{P}_{x,y}(H_\kappa^d) \text{ and } k \ge 0.$$

Since $M_0(\omega) = 0$, we obtain $M_k(\omega) \le \sum_{j=0}^{k-1} N_j(\omega)$, and thus

$$\sup_{t \in [0,1]} d(\omega(t), \gamma_{x,y}(t)) \ =\ \sup_{k \in \mathbf{N}} M_k(\omega) \ \le\ \sum_{j=0}^{\infty} N_j(\omega) \qquad \text{for all } \omega \in \mathcal{P}_{x,y}(H_\kappa^d).$$

Let $p = (d+3)/4$. By (3.6),

$$\begin{aligned} Q_{x,y}[N_j \ge \alpha] \ &\le\ \sum_{i=0}^{2^j-1} Q_{x,y}[d(\omega((i+1/2) \cdot 2^{-j}),\ \gamma_{\omega(i \cdot 2^{-j}), \omega((i+1) \cdot 2^{-j})}(1/2)\,) \ge \alpha\,] \\ &\le\ 2^j \cdot \sup_{x', y' \in H_\kappa^d} \mu_{2^{-j}}^{x',y'}[\, d(\,\cdot\,, \gamma_{x',y'}(1/2)) \ge \alpha\,] \ \le\ A_2 \cdot 2^{pj} \cdot e^{-\tilde{\delta} \cdot 2^j \cdot \alpha^2 \cdot (\alpha^2 \wedge 1)} \end{aligned}$$

for all $\alpha > 0$. Hence

$$\begin{aligned}
& Q_{x,y}[\{\omega \in \mathcal{P}_{x,y}(H^d_\kappa);\ \sup_{t\in[0,1]} d(\omega(t),\gamma_{x,y}(t)) \geq (1+2^{-1/8})^{-1}v\}] \\
&\leq \sum_{j=0}^{\infty} Q_{x,y}[N_j \geq 2^{-j/8}v] \ \leq\ A_2 \cdot \sum_{j=0}^{\infty} 2^{pj} e^{-\tilde{\delta}\cdot 2^j\cdot 2^{-j/4}v^2\cdot((2^{-j/4}v^2)\wedge 1)} \\
&\leq A_2 \cdot \sum_{j=0}^{\infty} 2^{pj} e^{-\tilde{\delta}\cdot 2^{j/2}v^2} \ \leq\ A_2 \cdot e^{-\tilde{\delta}v^2} \sum_{j=0}^{\infty} 2^{pj} e^{-\tilde{\delta}\cdot(2^{j/2}-1)}
\end{aligned}$$

for all $v \geq 1$. This shows that (3.1) holds for all $x, y \in H^d_\kappa$ and $u \geq (1-2^{-1/8})^{-1}$ with constants K_1, $c_1 \in (0,\infty)$ that do not depend on x, y and u. Clearly, by choosing K_1 large enough, we can ensure that (3.1) holds for $0 \leq u \leq (1-2^{-1/8})^{-1}$ as well.

(ii) Proof of (3.2) : This is slightly more involved. Fix $\alpha \in (0,1]$ and $x, y \in H^d_\kappa$ with $d(x,y) \geq 1$, and let $\ell = d(x,y)$. Let

$$\begin{aligned}
\Gamma \ = \ \{\, \omega \in \mathcal{P}_{x,y}(H^d_\kappa);\ & d(\omega(s),\omega(t)) \geq |t-s|\cdot \ell/2 \\
& \text{for all } s,t \in [0,1] \text{ with } |t-s| \geq 1/\sqrt{\ell}\,\}. \qquad (3.20)
\end{aligned}$$

By (3.1), and since $d(\gamma_{x,y}(s),\gamma_{x,y}(t)) = |t-s|\cdot\ell$ for all $s,t\in[0,1]$,

$$\begin{aligned}
Q_{x,y}[\mathcal{P}_{x,y}(H^d_\kappa)\setminus\Gamma] \ &\leq\ Q_{x,y}[\sup_{t\in[0,1]} d(\omega(t),\gamma_{x,y}(t)) \geq \sqrt{\ell}/4] \\
&\leq\ K_1 \cdot e^{-c_1\ell/16}\ . \qquad (3.21)
\end{aligned}$$

Hence to prove (3.2), it suffices to estimate $Q_{x,y}[\{\omega\in\Gamma;\ \sup_{t\in[0,1]} r_{x,y}(\omega(t)) \geq \alpha\}]$. Now let

$$\begin{aligned}
\bar{M}_k(\omega) \ &= \ \max_{0\leq i\leq 2^k} \operatorname{dist}(\omega(i\cdot 2^{-k}),\gamma_{x,y}([0,1])), \qquad \text{and} \\
\bar{N}_k(\omega) \ &= \ \max_{0\leq i< 2^k} \operatorname{dist}(\omega((i+1/2)\cdot 2^{-k}),\gamma_{\omega(i\cdot 2^{-k}),\omega((i+1)\cdot 2^{-k})}([0,1])\,).
\end{aligned}$$

Similarly as above, we have

$$\bar{M}_{k+1}(\omega) \ \leq\ \bar{M}_k(\omega) + \bar{N}_k(\omega) \qquad \text{for all } \omega\in\Gamma \text{ and } k\in\mathbf{N}\cup\{0\},$$

because

$$\operatorname{dist}(\gamma_1(t),\ \gamma_2([0,1])) \ \leq\ \max(\operatorname{dist}(\gamma_1(0),\gamma_2([0,1])),\ \operatorname{dist}(\gamma_1(1),\gamma_2([0,1]))\,)$$

holds for any two geodesics γ_1, $\gamma_2 : [0,1] \to H^d_\kappa$, and all $t \in [0,1]$. The latter fact can be seen for example from the expression of the metric on H^d_κ in Fermi coordinates based on a geodesic extending γ_2. Hence $\bar{M}_k(\omega) \le \sum_{j=0}^{k-1} \bar{N}_j(\omega)$, and

$$(3.22) \qquad \sup_{t\in[0,1]} r_{x,y}(\omega(t)) \;=\; \sup_{k\in\mathbf{N}} \bar{M}_k(\omega) \;\le\; \sum_{j=0}^{\infty} \bar{N}_j(\omega)\ .$$

For $\alpha \ge 0$ and $j \in \mathbf{N} \cup \{0\}$ with $2^{-j} \ge 1/\sqrt{\ell}$, we have by (3.5) and (3.20) :

$$\begin{aligned}(3.23) \quad & Q_{x,y}[\{\omega \in \Gamma;\ \bar{N}_j(\omega) \ge \alpha\}] \\ \le\;& 2^j \cdot \sup\left\{\mu^{x',y'}_{2^{-j}}[r_{x',y'} \ge \alpha]\,;\ x', y' \in H^d_\kappa \text{ with } d(x',y') \ge 2^{-j-1}\cdot \ell\right\} \\ \le\;& A_1 \cdot 2^{pj} \cdot e^{-\tilde{\delta}2^j\cdot(\alpha\wedge 1)^4\cdot(1+2^{-j-1}\ell)} \;=\; A_1 \cdot 2^{pj}\cdot e^{-\tilde{\delta}\cdot(2^j+\ell/2)\cdot(\alpha\wedge 1)^4},\end{aligned}$$

whereas for $\alpha \ge 0$ and $j \in \mathbf{N} \cup \{0\}$ with $2^{-j} < 1/\sqrt{\ell}$, we still have

$$\begin{aligned}(3.24) \quad & Q_{x,y}[\{\omega \in \Gamma;\ \bar{N}_j(\omega) \ge \alpha\}] \\ \le\;& 2^j \cdot \sup\left\{\mu^{x',y'}_{2^{-j}}[r_{x',y'} \ge \alpha]\,;\ x', y' \in H^d_\kappa\right\} \;\le\; A_1 \cdot 2^{pj} \cdot e^{-\tilde{\delta}\cdot 2^j\cdot(\alpha\wedge 1)^4}.\end{aligned}$$

Let $\bar{a} = a\cdot(1-2^{-1/8})$, i.e., $a = \bar{a}\cdot\sum_{j=0}^{\infty} 2^{-j/8}$. Since $\ell \ge 1$, we can find $k_0 \in \mathbf{N}\cup\{0\}$ such that $2^{-k_0} \ge 1/\sqrt{\ell} > 2^{-(k_0+1)}$. Notice that $\bar{a} < a \le 1$. By (3.22), (3.23) and (3.24),

$$\begin{aligned}(3.25) \quad & Q_{x,y}[\{\omega\in\Gamma\,;\ \sup_{t\in[0,1]} r_{x,y}(\omega(t)) \ge a\}] \\ \le\;& \sum_{j=0}^{\infty} Q_{x,y}[\{\omega\in\Gamma;\ \bar{N}_j(\omega) \ge 2^{-j/8}\bar{a}\}] \\ \le\;& A_1\cdot\sum_{j=0}^{k_0} 2^{pj} e^{-\tilde{\delta}\cdot(2^{j/2}+2^{-j/2}\cdot\ell/2)\bar{a}^4} \;+\; A_1\cdot\sum_{j=k_0+1}^{\infty} 2^{pj} e^{-\tilde{\delta}2^{j/2}\bar{a}^4} \\ \le\;& A_1\cdot e^{-\tilde{\delta}\bar{a}^4\ell^{3/4}}\cdot\sum_{j=0}^{\infty} 2^{pj}e^{-\tilde{\delta}2^{j/2}\bar{a}^4} \;+\; A_1\cdot\ell^{p/2}e^{-\tilde{\delta}\bar{a}^4\ell^{1/4}}\cdot\sum_{i=0}^{\infty} 2^{p\cdot(i+1)}e^{-\tilde{\delta}(2^{i/2}-1)\bar{a}^4}\end{aligned}$$

Here we have used in the last step that $2^{-j/2} \ge 2^{-k_0/2} \ge \ell^{-1/4}$ for all $j \le k_0$, $2^{pk_0} \le \ell^{p/2}$, $2^{(k_0+1+i)/2} - 2^{(k_0+1)/2} = 2^{(k_0+1)/2}\cdot(2^{i/2}-1) \ge 2^{i/2-1}$ for all $i \ge 0$, and $2^{(k_0+1)/2} > \ell^{1/4}$. Since $a > 0$, the sums on the right hand side of (3.25) are finite. By (3.21) and (3.25) we see that for every $a \in (0,1]$ and $\beta < 1/4$ there exists a constant $K_2 \in (0,\infty)$ such that (3.2) holds for all $x, y \in H^d_\kappa$ with $d(x,y) \geq 1$. If K_2 is chosen sufficiently large, then (3.2) holds for $x, y \in H^d_\kappa$ with $d(x,y) < 1$ as well. Finally, for $a > 1$, the estimate follows from the corresponding estimate for $a = 1$. □

4 Concentration of pinned Wiener and Bismut measures near energy minimizing loops

Let M be a compact connected Riemannian manifold satisfying (A 1). We fix $\varepsilon > 0$ as in Section 1. In this section we will apply the results from Section 3 to obtain concentration results for the pinned Wiener and Bismut measures on loop spaces over M, cf. Proposition 4.3.

Let $\sigma : \mathbf{R} \to M$ be the geodesic parametrized by arc length defined by $\sigma(s+kL(\gamma)) = \gamma(s)$ for all $k \in \mathbf{Z}$ and $s \in [0,1]$, where $L(\gamma)$ is the length of the closed geodesic γ. Let $\kappa < 0$ be the constant such that the sectional curvature is identically κ on U_ε. We fix a unit speed geodesic $\tilde{\sigma} : \mathbf{R} \to H^d_\kappa$, $d = \dim(M)$. Let $\{E_0(s), E_1(s), \ldots, E_{d-1}(s)\}$ and $\{\tilde{E}_0(s), \tilde{E}_1(s), \ldots, \tilde{E}_{d-1}(s)\}$, $s \in \mathbf{R}$, be parallel orthonormal frames along σ, $\tilde{\sigma}$ respectively such that $E_0(s) = \sigma'(s)$ and $\tilde{E}_0(s) = \tilde{\sigma}'(s)$ for all $s \in \mathbf{R}$. Let $u : H^d_\kappa \to \mathbf{R}$ and $v : H^d_\kappa \to \mathbf{R}^{d-1}$ be the coordinates given by $x = \exp(\sum_{i=1}^d v^i(x)\tilde{E}_i(u(x)))$ for all $x \in H^d_\kappa$. (u,v) is a global diffeomorphism between H^d_κ and $\mathbf{R}^d$, and $\operatorname{dist}(x, \tilde{\sigma}(\mathbf{R})) = |v(x)|$ for all $x \in H^d_\kappa$. Let $\tilde{U}_\varepsilon = \{x \in H^d_\kappa;\ \operatorname{dist}(x, \tilde{\sigma}(\mathbf{R})) < \varepsilon\}$. Since the curvature on U_ε is identically κ, the map

$$\pi : \tilde{U}_\varepsilon \to U_\varepsilon\,, \qquad \pi(x) \;=\; \exp\,(\sum_{i=1}^{d-1} v^i(x)E_i(u(x))),$$

is a *Riemannian covering map*. By the choice of ε, the exponential map is a diffeomorphism from the set of all vectors of length $< \varepsilon$ in the normal bundle along $\gamma(S^1)$ to U_ε. This implies that the map π is a bijection between $\tilde{U}_{\varepsilon,n} = \{x \in \tilde{U}_\varepsilon;\ n \cdot L(\gamma) \leq u(x) < (n+1)\cdot L(\gamma)\}$ and U_ε for every $n \in \mathbf{Z}$. Let l_n be the inverse of $\pi|_{\tilde{U}_{\varepsilon,n}}$. A loop $\omega \in LU_\varepsilon$ lifts to a unique continuous path $\hat{\omega} : [0,1] \to \tilde{U}_\varepsilon$ such that $\pi \circ \hat{\omega} = \omega$, $\hat{\omega}(0) = l_0(\omega(0))$, and $\hat{\omega}(1) = l_n(\omega(0))$ for some $n \in \mathbf{Z}$. For $x \in U_\varepsilon$, the sets

$$\Omega_{n,x} \;=\; \{\,\omega \in L_xU_\varepsilon\,;\,\hat{\omega}(1) = l_n(x)\,\}, \tag{4.1}$$

$n \in \mathbf{Z}$, are the homotopy classes in $L_x U_\varepsilon$. Let

$$\hat{\Omega}_{n,x} \;=\; \{\, \omega \in C([0,1], \tilde{U}_\varepsilon)\,;\, \omega(0) = l_0(x),\, \omega(1) = l_n(x)\,\}. \tag{4.2}$$

For every $n \in \mathbf{Z}$, the lifting map $\omega \mapsto \hat{\omega}$ is a homeomorphism between $\Omega_{n,x}$ and $\hat{\Omega}_{n,x}$. We now consider the image of pinned Wiener measure under this map. For $x, y \in M$ let $P_{x,y}$ denote the distribution of the Brownian bridge from x to y in time 1 on $C([0,1], M)$. Recall also that $Q_{x,y}$ denotes the distribution on $C([0,1], H^d_\kappa)$ of the hyperbolic Brownian bridge from x to y, and $q_t(x,y)$ is the heat kernel on H^d_κ.

Lemma 4.1 *Let $n \in \mathbf{Z}$ and $x \in U_\varepsilon$. Then for every Borel subset $B \subseteq \hat{\Omega}_{n,x}$,*

$$P_x\,[\,\{\omega \in L_x U_\varepsilon;\, \hat{\omega} \in B\}\,] \cdot p_1(x,x) \;=\; Q_{l_0(x), l_n(x)}[B] \cdot q_1(l_0(x), l_n(x)).$$

Proof. Let $(W_t)_{t \geq 0}$ be a Brownian motion on H^d_κ starting at $l_0(x)$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. For $\omega \in \tilde{\Omega}$ let $\tau(\omega) = \inf\{t \geq 0;\, W_t(\omega) \in \partial\hat{U}_\varepsilon\}$ where $\inf \emptyset = \infty$. The projection $\pi \circ W_t$ is defined for $0 \leq t \leq \tau$ if τ is finite and for $0 \leq t < \infty$ else, and $\tau = \inf\{t \geq 0;\, \pi \circ W_t \in \partial U_\varepsilon\}$ $\tilde{P}$–a.s. Since $\pi : \tilde{U}_\varepsilon \to U_\varepsilon$ is a local isometry, the U_ε–valued process $\pi \circ W_t$, $0 \leq t < \tau$, is (w.r.t. $\tilde{P}$) a Brownian motion on M starting at x, and stopped at the first hitting time of the boundary ∂U_ε. In other words, the process W_t, $0 \leq t < \tau$, is the lift of the Brownian motion $\pi \circ W_t$, $0 \leq t < \tau$, to the covering space $\tilde{U}_\varepsilon$.

Let P^{BM}_x, $Q^{\mathrm{BM}}_{l_0(x)}$ denote the distribution of Brownian motion starting at x, $l_0(x)$ on $C([0,1], M)$, $C([0,1], H^d_\kappa)$ respectively. By the considerations above, the restriction of $Q^{\mathrm{BM}}_{l_0(x)}$ to $C([0,1], \tilde{U}_\varepsilon)$ is the image of the restriction of P^{BM}_x to $C([0,1], U_\varepsilon)$ under the lifting map.

Now let $\mathrm{inj}\,(M)$ denote the infimum of the injectivity radii of all points in M. Since M is compact, $\mathrm{inj}\,(M) > 0$. We fix $r > 0$ such that $r < \mathrm{inj}\,(M)$ and $B_r(x) \subset U_\varepsilon$. Then $\pi^{-1}(B_r(x))$ is the disjoint union of the balls $B_r(l_n(x))$, $n \in \mathbf{Z}$, in $\tilde{U}_\varepsilon$. Let Ω'_x denote the set of all continuous paths $\omega : [0,1] \to U_\varepsilon$ such that $\omega(0) = x$ and $\omega(1) \in B_r(x)$. Let $\Omega'_{n,x}$ denote the connected component of Ω'_x that contains $\Omega_{n,x}$, and let $\hat{\Omega}'_{n,x} = \{\hat{\omega};\, \omega \in \Omega'_{n,x}\}$. Then $\hat{\Omega}'_{n,x}$ is the set of all continuous paths $p : [0,1] \to \tilde{U}_\varepsilon$ such that $p(0) = l(x_0)$ and $p(1) \in B_r(l(x_n))$. Now let $\tilde{F}$ be a bounded continuous function on $C([0,1], H^d_\kappa)$ that vanishes outside $\hat{\Omega}'_{n,x}$, and let F be the function on $C([0,1], M)$ defined by $F(\omega) = \tilde{F}(\hat{\omega})$ for $\omega \in \Omega'_{n,x}$, $F(\omega) = 0$ else. Since the lifting map is a homeomorphism, and $\partial\Omega'_{n,x} = \pi(\partial\hat{\Omega}'_{n,x})$, the function F is continuous as well. Moreover, for every bounded function g on M that vanishes outside $B_r(x)$,

$$\begin{aligned}
&\int_{B_r(x)} \Big(\int F\,dP_{x,y} \Big)\, g(y)\, p_1(x,y)\, V(dy) \\
&= \int F(\omega)\, g(\omega(1))\, P_x^{\mathrm{BM}}(d\omega) \\
&= \int \tilde{F}(p)\, g(\pi(p(1)))\, Q^{\mathrm{BM}}_{l_0(x)}(dp) \\
&= \int_{B_r(l_n(x))} \Big(\int \tilde{F}\, dQ_{l_0(x),z} \Big)\, g(\pi(z))\, q_1(l_0(x),z)\, V(dz) \\
&= \int_{B_r(x)} \Big(\int \tilde{F}\, dQ_{l_0(x),l_n(y)} \Big)\, g(y)\, q_1(l_0(x),l_n(y))\, V(dy).
\end{aligned}$$

In the last step we have used that l_n is an isometry from $B_r(x)$ to $B_r(l_n(x))$. We obtain

$$p_1(x,y)\cdot\int F\,dP_{x,y} \;=\; q_1(l_0(x),l_n(y))\cdot\int \tilde{F}\,dQ_{l_0(x),l_n(y)} \tag{4.3}$$

for a.e. $y \in B_r(x)$. On the other hand, the maps $(x,y) \mapsto P_{x,y}$ and $(x,y) \mapsto Q_{x,y}$ are weakly continuous on $M \times M$, $H^d_\kappa \times H^d_\kappa$ respectively, cf. e.g. [ES], 1.5. Thus both sides of (1.3) are continuous in y, whence (1.3) holds actually for every $y \in B_r(x)$. In particular,

$$p_1(x,x)\cdot\int F\,dP_x \;=\; q_1(l_0(x),l_n(x))\cdot\int \tilde{F}\,dQ_{l_0(x),l_n(x)}.$$

Since the equality holds in particular for every bounded continuous function $\tilde{F}$ on $C([0,1],H^d_\kappa)$ that vanishes outside $\hat{\Omega}_{n,x}$ with F defined as above, it implies the assertion of the lemma. □

Remarks. (i) By Lemma 4.1,

$$P_x[\{\omega \in L_xU_\varepsilon;\ \hat{\omega} \in B\} \mid \Omega_{n,x}] \;=\; Q_{l_0(x),l_n(x)}[B \mid \hat{\Omega}_{n,x}] \tag{4.4}$$

for every $x \in U_\varepsilon$, $n \in \mathbf{Z}$, and every Borel subset $B \subseteq \hat{\Omega}_{n,x}$.

(ii) The free homotopy classes Ω_n, $n \in \mathbf{N}$, introduced in Section 1.5 can be described by

$$\Omega_n \;=\; \{\omega \in LU_\varepsilon;\ \hat{\omega}(1) = l_n(\omega(0))\}. \tag{4.5}$$

In particular, Ω_n is the disjoint union of the sets $\Omega_{n,x}$, $x \in U_\varepsilon$. Let $\hat{\Omega}_n = \{\hat{\omega};\, \omega \in \Omega_n\}$, and let $Q^{(n)}$ denote the probability measure on $C([0,1], H^d_\kappa)$ defined by

(4.6)
$$Q^{(n)} \;=\; \int_{U_\varepsilon} Q_{l_0(x),l_n(x)} q_1(l_0(x), l_n(x))\, V(dx) \,/ \int q_1(l_0(x), l_n(x))\, V(dx).$$

Then

$$P[\,\{\omega \in LU_\varepsilon;\, \hat{\omega} \in B\} \,|\, \Omega_n\,] \;=\; Q^{(n)}[\,B \,|\, \hat{\Omega}_n\,] \tag{4.7}$$

for every Borel subset $B \subseteq \hat{\Omega}_n$. In fact, applying the lemma for every $x \in U_\varepsilon$ and integrating over x shows that $P[\{\omega \in LU_\varepsilon,\, \hat{\omega} \in \cdot\,\}]$ is proportional to $Q^{(n)}$ on the Borel σ–algebra of $\hat{\Omega}_n$, whence the normalized measures coincide.

For $x \in M$ let $r(x) = \mathrm{dist}\,(x, \gamma(S^1))$.

Lemma 4.2 *Let $r_1 \in (0, \varepsilon)$. There exist $r_2 \in (0, r_1)$ and $A,\ \lambda \in (0, \infty)$ such that*

$$q_1(l_0(x), l_n(x))/q_1(l_0(y), l_n(y)) \;\le\; A \cdot \exp(-\lambda n)$$

for all $x,\ y \in U_\varepsilon$ with $r(x) \ge r_1$ and $r(y) \le r_2$, and all $n \in \mathbf{N}$.

PROOF. For $a \ge 0$ let $g(a) = (2\pi)^{-d/2} e^{-a^2/2} e^{-(d-1)\sqrt{-\kappa}a/2}(1+a)^{(d-1)/2}$. By Lemma 3.2, there exists $K \in (1, \infty)$ such that

$$K^{-1} \cdot g(d(x,y)) \;\le\; q_1(x,y) \;\le\; K \cdot g(d(x,y)) \qquad \text{for all } x, y \in H^d_\kappa.$$

In particular, for $x, y \in U_\varepsilon$ and $n \in \mathbf{N}$

$$q_1(l_0(x), l_n(x))/q_1(l_0(y), l_n(y)) \;\le\; K^2 \cdot g(d_n(x))/g(d_n(y))$$

$$= \; K^2 \cdot e^{-(d_n(x)+d_n(y)+(d-1)\sqrt{-\kappa})(d_n(x)-d_n(y))/2} \left(\frac{1+d_n(x)}{1+d_n(y)}\right)^{(d-1)/2} \tag{4.8}$$

where

$$d_n(z) \;=\; d(l_0(z), l_n(z)) \qquad \text{for } z \in U_\varepsilon.$$

Now fix $n \in \mathbf{N}$ and $x \in U_\varepsilon$ with $r(x) \ge r_1$. I claim that

$$d_n(x) \;\ge\; \min\,(nL(\gamma)\cosh(\sqrt{-\kappa}r_1/2),\; nL(\gamma) + 2\log\cosh(\sqrt{-\kappa}r_1/2)\,). \tag{4.9}$$

To verify this, we introduce the Fermi coordinates $u(z)$, $\rho(z) = |v(z)|$, and $N(z) = v(z)/|v(z)|$, $z \in H^d_\kappa$, where u and v are defined as in the beginning of this section. Calculating the metric in these coordinates yields

$$ds^2 \;\geq\; d\rho^2 + \cosh^2(\sqrt{-\kappa}\rho)\, du^2, \tag{4.10}$$

cf. e.g. [C], Sect. 3.6. Note that $\rho(l_n(x)) = \rho(l_0(x)) = r(x) \geq r_1$. Since

$$u(l_n(x)) - u(l_0(x)) \;=\; n \cdot L(\gamma), \tag{4.11}$$

(4.9) clearly holds if the minimal geodesic $\gamma_{l_0(x),l_n(x)}$ satisfies

$$\rho(\gamma_{l_0(x),l_n(x)}(s)) \geq r_1/2 \quad \text{for all } s \in [0,1].$$

Otherwise, let p be a point on $\gamma_{l_0(x),l_n(x)}([0,1])$ with $\rho(p) = r_1/2$, and let z_0 and z_n be the orthogonal projections of $l_0(x)$ and $l_n(x)$ onto the cylinder $\mathcal{C} = \{z \in H^d_\kappa;\, \rho(z) \leq r_1/2\}$. In Fermi coordinates, $\rho(z_i) = r_1/2$, $u(z_i) = u(l_i(x))$, and $N(z_i) = N(l_i(x))$, $i = 0, n$. In particular, the geodesics $\gamma_{l_0(x),z_0}$ and $\gamma_{l_n(x),z_n}$ have length $r_1/2$ and hit ∂C orthogonally at z_0, z_n respectively. Moreover, $\mathcal{C}$ is convex, whence the geodesics $\gamma_{z_0,p}$ and $\gamma_{z_n,p}$ do not leave $\mathcal{C}$. Therefore, the angles of the triangles $(l_0(x), z_0, p)$ and $(l_n(x), z_n, p)$ at z_0, z_n respectively are greater or equal to $\pi/2$. Thus by Lemma 3.3 and the remark below,

$$\begin{aligned} d_n(x) &= d(l_0(x), l_n(x)) \;=\; d(l_0(x), p) + d(p, l_n(x)) \\ &\geq d(z_0, p) + \log\cosh(r_1/2) + d(z_n, p) + \log\cosh(r_1/2) \\ &\geq d(z_0, z_n) + 2\log\cosh(r_1/2). \end{aligned}$$

Since by (4.10),

$$d(z_0, z_n) \;\geq\; u(z_n) - u(z_0) \;=\; u(l_n(x)) - u(l_0(x)) \;=\; n \cdot L(\gamma),$$

this completes the proof of (4.9).

By (4.9), there exists $r_2 \in (0, r_1)$ such that $d_n(x) \geq nL(\gamma) + 3r_2$ holds for all $n \in \mathbf{N}$ and $x \in U_\varepsilon$ with $r(x) \geq r_1$. Now fix such an r_2. For $n \in \mathbf{N}$ and $y \in U_\varepsilon$ with $r(y) \leq r_2$, we have $\rho(l_0(y)) = \rho(l_n(y)) \leq r_2$ and $|u(l_n(y)) - u(l_0(y))| = n \cdot L(\gamma)$. Thus the piecewise geodesic connecting the points $l_0(y)$, $\tilde{\sigma}(u(l_0(y)))$, $\tilde{\sigma}(u(l_n(y)))$, and $l_n(y)$ has length $\leq nL(\gamma) + 2r_2$, whence

$$d_n(y) \;=\; d(l_0(y), l_n(y)) \;\leq\; n \cdot L(\gamma) + 2r_2 \;\leq\; d_n(x) - r_2 \tag{4.12}$$

for all $x \in U_\varepsilon$ with $r(x) \geq r_1$. Since $d_n(z) \geq nL(\gamma)$ for all $n \in \mathbf{N}$ and $z \in U_\varepsilon$, the assertion of the lemma follows from (4.12) and (4.8). □

For $x \in U_\varepsilon$ and $n \in \mathbf{N}$ let $\gamma_x^{(n)}$ denote the unique minimal geodesic in $\Omega_{n,x}$ parametrized proportional to arc length. For $z \in H_\kappa^d$ and $\omega \in C([0,1], H_\kappa^d)$ let

$$r_x^{(n)}(z) \;=\; \mathrm{dist}\,(z, \hat{\gamma}_x^{(n)}([0,1])) \qquad \text{and} \qquad R_x^{(n)}(\omega) \;=\; \sup_{0 \leq s \leq 1} r_x^{(n)}(\omega(s)).$$

Moreover, for $\omega \in LM$, we set $R(\omega) = \sup_{s \in S^1} r(\omega(s))$. As a consequence of Proposition 3.1 and the lemmas above we obtain :

Proposition 4.3 *Let $\delta > 0$ and $\beta < 1/4$. Then*

$$\lim_{n\to\infty} e^{n^\beta} \cdot P_x\,[\{\omega \in \Omega_{n,x};\; R_x^{(n)}(\hat{\omega}) \geq \delta\} \,|\, \Omega_{n,x}\,] \;=\; 0 \quad \textit{for every } x \in U_\varepsilon, \textit{ and}$$

$$\lim_{n\to\infty} e^{n^\beta} \cdot P\,[\,R \geq \delta \,|\, \Omega_n\,] \;=\; 0.$$

Proof. Let $x \in U_\varepsilon$ and $n \in \mathbf{N}$. By (4.4),

$$\begin{aligned} P_x\,[\{\omega \in \Omega_{n,x};\; R_x^{(n)}(\hat{\omega}) \geq \delta\} \,|\, \Omega_{n,x}] \;&=\; Q_{l_0(x),l_n(x)}[\,R_x^{(n)} \geq \delta \,|\, \hat{\Omega}_{n,x}\,] \\ &\leq\; Q_{l_0(x),l_n(x)}[\,R_x^{(n)} \geq \delta\,] \,/\, Q_{l_0(x),l_n(x)}[\,\hat{\Omega}_{n,x}]. \end{aligned} \tag{4.13}$$

Since $\pi : \tilde{U}_\varepsilon \to U_\varepsilon$ is a Riemannian covering, the lift $\hat{\gamma}_x^{(n)}$ of the geodesic $\gamma_x^{(n)}$ is the geodesic $\gamma_{l_0(x),l_n(x)}$ in H_κ^d, and $r_x^{(n)} = r_{l_0(x),l_n(x)}$ (cf. Section 3 for the notation). By (4.9), $d(l_0(x), l_n(x)) \geq n \cdot L(\gamma)$ for all $x \in U_\varepsilon$ and $n \in \mathbf{N}$, whence by Proposition 3.1,

$$\lim_{n\to\infty} e^{n^\beta}\, Q_{l_0(x),l_n(x)}[\,R_x^{(n)} \geq \delta\,] = 0 \quad \text{uniformly for } x \in U_\varepsilon. \tag{4.14}$$

To estimate $Q_{l_0(x),l_n(x)}[\hat{\Omega}_n^x]$ from below, we note that $\mathrm{dist}\,(l_0(x), \tilde{\sigma}(\mathbf{R})) = \mathrm{dist}\,(l_n(x), \tilde{\sigma}(\mathbf{R})) = r(x)$. Since $\gamma_{l_0(x),l_n(x)}$ and $\tilde{\sigma}$ are geodesics, this implies

$$\mathrm{dist}\,(\gamma_{l_0(x),l_n(x)}(s),\, \tilde{\sigma}(\mathbf{R})\,) \;\leq\; r(x), \tag{4.15}$$

and thus $\mathrm{dist}\,(\gamma_{l_0(x),l_n(x)}(s), H_\kappa^d \setminus \tilde{U}_\varepsilon) \geq \varepsilon - r(x)$ for all $s \in [0,1]$. Therefore

$$r_{l_0(x),l_n(x)} \;\geq\; \varepsilon - r(x) \qquad \text{on } H_\kappa^d \setminus \tilde{U}_\varepsilon,$$

and thus by (4.2),

$Q_{l_0(x),l_n(x)}[\hat{\Omega}_{n,x}] \geq Q_{l_0(x),l_n(x)}[\{\omega \in C([0,1], H^d_\kappa);\ \sup r_{l_0(x),l_n(x)} \circ \omega < \varepsilon - r(x)\}]$.

Since $U_\varepsilon = \{x \in M;\ r(x) < \varepsilon\}$, Proposition 3.1 implies

$$\lim_{n\to\infty} Q_{l_0(x),l_n(x)}[\hat{\Omega}_{n,x}] \ = \ 1 \qquad \text{for all } x \in U_\varepsilon, \tag{4.16}$$

where the convergence is uniform on $U_{\varepsilon'} = \{x \in M;\ r(x) < \varepsilon'\}$ for every $\varepsilon' < \varepsilon$. (4.13), (4.14) and (4.16) imply the first estimate in Proposition 4.3.

To prove the second estimate let $\rho(x) = \operatorname{dist}(x, \tilde{\sigma}(\mathbf{R}))$, $x \in H^d_\kappa$, and

$$\hat{R}(\omega) = \sup_{0\leq s\leq 1} \rho(\omega(s)), \qquad \omega \in C([0,1], H^d_\kappa).$$

Clearly, $\rho = r \circ \pi$ on $\tilde{U}_\varepsilon$ and $R(\omega) = \hat{R}(\hat{\omega})$ for all $\omega \in C([0,1], U_\varepsilon)$. Let $n \in \mathbf{N}$. For $\omega \in \hat{\Omega}_n$ and $s \in [0,1]$,

$$\rho(\omega(s)) \ \leq \ r_{\omega(0),\omega(1)}(\omega(s)) + \sup_{0\leq s\leq 1} \rho(\gamma_{\omega(0),\omega(1)}(s)) \ \leq \ r_{\omega(0),\omega(1)}(\omega(s)) + \rho(\omega(0)),$$

because $\tilde{\sigma}$ and $\gamma_{\omega(0),\omega(1)}$ are geodesics, and $\pi(\omega(0)) = \pi(\omega(1))$ implies $\rho(\omega(0)) = \rho(\omega(1))$. Since $\omega(0) = l_0(\pi(\omega(0)))$ and $\omega(1) = l_n(\pi(\omega(0)))$, we obtain

$$\hat{R}(\omega) \ \leq \ R^{(n)}_{\pi(\omega(0))}(\omega) \ + \ \rho(\omega(0)) \qquad \text{for all } \omega \in \hat{\Omega}_n, \tag{4.17}$$

whence by (4.7),

$$\begin{aligned} P[R \geq 2\delta | \Omega_n] \ &= \ Q^{(n)}[\hat{R} \geq 2\delta | \hat{\Omega}_n] \\ &\leq \ Q^{(n)}[\{\omega \in \hat{\Omega}_n;\ R^{(n)}_{\pi(\omega(0))}(\omega) \geq \delta \ \text{ or } \ \rho(\omega(0)) \geq \delta\}] \,/\, Q^{(n)}[\hat{\Omega}_n]. \end{aligned} \tag{4.18}$$

Since $Q^{(n)}$ is a mixture of measures $Q_{l_0(x),l_n(x)}$, $x \in U_\varepsilon$, (4.14) implies

$$\lim_{n\to\infty} \exp(n^\beta) \cdot Q^{(n)}[\{\omega \in \hat{\Omega}_n;\ R^{(n)}_{\pi(\omega(0))}(\omega) \geq \delta\}] \ = \ 0. \tag{4.19}$$

Moreover, by Lemma 4.2, there exist constants C, $\lambda \in (0, \infty)$ such that

$$\begin{aligned} &Q^{(n)}[\{\omega \in \hat{\Omega}_n;\ \rho(\omega(0)) \geq \delta\}] \\ &\leq \ \int_{U_\varepsilon \setminus U_\delta} q_1(l_0(x), l_n(x))\, V(dx) \,/ \int_{U_\varepsilon} q_1(l_0(x), l_n(x))\, V(dx) \ \leq \ C \cdot e^{-\lambda n} \end{aligned} \tag{4.20}$$

for all $n \in \mathbf{N}$. On the other hand, since the convergence in (4.16) is uniform on $U_{\varepsilon/2}$, we have

$$(4.21)\quad \liminf_{n\to\infty} Q^{(n)}[\hat{\Omega}_n] \geq \liminf_{n\to\infty} \int_{U_{\varepsilon/2}} q_1(l_0(x), l_n(x))\, V(dx) / \int_{U_\varepsilon} q_1(l_0(x), l_n(x))\, V(dx) \;=\; 1.$$

Since $\delta > 0$ has been chosen arbitrarily in the beginning, (4.18), (4.19), (4.20) and (4.21) imply the second estimate in Proposition 4.3. □

5 Proofs of the results on free loop spaces

Let M be a compact connected Riemannian manifold such that (A 1) holds. In this section we apply Lemma 2.1 and Proposition 4.3 to prove Theorem 1.3 and Corollary 1.4. The free loop space part in the assertion of Theorem 1.1 is a special case of the corollary. The results on based loop spaces will be proved in Section 6 after some additional preparations.

Consider the situation described in Section 1.5. The symmetric bilinear operator $\Gamma : \mathcal{F}C^\infty \times \mathcal{F}C^\infty \to L^1(LM;\, P)$ introduced there has a unique extension to $H^{1,2}_\Gamma(LM;\, P) \times H^{1,2}_\Gamma(LM;\, P)$ that is continuous w.r.t. the norm

$$(5.1)\qquad \|F\|_{1,2,\Gamma} \;=\; \left(\int (F^2 + \Gamma(F))\, dP\right)^{1/2}.$$

We denote this extension again by Γ. It is a carré du champ operator for the quadratic form $(\mathcal{E}^\Gamma, H^{1,2}_\Gamma(LM;\, P))$, and by (1.12) and continuity it satisfies the chain rule (2.3). In particular, $\mathcal{E}^\Gamma$ is a strongly local Dirichlet form. We set $\mathcal{E}^\Gamma(F) = \mathcal{E}^\Gamma(F, F)$ and $\Gamma(F) = \Gamma(F, F)$.

Let d_∞ be the metric on LM given by

$$d_\infty(\omega, \tilde{\omega}) \;=\; \sup_{s\in S^1} d(\omega(s), \tilde{\omega}(s)).$$

The distance w.r.t. d_∞ from an open subset $\tilde{\Omega} \subset LM$ will be denoted by $\mathrm{dist}_\infty(\,\cdot\,, \tilde{\Omega})$, i.e.,

$$\mathrm{dist}_\infty(\omega, \tilde{\Omega}) \;=\; \inf_{\tilde{\omega}\in\tilde{\Omega}} d_\infty(\omega, \tilde{\omega}) \qquad \text{for all } \omega \in \Omega.$$

We will need the following fact in the proof of Theorem 1.3 :

Lemma 5.1 *For every open set $\tilde{\Omega} \subset LM$, the function $\omega \mapsto \mathrm{dist}_\infty(\omega, \tilde{\Omega})$ is in $H_\Gamma^{1,2}(LM;\, P)$, and*

$$\Gamma(\mathrm{dist}_\infty(\,\cdot\,, \tilde{\Omega})\,) \leq \alpha \qquad P\text{–}a.s.$$

Here α is the P–integrable function appearing in the assumption (1.14).

REMARK. The lemma implies in particular that every indicator function χ_Λ of a connected component $\Lambda \subset LM$ is contained in $H_\Gamma^{1,2}(LM;\, P)$. In fact, $\mathrm{dist}_\infty(\Lambda, LM \setminus \Lambda) \geq \mathrm{inj}\,(M) > 0$. Let $\psi : \mathbf{R} \to [0,1]$ be a smooth function such that $\psi(r) = 1$ for $r \leq 0$ and $\psi(r) = 0$ for $r \geq \mathrm{dist}_\infty(\Lambda, LM \setminus \Lambda)$. Then $\chi_\Lambda = \psi \circ \mathrm{dist}_\infty(\,\cdot\,, \Lambda)$, which is in $H_\Gamma^{1,2}(LM;\, P)$ by Lemma 5.1. Moreover, by the chain rule,

$$\Gamma(\chi_\Lambda) \;=\; (\psi' \circ \mathrm{dist}_\infty(\,\cdot\,, \Lambda))^2 \cdot \Gamma(\mathrm{dist}_\infty(\,\cdot\,, \Lambda)) \;=\; 0 \qquad P\text{–a.e}$$

PROOF OF LEMMA 5.1. Fix $\tilde{\omega} \in \tilde{\Omega}$ and $s \in S^1$. Let $\varphi_n : \mathbf{R} \to (0, \infty)$, $n \in \mathbf{N}$, be smooth functions such that $\varphi_n(x) = |x|$ if $|x| \geq 1/n$ and $|\varphi_n'| \leq 1$. Then the function $G_{\tilde{\omega},s}^n : \omega \mapsto \varphi_n(d(\omega(s), \tilde{\omega}(s)))$ is in $\mathcal{FC}^\infty$, and by (1.14), $\Gamma(G_{\tilde{\omega},s}^n) \leq \alpha$ P–a.s. In particular, $(G_{\tilde{\omega},s}^n)_{n \in \mathbf{N}}$ is a bounded sequence in the Hilbert space $H_\Gamma^{1,2}(LM;\, P)$. Since the sequence converges P–a.s. to the function $G_{\tilde{\omega},s} : \omega \mapsto d(\omega(s), \tilde{\omega}(s))$, this function is in $H_\Gamma^{1,2}(LM;\, P)$ as well, and

$$\Gamma(G_{\tilde{\omega},s}) \;\leq\; \liminf_{n \to \infty} \Gamma(G_{\tilde{\omega},s}^n) \;\leq\; \alpha \qquad P\text{–a.s.} \tag{5.2}$$

In fact, the Césaro means of a subsequence of $(G_{\tilde{\omega},s}^n)_{n \in \mathbf{N}}$ converge in the Hilbert space $H_\Gamma^{1,2}(LM;\, P)$ by the theorems of Banach/Alaoglu and Banach/Saks, cf. [MR], Appendix 2. Because of the P–a.s. convergence to $G_{\tilde{\omega},s}$, the $H_\Gamma^{1,2}(LM;\, P)$ limit is $G_{\tilde{\omega},s}$ as well. Hence (5.2) holds by the continuity of $\Gamma : H_\Gamma^{1,2}(LM;\, P) \times H_\Gamma^{1,2}(LM;\, P) \to L^1(LM;\, P)$.

Since LM is separable w.r.t. d_∞, there exists a countable dense subset $\tilde{\Omega}_c$ of $\tilde{\Omega}$. Obviously,

$$\mathrm{dist}_\infty(\,\cdot\,, \tilde{\Omega}) \;=\; \inf_{\tilde{\omega} \in \tilde{\Omega}_c} \; \sup_{s \in (0,1) \cap \mathbf{Q}} G_{\tilde{\omega},s}\,. \tag{5.3}$$

Since the functions $G_{\tilde{\omega},s}$ are uniformly bounded by the diameter of M, and the measure P is finite, the assertion now follows from standard arguments for local Dirichlet forms, cf. Appendix A.2. $\square$

Recall the definition of R from Section 4, and the definition of the closed geodesics γ_n and the sets Ω_n and Ω_n^δ, $\delta > 0$, from Section 1.5. Note that $\Omega_n^\delta = \{\omega \in \Omega_n;\ R(\omega) < \delta\}$. We define $F_n : LM \to [0,1]$ by

$$F_n(\omega) \ = \ (2 - 3\varepsilon^{-1}R(\omega))^+ \wedge 1 \quad \text{for } \omega \in \Omega_n, \ 0 \text{ else},$$

where ε is chosen as in (A 1). We want to apply Lemma 2.1 with $A_n = \Omega_n^{\varepsilon/3}$ and $B_n = \Omega_n$. We first show :

Lemma 5.2 *For every $n \in \mathbf{N}$, the function F_n is in $H_\Gamma^{1,2}(LM;\ P)$, $\Gamma(F_n) \leq 3\varepsilon^{-1}\alpha$ P–a.e. on $\Omega_n^{2\varepsilon/3} \setminus \Omega_n^{\varepsilon/3}$, and $\Gamma(F_n) = 0$ P–a.e. else.*

Proof. We first remark that the function R is in $H_\Gamma^{1,2}(LM;\ P)$ and $\Gamma(R) \leq \alpha$ P–a.e. This follows by a similar argument as in the proof of Lemma 5.1, because

$$R(\omega) \ = \ \sup_{s \in (0,1) \cap \mathbf{Q}} d(\omega(s), \gamma(S^1)).$$

Hence the function $\tilde{F} = (2 - 3\varepsilon^{-1}R)^+ \wedge 1$ is in $H_\Gamma^{1,2}(LM;\ P)$ as well, and $\Gamma(\tilde{F}) \leq 3\varepsilon^{-1}\alpha$ P–a.e., cf. Appendix A.2.

Now fix a constant $\delta \in (0, \min(\mathrm{inj}\,(M),\ \varepsilon/3))$, and set

$$\Psi_n \ = \ (1 - \delta^{-1}\,\mathrm{dist}_\infty(\,\cdot\,, \Omega_n^{2\varepsilon/3})\,)^+. \tag{5.4}$$

For every $n \in \mathbf{N}$, Ψ_n is a function on LM with $\Psi_n(\omega) = 1$ for all $\omega \in \Omega_n^{2\varepsilon/3}$. I claim that Ψ_n vanishes outside Ω_n. To see this, suppose the contrary. Let $\omega \in LM \setminus \Omega_n$ and $\sigma \in \Omega_n^{2\varepsilon/3}$ such that $d_\infty(\omega, \sigma) < \delta$. Since $\delta < \mathrm{inj}\,(M)$, the vector field $X(s) = \exp_{\sigma(s)}^{-1}(\omega(s))$, $s \in S^1$, along σ is well–defined and continuous. Hence $H(t,s) = \exp_{\sigma(s)}(tX(s))$, $s \in S^1$, $t \in [0,1]$, defines a homotopy between $H(0,\cdot) = \sigma$ and $H(1,\cdot) = \omega$. Since σ is contained in Ω_n but ω is not, there exists $t_0 \in (0,1]$ such that $H(t_0,\cdot)$ is in $\partial\Omega_n$. Since Ω_n is a connected component of LU_ε, $H(t_0,s_0)$ is in ∂U_ε for some $s_0 \in S^1$, i.e., $d(H(t_0,s_0), \gamma(S^1)) = \varepsilon$. This is a contradiction, because on the other hand,

$$\begin{aligned} \mathrm{dist}\,(H(t_0,s_0), \gamma(S^1)) \ &\leq \ d(H(t_0,s_0), \sigma(s_0)) \ + \ \mathrm{dist}\,(\sigma(s_0), \gamma(S^1)) \\ &\leq \ t_0 \cdot d_\infty(\omega,\sigma) \ + \ \mathrm{dist}_\infty(\sigma, \gamma(S^1)) \ < \ \delta + 2\varepsilon/3 \ < \ \varepsilon. \end{aligned}$$

Hence $\Psi_n(\omega) = 0$ for $\omega \in LM \setminus \Omega_n$, and thus $F_n = \tilde{F} \cdot \Psi_n$.

By the considerations above and by Lemma 5.1, both $\tilde{F}$ and Ψ_n are bounded functions in $H^{1,2}_\Gamma(LM;\, P)$. Hence F_n is in $H^{1,2}_\Gamma(LM;\, P)$ as well. Moreover, F_n vanishes on $LM \setminus \Omega_n^{2\varepsilon/3}$ and $F_n - 1$ vanishes on $\Omega_n^{\varepsilon/3}$, whence $\Gamma(F_n) = 0$ P–a.e. on $(LM \setminus \Omega_n^{2\varepsilon/3}) \cup \Omega_n^{\varepsilon/3}$, cf. (A.5) in the appendix. On the other hand, $F_n = \tilde{F}$ on $\Omega_n^{2\varepsilon/3}$, so $\Gamma(F_n) = \Gamma(\tilde{F}) \le 3\varepsilon^{-1}\alpha$ P–a.e. on $\Omega_n^{2\varepsilon/3}$. $\square$

Proof of Theorem 1.3. Let $\beta < 1/4$ such that (1.15) holds. By Proposition 4.3,

$$e^{n^\beta}\, P[\Omega_n \setminus \Omega_n^{\varepsilon/3}]/P[\Omega_n^{\varepsilon/3}] \;\le\; e^{n^\beta}\, P[R \ge \varepsilon/3 \,|\, \Omega_n] \;\longrightarrow 0$$

as $n \to \infty$. On the other hand,

$$E_P[\,\Gamma(F_n) \,|\, \Omega_n \setminus \Omega_n^{\varepsilon/3}\,] \;\le\; 3\,\varepsilon^{-1}\, E_P[\,\alpha \,|\, \Omega_n \setminus \Omega_n^{\varepsilon/3}]$$

by Lemma 5.2. The assertion now follows by Lemma 2.1. $\square$

Proof of Corollary 1.4. W.l.o.g. we may assume that $A \ge 1$ P–a.e. Otherwise, we may apply the corollary with $\tilde{A} = 1 + A$. Clearly, the assumption for A implies the assumption for $\tilde{A}$, and the non–validity of a Poincaré inequality w.r.t. $\tilde{A}$ implies the non–validity of the corresponding inequality w.r.t. A.

Now suppose $A \ge 1$ P–a.e. Then the symmetric bilinear form

$$\mathcal{E}^A(F,G) \;=\; \int_{LM} A(\omega)\, \langle DF, DG\rangle_{T^1_\omega LM}\, P(d\omega), \qquad F,G \in \mathcal{FC}^\infty,$$

is closable on $L^2(LM;\, P)$, cf. Appendix 1. The closure $(\mathcal{E}^A, \mathrm{Dom}(\mathcal{E}^A))$ is a strongly local Dirichlet form with $\mathrm{Dom}\,(\mathcal{E}^A) \subset H^{1,2}(LM;\, P)$ and carré du champ operator $\Gamma^A(F,G)(\omega) = A(\omega)\langle DF, DG\rangle_{T^1_\omega LM}$. Let F be a function on LM of type $F(\omega) = f(\omega(s))$ with $s \in [0,1]$ and $f \in C^\infty(M)$. Note that for $\omega \in LM$ and $X \in T_\omega LM$, we have

$$|X_s|_{T_{\omega(s)}M} \;\le\; |X_t|_{T_{\omega(t)}M} + \int_0^1 \left|\frac{\nabla X}{du}(u)\right|_{T_{\omega(u)}M} du \quad \text{for all } t \in [0,1],$$

and thus

$$|X_s|_{T_{\omega(s)}M} \;\le\; \int_0^1 \left(|X_u|_{T_{\omega(u)M}} + \left|\frac{\nabla X}{du}(u)\right|_{T_{\omega(u)M}} \right) du \;\le\; \sqrt{2}\cdot |X|_{T^1_\omega LM}\,.$$

Hence

$$
\begin{aligned}
|DF(\omega)|_{T^1_\omega LM} &= \sup\{\, XF;\ X \in T^1_\omega LM,\ |X|_{T^1_\omega LM} = 1\} \\
&\leq \sqrt{2} \cdot |\mathrm{grad}_{\omega(s)} f|_{T_{\omega(s)}M}, \qquad \text{and thus} \\
\Gamma^A(F,F)(\omega) &\leq \sqrt{2} A(\omega) \cdot |\mathrm{grad}_{\omega(s)} f|^2_{T_{\omega(s)}M} \qquad (5.5)
\end{aligned}
$$

for every $\omega \in LM$, and f, s, F as above. By the assumption in Corollary 1.4 and (5.5), Theorem 1.3 implies

$$\inf\{\mathcal{E}^A(F,F)/\mathrm{Var}_P(F);\ F \in \mathrm{Dom}\,(\mathcal{E}^A),\ \mathrm{Var}_P(F) \neq 0\} \;=\; 0.$$

The assertion of Corollary 1.4 follows because $\mathcal{FC}^\infty$ is dense in $\mathrm{Dom}\,(\mathcal{E}^A)$ by the definition of $\mathcal{E}^A$. □

Proof of the free loop space parts of Theorem 1.1 and Corollary 1.2. Theorem 1.1 is a special case of Corollary 1.4. Actually, the proof of Corollary 1.4 and the remark below Lemma 2.1 show that more generally, (A 1) implies

$$\inf\{\, \mathcal{E}(F,F);\ F \in \mathcal{FC}^\infty,\ E_P[(F - \mathrm{pr}_{\ker \mathcal{L}} F)^2] = 1\,\} \;=\; 0$$

if the kernel of $\mathcal{L}$ is finite dimensional, and $\mathrm{pr}_{\ker \mathcal{L}}$ denotes the orthogonal projection onto the kernel in $L^2(LM;\, P)$. This proves the assertion of Corollary 1.2. □

6 Proofs on based loop spaces

Let M be a compact connected Riemannian manifold satisfying (A 1). We fix $x \in U_\varepsilon$. The aim of this section is the proof of the based loop space parts of Theorem 1.1 and Corollary 1.2. We will essentially use the same techniques as in the free loop space case (cf. Section 5), but now the functions F_n we choose will depend on the lift $\hat{\omega}$ of a loop ω, and not only on ω itself. Therefore, we need some additional preparations.

Recall that $r(x) = \mathrm{dist}\,(x, \gamma(S^1)) < \varepsilon$. We fix a constant $\delta \in (0, \mathrm{inj}\,(M))$ such that $3\delta < \varepsilon - r(x)$, and define functions Ψ_N, $n \in \mathbf{N}$, on $L_x M$ by

$$\Psi_n \;=\; (1 - \delta^{-1}\, \mathrm{dist}_\infty(\,\cdot\,, \Omega_n^{r(x)+2\delta}))^+. \qquad (6.1)$$

Here $\Omega^a_{n,x} = \Omega^a_n \cap L_xM$ for $a > 0$. Since $r(x) + 3\delta < \varepsilon$, we can show by the same argument as in the proof of Lemma 5.2, that the extension of Ψ_n to LM vanishes on $LM \setminus \Omega_n$. Hence Ψ_n vanishes on $L_xM \setminus \Omega_{n,x}$. Moreover, in the same way as in the proof of Lemma 5.1, we see that the function $\mathrm{dist}_\infty(\,\cdot\,, \Omega_n^{r(x)+2\delta})$ is in $H^{1,2}(L_xM;\, P_x)$, and $|D^0 \mathrm{dist}_\infty(\,\cdot\,, \Omega_n^{r(x)+2\delta})|_{T^1_\omega L_xM} \leq 1$ for P_x–a.e. $\omega \in L_xM$. Hence Ψ_n is in $H^{1,2}(L_xM;\, P_x)$ as well, and

$$(6.2) \qquad |D^0\Psi_n|_{T^1_\omega L_xM} \leq \delta^{-1} \qquad \text{for } P_x\text{–a.e. } \omega.$$

Lemma 6.1 *Let $s \in (0,1)$, and let f be a bounded Lipschitz continuous function on $\tilde{U}_\varepsilon$. Fix $n \in \mathbf{N}$, and let F be a function on L_xM such that $F(\omega) = f(\hat{\omega}(s))$ for all $\omega \in \Omega_{n,x}$. Then $F \cdot \Psi_n$ is in $H^{1,2}(L_xM;\, P_x)$, and*

$$(6.3) \qquad D^0(F \cdot \Psi_n)(\omega) \;=\; F(\omega)\,(D^0\Psi_n)(\omega) \;+\; \Psi_n(\omega)\,Y(\omega)$$

for P_x–a.e. $\omega \in L_xM$, where $Y(\omega) \in T^1_\omega L_xM$ is defined arbitrarily for $\omega \in L_xM \setminus \Omega_{n,x}$, and

$$(Y(\omega))(t) \;=\; (s \wedge t - st) \cdot U_t(\omega)U_s(\omega)^{-1}(d_{\hat{\omega}(s)}\pi)(\mathrm{grad}_{\hat{\omega}(s)} f)$$

for all $t \in [0,1]$ and $\omega \in \Omega_{n,x}$.

Proof. We fix $\alpha \in (0, \mathrm{inj}\,(M)/3)$. For $x \in U_\varepsilon$ let $B_\alpha(x) = \{z \in U_\varepsilon;\, d'(z,x) < \alpha\}$, where d' is the intrinsic distance function on $\bar{U}_\varepsilon$. Note that d' generates the same topology as the distance function d on M, but d can be smaller than d' in general. For $x \in \tilde{U}_\varepsilon$ let $\tilde{B}_\alpha(x) = \{z \in \tilde{U}_\varepsilon;\, d(z,x) < \alpha\}$. Since $\tilde{U}_\varepsilon$ is a convex subset of H^d_κ, the intrinsic distance on $\tilde{U}_\varepsilon$ coincides with the restriction of the hyperbolic distance. We now proceed in two steps :

Step 1 : In this step we prove the assertion under the additional assumption that there exists $y \in U_\varepsilon$ such that f vanishes on $\tilde{U}_\varepsilon \setminus \pi^{-1}(B_\alpha(y))$.

Note that $d(l_m(y), l_p(y)) \geq \mathrm{inj}\,(M)$ for all $m, p \in \mathbf{Z}$, $m \neq p$. Since $\alpha < \mathrm{inj}\,(M)/3$, $\pi^{-1}(B_\alpha(y))$ is the *disjoint* union of the sets $\tilde{B}_\alpha(l_m(y))$, $m \in \mathbf{Z}$, and the distance between any two of these sets is greater than α. Consequently, the set

$$\Omega'_{n,x} \;=\; \{\, \omega \in \Omega_{n,x};\, \omega(s) \in B_\alpha(y) \,\}$$

is the *disjoint* union of the sets

$$\Omega^{(m)}_{n,x} \;=\; \{\, \omega \in \Omega_{n,x};\, \hat{\omega}(s) \in \tilde{B}_\alpha(l_m(x)) \,\}, \quad m \in \mathbf{Z}.$$

Moreover :

Claim : There exists a constant $r > 0$ such that $d_\infty(\omega, \sigma) \geq r$ holds for all $\omega \in \Omega_{n,x}^{(m)}$ and $\sigma \in \Omega_{n,x}^{(p)}$ with $m,\, p \in \mathbf{Z}$, $m \neq p$.

Here $d_\infty(\omega, \tilde{\omega}) = \sup_{t\in[0,1]} d(\omega(t), \tilde{\omega}(t))$ as above, where d is the distance function on M. To prove the claim, we first remark that the function $(x, y) \mapsto d(x, y)$ is continuous and strictly positive on the compact set $\{(x, y) \in \bar{U}_\varepsilon \times \bar{U}_\varepsilon;\, d'(x, y) \geq \alpha\}$. Hence there exists $r > 0$ such that $d(x, y) \geq r$ for all $x, y \in U_\varepsilon$ with $d'(x, y) \geq \alpha$. Now fix $m, p \in \mathbf{Z}$ with $m \neq p$, as well as $\omega \in \Omega_{n,x}^{(m)}$ and $\sigma \in \Omega_{n,x}^{(p)}$. For every $t \in [0, 1]$, the minimal geodesic from $\omega(t)$ to $\sigma(t)$ in $\bar{U}_\varepsilon$ has length $d'(\omega(t), \sigma(t))$, whereas all the other (non–homotopic) geodesics from $\omega(t)$ to $\sigma(t)$ in U_ε have length greater than 3α. Since the minimal geodesic in $\tilde{U}_\varepsilon$ between $\hat{\omega}(t)$ and $\hat{\sigma}(t)$ is the lift of one of the geodesics above, we have

$$d(\hat{\omega}(t), \hat{\sigma}(t)) \;=\; d'(\omega(t), \sigma(t)) \quad \text{or} \quad d(\hat{\omega}(t), \hat{\sigma}(t)) \geq 3\alpha$$

for every $t \in [0, 1]$. Moreover, the function $t \mapsto d(\hat{\omega}(t), \hat{\sigma}(t))$ is continuous, $d(\hat{\omega}(0), \hat{\sigma}(0)) = 0$, and

$$d(\hat{\omega}(s), \hat{\sigma}(s)) \;\geq\; \operatorname{dist}(\tilde{B}_\alpha(l_m(y)), \tilde{B}_\alpha(l_p(y))) \;>\; \alpha.$$

This is not possible if $\sup_{t\in[0,1]} d'(\omega(t), \sigma(t)) \leq \alpha$, since then the function $t \mapsto d(\hat{\omega}(t), \hat{\sigma}(t))$ could only take values in $[0, \alpha] \cup [3\alpha, \infty)$, and thus (by continuity) not both in $[0, \alpha]$ and in (α, ∞). Hence $\sup_{t\in[0,1]} d'(\omega(t), \sigma(t)) > \alpha$, and thus $d_\infty(\omega, \sigma) = \sup_{t\in[0,1]} d(\omega(t), \sigma(t)) \geq r$. This proves the claim.

Now fix $r > 0$ as in the claim. Let $m \in \mathbf{Z}$, and let $\Phi_{n,m}$ be the function on L_xM defined by

$$\Phi_{n,m}(\omega) \;=\; (1 - r^{-1} \cdot \operatorname{dist}_\infty(\omega, \Omega_{n,x}^{(m)})\,)^+.$$

As in the proof of Lemma 5.1, one shows that $\Phi_{n,m}$ is in $H^{1,2}(L_xM;\, P_x)$ and $|D^0\Phi_{n,m}|_{T^1_\omega L_xM} \leq r^{-1}$ for P_x–a.e. $\omega \in L_xM$. Moreover, $\Phi_{n,m} = 1$ on $\Omega_{n,x}^{(m)}$, and (by the claim) $\Phi_{n,m}$ vanishes on $\Omega_{n,x}^{(p)}$ for every $p \in \mathbf{Z} \setminus \{m\}$.

Let $G_m = F \cdot \Psi_n \cdot \Phi_{n,m}$. We now show first that G_m is in $H^{1,2}(L_xM;\, P_x)$. Note that, by our assumption on f, F vanishes on $\Omega_{n,x} \setminus \Omega'_{n,x}$. Hence G_m vanishes on $L_xM \setminus \Omega_{n,x}^{(m)}$. On the other hand, for every $1 \leq i \leq k$, the restriction of the covering map $\pi : \hat{U} \to U$ to $\tilde{B}_\alpha(l_m(y))$ is an isometry onto

$B_\alpha(y)$. Hence there exists a Lipschitz continuous function $\tilde{f} : M \to \mathbf{R}$ such that

$$
\tilde{f}(\pi(x)) \; = \; f(x) \qquad \text{for all } x \in \tilde{B}_\alpha(l_m(y)). \tag{6.4}
$$

Let $\tilde{F}$ be the function on $L_x M$ given by $\tilde{F}(\omega) = \tilde{f}(\omega(s))$. Then $F(\omega) = f(\hat{\omega}(s)) = \tilde{F}(\omega)$ for all $\omega \in \Omega_{n,x}^{(m)}$, whence $G_m = \tilde{F} \cdot \Psi_n \cdot \Phi_{n,m}$. Since $\tilde{F}$, Ψ_n, and $\Phi_{n,m}$ are bounded elements in $H^{1,2}(L_x M;\, P_x)$, G_m is in $H^{1,2}(L_x M;\, P_x)$ as well. Moreover, since G_m vanishes on $L_x M \setminus \Omega_{n,x}^{(m)}$ and $G_m = \tilde{F} \cdot \Psi_n$ on $\Omega_{n,x}^{(m)}$, we have

$$
D^0 G_m \; = \; 0 \qquad P_x\text{–a.e. on } L_x M \setminus \Omega_{n,x}^{(m)}, \qquad \text{and} \tag{6.5}
$$

$$
\begin{aligned}
D^0 G_m \; &= \; D^0(\tilde{F} \cdot \Psi_n) \; = \; \tilde{F} \cdot D^0 \Psi_n \; + \; \Psi_n \cdot D^0 \tilde{F} \\
&= \; F \cdot D^0 \Psi_N \; + \; \Psi_n \cdot D^0 \tilde{F} \qquad P_x\text{–a.e. on } \Omega_{n,x}^{(m)}.
\end{aligned} \tag{6.6}
$$

For P_x–a.e. $\omega \in \Omega_{n,x}^{(m)}$, $(D^0 \tilde{F})(\omega)$ is given by

$$
\begin{aligned}
((D^0 \tilde{F})(\omega))(t) \; &= \; U_t(\omega) U_s(\omega)^{-1} \mathrm{grad}_{\omega(s)} \tilde{f} \\
&= \; U_t(\omega) U_s(\omega)^{-1} (d_{\hat{\omega}(s)} \pi)(\mathrm{grad}_{\hat{\omega}(s)} f)
\end{aligned} \tag{6.7}
$$

The last equality holds by (6.4) and because π is a local isometry.

The function $F \cdot \Psi_n$ coincides with G_m on $\Omega_{n,x}^{(m)}$ for every $m \in \mathbf{Z}$, and vanishes on $L_x M \setminus \Omega'_{n,x}$. In particular, $F \cdot \Psi_n = \lim_{M \to \infty} S_M$ where $S_M = \sum_{m=-M}^{M} G_m$. The functions S_M, $M \in \mathbf{N}$, are in $H^{1,2}(L_x M;\, P_x)$. For $m \in \mathbf{Z}$, $S_M = G_m$ or $S_M = 0$ on $\Omega_{n,x}^{(m)}$, whence by (6.6) and (6.7),

$$
\begin{aligned}
|S_M(\omega)| \; &\leq \; |G_m(\omega)| \; \leq \; \sup_{x \in \tilde{U}_\varepsilon} |f(x)| \qquad \text{and} \\
|D^0 S_M|_{T^1_\omega L_x M} \; &\leq \; |D^0 G_m|_{T^1_\omega L_x M} \; \leq \; \delta^{-1} \cdot \sup_{x \in \tilde{U}_\varepsilon} |f(x)| \; + \; \mathrm{esssup}_{x \in \tilde{U}_\varepsilon} |\mathrm{grad}\, f|_x
\end{aligned}
$$

hold for all $M \in \mathbf{N}$ and P_x–a.e. $\omega \in \Omega'_{n,m}$. Since the functions S_M (and hence their gradients) vanish outside $\Omega'_{n,x}$, their $H^{1,2}(L_x M;\, P_x)$ norms are bounded. Hence the pointwise limit $F \cdot \Psi_n$ of the sequence $(S_M)_{M \in \mathbf{N}}$ is in $H^{1,2}(L_x M;\, P_x)$ as well. Since $F \cdot \Psi_n = 0$ on $L_x M \setminus \Omega'_{n,x}$ and $F \cdot \Psi_n = G_m$ on $\Omega_{n,x}^{(m)}$ for every $m \in \mathbf{Z}$, we finally obtain

$$
\begin{aligned}
D^0(F \cdot \Psi_n) \; &= \; 0 \qquad P_x\text{–a.e. on } L_x M \setminus \Omega'_{n,x}, \qquad \text{and, by (6.6),} \\
D^0(F \cdot \Psi_n) \; &= \; D^0 F_m \; = \; F \cdot D^0 \Psi_n + \Psi_n \cdot D^0 \tilde{F} \quad P_x\text{–a.e. on } \Omega_{n,x}^{(m)}, \; m \in \mathbf{Z}.
\end{aligned}
$$

By (6.7), this implies the assertion of the lemma for the special class of functions f and F we have considered in Step 1.

Step 2. We now prove the assertion for arbitrary bounded Lipschitz continuous functions f on $\tilde{U}_\varepsilon$. Fix such a function f, and let F be a function as in the statement of the lemma. The balls $B'_\alpha(y) = \{z \in \bar{U}_\varepsilon;\, d'(z,y) < \alpha\}$, $y \in U_\varepsilon$, form an open covering of the compact set $\bar{U}_\varepsilon$. Let $\{B'_\alpha(y_1), \ldots, B'_\alpha(y_N)\}$, $N \in \mathbf{N}$, be a finite sub–covering, and let V_i, $1 \le i \le N$, be open sets in M such that $V_i \cap \bar{U}_\varepsilon = B'_\alpha(y_i)$. Such sets exist, since $B'_\alpha(y_i) \cap \partial U_\varepsilon$ is a relatively open subset of ∂U_ε. Let $\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_N \in C^\infty(M)$ be a partition of unity on M adapted to the open covering $\{M \setminus \bar{U}_\varepsilon, V_1, \ldots, V_N\}$ of M, i.e., $\varphi_i \ge 0$ for all $0 \le i \le N$, $\sum_{i=0}^N \varphi_i = 1$, φ_0 vanishes on $\bar{U}_\varepsilon$, and φ_i vanishes outside V_i (and thus on $U_\varepsilon \setminus B_\alpha(y_i)$) for every $1 \le i \le N$. Note that $\sum_{i=1}^N \varphi_i(x) = 1$ for $x \in U_\varepsilon$. Fix $i \in \{1, 2, \ldots, N\}$ for the moment. We define functions h_i on $\tilde{U}_\varepsilon$ and H_i on L_xM by $h_i(x) = f(x) \cdot \varphi_i(\pi(x))$ and $H_i(\omega) = F(\omega) \cdot \varphi_i(\omega(s))$. Clearly, $H_i(\omega) = h_i(\hat{\omega}(s))$ for all $\omega \in \Omega_{n,x}$. Moreover, h_i is bounded and Lipschitz continuous, and vanishes on $\tilde{U}_\varepsilon \setminus \pi^{-1}(B_\alpha(y_i))$. Thus by Step 1, $H_i \cdot \Psi_n$ is in $H^{1,2}(L_xM;\, P_x)$ and (6.3) holds with F, f replaced by H_i, h_i respectively. But $F(\omega) = \sum_{i=1}^N H_i(\omega)$ for all $\omega \in \Omega_{n,x}$, beacause $\sum_{i=1}^N \varphi_i = 1$ on U_ε. Since $\Psi_n = 0$ on $L_xM \setminus \Omega_{n,x}$, we have

$$\begin{aligned} F \cdot \Psi_n &= \sum_{i=1}^N H_i \cdot \Psi_n \;\in\; H^{1,2}(L_xM;\, P_x), \qquad \text{and} \\ D^0(F \cdot \Psi_n) &= \sum_{i=1}^N D^0(H_i \cdot \Psi_n). \end{aligned}$$

Now (6.3) follows from the corresponding formulae for H_i, $1 \le i \le N$. □

Recall the definition of the function $R_x^{(n)}$ from Section 4. We define functions $F_n : L_xM \to [0,1]$, $n \in \mathbf{N}$, by

$$F_n(\omega) \;=\; \begin{cases} (2 - \delta^{-1} R_x^{(n)}(\hat{\omega}))^+ \wedge 1 & \text{for } \omega \in \Omega_{n,x}, \\ 0 & \text{else.} \end{cases}$$

Lemma 6.2 *For every $n \in \mathbf{N}$, the function F_n is in $H^{1,2}(L_xM;\, P_x)$, and $|D^0 F_n|_{T^1_\omega L_xM} \le \delta^{-1}$ for P_x–a.e. $\omega \in L_xM$.*

Proof. Let $n \in \mathbf{N}$. We first remark that F_n vanishes on $L_xM \setminus \Omega_{n,x}^{r(x)+2\delta}$. In fact, for $\omega \in \Omega_{n,x} \setminus \Omega_{n,x}^{r(x)+2\delta}$ there exists $s \in [0,1]$ with $r(\pi(\hat{\omega}(s))) =$

$r(\omega(s)) \geq r(x) + 2\delta$. On the other hand, for all $t \in [0,1]$, $r(\pi(\hat{\gamma}_x^{(n)}(t))) \leq r(x)$ (cf. (4.15)). Hence $d(\hat{\omega}(s), \hat{\gamma}_x^{(n)}(t)) \geq 2\delta$, and thus $R_x^{(n)}(\hat{\omega}) \geq 2\delta$ and $F_n(\omega) = 0$ for all $\omega \in \Omega_{n,x} \setminus \Omega_{n,x}^{r(x)+2\delta}$. Outside $\Omega_{n,x}$, F_n vanishes by definition. Since $\Psi_n = 1$ on $\Omega_{n,x}^{r(x)+2\delta}$ and $R_x^{(n)}(\hat{\omega}) = \sup r_x^{(n)} \circ \hat{\omega}$, we have

$$(6.8) \qquad F_n(\omega) \;=\; (2 - \delta^{-1} \sup_{s\in[0,1]} (\Psi_n(\omega)\, r_x^{(n)}(\hat{\omega}(s)))^+ \wedge 1 \qquad \text{for } \omega \in \Omega_{n,x}$$

Let G_s, $s \in [0,1]$, be uniformly bounded functions on L_xM with $G_s(\omega) = r_x^{(n)}(\hat{\omega}(s))$ for $\omega \in \Omega_{n,x}$. By Lemma 6.1 and (6.2), the functions $\Psi_n \cdot G_s$, $0 \leq s \leq 1$, are in $H^{1,2}(L_xM;\, P_x)$. Moreover, by (6.3), (6.2), and since $D^0\Psi_n = 0$ P_x–a.e. on $\Omega_{n,x}^{r(x)+2\delta}$, we have

$$|D^0(\Psi_n \cdot G_s)|_{T^1_\omega L_xM} \leq 1 + \delta^{-1} \cdot \sup |G_s| \; \chi_{L_xM \setminus \Omega_{n,x}^{r(x)+2\delta}}(\omega)$$

for all $\omega \in L_xM$ and $s \in [0,1]$. The assertion now follows by standard arguments, because $F_n = (2 - \delta^{-1} \sup\{\Psi_n \cdot G_s;\, s \in [0,1] \cap \mathbf{Q}\}\,)^+ \wedge 1$, and F_n vanishes outside $\Omega_{n,x}^{r(x)+2\delta}$. $\square$

Proof of Theorem 1.1, based loop space part. Let $x \in U_\varepsilon$. By Proposition 4.3 and Lemma 6.2,

$$\operatorname{esssup}_{\omega \in L_xM} \left(|D^0F_n|^2_{T^1_\omega L_xM} \cdot \frac{P_x[\{\omega \in \Omega_{n,x};\, R_x^{(n)}(\hat{\omega}) \geq \delta\}]}{P_x[\{\omega \in \Omega_{n,x};\, R_x^{(n)}(\hat{\omega}) < \delta\}]} \right) \;\to\; 0$$

as $n \to \infty$. The assertion now follows by Lemma 2.1. $\square$

Appendix A.1 Closability of gradients on loop spaces

Let H_1, H_2 be Hilbert spaces, and let A be a dense subset of H_1. A linear operator $\mathcal{D} : \mathcal{A} \subset H_1 \to H_2$ is called *closable* if $DF \to 0$ in H_2 for every sequence $F_n \in \mathcal{A}$, $n \in \mathbf{N}$, such that $F_n \to 0$ in H_1 and DF_n is Cauchy in H_2. A non–negative definite symmetric bilinear form $\mathcal{E} : \mathcal{A} \times \mathcal{A} \to \mathbf{R}$ is called closable if $\mathcal{E}(F_n, F_n) \to 0$ for every sequence $F_n \in \mathcal{A}$, $n \in \mathbf{N}$, such that $F_n \to 0$ in H_1 and $\mathcal{E}(F_n - F_m, F_n - F_m) \to 0$ as $n, m \to \infty$. Closability of an operator $\mathcal{D}$ as above is hence equivalent to closability of the form $\mathcal{E}(F,G) = (\mathcal{D}F, \mathcal{D}G)_{H_2}$. We now consider the situation described in the introduction with an arbitrary compact Riemannian manifold M.

Proposition *The operators* $D^0 : \mathcal{FC}^\infty \subset L^2(L_xM;\, P_x) \to L^2(T^1L_xM;\, P_x)$, $x \in M$, *and* $D : \mathcal{FC}^\infty \subset L^2(LM;\, P) \to L^2(T^1LM;\, P)$ *are closable.*

The closability of D^0 has first been shown in [DR]. We now recall the proof from this article, and we sketch a simple proof for the closability of D :

Proof. We fix an orthonormal basis $\{h_n;\, n \in \mathbf{N}\}$ of $H_0^{1,2}([0,1], \mathbf{R}^d)$ with Cameron Martin type inner product. Then $\{X^{h_n}(\omega);\, n \in \mathbf{N}\}$ (cf. (1.1)) is an orthonormal basis of $T^1_\omega L_xM$ for every $\omega \in L_xM$. Hence the symmetric bilinear form $\mathcal{E}^0_x$ on $L^2(L_xM;\, P_x)$ defined by (1.8) is given by

$$\mathcal{E}^0_x(F,G) \;=\; \sum_{n=1}^\infty \int (X^{h_n}F)(\omega)\,(X^{h_n}G)(\omega)\; P_x(d\omega) \quad \text{for all } F,G \in \mathcal{FC}^\infty.$$

To prove closability of this form (and thus of D^0) it suffices to show that each of the forms $(F,G) \mapsto \int X^hF\, X^hG\;dP_x$, $h \in H_0^{1,2}([0,1], \mathbf{R}^d)$, with domain $\mathcal{FC}^\infty$ is closable on $L^2(L_xM;\, P_x)$, cf. [MR]. This, however, is a direct consequence of the *integration by parts identity* (1.2) : If $(F_n)_{n \in \mathbf{N}}$ is a null sequence in $L^2(L_xM;\, P_x)$ then by (1.2), $\int X^hF_n\, G\, dP_x \to 0$ for every $G \in \mathcal{FC}^\infty$. So, if in addition $(X^hF_n)_{n\in\mathbf{N}}$ is Cauchy in $L^2(L_xM;\, P_x)$, then $X^hF_n \to 0$ in $L^2(L_xM;\, P_x)$.

To show the closability of D, we fix an $O(M)$ valued stochastic process $(U_s(\omega))_{0 \le s \le 1}$, $\omega \in LM$, such that $\pi(U_s(\omega)) = \omega(s)$ for all $\omega \in LM$ and $s \in [0,1]$, and U is a version of the stochastic horizontal lift w.r.t. P_x for a.e. $x \in M$. In particular, it is a version of the stochastic horizontal lift w.r.t. P. Let X^h, $h \in H_0^{1,2}([0,1], \mathbf{R}^d)$, be the vector fields on LM given by $X^h_s(\omega) = U_s(\omega)h(s)$, $0 \le s \le 1$, $\omega \in LM$. Integrating the integration by parts identity (1.2) over x w.r.t. the measure $p_1(x,x)\,V(dx)$ now yields the same kind of integration by parts formula with L_xM and P_x replaced by LM and P (provided h is in $H_0^{1,2}([0,1], \mathbf{R}^d)$). Moreover, the divergence term β_h has the same expression as in the based loop space case, and is contained in $L^2(LM;\, P)$. The square–integrability w.r.t. P can be seen by the same arguments as that w.r.t. P_x, cf. e.g. [H2], Ch. 4, Prop. 3.4. Let g_n, $n \in \mathbf{N}$, be an orthonormal basis of $H_0^{1,2}([0,1], \mathbf{R}^d)$ w.r.t. the inner product $(h,g)_{1,2} = \int_0^1 (h'g' + hg)\, ds$. Then the symmetric bilinear form

$$\mathcal{E}^0(F,G) \;=\; \sum_{n=1}^\infty \int_{LM} (X^{g_n}F)(\omega)\,(X^{g_n}G)(\omega)\; P(d\omega), \quad F,G \in \mathcal{FC}^\infty,$$

on $L^2(LM;P)$ is closable by a similar argument as above. Notice that

$$\mathcal{E}^0(F,G) \;=\; \int \langle \tilde{D}^0 F, \tilde{D}^0 G\rangle_{T^1_\omega LM}\; P(d\omega),$$

where $\tilde{D}^0$ denotes the gradient on the measurable bundle $T^{1,0}_\omega LM = \{X \in T^1_\omega LM;\; X(0) = 0\}$, $\omega \in LM$, endowed with the $H^{1,2}$ metric defined (1.4) and (1.3). Clearly, $(\tilde{D}^0 F)(\omega)$ is the orthogonal projection of $(DF)(\omega)$ onto the closed subspace $T^{1,0}_\omega LM$ of $T^1_\omega LM$.

Next, fix $s \in (0,1)$. For $F \in \mathcal{FC}^\infty$ and $\omega \in LM$ let $(\tilde{D}^s F)(\omega)$ denote the orthogonal projection of $(DF)(\omega)$ onto the closed subspace

$$T^{1,s}_\omega LM \;=\; \{\, X \in T^1_\omega LM;\; X(s) = 0 \,\}$$

of $T^1_\omega LM$. Then the symmetric bilinear form

$$\mathcal{E}^s(F,G) \;=\; \int \langle \tilde{D}^s F, \tilde{D}^s G\rangle_{T^1_\omega LM}\; P(d\omega), \quad F, G \in \mathcal{FC}^\infty,$$

on $L^2(LM;\, P)$ is closable as well. In fact, because of the invariance of the Bismut measure under the natural action of S^1 on LM by reparametrization, the closability of $\mathcal{E}^s$ follows from the closability of $\mathcal{E}^0$ and straightforward considerations concerning the behaviour of the stochastic parallel transport under reparametrization. Note that

$$\text{(A.1)} \quad \mathcal{E}^t(F,F) \;\leq\; \int \langle DF, DF\rangle_{T^1_\omega LM}\, P(d\omega) \;=\; \mathcal{E}(F,F) \quad \text{for all } F \in \mathcal{FC}^\infty$$

holds for $t = 0$ and $t = s$. In particular,

$$\text{(A.2)} \qquad \mathcal{E}^0(F,F) + \mathcal{E}^s(F,F) \;\leq\; 2\,\mathcal{E}(F,F) \quad \text{for all } F \in \mathcal{FC}^\infty.$$

On the other hand, there exists a finite constant C such that

$$\text{(A.3)} \qquad \mathcal{E}(F,F) \;\leq\; C \cdot (\,\mathcal{E}^0(F,F) + \mathcal{E}^s(F,F)\,) \quad \text{for all } F \in \mathcal{FC}^\infty.$$

To see this, let $f, g : [0,1] \to [0,1]$ be Lipschitz continuous functions such that $f(1) = f(0) = 0$, $g(s) = 0$, and $f(t) + g(t) = 1$ for all $t \in [0,1]$. By the Lipschitz continuity of f and g, there exists a finite constant C_1 such that

$$|fX|_{T^1_\omega LM} \;\leq\; C_1 \cdot |X|_{T^1_\omega LM} \qquad \text{and} \qquad |gX|_{T^1_\omega LM} \;\leq\; C_1 \cdot |X|_{T^1_\omega LM}$$

for all $\omega \in LM$ and $X \in T^1_\omega LM$. In particular, since $X = fX + gX$,

$$\begin{aligned}
|XF| &\leq |f\,XF| + |g\,XF| \\
&\leq |fX|_{T^1_\omega LM} \cdot |(\tilde{D}^0 F)(\omega)|_{T^1_\omega LM} + |gX|_{T^1_\omega LM} \cdot |(\tilde{D}^s F)(\omega)|_{T^1_\omega LM} \\
&\leq C_1 \cdot |X|_{T^1_\omega LM} \cdot (|(\tilde{D}^0 F)(\omega)|_{T^1_\omega LM} + |(\tilde{D}^s F)(\omega)|_{T^1_\omega LM})
\end{aligned}$$

for all $F \in \mathcal{F}\mathcal{C}^\infty$, $\omega \in LM$ and $X \in T^1_\omega LM$, whence

$$|(DF)(\omega)|_{T^1_\omega LM} \;\leq\; C_1 \cdot (|(\tilde{D}^0 F)(\omega)|_{T^1_\omega LM} + |(\tilde{D}^s F)(\omega)|_{T^1_\omega LM}).$$

This proves (A.3).

Since the non–negative definite symmetric bilinear forms $(\mathcal{E}^0, \mathcal{F}\mathcal{C}^\infty)$ and $(\mathcal{E}^s, \mathcal{F}\mathcal{C}^\infty)$ are closable on $L^2(LM;\, P)$, the form $(\mathcal{E}^0 + \mathcal{E}^s, \mathcal{F}\mathcal{C}^\infty)$ is closable as well. (A.2) and (A.3) now imply the closability of $(\mathcal{E}, \mathcal{F}\mathcal{C}^\infty)$ on $L^2(LM;\, P)$, and hence that of the gradient D. $\square$

Corollary *Let A be a function in $L^1(LM;\, P)$ such that $A \geq a_0$ P–a.e. for some constant $a_0 > 0$. Then the symmetric bilinear form*

$$\mathcal{E}^A(F,G) \;=\; \int A(\omega)\, \langle (DF)(\omega), (DG)(\omega) \rangle_{T^1_\omega LM}\, P(d\omega),$$

$F, G \in \mathcal{F}\mathcal{C}^\infty$, is closable on $L^2(LM;\, P)$.

Proof. Let $(F_n)_{n \in \mathbf{N}}$ be a sequence in $\mathcal{F}\mathcal{C}^\infty$ with $F_n \to 0$ in $L^2(LM;\, P)$ and $\mathcal{E}^A(F_n - F_m, F_n - F_m) \to 0$ as $n, m \to \infty$. Since $\mathcal{E}^A(F,F) \geq a_0 \cdot \mathcal{E}(F,F)$ for all $F \in \mathcal{F}\mathcal{C}^\infty$, $|DF_n|_{T^1_\omega LM} \to 0$ in $L^2(LM;\, P)$ by the proposition above. In particular, $|DF_{n_k}|_{T^1_\omega LM} \to 0$ P–a.e. for a sequence $n_k \to \infty$, whence

$$\begin{aligned}
\mathcal{E}^A(F_n, F_n) &= \int \lim_{k \to \infty} \left(A(\omega)\, |DF_n - DF_{n_k}|^2_{T^1_\omega LM}\right) P(d\omega) \\
&\leq \liminf_{k \to \infty} \mathcal{E}^A(F_n - F_{n_k}, F_n - F_{n_k}).
\end{aligned}$$

Thus $\lim_{n \to \infty} \mathcal{E}^A(F_n, F_n) = 0$. $\square$

Appendix A.2 Local Dirichlet forms

We recall some basic facts about local Dirichlet forms, cf. [BH], Ch. I, for details.

Let $(\Omega, \mathcal{F}, \mu)$ be a σ–finite measure space. A densely defined closed symmetric bilinear form $(\mathcal{E}, \mathrm{Dom}\,(\mathcal{E}))$ on $L^2(\Omega;\mu)$ is called a *Dirichlet form* if $F^+ \wedge 1$ is in $\mathrm{Dom}\,(\mathcal{E})$ and $\mathcal{E}(F^+ \wedge 1, F^+ \wedge 1) \leq \mathcal{E}(F,F)$ for every $F \in \mathrm{Dom}\,(\mathcal{E})$. This property implies that the composition $\phi(F_1, \ldots, F_n)$ is in $\mathrm{Dom}\,(\mathcal{E})$ for all $n \in \mathbf{N}$, $F_1, \ldots, F_n \in \mathrm{Dom}\,(\mathcal{E})$, and all Lipschitz continuous functions $\Phi : \mathbf{R}^n \to \mathbf{R}$ such that $\phi(0) = 0$. In particular, FG is in $\mathrm{Dom}\,(\mathcal{E})$ for all $F, G \in \mathrm{Dom}\,(\mathcal{E}) \cap L^\infty(\Omega;\mu)$. A Dirichlet form $(\mathcal{E}, \mathrm{Dom}\,(\mathcal{E}))$ is called *strongly local* if $\mathcal{E}(F,G) = 0$ for all $F, G \in \mathrm{Dom}\,(\mathcal{E})$ such that $(F+c)G = 0$ for some $c \in \mathbf{R}$. One says that $(\mathcal{E}, \mathrm{Dom}\,(\mathcal{E}))$ *admits a carré du champ* if there exists a symmetric continuous bilinear operator $\Gamma : \mathrm{Dom}\,(\mathcal{E}) \times \mathrm{Dom}\,(\mathcal{E}) \to L^1(\Omega;\mu)$ such that

$$\text{(A.4)} \qquad \mathcal{E}(FH,G) + \mathcal{E}(GH,F) - \mathcal{E}(H,FG) \;=\; 2\int H\,\Gamma(F,G)\,d\mu$$

holds for all F, G, $H \in \mathrm{Dom}\,(\mathcal{E}) \cap L^\infty(\Omega;\mu)$. Γ is uniquely determined by (A.4). The carré du champ of a strongly local Dirichlet form with $1 \in \mathrm{Dom}\,(\mathcal{E})$ satisfies the chain rule (2.3). On the other hand, if $(\mathcal{E}, \mathrm{Dom}\,(\mathcal{E}))$ is a closed quadratic form on $L^2(\Omega;\mu)$, and there exists a symmetric bilinear operator $\Gamma : \mathrm{Dom}\,(\mathcal{E}) \times \mathrm{Dom}\,(\mathcal{E}) \to L^1(\Omega;\mu)$ satisfying (2.1), (2.2) and (2.3), then $(\mathcal{E}, \mathrm{Dom}\,(\mathcal{E}))$ is a strongly local Dirichlet form, and Γ is the corresponding carré du champ.

The carré du champ operator of a strongly local Dirichlet form satisfies

$$\text{(A.5)} \qquad \Gamma(F) = 0 \quad \mu\text{–a.e. on } \{\omega \in \Omega;\ F(\omega) = c\}$$

for every $F \in \mathrm{Dom}\,(\mathcal{E})$ and $c \in \mathbf{R}$, where $\Gamma(F) = \Gamma(F,F)$, cf. [BH], Ch. I, Thm. 7.1.1. Moreover :

Lemma *Let $(\mathcal{E}, \mathrm{Dom}\,(\mathcal{E}))$ be a strongly local Dirichlet form with carré du champ Γ.*
(i) If $1 \in \mathrm{Dom}\,(\mathcal{E})$ then $\phi \circ F$ is in $\mathrm{Dom}\,(\mathcal{E})$, and

$$\Gamma(\phi \circ F) \;=\; (\phi' \circ F)^2\,\Gamma(F) \qquad \mu\text{–a.e.}$$

for all $F \in \mathrm{Dom}\,(\mathcal{E})$ and every Lipschitz continuous function $\phi : \mathbf{R} \to \mathbf{R}$.

(ii) Let F_k, $k \in \mathbf{N}$, be functions in $\mathrm{Dom}\,(\mathcal{E})$ *such that* $\sup_{k\in\mathbf{N}}(|F_k|^2 + \Gamma(F_k))$ *is in* $L^1(\Omega;\mu)$. *Then* $\sup_{k\in\mathbf{N}} F_k$ *and* $\inf_{k\in\mathbf{N}} F_k$ *are in* $\mathrm{Dom}\,(\mathcal{E})$,

$$
\begin{aligned}
\Gamma(\sup F_k) &\leq \sup \Gamma(F_k) \qquad \mu\text{–}a.e., \qquad \textit{and} \\
\Gamma(\inf F_k) &\leq \sup \Gamma(F_k) \qquad \mu\text{–}a.e.
\end{aligned}
$$

Proof. (i) See [BH], Ch. I, Cor. 7.1.2.

(ii) The assertion follows from [BH], Ch. I, Prop. 4.14, and [MR], Ch. I, Lem. 2.12. □

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