SK_1-LIKE FUNCTORS FOR DIVISION ALGEBRAS

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Abstract. We investigate the group valued functor $G(D) = D^*/F^*D'$ where D is a division algebra with center F and D' the commutator subgroup of D^*. We show that G has the most important functorial properties of the reduced Whitehead group SK_1. We then establish a fundemental connection between this group, its residue version and relative value group when D is a Henselian division algebra. The structure of $G(D)$ turns out to carry significant information about the arithmetic of D. Along these lines, we employ $G(D)$ to compute the group $SK_1(D)$. As an application, we obtain theorems of reduced K-theory which require heavy machinery, as simple examples of our method.

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1. Introduction

Let D be a division algebra with center F. The non-triviality of the important group $SK_1(D)$ is shown by V. P. Platonov who developed a so-called *Reduced K-Theory* to compute $SK_1(D)$ for certain division algebras. The group $SK_1(D)$ enjoys some interesting properties which distinguish it from the *K-Theory* functor $K_1(D)$. An interesting characteristic of the group $SK_1(D)$ is its behavior under extension of the ground field. Namely for any field extension L/F one has a homomorphism $SK_1(D) \to SK_1(D \otimes_F L)$. On the other hand SK_1 enjoys a transfer map, that is, if L/F is a finite extension, then there exist a norm homomorphism $SK_1(D \otimes_F L) \to SK_1(D)$. Since $SK_1(M_n(L)) = 1$, one can then deduce that SK_1 is a torsion abelian group of bounded exponent $i(D)$ and if the degree $[L : F]$ is relatively prime to index of D, then $SK_1(D) \hookrightarrow SK_1(D \otimes_F L)$. Moreover the primary decomposition of a division algebra induces a corresponding decomposition of $SK_1(D)$. Furthermore in the case of a valued division algebra SK_1 is stable, namely $SK_1(D) = SK_1(\overline{D})$, where D is unramified division algebra. (See [12] for the complete list of the properties of SK_1 and [3] for the proofs).

In this note we investigate the group $G(D) = D^*/F^*D'$ where D is a division algebra with center F and D' the commutator subgroup of D^*. We shall show that

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G enjoys most important functorial properties of the reduced Whitehead group SK_1. We show that the functor G may grow "pathologically" for an algebraic extension of the ground field whose degree is prime to the index of D. It is then shown that this functor satisfies a decomposition property analogous to one for $SK_1(D)$. To be more precise, we will show the following properties:

i. For any field extension L/F one has a homomorphism $G(D) \to G(D \otimes_F L)$.

ii. If L/F is a finite extension, then there exist a transfer homomorphism $G(D \otimes_F L) \to G(D)$. (Proposition 2.3)

iii. $G(D)$ is a torsion group of bounded exponent $i(D)$ (Corollary 2.4, Lemma 2.9)

iv. If $[L : F]$ is relatively prime to $i(D)$, then $G(D) \hookrightarrow G(D \otimes_F L)$. (Corollary 2.5).

v. If $D = D_1 \otimes_F D_2 \otimes_F \cdots \otimes_F D_K$ and the $i(D_i)$ are relatively prime, then $G(D) \simeq \coprod G(D_i)$. (Theorem 2.8)

vi. If D is unramfied tame Henselian division algebra, then $G(D) \simeq G(\overline{D})$. (Theorem 3.2 i)

It turns out that there is a close connection between the group structure of $G(D)$ and algebraic structure of D. For example in section 3, after establishing a fundemental connection between $G(D)$, its residue version and relative value group when D admits a Henselian valuation, we show that if D is a totally ramified division algebra, then there is a one to one correspondence between the isomorphism classes of F-subalgebras of D and the subgroups of $G(D)$.

We then use $G(D)$ to compute $SK_1(D)$ for certain division algebras. We show that if $G(D)$ canonically coincides with the relative value group, then there is an explicit formula for the group $SK_1(D)$ (Theorem 3.8). It turns out that some theorems and examples of reduced K-theory which require heavy machinery can all be viewed as simple examples of our case (Example 3.10, 3.11 and 3.13). Section 4 is devoted to the unitary version of the group $G(D)$.

We fix some notation. Let D be a division algebra over its center F with index $i(D) = n$. Then $Nrd_{D/F} : D^* \longrightarrow F^*$ is the reduced norm function and $SK_1(D) = D^{(1)}/D'$ is the reduced Whitehead group where $D^{(1)}$ is the kernel of $Nrd_{D/F}$. Put $SH^0(D)$ for the cokernel of $Nrd_{D/F}$. we take $\mu_n(F)$ for the group of $n-th$ roots of unity in F, and $Z(D')$ for the center of the group D'. Observe that $\mu_n(F) = F^* \cap D^{(1)}$ and $Z(D') = F^* \cap D'$. If G is a group, denote by G^n the subgroup of G generated by the $n-th$ powers of elements of G. Let $exp(G)$ stands for the exponent of the group G. If H and K are subgroups of G, denote by $[H, K]$ the subgroup of G generated by mixed-commutators $[h, k] = hkh^{-1}k^{-1}$, where $h \in H$ and $k \in K$. For convenience we denote $[D^*, D^*]$ by D'. Denote by $det : GL_n(D)/SL_n(D) \longrightarrow D^*/D'$ the Dieudonne determinant, where $GL_n(D)$ is the general linear group and $SL_n(D)$ is its commutator subgroup (See [3]).

2. Functor $G(D) = D^*/F^*D'$

Let $\mathcal{C}$ be the category of all central simple algebras and $G : \mathcal{C} \longrightarrow \mathcal{A}b$ be a covariant functor from $\mathcal{C}$ to the category of abelian groups such that for any central simple algebra A with center F, $G(A) = A^*/F^*A'$.

It is easy to observe that the functor G has the following properties:

D1. There is a collection of homomorphisms $d_n : G(M_n(D)) \longrightarrow G(D)$ for each division algebra D and each positive integer n such that for each $x \in G(D)$, $d_n i_n(x) = x^n$ where $i_n : G(D) \longrightarrow G(M_n(D))$ is the homomorphism induced by the natural embedding $D \longrightarrow M_n(D)$ and d_n induced by Dieudonne determinant [3, §20].

D2. For any field F, $G(F)$ is trivial.

D3. If $x \in Ker(G(M_n(D)) \xrightarrow{d_n} G(D))$, where D is a division algebra and $n \in \mathbb{N}$, then $x^n = 1$.

On the other hand there have been other groups associated with a division ring D which have been used to study the arithmetic and algebraic structure of D. For example the square class group D^*/D^{*2} in [10] in connection with Witt ring of a division algebra or the group $D^*/Nrd(D^*)D'$ in [1]. The following examples show that some important groups already associated to D share the three conditions above.

Example 2.1. Let $A \in \mathcal{C}$ with center F, then it is easy to observe that functors $\mathfrak{G}(A) = (A^*)^2/(F^*)^2A'$, and $\mathfrak{G}(A) = A^*/F^*A'_r$ where $A'_r = \{x \in A^* | x^r \in A'\}$ and $r \in \mathbb{N}$ also satisfy the properties **D1,D2** and **D3** above.

Example 2.2. Let $A \in \mathcal{C}$ be a central simple algebra finite over its center. The following commutative diagram with exact rows shows that $SK_1(D) = D^{(1)}/D'$ and $SH^0(D) = F^*/Nrd_{D/F}(D^*)$ satisfy the three conditions above,

$$\begin{array}{ccccccccc}
1 & \longrightarrow & SK_1(D) & \longrightarrow & K_1(D) & \xrightarrow{Nrd_{D/F}} & F^* & \longrightarrow & SH^0(D) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \eta_n & & \downarrow & & \\
1 & \longrightarrow & SK_1(M_n(D)) & \longrightarrow & K_1(M_n(D)) & \xrightarrow{Nrd_{D/F}} & F^* & \longrightarrow & SH^0(M_n(D)) & \longrightarrow & 1 \\
 & & \downarrow & & det \downarrow & & \downarrow 1 & & \downarrow & & \\
1 & \longrightarrow & SK_1(D) & \longrightarrow & K_1(D) & \xrightarrow{Nrd_{D/F}} & F^* & \longrightarrow & SH^0(D) & \longrightarrow & 1
\end{array}$$

where $\eta_n(x) = x^n$ for any $x \in F^*$ and D is a division algebra with center F. Note that in order to consider SK_1 and SH^0 as functors, we should limit the objects of our category (See §22, §23 in [3]).

In the same way, it can be seen that $\mathfrak{G}(A) = A^*/Nrd_{A/F}(A^*)A'$ and $\mathfrak{G}(A) = A^*/F^*A^{(1)} \simeq Nrd(A^*)/F^{*\,DegA}$ also satisfy **D1,D2** and **D3**.

For the rest of this section we restrict our attention to the functor $G(A) = A^*/F^*A'$, although the results we get can be formulated and proved *mutatis mutandis* for all the functors above.

Our primary aim in this section is to show that the functor G shares almost all important functorial properties of SK_1. Clearly the natural embedding of D in $D \otimes_F L$ where L is a finite field extension of F, induces a group homomorphism $\mathcal{I} : G(D) \longrightarrow G(D \otimes_F L)$. The following proposition provides us with a homomorphism in the opposite direction.

Proposition 2.3. **(Transfer map)** *Let D be a division ring with center F and L be a finite extension of F such that $[L : F] = m$. Then there is a homomorphism $\mathcal{P} : G(D \otimes_F L) \longrightarrow G(D)$ such that $\mathcal{PI} = \eta_m$, where $\eta_m(x) = x^m$.*

Proof. Consider the regular representation $L \xrightarrow{\iota} M_m(F)$ and the corresponding sequence when we tensor over F with D:

$$D \longrightarrow D \otimes_F L \xrightarrow{1\otimes\iota} D \otimes_F M_m(F) \longrightarrow M_m(D)$$

$$a \longmapsto a \otimes 1 \longmapsto a \otimes 1 \longmapsto aI_m \tag{2.1}$$

$$1 \otimes \ell \longmapsto 1 \otimes \iota(\ell) \longmapsto \iota(\ell).$$

Thanks to the Dieudonne determinant, there is a homomorphism $K_1(D \otimes_F L) \to K_1(D)$ which maps the center of $D \otimes_F L$ into the center of D. Therefore $G(D \otimes_F L) \to G(D)$. Again the sequence (2.1) shows that $\mathcal{PI}(x) = x^m$. $\square$

Note that in the above proposition D could be an infinite dimensional division algebra. If D is finite dimension over its center F, then it turns out that $G(D)$ is a torsion group.

Corollary 2.4. *Let D be a division algebra of index n. Then $G(D)$ is a torsion group of bounded exponent $n^2 = [D : Z(D)]$.*

Proof. Thanks to Proposition 2.3, for any finite field extension L of $F = Z(D)$, we have the sequence of homomorphisms $G(D) \xrightarrow{\mathcal{I}} G(D \otimes_F L) \xrightarrow{\mathcal{P}} G(D)$, such that $\mathcal{PI}(x) = x^m$, where $x \in \mathfrak{G}(D)$ and $[L : F] = m$. Now let L be a maximal subfield of D. Since L is a splitting field for D, we get the sequence of homomorphisms $G(D) \xrightarrow{\mathcal{I}} G(M_n(L)) \xrightarrow{\mathcal{P}} G(D)$. From **D2** and **D3** it follows that $G(M_n(L))$ is a torsion group of bounded exponent n. Now the fact that for any $x \in G(D)$, $\mathcal{PI}(x) = x^n$, shows that $G(D)$ is a torsion group of bounded exponent $n^2 = [D : Z(D)]$. $\square$

It is now immediate that if A is a central simple algebra, then $G(A)$ is also torsion. Later in this section we show that the bound can be reduced to n, the index of D.

The following corollary shows that the analogous result for the behavior of SK_1 under extension of the ground field holds for G too. Namely, we show that $G(D)$ embeds in $G(D \otimes_F L)$ when the index of D and $[L : F]$ are relatively prime.

Corollary 2.5. *Let D be a division ring over its center F and L/F be a finite field extension such that $[L : F]$ is relatively prime to the index of D. Then the canonical homomorphism $G(D) \xrightarrow{\mathcal{I}} G(D \otimes_F L)$ is injective.*

Proof. Let $i(D) = n$ and $[L : F] = m$. Suppose $\mathcal{I}(x) = 1$ for some $x \in G(D)$. By Proposition 2.3, $\mathcal{P}\mathcal{I}(x) = x^m = 1$. But $G(D)$ is torsion of bounded exponent n^2. Hence $x^{n^2} = 1$. Since m and n are relatively prime, $x = 1$ and the proof is complete. □

In the next section we compute the functor G for certain division algebras. But before we continue with the functorial properties of G, let us consider the case when the group $G(D)$ is trivial. Besides **D1, D2** and **D3**, the functor G enjoys an additional property, namely there is a natural transformation $\tau : K_1 \longrightarrow G$ such that,

(1) For any object A in $\mathcal{C}$, $\tau_A : K_1(A) \longrightarrow G(A)$ is an epimorphism.

(2) For any division algebra D and any positive integer n, the following diagram commutes,

$$\begin{array}{ccc} K_1(M_n(D)) & \xrightarrow{\tau} & G(M_n(D)) \\ \Big\downarrow{\scriptstyle det} & & \Big\downarrow{\scriptstyle d} \\ K_1(D) & \xrightarrow{\tau} & G(D). \end{array}$$

Note that the functors of Example 2.1, or $\mathfrak{G}(D) = Nrd_{D/F}(D^*)/F^{*i(D)}$ and $\mathfrak{G}(A) = A^*/Nrd_{A/F}(A^*)A'$ satisfy the above property as well.

The following is almost the only known example where $G(D)$ (and above functors) is trivial.

Corollary 2.6. *Let D be a division algebra of quaternions over a real-closed field. Then $G(D) = 1$.*

Proof. For any finite field extension L of $F = Z(D)$, the following diagram is commutative,

$$\begin{array}{ccc} K_1(D \otimes_F L) & \xrightarrow{\mathcal{P}} & K_1(D) \\ \Big\downarrow{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle \tau} \\ G(D \otimes_F L) & \xrightarrow{\mathcal{P}} & G(D). \end{array}$$

Now since D is algebraically closed (See [9], Section 16), thanks to Proposition 2.3 and above diagram, $G(D\otimes_F L)\xrightarrow{\mathcal{P}} G(D)$ is an epimorphism. Replace L by $\overline{F}$, the algebraic closure of F. Because $\overline{F}$ is a splitting field for D, $G(D\otimes_F\overline{F})=G(M_2(\overline{F}))$. We show that $G(M_2(\overline{F}))$ is a trivial group and hence the corollary follows. Since $\tau: K_1\longrightarrow G$ is a natural epimorphism, there is a composite homomorphism

$$\psi: K_1(\overline{F})=\overline{F}^*\xrightarrow{\simeq} K_1(M_2(\overline{F}))\xrightarrow{epi.} G(M_2(\overline{F})).$$

Take $x\in G(M_2(\overline{F}))$. Since $\overline{F}$ is algebraically closed, there exist $y\in K_1(\overline{F})=\overline{F}^*$ such that $\psi(y^2)=x$. But $G(M_2(\overline{F}))$ is a torsion group of bounded exponent 2, hence $x=1$. This shows that $G(M_2(\overline{F}))$ is trivial and the proof is complete. $\square$

Back to the functorial properties of G, the next step is to replace the field L in Proposition 2.3 by a division ring. The following proposition shows that the same result holds here too.

Proposition 2.7. *Let A and B be division algebras with center F such that $[B:F]$ is finite. Then there is a homomorphism $\mathcal{P}: G(A\otimes_F B)\longrightarrow G(A)$ such that $\mathcal{P}\mathcal{I}=\eta_{[B:F]}$.*

Proof. Let $[B:F]=m$. We have the following sequence of F-algebra homomorphisms,

$$A\longrightarrow A\otimes_F B\longrightarrow A\otimes_F B\otimes_F B^{op}\longrightarrow A\otimes_F M_m(F)\longrightarrow M_m(A).$$

This implies the group homomorphism $\mathcal{P}: G(A\otimes_F B)\longrightarrow G(M_m(A))\xrightarrow{d} G(A)$. The rest of the proof follows from **D1**. $\square$

Note that in the above proposition A could be of infinite dimension over its center F. A same statement as Corollary 2.5 could be obtained here too. In particular if $(i(A),i(B))=1$ then $G(A)$ embeds in $G(A\otimes_F B)$ and similarly for B. Employing torsion theory of groups and sequences which appeared in the above propositions, we can write the primary decomposition for $G(D)$. The proof follows more or less the same pattern as for $SK_1(D)$.

Theorem 2.8. *Let A and B be division algebras with center F such that $(i(A),i(B))=1$. Then $G(A\otimes_F B)=G(A)\times G(B)$.*

Proof. By Corollary 2.4, $G(A\otimes_F B)$ is a torsion group of bounded exponent m^2n^2 where $m=i(A)$ and $n=i(B)$. Therefore $G(A\otimes_F B)\simeq\mathcal{G}\times\mathcal{H}$, where $exp(\mathcal{G})|m^2$ and $exp(\mathcal{H})|n^2$. By Proposition 2.7, we have the sequence:

$$G(A)\xrightarrow{\phi} G(A\otimes_F B)\xrightarrow{\psi} G(A\otimes B\otimes B^{op})\xrightarrow{\theta} G(A) \tag{2.2}$$

such that $\theta\psi\phi=\eta_{n^2}$. Hence $G(A)=\eta_{n^2}\eta_{n^2}(G(A))=\eta_{n^2}\theta\psi\phi(G(A))\subseteq\theta\psi\eta_{n^2}(\mathcal{G}\times\mathcal{H})=\theta\psi(\mathcal{G})\subseteq G(A)$. This shows that $\theta\psi|_{\mathcal{G}}:\mathcal{G}\longrightarrow G(A)$ is surjective. Next we show that $\theta\psi|_{\mathcal{G}}$ is injective. Considering the regular representation $B^{op} \longrightarrow M_{n^2}(F)$. As Proposition 2.3, we have the following sequence

$$G(A \otimes_F B) \xrightarrow{\psi} G(A \otimes B \otimes B^{op}) \xrightarrow{\psi'} G(A \otimes B \otimes M_{n^2}(F)) \xrightarrow{\theta'} G(A \otimes_F B)$$

such that $\theta'\psi'\psi = \eta_{n^2}$. Now if $1 \neq w \in \mathcal{G}$, then $\theta'\psi'\psi(w) = \eta_{n^2}(w) = w^{n^2} \neq 1$. Therefore $\psi|_{\mathcal{G}}$ is injective. Rewrite the sequence (2.2) as follows:

$$G(A \otimes_F B) \xrightarrow{\psi} G(A \otimes B \otimes B^{op}) \xrightarrow{iso.} G(M_{n^2}(A)) \xrightarrow{d} G(A).$$

Suppose $x \in \mathcal{G}$ such that $\theta\psi(x) = 1$. The above sequence and **D3** shows that $\psi(x)^{n^2} = 1$. Since $\psi|_{\mathcal{G}}$ is injective, $x^{n^2} = 1$. On the other hand because $exp(\mathcal{G})|m^2$ then $x^{m^2} = 1$. Since m and n are relatively prime, $x = 1$. This shows that $\theta\psi$ is an isomorphism and so $G(A) \simeq \mathcal{G}$. In the similar way it can be shown that $G(B) \simeq \mathcal{H}$. Therefore the proof is complete. $\square$

Let $A = M_m(D)$ be a central simple algebra. From Corollary 2.4 and **D3** it is immediate that $G(A)$ is a torsion group of bounded exponent $m[D : Z(D)]$. The following lemma, which is interesting in its own right, will be used to reduce the bound of the group $G(D)$. Also we will use the the lemma in Section 3 for normal subgroups of D^* which arise from a valuation on D.

Lemma 2.9. **[5]** *Let D be a division algebra over its center F with index n. Let N be a normal subgroup of D^*. Then $N^n \subseteq Z(N)[D^*, N]$.*

If in the above lemma we take $N = D^*$, then for any $x \in D^*$, $x^n \in F^*D'$. This in effect shows that $G(D) = D^*/F^*D'$ is a torsion group of bounded exponent n.

In the next section we will show yet another SK_1-like property for the group $G(D)$. Namely $G(D)$ satisfy the following stability, $G(D) \simeq G\big(D((x))\big)$ where $D((x))$ is the division ring of formal Laurent series (Corollary 3.7). We close this section by the following theorem, which shows that the group $G(D) = D^*/F^*D'$ does not always follow the same pattern as the reduced Whitehead group $SK_1(D)$. Namely $G(D)$ is not "homotopy invariant".

Theorem 2.10. **(J. -P. Tignol)** *Let D be a division algebra over its center F with index n. Then the following sequence where $\wp$ runs over the irreducible monic polynomials of $F[x]$ and $n_\wp$ is the index of $D \otimes_F F[x]/\wp$, is split exact.*

$$1 \longrightarrow G(D) \longrightarrow G(D(x)) \longrightarrow \bigoplus_{\wp} \frac{\mathbb{Z}}{n/n_\wp\mathbb{Z}} \longrightarrow 1.$$

Proof. By Proposition 7 in [10], the sequence

$$1 \longrightarrow K_1(D) \longrightarrow K_1(D(x)) \longrightarrow \bigoplus_{\wp} n_\wp/n\mathbb{Z} \longrightarrow 1$$

which is obtained from the localization exact sequence of algebraic K-theory is split exact. Now since the group $G(D)$ is the cokernel of the natural map $K_1(F) \longrightarrow K_1(D)$, applying the snake lemma to the commutative diagram,

$$\begin{tikzcd} 1 \arrow[r] & K_1(F) \arrow[r] \arrow[d] & K_1(F(x)) \arrow[r] \arrow[d] & \bigoplus_{\wp} \mathbb{Z} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & K_1(D) \arrow[r] & K_1(D(x)) \arrow[r] & \bigoplus_{\wp} n_{\wp}/n\mathbb{Z} \arrow[r] & 1 \end{tikzcd}$$

the result follows. $\square$

3. On The Group $G(D)$ Over Henselian Division Algebras

In this section we assume that D is a finite dimensional division algebra over a Henselian field $F = Z(D)$. Recall that a valuation v on F is called *Henselian* if and only if v has a unique extension to each field algebraic over F. Therefore v has a unique extension denoted also by v to D (see [8] and [17]). Denote by V_D, V_F the valuation rings, M_D, M_F their maximal ideals, and $\overline{D}, \overline{F}$ the residue division algebra and the residue field of D and F respectively. We let Γ_D, Γ_F denote the value groups of v on D and F respectively and U_D, U_F for the groups of units of V_D, V_F respectively. Furthermore, we assume that D is a *tame* division algebra, i.e., $Z(\overline{D})$ is separable over $\overline{F}$ and $Char\overline{F}$ does not divide $i(D)$. The quotient group Γ_D/Γ_F is called the *relative value group* of the valuation. In this setting it turns out that D is *defectless*, namely we have $[\overline{D} : \overline{F}][\Gamma_D : \Gamma_F] = [D : F]$. D is said to be *unramified* over F if $[\Gamma_D : \Gamma_F] = 1$. At the other extreme D is said to be *totally ramified* if $[D : F] = [\Gamma_D : \Gamma_F]$. D is called *semiramified* if $\overline{D}$ is a field and $[\overline{D} : \overline{F}] = [\Gamma_D : \Gamma_F] = i(D)$. Since the valuation is Henselian, Hensel's lemma can be used to obtain a relation between the reduced norm of D and that of its residue algebra, i.e.

$$\overline{Nrd_D(a)} = N_{Z(\overline{D})/\overline{F}} Nrd_{\overline{D}}(\overline{a})^{n/mm'}, \tag{3.1}$$

where $a \in U_D$ and $m = i(\overline{D})$ and $m' = [Z(\overline{D}) : \overline{F}]$ (see [4]).

For a recent account of the theory of Henselian valued division algebras see [8]. We start with the following theorem which describes a fundemental connection between the group $G(D)$ and its residue version.

Theorem 3.1. *Let D be a tame division algebra over a Henselian field $F = Z(D)$ with index n. Let L/F be a subfield of D. Then the following sequence is exact.*

$$1 \longrightarrow \overline{D}^*/\overline{L}^*\overline{D'} \longrightarrow D^*/L^*D' \longrightarrow \Gamma_D/\Gamma_L \longrightarrow 1. \tag{3.2}$$

Proof. Consider the normal subgroup $1+M_D$ of D^*. Thanks to Lemma 2.9, we have

$$(1+M_D)^n \subseteq ((1+M_D)\cap F^*)[D^*, 1+M_D]. \tag{3.3}$$

We will show that $(1+M_D) = (1+M_D)^n$. Let $a \in 1+M_D$. Consider the field $F(a)$ and $a \in 1+M_{F(a)}$. Since F is a Henselian field, so is $F(a)$. The polynomial $f(x) = x^n - a$ has 1 as a simple root modulo $M_{F(a)}$, because $Char\overline{F(a)}$ does not divide n. Applying Hensel's lemma to the polynomial $f(x) = x^n - a$, we obtain an element $b \in 1+M_{F(a)}$ such that $b^n = a$. This shows that $a \in (1+M_D)^n$. Thus $1+M_D$ is *n-divisible*, namely, $(1+M_D) = (1+M_D)^n$. Hence from (3.3) it follows that $1+M_D \subseteq (1+M_F)D'$. Now consider the reduction map $U_D \longrightarrow \overline{D}^*$. We have the following sequence:

$$\overline{D}^* \xrightarrow{\simeq} U_D/1+M_D \xrightarrow{nat.} U_D/(1+M_F)D' \xrightarrow{nat.} U_D/(1+M_L)D' \xrightarrow{nat.}$$
$$\xrightarrow{nat.} U_D/U_LD' \xrightarrow{\simeq} L^*U_D/L^*D'.$$

Therefore $\psi : \overline{D}^*/(\overline{L}^*)\overline{D'} \longrightarrow L^*U_D/L^*D'$ is an isomorphism. Considering the fact that $D^*/L^*U_D \simeq \Gamma_D/\Gamma_L$, the theorem follows. □

Now we are ready to compute $G(D)$ for some certain cases. The statements *i.* and *ii.* of the following theorem first appeared in [7] using results from reduced K-theory.

Theorem 3.2. *Let D be a Henselian division algebra tame over its center F with index n. Then*

i. If D is unramified over F then $G(D) \simeq G(\overline{D})$.
ii. If D is totally ramified over F then $G(D) = \Gamma_D/\Gamma_F$.
iii. If D is semiramified and $\overline{D}$ is cyclic over $\overline{F}$ then the following sequence where $N_{\overline{D}/\overline{F}}$ is the norm function, is exact.

$$1 \longrightarrow N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*n} \longrightarrow G(D) \longrightarrow \Gamma_D/\Gamma_F \longrightarrow 1.$$

Proof. *i.* Writing (3.2) for $L=F$, we have:

$$1 \longrightarrow \overline{D}^*/\overline{F}^*\overline{D'} \longrightarrow G(D) \longrightarrow \Gamma_D/\Gamma_F \longrightarrow 1.$$

Now if (D,v) is unramified, namely $[\Gamma_D : \Gamma_F] = 1$, then $\overline{D}^*/\overline{F}^*\overline{D'} \simeq D^*/F^*D'$. On the other hand $Z(\overline{D}) = \overline{F}$ and $D^* = F^*U_D$. Therefore, for $a, b \in D^*$, the element $c = aba^{-1}b^{-1}$ may be written in the form $c = \alpha\beta\alpha^{-1}\beta^{-1}$, where α and $\beta \in U_D$. This shows $\overline{D'} = \overline{D}'$, so $G(D) \simeq G(\overline{D})$.

ii. If D is totally ramified over F then $\overline{D} = \overline{F}$. Writing (3.2) for $L = F$, since the group $\overline{D}^*/\overline{F}^*\overline{D'}$ is trivial $G(D) = \Gamma_D/\Gamma_F$.

iii. Let D be semiramified and $\overline{D}$ be cyclic over $\overline{F}$. Consider the norm function $N_{\overline{D}/\overline{F}} : \overline{D}^* \longrightarrow \overline{F}^*$. Moreover for any $x \in U_D$, from (3.1) it follows that, $\overline{Nrd_{D/F}(x)} = N_{\overline{D}/\overline{F}}(\overline{x})$. This shows that $\overline{D'} \subseteq KerN_{\overline{D}/\overline{F}}$. But if $x \in KerN_{\overline{D}/\overline{F}}$ then by Hilbert theorem 90, there exists $\overline{a}$ such that $x = \overline{a}\sigma(\overline{a})^{-1}$, where σ is the generator of $Gal(\overline{D}/\overline{F})$. It is well known that the *fundemental homomorphism* $D^* \longrightarrow Gal(Z(\overline{D})/\overline{F})$ is surjective. Therefore $\sigma : \overline{D} \longrightarrow \overline{D}$ is of the form $\sigma(\overline{a}) = \overline{cac^{-1}}$, for some $c \in D^*$. This shows that $x \in \overline{D'}$. Therefore $KerN_{\overline{D}/\overline{F}} = \overline{D'}$. Now it is easy to see that $\overline{D}^*/\overline{F}^*\overline{D'} \simeq N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*n}$. So thanks to (3.2), $1 \longrightarrow N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*n} \longrightarrow G(D) \longrightarrow \Gamma_D/\Gamma_F \longrightarrow 1$ is exact. □

Remark 3.3. If $\overline{D}$ is a cyclic field extension of $\overline{F}$, a similar proof as *iii.* above shows that $KerN_{\overline{D}/\overline{F}} \subseteq \overline{D'}$. In particular it follows that $N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*f} \longrightarrow \overline{D}^*/\overline{F}^*\overline{D'}$, where $[\overline{D} : \overline{F}] = f$, is always surjective. Therefore if $N_{\overline{D}/\overline{F}}(\overline{D}^*)/\overline{F}^{*f} = 1$ then $G(D) = \Gamma_D/\Gamma_F$. This will be used in Example 3.10.

Example 3.4. Let $\mathbb{C}$ be the field of complex numbers. Let $1 \neq \sigma \in Gal(\mathbb{C}/\mathbb{R})$ where $\mathbb{R}$ is real numbers. Then by Hilbert construction (See [3], §1), $D = \mathbb{C}((x, \sigma))$ is a division ring with center $F = \mathbb{R}((x^2))$. We show that $G(D) = \mathbb{Z}_2$. D has a natural valuation such that $\Gamma_D/\Gamma_F = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$. Clearly $\overline{D} = \mathbb{C}$ and $\overline{F} = \mathbb{R}$. Since $N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}) = \mathbb{R}^2$ by Theorem 3.2 *iii.*, $G(D) = \Gamma_D/\Gamma_F = \mathbb{Z}_2$.

Example 3.5. Let $q \geq 3$ be a prime number. Take a prime number $p \neq q$ such that $(p-1, q) = 1$. Consider the cyclic extension $\mathbb{F}_{p^q}/\mathbb{F}_p$ where $\mathbb{F}_p$ and $\mathbb{F}_{p^q}$ are fields with p and p^q elements respectively. Let σ be a generator of the cyclic group $Gal(\mathbb{F}_{p^q}/\mathbb{F}_p)$. By Hilbert construction $D = \mathbb{F}_{p^q}((x, \sigma))$ is a division algebra with center $F = Z(D) = \mathbb{F}_p((x^q))$ and $i(D) = ord(\sigma) = q$. D has a natural valuation which is tame and Henselian. It is easy to see that with this valuation D is semiramified, $\overline{D} = \mathbb{F}_{p^q}$ and $\overline{F} = \mathbb{F}_p$. Since $N_{\overline{D}/\overline{F}}$ is surjective, by Theorem 3.2 *iii*, it follows that $G(D) = \mathbb{Z}_q$.

There have been significant results on the structure of the relative value group in the case of a totally ramified algebra. Using Theorem 3.2 we can write interesting statements relating the group structure of $G(D)$ to the algebraic structure of D. Recall that the group $G(D)$ is torsion of bounded exponent n.

Theorem 3.6. *Let D be a valued division algebra tame and totally ramified over a Henselian field $F = Z(D)$ of index n. Then,*

i. There is a one to one correspondence between the isomorphism classes of F-subalgebras of D and the subgroups of $G(D)$.

ii. $exp(G(D))$ divides the exponent of D, i.e., the order of $[D]$ in $Br(F)$, the Brauer group of F.

iii. D is a cyclic division algebra if and only if $exp(G(D)) = n$.

Proof. The theorem follows by comparing Theorem 3.2 *ii.*, with the results on the relative value group in the case of a totally ramified valuation (See for example [17]). $\square$

Corollary 3.7. *Let D be a finite dimensional division algebra over its center F. If $Char F \nmid i(D)$ then $G(D) \backsimeq G(D((x)))$.* $\square$

Now we are in a position to use the group $G(D)$ to compute $SK_1(D)$. The following theorem enables us to compute $SK_1(D)$ when, roughly speaking, $G(\overline{D})$ is trivial. Note that we do not use any results from reduced K-theory.

Theorem 3.8. *Let D be a tame division algebra over a Henselian field $F = Z(D)$ of index n.*

i. If $\overline{D}^*/\overline{F}^*\overline{D'} = 1$ then $SK_1(D) = \mu_n(F)/Z(D')$.

ii. If D is a cyclic division algebra with a maximal cyclic extension L/F such that $\overline{D}^*/\overline{L}^*\overline{D'} = 1$ then $SK_1(D) = 1$.

Proof. i. As the proof of Theorem 3.1 shows, we have a natural isomorphism,

$$\psi : \overline{D}^*/(\overline{F}^*)\overline{D'} \longrightarrow U_D/U_F D'.$$

Now if $\overline{D}^*/\overline{F}^*\overline{D'} = 1$ then $U_D = U_F D'$. But $D^{(1)} \subseteq U_D$. This shows that $D^{(1)} = \mu_n(F)D'$. Using the fact that $\mu_n(F) \cap D' = Z(D')$ the theorem follows.

ii. The same proof as *i.* shows that if $\overline{D}^*/\overline{L}^*\overline{D'} = 1$ then $U_D = U_L D'$. Therefore $D^{(1)} \subseteq U_L D'$. Let $x \in D^{(1)}$. Then $x = ld$ where $l \in L$ and $d \in D'$. So $Nrd_{D/F}(x) = N_{L/F}(l) = 1$. Hilbert theorem 90 for the cyclic extension L/F guarantee that $l = a\sigma(a)^{-1}$, where σ is a generator of $Gal(L/F)$. Now the Skolem-Noether theorem implies that $\sigma(a) = cac^{-1}$ where $c \in D^*$. Therefore $l = aca^{-1}c^{-1}$. This shows that $D^{(1)} = D'$. $\square$

Remark 3.9. Note that if D is tame and Henselian over its center F with index n, using the Hensel's lemma, it is easy to see that $\mu_n(F) \to \mu_n(\overline{F}), a \mapsto \overline{a}$ is an isomorphism Also using the formula (3.1), it is not difficult to show that $Z(D') \simeq \overline{Z(D')} \simeq \overline{D'} \cap \overline{F}$. Therefore in the above theorem the formula for the group $SK_1(D)$ can be written as $SK_1(D) = \mu_n(\overline{F})/(\overline{D'} \cap \overline{F})$.

Part *i.* of the above theorem shows that if D is totally ramified, then $SK_1(D) = \mu_n(F)/Z(D')$. This and the above remark shows that Tignol's formula for $SK_1(D)$ is a special case of Theorem 3.8 (See [10]).

We deduce both theorems of Lipnitskii [11] which are obtained by using heavy machinery of reduced K- theory, as natural examples of the above theorem.

Example 3.10. *For any division algebra D with center $F = \mathbb{R}((x_1, \cdots, x_m))$ where $\mathbb{R}$ is the real numbers, $SK_1(D)$ is trivial.*

Proof. From number theory, it is well known that $[D : F] = 2^s$ where $s \leq m$. Since the complete field $F = \mathbb{R}((x_1, \cdots, x_m))$ has a natural valuation, then D admits a valuation which is obviously tame. It is clear that $\overline{F} = \mathbb{R}$. Because the only division algebras over real numbers are either the quaternion $\mathbb{H}_{\mathbb{R}}$ or the field $\mathbb{C}$ of complex numbers, therefore $\overline{D} = \mathbb{H}_{\mathbb{R}}$ or $\overline{D} = \mathbb{C}$. Now Corollary 2.6 and Remark 3.3 show that in either case $\overline{D}^* / \overline{F}^* \overline{D'} = 1$. Now by Theorem 3.8, $SK_1(D) \simeq \mu_{i(D)}(F)/Z(D')$. But clearly $\mu_{i(D)}(F) = \{1, -1\}$. Now from the Remarks 3.9 and 3.3, it follows that $-1 \in D'$. Thus $SK_1(D) = 1$.

Example 3.11. *For any division algebra with center $F = C((x_1, \cdots, x_m))$ where C is an algebraically closed field, $SK_1(D)$ is cyclic.*

Example 3.12. *Hilbert classical construction of division algebras.* Let L be a field and $\sigma \in Aut(L)$ such that $o(\sigma) = n$. Let $F = Fix(\sigma)$ be the fixed field of σ. Hence $Gal(L/F)$ is a cyclic group with the generator σ. Let $D = L((x, \sigma))$ be the division ring of formal Laurent series. It follows that $Z(D) = F((x^n))$ and $i(D) = n$. D has a natural valuation, and it is easy to see that with this valuation D is semiramified and $L((x^n))$ is a maximal subfield of D. Now by Theorem 3.8 *ii.*, $SK_1(D)$ is trivial.

Example 3.13. From Theorem 3.8 *ii.*, it is immediate that reduced Whitehead group of a division algebra over a local field is trivial.

Because most of the interesting valued division algebras arise from the iterated formal power series fields, we may consider *r-iterated Henselian division algebras*. Following Platonov in [15], we define inductively an r-iterated Henselian field F if its residue field $\overline{F}$ is an $(r-1)$-iterated Henselian field.[1] Let $(D_i, v_i), 0 \leq i \leq r-1$ be an r iterated Henselian division algebra ($\overline{D_i} = D_{i+1}$). Let $\Phi_i : U_{D_{i-1}} \longrightarrow D_i$ be the i-th natural reduction map. Then $\Phi_i(\Phi_{i-1}(\cdots(\Phi_1(a))\cdots))$ is called an i-iterated reduction, if it is defined. Denote the r iterated Henselian division algebra by D, $(D = D_0, Z(D) = F = F_0)$. We also need the following notations in order to state the following lemma. By $[U_D]_i$ and $[U_F]_i$ we denote the set of all elements of D and F respectively, such that i iterated reduction is defined. Also by $[1 + M_D]_i$ and $[1 + M_F]_i$, we denote the subsets of $[U_D]_i$ and $[U_F]_i$ such that the i iterated reduction equals one. Clearly $[1 + M_F]_1 = 1 + M_F$. we can write the main lemma of [6] in this setting.

Lemma 3.14. *Let D be an i iterated tame division algebra of finite dimension over a Henselian field $F = Z(D)$ with index n.*

[1] See Ershov's comment in [3] on iterated valued field. Among other things, considering iterated valued field, enables us to have more insight in each step of reduction.

i. For each $a \in [1+M_D]_i$ there is $b \in [1+M_F]_i$ such that $ab \in D'$.
ii. $[1+M_D]_i \subsetneqq [1+M_F]_i D'$.

Proof. *i.* Let $a \in [1+M_D]_i$. Then a is contained in a maximal subfield of D, say L. Therefore $a \in [1+M_L]_i$. By lemma 3 of [15], we have $N_{L/F}([1+M_L]_i) = [1+M_F]_i$. So $Nrd_{D/F}(a) = N_{L/F}(a) \in [1+M_F]_i$. Let $t = Nrd_{D/F}(a)$. Using an inductive argument for Hensel's lemma, we will show that there exists $c \in [1+M_F]_i$ such that $c^n = t$. Let $s \in 1+M_F = [1+M_F]_1$. Applying Hensel's lemma for $f(x) = x^n - s$ gives $c \in 1+M_F$ such that $c^n = s$. Now it is not hard to see that $\Phi_1([1+M_F]_i) = [1+M_{\overline{F}}]_{i-1}$. Therefore $[1+M_F]_i/[1+M_F]_1 \simeq [1+M_{\overline{F}}]_{i-1}$. Now by induction, we conclude that $[1+M_F]_i$ is n-divisible. Therefore exist $c \in [1+M_F]_i$ such that $c^n = t$. Now $Nrd_{D/F}(a) = c^n$. So $Nrd_{D/F}(ac^{-1}) = 1$. Hence $ac^{-1} \in D^{(1)} \cap [1+M_D]_i$. Applying Platonov's generalized congruence theorem (cf. [15] and [4]) , we obtain $ac^{-1} \in D'$. Take $b = c^{-1}$ and the proof is complete.

ii. Applying the first part of the lemma for $i = 1$, in each step of reduction we have, $1+M_{D_i} \subseteq (1+M_{K_i}){D_i}'$ where $K_i = Z(D_i)$. First we show that in each step of reduction, ${D_i}' \nsubseteq 1+M_{D_i}$. Consider the groups $\Delta = {D_i}^*/1+M_{D_i}$ and $P(D_i) = (1+M_{K_i}){D_i}'/(1+M_{D_i})$. One can easily observe that $P(D_i) = \Delta'$ and as Theorem 2.11 of [2] shows, the center of Δ is ${K_i}^*(1+M_{D_i})/(1+M_{D_i})$. We claim that Δ is not an abelian group, for otherwise $U_{D_i} = U_{K_i}(1+M_{D_i})$ which implies that D_i is totally ramified. Thus $\overline{{D_i}'} = \mu_e(\overline{K_i})$, where $e = exp(\Gamma_{D_i}/\Gamma_{K_i})$, (cf. the proof of Theorem 3.1 of [17]) which leads us to a contradiction. Therefore ${D_i}'$ is not in $1+M_{D_i}$ and Δ is not abelian. But $\Phi^{-1}(1+M_{D_i}) = [1+M_D]_{i+1}$. If $D' \subseteq [1+M_D]_{i+1}$ then $\Phi(D') \subseteq \Phi([1+M_D]_{i+1}) = 1+M_{D_i}$. But ${D_i}' \subseteq \Phi(D')$ so ${D_i}' \subseteq 1+M_{D_i}$ a contradiction. □

Remark 3.15. In the proof of *i.* above, we could use Lemma 2.9 and avoid the Platonov congruence theorem.

Theorem 3.16. *Let D be an r iterated tame division algebra over a Henselian field F of index n. If there is an $0 \le \ell \le r-1$ such that $\overline{D}_\ell/\overline{F}_\ell\overline{D'}_\ell = 1$ then $SK_1(D) \simeq \mu_n(F)/Z(D')$.*

Proof. For any $0 \le k \le r-1$, consider the $k+1-th$ reduction map

$$[U_D]_{k+1} \overset{\Phi_{k+1}\Phi_k\cdots\Phi_1}{\longrightarrow} \overline{D_k}^*.$$

Thanks to Lemma 3.14 *i.*, we have:

$$\overline{D_k}^* \overset{\simeq}{\to} [U_D]_{k+1}/[1+M_D]_{k+1} \overset{nat.}{\to} [U_D]_{k+1}/[1+M_F]_{k+1}D' \overset{nat.}{\to} [U_D]_{k+1}/[U_F]_{k+1}D'.$$

Therefore,

$$\overline{D_k}^*/\overline{F_k}^*\overline{D_k'} \overset{\simeq}{\longrightarrow} [U_D]_{k+1}/[U_F]_{k+1}D'.$$

Hence if there is a ℓ such that $\overline{D_\ell}^*/\overline{F_\ell}^*\overline{D'_\ell}^* = 1$ then $[U_F]_{\ell+1}D' = [U_D]_{\ell+1}$. By lemma 1 in [15] $D^{(1)} \subseteq [U_D]_{\ell+1}$ so $D^{(1)} = \mu_n(F)D'$. Using the fact that $\mu_n(F) \cap D' = Z(D')$ the theorem follows. $\square$

Considering the fact that each Henselian division algebra is a 1-iterated division algebra, we recover Theorem 3.8 from the above theorem.

4. On The Unitary Setting

In this section we introduce the unitary version of the group $G(D)$ and obtain similar results in the unitary setting. Let D be a division ring with an involution τ over its center F with index n. Let $S_\tau(D) = \{a \in D | a^\tau = a\}$ be the subspace of symmetric elements and $\Sigma_\tau(D)$ the subgroup of D^* generated by nonzero symmetric elements. Here we concentrate on involutions of the first kind, i.e. $\Sigma_\tau(D) \cap F^* = F^*$.

Definition 4.1. Let D be a division ring with an involution τ. Then the group $KU_1(D) = D^*/\Sigma_\tau(D)D'$ is called unitary Whitehead group and the $GU(D) = \Sigma_\tau(D)D'/F^*D'$ the unitary version of $G(D)$.

We will prove that there is a stability theorem for $GU(D)$ similar to one in Corollary 3.7. The first part of the following theorem was first proved by Platonov and Yanchevskii [16].

Theorem 4.2. *Let D be a finite dimensional tame and unramified division algebra with an involution of the first kind over a Henselian field Z(D)=F. Then* $KU_1(\overline{D}) \backsimeq KU_1(D)$ *and* $GU(\overline{D}) \backsimeq GU(D)$.

Proof. Consider the following sequence:

$$\overline{D}^* \longrightarrow U_D/1+M_D \longrightarrow F^*U_D/F^*(1+M_D) \longrightarrow D^*/\Sigma_\tau(D)D'.$$

Because the valuation is unramified, we have $\Sigma_{\overline{\tau}}(\overline{D}) = \overline{\Sigma_\tau(D) \cap U_D}$ (See [16]), and $\overline{D}' = \overline{D'}$ (See Theorem 3.2 *i.*). Therefore we have the following isomorphism: $\overline{D}^*/\Sigma_{\overline{\tau}}(\overline{D})\overline{D}' \xrightarrow{\backsimeq} D^*/\Sigma\tau(D)D'$.

For the second part, consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & GU(\overline{D}) \arrow[r] \arrow[d] & G(\overline{D}) \arrow[r] \arrow[d, "iso."] & KU_1(\overline{D}) \arrow[r] \arrow[d, "iso."] & 1 \\
1 \arrow[r] & GU(D) \arrow[r] & G(D) \arrow[r] & KU_1(D) \arrow[r] & 1.
\end{tikzcd}$$

The two of the vertical arrows are isomorphisms, thanks to the first part of this theorem and Theorem 3.2 *i.*. Therefore the third one is also an isomorphism which completes the proof. $\square$

If D has an involution of the first kind, then $D((x))$ enjoys a natural involution which is induced by the one from D. Therefore if $CharF \nmid i(D)$ then thanks to the above theorem, we have $GU(D) \backsimeq GU(D((x)))$ which is a stability theorem for $GU(D)$.

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