Mixed representations of quivers and relative problems I

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Introduction

The notion of quiver representations introduced in [Gab] provides a framework for a wide range of linear algebra problems. Roughly speaking, a representation of a quiver is a collection of linear maps, and two representations are equivalent if they lie in the same orbit under a natural action of the product of some general linear groups. So the problem of classifying representations of quivers is the same as describing the orbit spaces of certain linear actions of products of general linear groups. To approximate these orbit spaces by algebraic varieties one has to present polynomial invariants.

For the first time invariants of quivers were described in the characteristic zero case in [PrB1, PrB2]. This result was applied to investigate an etale local structure of categorical quotients of quiver representation spaces [PrB1, PrB2].

The modular case was explored in [Don1, Zub4]. Namely, in [Don1] invariants of any quiver were described over arbitrary infinite field and in [Zub4] – all relations between them. Notice that the last result was proved independently in [Dom] for the characteristic zero case.

Finally, in [DZ2] main results from [PrB1, PrB2] concerning an etale local structure of invariants of any quiver were extended for any algebraically closed field case too.

Without doubt the next step must be to generalize all these statements for another classical groups – orthogonal or symplectic ones. It is clear that one has to start with the action of $O(n), SO(n)$ or $Sp(n)$ on m-tuples of $n \times n$ matrices by simultaneous conjugation. Using so-called transfer principle [Gr] one can reduce this problem to the special kind representations of some new quiver with respect to the diagonal action of $GL(n)$. This example shows that one has to introduce new type representations of quivers. We call them “mixed” representations.

In this article we will find the generating invariants of ”mixed” representations of any quiver. For the first time it was done in [Zub5] for the quiver mentioned above but without any referring to these terms – quivers, "mixed" representations and etc.

Following the same ideas as in [Zub5] we find some "suitable" generators having the nice property – they can be considered as generators of a free invariant algebra of a given quiver. Any invariant algebra of "mixed" representations of this quiver of fixed dimension is a quotient of one modulo some ideal which is a T-ideal with respect to some special kind substitutions of matrix variables. Besides, a part of "suitable" generators span this ideal. Simplifying the last generators one can get some generators of this ideal as a T-one. We will do it in the next article.

Further, we represent some useful correspondence between invariants of "mixed" representations of any quiver and invariants of ordinary representations of some "doubled" one. This correspondence explains all partial results from [Zub5] in more clear way. Finally, we introduce more general "supermixed" representations of quivers and show that invariants of ones with respect to the action of products of some classical groups as well as defining relations between them can be produced from invariants and defining relations between them of "mixed" representations of some modified quivers in almost all characteristics. In particular, this result affords to describe invariants of orthogonal or symplectic representations of symmetric quivers introduced in [DW3].

1 Motivations

Let us remind some definitions and notations (see [Gab, Don1, PrB1, PrB2]). A quiver is a quadruple $Q = (V, A, h, t)$, where V is a vertex set and A is an arrow set of Q. Let the maps $h, t : A \to V$ associate each arrow $a \in A$ its origin $h(a) \in V$ and its end $t(a) \in V$. We enumerate elements of the vertex set as $V = \{1, \ldots, n\}$.

Let us consider a collection of vector spaces $E_1, \ldots, E_n$ over an algebraically closed field K. Let $\dim E_1 = k_1, \ldots, \dim E_n = k_n$. Denote by $\bar{k}$ the vector $(k_1, \ldots, k_n)$. This vector is called a dimensional vector. For two dimensional vectors $\bar{k}(1), \bar{k}(2)$ we write $\bar{k}(1) \geq \bar{k}(2)$ iff $\forall i \in V, \bar{k}(1)_i \geq \bar{k}(2)_i$.

Denote by $GL(\bar{k})$ the group $GL(E_1) \times \ldots \times GL(E_n) = GL(k_1) \times \ldots \times GL(k_n)$. A $\bar{k}$-representations space of quiver Q is the space $R(Q, \bar{k}) = \prod_{a \in A} \mathrm{Hom}_K(E_{h(a)}, E_{t(a)})$. The group $GL(\bar{k})$ acts on $R(Q, \bar{k})$ by the rule:

$$(y_a)^g_{a \in A} = (g_{t(a)} y_a g^{-1}_{h(a)})_{a \in A}, g = (g_1, \ldots, g_n) \in GL(\bar{k})$$

For example, if our quiver Q has one vertex and m loops which are incidenced to this vertex then the k-representations space of this quiver is isomorphic to the space of m $k \times k$-matrices with respect to the diagonal action of the group $GL(k)$ by conjugations.

The coordinate ring of the affine variety $R(Q, \bar{k})$ is isomorphic to $K[y_{ij}(a) \mid 1 \leq j \leq k_{h(a)}, 1 \leq i \leq k_{t(a)}, a \in A]$. For any $a \in A$ denote by $Y_{\bar{k}}(a)$ the general matrix $(y_{ij}(a))_{1 \leq j \leq k_{h(a)}, 1 \leq i \leq k_{t(a)}}$. The action of $GL(\bar{k})$ on $R(Q, \bar{k})$ induces the action on the coordinate ring by the rule $Y_{\bar{k}}(a) \to g_{t(a)}^{-1} Y_{\bar{k}}(a) g_{h(a)}, a \in A$. We omit the lower index $\bar{k}$ if it does not lead to confusion. For example, we write just $Y(a)$ instead of $Y_{\bar{k}}(a)$.

To explain why we should generalize ordinary representations of quivers let us start again with the quiver Q with one vertex. Let $G_n = O(n), Sp(n)$ acts on its representation space of dimension n, i.e. on the space of m-tuples of $n \times n$ matrices $M(n)^m$, diagonally by conjugations. Finally, let us suppose that in the $O(n)$-action case the characteristic of K is add (or zero).

Proposition 1.1 *([Gr]) Let G be an algebraic group and H its closed subgroup. If G acts on an affine variety X then the invariant algebra $K[X]^H$ is isomorphic to $(K[X] \otimes k[G/H])^G$, where G acts on $K[G/H]$ by left translation. The isomorphism is given by $a \otimes f \to af(eH)$.*

Using this proposition one can replace the invariant algebra $K[M(n)^m]^{G_n}$ by its isomorphic copy $K[M(n)^m \times GL(n)/G_n]^{GL(n)}$.

It is not hard to prove that $GL(n)/G_n$ is isomorphic to the affine variety $S(n)$ consisting of all non-degenerate symmetric or skew-symmetric with zero diagonal matrices with respect to which case is considered – $G_n = O(n)$ or $G_n = Sp(n)$. The group $GL(n)$ acts on $S(n)$ by the rule $s^g = gsg^t, s \in S(n), g \in GL(n)$.

One can embed the variety $S(n)$ into $M(n) \times M(n)$ by the rule $x \to (x, x^{-1}), x \in S(n)$. Moreover, it will be a $GL(n)$- equivariant map if we define the action of $GL(n)$ on $M(n) \times M(n)$ as $(x, y)^g = (gxg^t, (g^t)^{-1} y g^{-1}), x, y \in M(n), g \in GL(n)$.

Using the notation of a good pair of varieties [Don7] one can prove that the pair $(M(n) \times M(n), S(n))$ is a good one [Zub5]. In particular, the invariant algebra $k[M(n)^m \times S(n)]^{GL(n)}$ is an epimorphic image of the algebra $k[M(n)^m \times M(n)^2]^{GL(n)}$, where $GL(n)$ acts on the second factor $M(n)^2$ by the rule given above [Zub5]. It is clear that this epimorphism is defined by specializations $(x, y) \to (I_n, I_n)$ or $(x, y) \to (J_n, J_n^{-1})$ respectively, where I_n is the unit $n \times n$ matrix and J_n is a $n \times n$ skew-symmetric matrix of the bilinear form defining the group $Sp(n)$.

The space $M = M(n)^m \times M(n)^2$ can be interpreted as a new type representation space of some quiver which corresponds to the original one.

Namely, let us define the quiver Q' with two vertexes, say $1, 2$, and $m+2$ arrows. The first m arrows are loops which are incidenced to the vertex 1. The $m+1$-th and $m+2$-th arrows connect both vertexes and have opposite directions.

Let us consider the representation space of this quiver of dimension (n, n) and replace the standard action of $GL(n)$ on the vector space E_2 by $v^g = (g^t)^{-1} v, v \in E_2, g \in GL(n)$. In other words, this is the standard action of $GL(n)$ on E_2^* with respect to a dual base, i.e. we just replace the space E_2 by its dual E_2^*. It is clear that the $GL(n)$-variety M is isomorphic to this new representation space under the diagonal action of $GL(n)$.

In fact, if we would consider the representation space $R(Q, (n, n))$ even after replacing E_2 by E_2^* relative to the action of its automorphism group $GL(n) \times GL(n)$ we will get nothing new. But if we replace this group by its diagonal subgroup, i.e.

by $GL(n)$, we get absolutely another class of representations. We call them "mixed" ones. The general definition for any quiver will be given below.

Roughly speaking, "mixed" representations of quivers model orthogonal or symplectic representations of ones.

Replacing the products of general linear groups by the products of special linear ones one can set the problems to find the generators and defining relations between them for semi-invariants of "mixed" representation of quivers. It is very important to include in all our considerations the special orthogonal group case too.

As for the ordinary representations of quivers it was very popular theme during the last 20 years starting with the remarkable Kac's article [Ka]. Important results were obtained in [S1, S2]. There is also an extensive literature on semi-invariants of Dynkin and Eucledian (or extended Dynkin) quivers, see [As], [Ri], [Hap1], [Hap2], [Ko1], [Ko2], [HH], [SwWl], [SkW]. The complete descriptions of semi-invariants for arbitrary quiver case were obtained in [DW1, DW2] and [DZ]. In the characteristic zero case the similar result was proved in [SV].

2 Definitions and auxiliary results

Keeping in mind the example from the previous section let us generalize our definitions concerning representations of quivers. Fix some quiver Q and assign to each vertex i either some k_i-dimensional space E_i considered as the standard $GL(E_i)$-module as above, or E_i^* with respect to the standard action of the same group $GL(E_i)$. To be precise, $f^g(v) = f(g^{-1}v), f \in E_i^*, g \in GL(E_i), v \in E_i$.

To remark that some vertexes are occupied by dual spaces we introduce a "new" dimensional vector $\bar{t} = (t_1, \dots, t_l)$, where $t_i = k_i$ iff we assign to i the space E_i, otherwise $t_i = k_i^*$. We call the vector $\bar{k}$ underlying relative to $\bar{t}$ and use both ones in our notations.

For the sake of convenience the space assigned to i denote by $W_i, 1 \leq i \leq l$. That is $W_i = E_i$ or $W_i = E_i^*$.

By definition, the $\bar{t}$(or $\bar{k}$)-dimensional "mixed" representation space of the quiver Q is equal to the space $R(Q, \bar{t}) = \prod_{a \in A} \operatorname{Hom}_K(W_{h(a)}, W_{t(a)})$. When each W_i coincides with E_i we have an ordinary representation space.

The space $R(Q, \bar{t})$ is a $G = GL(E_1) \times \ldots \times GL(E_l)$-module under the same action

$$(y_a)_{a \in A}^g = (g_{t(a)} y_a g_{h(a)}^{-1})_{a \in A}$$
$$g = (g_1, \dots, g_l) \in G, (y_a)_{a \in A} \in R(Q, \bar{t})$$

Let us divide the vertex set of the quiver Q into several disjoint subsets. To be precise, let $V = \bigsqcup_{i=1}^{i=l} V_i$.

Definition 1 *A dimensional vector $\bar{t}$ is said to be compatible with this partition of V into disjoint subsets iff $\forall i \ \forall k, s \in V_i, \ \dim W_k = \dim W_s = d_i$. In other words, all coordinates of the vector $\bar{k}$ with numbers from the same V_i are equal one to another.*

From now on all dimension vectors are compatible with some fixed division $V = \bigsqcup_{i=1}^{i=l} V_i$ unless otherwise stated.

Decompose the group $GL(\bar{k})$ into blocks by the rule

$$G = GL(\bar{k}) = \times_{i=1}^{i=l}(\times_{s\in V_i} GL(E_s))$$

.

Denote each block $\times_{s\in V_i} GL(E_s)$ by G_i, $1 \le i \le l$. Then $G = \times_{i=1}^{i=l} G_i$. Let $H_i \cong GL(d_i)$ be a diagonal subgroup of G_i and $H(\bar{k}) = H = \times_{i=1}^{i=l} H_i$. Since the definition of the group $H(\bar{k})$ does not depend on are the coordinate of $\bar{t}$ the same as coordinates of $\bar{k}$ or not we will use a notation $H(\bar{t})$ too.

Definition 2 *The space $R(Q,\bar{t})$ with respect to the action of the group $H(\bar{k})$ is called a "mixed" representation space of the quiver Q of dimension $\bar{t}$ (or $\bar{k}$) relative to the division $V = \bigsqcup_{i=1}^{i=l} V_i$.*

We formulate

Problem 1 *What are the generators and the defining relations between them for the ring $K[R(Q,\bar{t})]^{H(\bar{k})}$?*

Without loss of generality we can identify the coordinate algebras $K[R(Q,\bar{t})]$ and $K[R(Q,\bar{k})]$. For given $\bar{k}(1) \ge \bar{k}(2)$ define an epimorphism

$$p_{\bar{t}(1),\bar{t}(2)} : K[R(Q,\bar{t}(1))] \to K[R(Q,\bar{t}(2))]$$

as follows.

Take any arrow $a \in A$. Let $h(a) = i$ and $t(a) = j$. For the sake of simplicity denote $k_i(t)$ and $k_j(t)$ by m_t and l_t respectively, $t = 1, 2$. We know that $m_1 \ge m_2$ and $l_1 \ge l_2$. Then our epimorphism takes $y_{sr}(a)$ to zero iff either $s > l_2$ or $r > m_2$. On the rest variables our epimorphism is the identical map.

On the other hand, one can define the isomorphism $i_{\bar{t}(2),\bar{t}(1)}$ of the variety $R(Q,\bar{t}(2))$ onto a closed subvariety of $R(Q,\bar{t}(1))$ by the dual rule, that is the epimorphism defined above is the comorphism $i^*_{\bar{t}(2),\bar{t}(1)}$.

By almost the same way as $i_{\bar{t}(2),\bar{t}(1)}$ one can define the isomorphism $j_{\bar{k}(2),\bar{k}(1)}$ of the group $H(\bar{k}(2))$ onto a closed subgroup of the group $H(\bar{k}(1))$ just bordering any invertible $k_i(2)\times k_i(2)$ matrix by the $k_i(1)-k_i(2)$ additional rows and columns which are zero outside of the diagonal tail of length $k_i(1) - k_i(2)$. The last one must be occupied by units.

It is not hard to check that $i_{\bar{t}(2),\bar{t}(1)}(\phi^g) = i_{\bar{t}(2),\bar{t}(1)}(\phi)^{j_{\bar{k}(2),\bar{k}(1)}(g)}$ for any $g \in H(\bar{k}(2))$ and $\phi \in R(Q,\bar{t}(2))$. The analogous equation is valid for the epimorphism $p_{\bar{t}(1),\bar{t}(2)}$.

All one need to know about modules with good filtration (briefly – modules with GF) the reader can find in [Jan, Don3, Zub1]. Let us remind some basic definitions.

Let G be a reductive group. Fix some maximal torus of the group G, say T, and a Borel subgroup B containing T. The group B has a semi-direct decomposition $B = T \bowtie U$, where U is a maximal unipotent subgroup of the group B.

A G-module V is said to be algebraic iff any finite subset of V is contained in some finitely dimensional G-submodule W of V. Besides, the restriction map $g \to g\mid_W$ is a morphism of the algebraic groups $G \to GL(W)$. From now on we suppose that all modules are algebraic if otherwise stated.

Denote by $X(T)$ the character group of the torus T and by $X(T)^+$ the dominate weight subset of one corresponding to B. If $\mu \in X(T)^+$ then denote by $\bigtriangledown(\mu)$ the induced module $\mathrm{ind}_{B^-}^G K_\mu$, where B^- is the opposite Borel subgroup and K_μ is the one-dimensional B^--module with respect to the action $(tu)\circ x = \mu(t)x, t \in T, u \in U^-, x \in K_\mu$.

A G-module V is called a module with GF iff there is some filtration with at most countable number of members

$$0 \subseteq V_1 \subseteq V_2 \subseteq \ldots, \bigcup_{i=1}^{\infty} V_i = V$$

such that $\forall i \geq 1, V_i/V_{i-1} \cong \bigtriangledown(\mu_i)$.

Respectively, a G-module W is called a module with Weyl filtration (briefly – module with WF) iff there is some filtration with at most countable number of members

$$0 \subseteq W_1 \subseteq W_2 \subseteq \ldots, \bigcup_{i=1}^{\infty} W_i = W$$

such that $\forall i \geq 1, W_i/W_{i-1} \cong \triangle(\mu_i)$, where $\triangle(\mu) \cong \bigtriangledown(\mu^*)^*, \mu^* = -w_0(\mu)$ and w_0 is the longest element of the Weyl group $W(G,T) = N_G(T)/T$.

It is clear that a finitely dimensional G -module V with WF iff the dual module V^* is a module with GF. A finitely dimensional module V is called a tilting one if both V and V^* are with GF. In other words, V has good and Weyl filtrations simultaneously.

Let us list some standard properties of modules with GF.

1. If

$$0 \to V \to W \to S \to 0$$

is a short exact sequence of G-modules and V with GF then the diagram

$$0 \to V^G \to W^G \to S^G \to 0$$

is exact too.

2. The next property is a consequence of Donkin's criterion [Don6]. Namely, if W is a G-module with GF and V is its submodule with GF then the quotient W/V is a G-module with GF too.

3. The most important property is the Donkin-Mathieu's theorem [Don5, Mat1]: for given G-modules with GF their tensor product with respect to the diagonal action of the group G is a module with GF too.

For example, let $G = GL(k)$ and $T(k) = \{diag(t_1, \ldots, t_k) \mid t_1, \ldots, t_k \in K^*\}$ is the standard torus of G. We fix the Borel subgroup $B(k)$ consisting of all upper triangular matrices. It is clear that $B^-(k)$ consists of all lower triangular matrices.

Any character $\lambda \in X(T(k))$ can be regarded as a vector $(\lambda_1, \ldots, \lambda_k)$ with integer coordinates. By definition $\lambda(t) = t_1^{\lambda_1} \ldots t_k^{\lambda_k}, t \in T(k)$.

It is known that $\lambda \in X(T(k))^+$ iff $\lambda_1 \geq \ldots \geq \lambda_k$ [Don2]. If $\lambda_k \geq 0$ then $(\lambda_1, \ldots, \lambda_k)$ is called an ordered partition and $\bigtriangledown(\lambda)$ is isomorphic to so-called Schur module $L_{\tilde{\lambda}}(K^k)$ (see below), where $\tilde{\lambda}$ is conjugated to λ. For example, if $\lambda = (\underbrace{1, \ldots, 1}_{t}, 0, \ldots, 0)$ then $\bigtriangledown(\lambda) = L_{\tilde{\lambda}}(K^k) = \Lambda^t(K^k)$.

The Weyl group $W(GL(k), T(k))$ is isomorphic to the group S_k consisting of all permutations on k symbols.

More generally, one can describe some fragment of the representation theory of any group $GL(\bar{k})$. A maximal torus of the group $GL(\bar{k})$ can be defined as $T(\bar{k}) = T(k_1) \times \ldots \times T(k_n)$. Respectively, $B(\bar{k}) = B(k_1) \times \ldots \times B(k_n)$ is a Borel subgroup and then $B^-(\bar{k}) = B^-(k_1) \times \ldots \times B^-(k_n)$.

The characters of the group $T(\bar{k})$ can be defined as collections $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)$, where each λ_i is a character of the corresponding torus $T(k_i)$, $i = 1, 2, \ldots, n$. It is obvious that the root data of $GL(\bar{k})$ is the direct product of the root datas of the groups $GL(k_i)$. In particular, $X(T(\bar{k}))^+$ coincides with $X(T(k_1))^+ \times \ldots \times X(T(k_n))^+$. Moreover, for any weight $\bar{\lambda} \in X(T(\bar{k}))^+$ we have an isomorphism $\bigtriangledown_{\bar{k}}(\bar{\lambda}) \cong \bigtriangledown(\lambda_1) \otimes \ldots \otimes \bigtriangledown(\lambda_n)$ and $\triangle_{\bar{k}}(\bar{\lambda}) \cong \triangle(\lambda_1) \otimes \ldots \otimes \triangle(\lambda_n)$. Therefore, if all λ_i are ordered partitions we see that $\bigtriangledown_{\bar{k}}(\bar{\lambda}) \cong L_{\tilde{\lambda}_1}(E_1) \otimes \ldots L_{\tilde{\lambda}_n}(E_n)$.

Finally, note that the Weyl group $W(GL(\bar{k}, T(\bar{k}))$ is the direct product of the Weyl groups of all factors $GL(E_i)$ too. In particular, we have $\bar{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$.

Let us consider the dimensional vectors $\bar{t}(1)$, $\bar{t}(2)$ such that $\bar{k}(1) \geq \bar{k}(2)$. We define a Schur functor $d_{\bar{k}(1), \bar{k}(2)}$ by the following rule. For any $GL(\bar{k}(1))$-module V we suppose that $d_{\bar{k}(1), \bar{k}(2)}(V) = \sum_{\bar{\mu} \in L} V_{\bar{\mu}}$, where the set L consists of all $\bar{\mu} = (\mu_1, \ldots, \mu_n)$ such that for any i all coordinates of μ_i beginning with $k(2)_i + 1$-th one are equal to zero and $\sum_{\bar{\mu} \in X(T(\bar{k}(1)))} V_{\bar{\mu}}$ is the weight decomposition of V.

Identifying the group $GL(\bar{k}(2))$ with a subgroup of $GL(\bar{k}(1))$ as above we see that $d_{\bar{k}(1), \bar{k}(2)}(V)$ is a $GL(\bar{k}(2))$-module. Besides, one can define a linear endomorphism of V which takes any $v = \sum_{\bar{\mu} \in X(T(\bar{k}(1)))} v_{\bar{\mu}} \in V$ to $\sum_{\bar{\mu} \in L} v_{\bar{\mu}}$. Denote this endomorphism by the same symbol $d_{\bar{k}(1), \bar{k}(2)}$.

It is not hard to prove that if all coordinates λ_i of $\bar{\lambda}$ are some ordered partitions then $d_{\bar{k}(1), \bar{k}(2)}(\triangle_{\bar{k}(1)}(\bar{\lambda})) \neq 0$ iff each "component" λ_i has not any non-zero coordinate with a number $\geq k(2)_i + 1$. In the last case we have $d_{\bar{k}(1), \bar{k}(2)}(\triangle_{\bar{k}(1)}(\bar{\lambda})) = \triangle_{\bar{k}(2)}(\bar{\lambda})$. The same is valid for the induced modules $\bigtriangledown_{\bar{k}(1)}(\bar{\lambda})$ as well as for simple ones. The reader can find the detailed proof in [Green] for the case $n = 1$. The general case is a trivial consequence of the case $n = 1$.

If the interval $[1, k] = \{1, \ldots, k\}$ is decomposed into some disjoint subsets, say $[1, k] = \bigsqcup_{1 \leq j \leq m} T_j$, one can define the Young subgroup $S_{\mathcal{T}} = S_{T_1} \times \ldots \times S_{T_m}$ of the group S_k consisting of all substitutions $\sigma \in S_k$ such that $\sigma(T_j) = T_j, 1 \leq j \leq m$. By definition, $S_T = \{\sigma \in S_k \mid \sigma(T) = T, \forall j \notin T \; \sigma(j) = j\}$ for arbitrary subset T. The subsets $T_1, \ldots, T_m$ is said to be the layers of the group $S_{\mathcal{T}}$ [Zub1, Zub4].

The group $S_{\mathcal{T}}$ can be defined by another way. In fact, let f be a map from $[1, k]$ onto $[1, m]$ defined by the rule $f(T_j) = j, j = 1, \ldots, m$. Then $S_{\mathcal{T}1} = \{\sigma \in S_k \mid f \circ \sigma = f\}$. Sometimes we will denote the group $S_{\mathcal{T}}$ by S_f.

For any group G and its subgroup H we denote by G/H some fixed representative set of the left H cosets if it does not lead to confusion. Besides, for any $g \in G$ denote by $\bar{g} \in G/H$ the representative of the left H coset gH.

For any vector $\lambda = (\lambda_1, \ldots, \lambda_s)$ denote by $\mid \lambda \mid$ its degree or weight $\lambda_1 + \ldots + \lambda_s$. If all coordinates of λ are non-negative integral numbers we denote by $\Lambda^{\lambda}(V)$ the tensor product $\Lambda^{\lambda_1}(V) \otimes \ldots \Lambda^{\lambda_s}(V)$.

Let us remind the standard embedding of an exterior power $\Lambda^p(V)$ into $V^{\otimes p}$. This map is defined by the rule

$$i_p : v_1 \bigwedge \ldots \bigwedge v_p \to \sum_{\sigma \in S_p} (-1)^{\sigma} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(p)}, v_1, \ldots, v_p \in V$$

Obviously, it is a $GL(V)$-equivariant one. By the same way, one can define more general embedding $i_{\lambda} : \Lambda^{\lambda}(V) \to V^{\otimes p}$, where $\lambda = (\lambda_1, \ldots, \lambda_l)$ is any (non-ordered) partition, $p = \mid \lambda \mid$ and $i_{\lambda} = \otimes_{1 \leq q \leq l} i_{\lambda_q}$. The canonical epimorphism from $V^{\otimes p}$ onto $\Lambda^{\lambda}(V)$ denote by p_{λ}.

Let $S^r(V \otimes W)$ be a homogeneous component of degree r of the symmetric algebra $S(V \otimes W)$, where V, W are any vector spaces.

For any ordered partition λ of degree r we define the map

$$d_{\lambda} : \Lambda^{\lambda}(V) \otimes \Lambda^{\lambda}(W) \to S^r(V \otimes W)$$

as $d_{\lambda} = d_{\lambda_1} \bar{\otimes} ... \bar{\otimes} d_{\lambda_s}$, where $d_{\lambda_i} : \Lambda^{\lambda_i}(V) \otimes \Lambda^{\lambda_i}(W) \to S^{\lambda_i}(V \otimes W), i = 1, \ldots, s$, and the symbol $\bar{\otimes}$ means the product map $S^{\lambda_1}(V \otimes W) \otimes ... \otimes S^{\lambda_s}(V \otimes W) \to S^r(V \otimes W)$.

Besides, for any $t \in N$ $d_t : \Lambda^t(V) \otimes \Lambda(W) \to S^t(V \otimes W)$ is defined by the rule

$$d_t((v_1 \wedge ... \wedge v_t) \otimes (w_1 \wedge ... \wedge w_t)) = \sum_{\sigma \in S_t} (-1)^{\sigma} v_1 \otimes w_{\sigma(1)} ... v_t \otimes w_{\sigma(t)}$$

$$v_i \in V, w_i \in W, 1 \leq i \leq t.$$

Let $M_{\lambda} = \sum_{\gamma \succeq \lambda} \mathrm{Im} d_{\gamma}$ and $\dot{M}_{\lambda} = \sum_{\gamma \succ \lambda} \mathrm{Im} d_{\gamma}$. The symbol $\succeq$ means the lexicographical order from the left to the right on the set of partitions. The $GL(V) \times GL(W)$-module $S^r(V \otimes W)$ has the filtration

$$0 \subseteq M_{(r)} \subseteq M_{(r-1,1)} \subseteq ... \subseteq M_{\underbrace{(1, ..., 1)}_{r}} = S^r(V \otimes W)$$

with quotients

$$M_\lambda/\dot{M}_\lambda \cong L_\lambda(V)\otimes L_\lambda(W)$$

, where $L_\lambda(V)$ is a Schur module (see [Ak]). We call this filtration Akin-Buchsbaum-Weyman filtration or briefly – ABW-one. Any $GL(V)$-module $L_\lambda(V)$ has a finite resolution

$$0\to C_m\to C_{m-1}\to\ldots\to C_1\to L_\lambda(V)\to 0$$

such that all its members are direct sums of tensor products of some exterior powers $\Lambda^t(V)$ and $C_1=\Lambda^\lambda(V)$. In particular, all members of this resolution with GF as $GL(V)$-modules [Ak, Don6]. Moreover, all these modules are tilting ones.

Let us consider any arrow $a\in A$ and enumerate all possible cases to occupy its limiting vertexes by spaces $E_?$ or $E_?^*$.

1. Let $h(a)=i, t(a)=j$. If $W_i=E_i, W_j=E_j$ then $H=H(\bar{k})$ acts on the component $K[\mathrm{Hom}_K(E_i,E_j)]=K[Y(a)]=K[y_{lt}(a)\mid 1\leq l\leq k_j, 1\leq t\leq k_i]$ by the rule $Y(a)\to g^{-1}Y(a)h, g\in GL(k_j), h\in GL(k_i)$ and $g=h$ if there is some s such that $i,j\in V_s$. It can easily be checked that $K[Y(a)]\cong S(E_j^*\otimes E_i)$ and this isomorphism of $GL(k_j)\times GL(k_i)$-modules is defined by the rule $y_{lt}(a)\longleftrightarrow e_l^*\otimes f_t$, where $e_1,\ldots,e_{k_j}$ and $f_1,\ldots,f_{k_i}$ are some fixed bases of the spaces E_j and E_i respectively. The base $e_1^*,\ldots,e_{k_j}^*$ is a dual one relative to $e_1,\ldots,e_{k_j}$.

2. Let $W_i=E_i, W_j=E_j^*$ then $K[Y(a)]\cong S(E_j\otimes E_i)$ with respect to the identification $y_{lt}(a)\longleftrightarrow e_l\otimes f_t$. In other words, H acts on $Y(a)$ by the rule $Y(a)\to g^tY(a)h$.

   Other cases are listed without any comments.

3. $W_i=E_i^*, W_j=E_j,\ K[Y(a)]\cong S(E_j^*\otimes E_i^*), y_{lt}(a)\longleftrightarrow e_l^*\otimes f_t^*,\ Y(a)\to g^{-1}Y(a)(h^t)^{-1}$.

4. $W_i=E_i^*, W_j=E_j^*,\ K[Y(a)]\cong S(E_j\otimes E_i^*), y_{lt}(a)\longleftrightarrow e_l\otimes f_t^*,\ Y(a)\to g^tY(a)(h^t)^{-1}$.

Let us consider the fourth case. Add to V new vertexes i',j' and redefine the maps h,t on any arrow a which goes from i to j by the opposite way: $h(a)=j', t(a)=i'$. We get the new quiver Q'. Let us consider the representation space of this quiver of dimension $\bar{t}'=(t_1,\ldots,t_n,\underbrace{k_i}_{i'},\underbrace{k_j}_{j'})$. In other words, we drop the components $\mathrm{Hom}_K(E_i^*,E_j^*)$ from the space $\prod_{a\in A}\mathrm{Hom}_K(W_{h(a)},W_{t(a)})$ and put on the freed "places" new components $\mathrm{Hom}_K(E_{j'},E_{i'})$. By definition $\dim E_{i'}=k_i=\dim E_i, \dim E_{j'}=k_j=\dim E_j$. Besides, i' or j' must belong to the same V_s as i or j respectively.

It is clear that the group H remains the same up to some obvious identification. Moreover, the algebra $K[R(Q,\bar{t})]$ is isomorphic to the algebra $K[R(Q,\bar{t'})]$. To be precise, we must take each $y_{lt}(a)$ to $z_{tl}(a)$, where $Z(a)=Z_{\bar{k'}}(a), h(a)=i, t(a)=j$. The rest generators of $K[R(Q,\bar{t})]$ and $K[R(Q,\bar{t'})]$ coincide one to another. It can easily be checked that this isomorphism is a H-equivariant one. After repeating this procedure as many times as we need one can assume that the fourth case does not happen at all.

Let us decompose each set $V_i, 1\leq i\leq l$ into two subsets $U_i=\{d\in V_i \mid t_d=k_d\}$ and $U_i^*=\{d\in V_i \mid t_d=k_d^*\}$. In other words, $U_i=\{d\in V_i \mid W_d=E_d\}$ and $U_i^*=\{d\in V_i \mid W_d=E_d^*\}$.

All dimensions $\dim W_d, d\in V_i$ are the same. In particular, we can identify all spaces E_d (E_d^*) one to another. It means that we can contract all vertexes from U_i (U_i^*) to one vertex. We get the new quiver which has the same arrow set as the previous one. Therefore, their representation spaces are the same too. Moreover, it is clear that the group H remains the same up to some trivial identification.

Summarizing all we said above one can assume that $\mid V_i\mid\leq 2, 1\leq i\leq l$, and there is not any arrow a which connects two vertexes occupied by dual spaces. Moreover, if some V_i contains two vertexes then $V_i=U_i\bigcup U_i^*, \mid U_i\mid=\mid U_i^*\mid=1$.

Let us decompose the arrow set A into three subsets $A_i, i=1,2,3$, where $A_1=\{a\in A \mid W_{h(a)}=E_{h(a)}, W_{t(a)}=E_{t(a)}\}$, $A_2=\{a\in A \mid W_{h(a)}=E_{h(a)}, W_{t(a)}=E_{t(a)}^*\}$ and $A_3=\{a\in A \mid W_{h(a)}=E_{h(a)}^*, W_{t(a)}=E_{t(a)}\}$.

The algebra $K[R(Q,\bar{t})]$ is isomorphic to the tensor product

$$\prod_{1\leq k\leq 3}\otimes(\otimes_{a\in A_k}K[Y(a)])$$

or to

$$\prod_{1\leq k\leq 3}\otimes(\otimes_{a\in A_k}(\oplus_{r_a}K[Y(a)](r_a)))\cong(\prod_{a\in A_1}\otimes(\oplus_{r_a}S^{r_a}(E_{t(a)}^*\otimes E_{h(a)}))\otimes$$

$$(\prod_{a\in A_2}\otimes(\oplus_{r_a}S^{r_a}(E_{t(a)}\otimes E_{h(a)}))\otimes(\prod_{a\in A_3}\otimes(\oplus_{r_a}S^{r_a}(E_{t(a)}^*\otimes E_{h(a)}^*))$$

as a H-module.

Fix a multidegree $\bar{r}=(r_a)_{a\in A}$. Sometimes we will rewrite it as $(\bar{r}_1,\bar{r}_2,\bar{r}_3)$, where $\bar{r}_i=(r_a)_{a\in A_i}, i=1,2,3$. Denote $\sum_{a\in A}r_a$ by r and $\sum_{a\in A_i}r_a$ by $r_i, i=1,2,3$.

Tensoring ABW filtrations of all factors in the previous tensor products we see that the $\bar{r}$-homogeneous component of the algebra $K[R(Q,\bar{t}]$ has a filtration with quotients

$$\prod_{a\in A_1}\otimes(L_{\lambda_a}(E_{t(a)}^*)\otimes L_{\lambda_a}(E_{h(a)}))\otimes\prod_{b\in A_2}\otimes(L_{\mu_b}(E_{t(a)})\otimes L_{\mu_b}(E_{h(a)}))\otimes$$

$$\prod_{c\in A_3}\otimes(L_{\gamma_c}(E_{t(c)}^*)\otimes L_{\gamma_c}(E_{h(c)}^*))$$

as $\times_{1\leq l\leq 3}(\prod_{a\in A_l}(GL(k_{t(a)})\times GL(k_{h(a)}))$-module, where by definition $\forall a\in A_1, \forall b\in A_2, \forall c\in A_3, \mid \lambda_a\mid = r_a, \mid \mu_b \mid = r_b, \mid \gamma_c \mid = r_c$.

Let us enumerate the members of this filtration by the triples $\Theta = (\lambda_{A_1}, \mu_{A_2}, \gamma_{A_3})$, where $\lambda_{A_1} = (\lambda_a)_{a\in A_1}, \mu_{A_2} = (\mu_a)_{a\in A_2}, \gamma_{A_3} = (\gamma_a)_{a\in A_3}$, say

$$\ldots \subseteq M_\Theta(\bar{t}) = M_\Theta \subseteq \ldots$$

Sometimes we will omit the indexes $\bar{t}(\bar{k})$ or Θ if it does not lead to confusion.

Denote by $\Lambda^{\lambda_{A_1}} \otimes \Lambda^{\mu_{A_2},\mu_{A_2}}(\bar{t})$ and $\Lambda^{\lambda_{A_1}}(\triangleright) \otimes \Lambda^{\gamma_{A_3},\gamma_{A_3}}(\bar{t})$ the spaces

$$\prod_{a\in A_1} \otimes(\Lambda^{\lambda_a}(E_{h(a)})) \otimes \prod_{a\in A_2} \otimes(\Lambda^{\mu_a}(E_{t(a)}) \otimes \Lambda^{\mu_a}(E_{h(a)}))$$

and

$$\prod_{a\in A_1} \otimes(\Lambda^{\lambda_a}(E_{t(a)})) \otimes \prod_{a\in A_3} \otimes(\Lambda^{\gamma_a}(E_{t(a)}) \otimes \Lambda^{\gamma_a}(E_{h(a)}))$$

respectively.

We identify the dual space $(\Lambda^{\lambda_{A_1}}(\triangleright) \otimes \Lambda^{\gamma_{A_3},\gamma_{A_3}})^*$ with the space

$$\prod_{a\in A_1} \otimes(\Lambda^{\lambda_a}(E^*_{t(a)})) \otimes \prod_{a\in A_3} \otimes(\Lambda^{\gamma_a}(E^*_{t(a)}) \otimes \Lambda^{\gamma_a}(E^*_{h(a)}))$$

Let us arrange the tensor factors of the quotient $M_\Theta/\dot{M}_\Theta$ into groups by the following rule:

$$\prod_{a\in A_1} \otimes(L_{\lambda_a}(E_{h(a)})) \otimes \prod_{a\in A_2} \otimes(L_{\mu_a}(E_{t(a)}) \otimes L_{\mu_a}(E_{h(a)}))\otimes$$
$$\prod_{a\in A_1} \otimes(L_{\lambda_a}(E^*_{t(a)})) \otimes \prod_{a\in A_3} \otimes(L_{\gamma_a}(E^*_{t(a)}) \otimes L_{\gamma_a}(E^*_{h(a)}))$$

The first factor

$$\prod_{a\in A_1} \otimes(L_{\lambda_a}(E_{h(a)})) \otimes \prod_{a\in A_2} \otimes(L_{\mu_a}(E_{t(a)}) \otimes L_{\mu_a}(E_{h(a)}))$$

is a $(\times_{a\in A_1} GL(k_{h(a)})) \times (\times_{a\in A_2} GL(k_{t(a)}) \times GL(k_{h(a)}))$-module. Denote this group by $G_1 = G_1(\bar{k})$.

Analogously, the second factor

$$\prod_{a\in A_1} \otimes(L_{\lambda_a}(E^*_{t(a)})) \otimes \prod_{a\in A_3} \otimes(L_{\gamma_a}(E^*_{t(a)}) \otimes L_{\gamma_a}(E^*_{h(a)}))$$

is a $(\times_{a\in A_1} GL(k_{t(a)})) \times (\times_{a\in A_3} GL(k_{t(a)}) \times GL(k_{h(a)}))$-module. Denote this group by $G_2 = G_2(\bar{k})$.

Notice that by the definition of ABW-filtrations we have an epimorphism

$$d_\Theta : (\Lambda^{\lambda_{A_1}} \otimes \Lambda^{\mu_{A_2},\mu_{A_2}}) \otimes (\Lambda^{\lambda_{A_1}}(\triangleright) \otimes \Lambda^{\gamma_{A_3},\gamma_{A_3}})^* \to M_\Theta/\dot{M}_\Theta \to 0$$

To define its kernel one can use some arguments from [Zub4]. Namely, let us consider some collection of ordered partitions ("superpartition"), say $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{i,s_i}), \lambda_{i1} \geq \ldots \geq \lambda_{i,s_i} \geq 0, i = 1, \ldots, n$.

One can endow the space

$$\Lambda^{\bar{\lambda}}(\bar{f}) = \prod_{i=1}^{i=n} \otimes(\Lambda^{\lambda_i}(E_i)) = \prod_{i=1}^{i=n} \otimes(\prod_{j=1}^{j=s_i} \otimes \Lambda^{\lambda_{ij}}(E_i))$$

with a $GL(\bar{f})$-module structure, where $\bar{f} = (f_1, \ldots, f_n)$, $\dim E_i = f_i, 1 \leq i \leq n$. To be precise, each factor $GL(E_i)$ of the group $GL(\bar{f})$ acts on the corresponding tensor product $\prod_{j=1}^{j=s_i} \otimes \Lambda^{\lambda_{ij}}(E_i)$ diagonally.

It is not hard to prove that $\tilde{\bar{\lambda}} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$ is a higher weight of the $GL(\bar{f})$-module $\Lambda^{\bar{\lambda}}(\bar{f})$. Moreover, its multiplicity is equal to 1. Since $\Lambda^{\bar{\lambda}}(\bar{f})$ is a tilting $GL(\bar{f})$-module there are good and Weyl filtrations of this module such that the last quotient of the first filtration, respectively – the first quotient of the second one, is isomorphic to $\bigtriangledown_{\bar{f}}(\tilde{\bar{\lambda}})$, respectively – to $\bigtriangleup_{\bar{f}}(\tilde{\bar{\lambda}})$ [Zub3, Zub4].

Denote by $R_{\bar{f}}(\bar{\lambda})$ the kernel of the corresponding epimorphism

$$\Lambda^{\bar{\lambda}}(\bar{f}) \to \bigtriangledown_{\bar{f}}(\tilde{\bar{\lambda}})$$

and by $S_{\bar{f}}(\bar{\lambda})$ the cokernel of the inclusion

$$\bigtriangleup_{\bar{f}}(\tilde{\bar{\lambda}}) \to \Lambda^{\bar{\lambda}}(\bar{f})$$

The $GL(\bar{f})$-modules $R_{\bar{f}}(\bar{\lambda})$ and $S_{\bar{f}}(\bar{\lambda})$ are with GF and WF respectively. Besides, the module $R_{\bar{f}}(\bar{\lambda})$ and the inclusion of $\bigtriangleup_{\bar{f}}(\tilde{\bar{\lambda}})$ are uniquely defined ([Zub4], the proposition 1.1). In particular, we have a short exact sequence

$$0 \to S_{\bar{f}}(\bar{\lambda})^* \to \Lambda^{\bar{\lambda}}(\bar{f})^* \to \bigtriangleup_{\bar{f}}(\tilde{\bar{\lambda}})^* \to 0$$

By definition, $\bigtriangleup_{\bar{f}}(\tilde{\bar{\lambda}})^* \cong \bigtriangledown_{\bar{f}}(\tilde{\bar{\lambda}}^*)$. The unique higher weight of the module $\Lambda^{\bar{\lambda}}(\bar{f})^* \cong \Lambda^{\lambda_1}(E_1^*) \otimes \ldots \otimes \Lambda^{\lambda_n}(E_n^*)$ is equal to $\tilde{\bar{\lambda}}^*$ and since $\Lambda^{\bar{\lambda}}(\bar{f})$ is a tilting module we get that the module $S_{\bar{f}}(\bar{\lambda})^*$ is uniquely defined by the same proposition 1.1 from [Zub4].

Let us consider another group $GL(\bar{g}), \bar{g} = (g_1, \ldots, g_m)$ and some "superpartition" $\bar{\mu} = (\mu_1, \ldots, \mu_m), i = 1, \ldots, m$. We have a short exact sequence of $GL(\bar{f}) \times GL(\bar{g})$-modules

$$0 \to D_{\bar{f},\bar{g}}(\bar{\lambda}, \bar{\mu}) \to \Lambda^{\bar{\lambda}}(\bar{f}) \otimes \Lambda^{\bar{\mu}}(\bar{g})^* \to \bigtriangledown_{\bar{f}}(\tilde{\bar{\lambda}}) \otimes \bigtriangledown_{\bar{g}}(\tilde{\bar{\mu}}^*) \cong \bigtriangledown_{\bar{f}}(\tilde{\bar{\lambda}}) \otimes \bigtriangleup_{\bar{g}}(\tilde{\bar{\mu}})^* \to 0,$$

where

$$D_{\bar{f},\bar{g}}(\bar{\lambda}, \bar{\mu}) = R_{\bar{f}}(\bar{\lambda}) \otimes \Lambda^{\bar{\mu}}(\bar{g})^* + \Lambda^{\bar{\lambda}}(\bar{f}) \otimes S_{\bar{g}}(\bar{\mu})^*$$

By the same proposition 1.1 from [Zub4] it follows that the kernel $D_{\bar f,\bar g}(\bar\lambda,\bar\mu)$ is uniquely defined.

Notice that $L_\lambda(V^*)\cong\triangle(\tilde\lambda)^*$ [Zub1] as a $GL(V)$-module. In particular, the $GL(\bar g)$-module $\triangle_{\bar g}(\tilde{\bar\mu})^*$ is isomorphic to $L_{\tilde\lambda_1}(E_1^*)\otimes\ldots L_{\tilde\lambda_m}(E_m^*)$, where $\dim E_j=g_j, 1\le j\le m$.

Slightly abusing our notations one can say that the $G_1(\bar k)$-module

$$(\prod_{a\in A_1}\otimes L_{\lambda_a}(E_{h(a)}))\otimes\prod_{a\in A_2}\otimes(L_{\mu_a}(E_{t(a)})\otimes L_{\mu_a}(E_{h(a)}))$$

coincides with $\bigtriangledown_{\bar k}=\bigtriangledown_{\bar k}(\tilde\Theta)$. Analogously, the $G_2(\bar k)$-module

$$(\prod_{a\in B_1}\otimes L_{\lambda_a}(E^*_{t(a)}))\otimes\prod_{a\in A_3}\otimes(L_{\gamma_a}(E^*_{t(a)})\otimes L_{\gamma_a}(E^*_{h(a)}))$$

coincides with $\triangle^*_{\bar k}=\triangle^*_{\bar k}(\tilde\Theta)$.

As a consequence we have the uniquely defined short exact sequence of $G_1\times G_2$-modules

$$0\to D_{\bar k}(\Theta)=D(\Theta)\to(\Lambda^{\lambda_{A_1}}\otimes\Lambda^{\mu_{A_2},\mu_{A_2}})\otimes(\Lambda^{\lambda_{A_1}}(\triangleright)\otimes\Lambda^{\gamma_{A_3},\gamma_{A_3}})^*\to M_\Theta/\dot M_\Theta\to 0,$$

where $D_{\bar k}(\Theta)=D_{\bar k,\bar k}(\Theta,\Theta)$.

If we want to turn to the group $H=H(\bar k)$ we have to replace the group $G_1(G_2)$ by some its subgroup. Indeed, let us represent, say G_1, as $\times_{i\in V}GL(k_i)^{w_i}$, where w_i is the number of factors of G_1 coinciding with $GL(k_i), i\in V$. The next step is to replace any subproduct $\times_{i\in V_s}GL(k_i)^{w_i}$ by its diagonal subgroup which is isomorphic to $H(s), s=1,\ldots,l$.

Using Donkin-Mathieu's theorem we see that any G_i-module with GF (respectively – any G_i-module with WF) remains the same one under the restriction to the group H, $i=1,2$. For example, $\Lambda^{\lambda_{A_1}}\otimes\Lambda^{\mu_{A_2},\mu_{A_2}}$ and $\Lambda^{\lambda_{A_1}}(\triangleright)\otimes\Lambda^{\gamma_{A_3},\gamma_{A_3}}$ are tilting H-modules.

Further, we have the short exact sequence

$$0\to D(\Theta)^H\to((\Lambda^{\lambda_{A_1}}\otimes\Lambda^{\mu_{A_2},\mu_{A_2}})\otimes(\Lambda^{\lambda_{A_1}}(\triangleright)\otimes\Lambda^{\gamma_{A_3},\gamma_{A_3}})^*)^H\to Z(\Theta)\to 0$$

Here, $Z(\Theta)=(M_\Theta/\dot M_\Theta)^H=M_\Theta^H/\dot M_\Theta^H$.

The filtration $\ldots\subseteq M_\Theta\subseteq\ldots$ is good one for the component $K[R(Q,\bar t)](\bar r)$ considered as a H-module. One can rewrite the above sequence as

$$0\to D(\Theta)^H\to\mathrm{Hom}_H(\Lambda^{\lambda_{A_1}}(\triangleright)\otimes\Lambda^{\gamma_{A_3},\gamma_{A_3}},\Lambda^{\lambda_{A_1}}\otimes\Lambda^{\mu_{A_2},\mu_{A_2}})\to Z(\Theta)\to 0$$

To simplify our notations denote $(\Lambda^{\lambda_{A_1}}\otimes\Lambda^{\mu_{A_2},\mu_{A_2}})(\bar t)$ and $(\Lambda^{\lambda_{A_1}}(\triangleright)\otimes\Lambda^{\gamma_{A_3},\gamma_{A_3}})(\bar t)$ by $V(\bar t)=V_\Theta(\bar t)$ and $W(\bar t)=W_\Theta(\bar k)$ respectively.

For given pair of compatible dimensional vectors $\bar{k}(1) \geq \bar{k}(2)$ one can define at least three Schur functors d, d_1, d_2 for the groups H, G_1, G_2 correspondently (we omit all sub-indexes). Nevertheless, it is not hard to see that the action of the Schur functor d coincides with the actions of both functors $d_i, i = 1, 2$ on the short exact sequences

$$0 \to R_{\bar{k}(1)}(\Theta) \to V(\bar{t}(1)) \to \bigtriangledown_{\bar{k}(1)}(\tilde{\Theta}) \to 0$$

and

$$0 \to \bigtriangleup_{\bar{k}(1)}(\tilde{\Theta}) \to W(\bar{t}(1)) \to S_{\bar{k}(1)}(\Theta) \to 0$$

Therefore, we can identify the exact sequences

$$0 \to d(R_{\bar{k}(1)}(\Theta)) \to d(V(\bar{t}(1))) \to d(\bigtriangledown_{\bar{k}(1)}(\tilde{\Theta})) \to 0$$

and

$$0 \to d(\bigtriangleup_{\bar{k}(1)}(\tilde{\Theta})) \to d(W(\bar{t}(1))) \to d(S_{\bar{k}(1)}(\Theta)) \to 0$$

with

$$0 \to R_{\bar{k}(2)}(\Theta) \to V(\bar{t}(2)) \to \bigtriangledown_{\bar{k}(2)}(\tilde{\Theta}) \to 0$$

and

$$0 \to \bigtriangleup_{\bar{k}(2)}(\tilde{\Theta}) \to W(\bar{t}(2)) \to S_{\bar{k}(2)}(\Theta) \to 0$$

respectively since R and S are uniquely defined in all these sequences.

Let $\psi : V \otimes W^* \to d(V) \otimes d(W)^*$ be a map given by $\psi(v \otimes \alpha) = d(v) \otimes \alpha \mid_{d(W)}$, $v \in V, \alpha \in W^*$. In other notations, it is the map $\mathrm{Hom}_K(W, V) \to \mathrm{Hom}_K(d(W), d(V))$ defined as $\phi \to d \circ \phi \mid_{d(W)}$.

If $\phi \in \mathrm{Hom}_H(W, V)$ then $\phi(d(W)) \subseteq d(V)$ since ϕ commutes with the torus action. In particular, ψ is the restriction map on $\mathrm{Hom}_H(W, V)$. Moreover, $\psi(D_{\bar{k}(1)}) \subseteq D_{\bar{k}(2)}$.

Indeed, it is clear for the summand $R \otimes W^*$. Let $v \otimes \alpha \in V \otimes S^*$. The space S^* is identified with a subspace of W^* by the rule $\alpha \to \alpha \circ p$, where p is the epimorphism of the G_2-modules $W \to S \to 0$. In particular, $p(d(W)) = d(S)$ and $(\alpha \circ p) \mid_{d(W)} = \alpha \mid_{d(S)} \circ p \mid_{d(W)}$.

Let us consider the following filtration of a H-module D

$$0 \subseteq R \otimes S^* \subseteq D$$

with quotients $R \otimes S^*$ and $(R \otimes \bigtriangleup^*) \oplus (\bigtriangledown \otimes S^*)$. These quotients can be identified with $\mathrm{Hom}_K(S, R)$ and $\mathrm{Hom}_K(\bigtriangleup, R) \oplus \mathrm{Hom}_K(S, \bigtriangledown)$ respectively and the map ψ induces on ones the same kind maps

$$\mathrm{Hom}_K(S,R)\to \mathrm{Hom}_K(d(S),d(R)), \mathrm{Hom}_K(\triangle,R)\to \mathrm{Hom}_K(d(\triangle),d(R)),$$

$$\mathrm{Hom}_K(S,\bigtriangledown)\to \mathrm{Hom}_K(d(S),d(\bigtriangledown))$$

as above.

All these arguments show that we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & D & \to & \mathrm{Hom}_K(W,V) & \to & \bigtriangledown\otimes\triangle^* & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & d(D) & \to & \mathrm{Hom}_K(d(W),d(V)) & \to & d(\bigtriangledown)\otimes d(\triangle)^* & \to & 0
\end{array}$$

If we identify the last right members of the horizontal sequences with the corresponding quotients of the filtrations of $K[R(Q,\bar{t}(1))](\bar{r})$ and $K[R(Q,\bar{t}(2))](\bar{r})$ respectively then the last right vertical arrow is induced by the epimorphism $p_{\bar{t}(1),\bar{t}(2)}$.

Indeed, the map d takes a base vector of V or W to zero if its record contains at least one vector $e_j^{(i)}$ or $(e_j^{(i)})^*$, where $j\geq k(2)_i+1$ and $e_1^{(i)},\ldots,e_{k(1)_i}^{(i)}$ is a fixed base of E_i, $1\leq i\leq n$. It remains to remember the rule of the identification of the algebra $K[R(Q,\bar{t}]$ with the corresponding symmetric algebra.

Finally, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & D_{\bar{k}(1)}^{H(\bar{k}(1))} & \to & \mathrm{Hom}_{H(\bar{k}(1))}(W(\bar{t}(1)),V(\bar{t}(1))) & \to & Z_{\bar{t}(1)} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & D_{\bar{k}(2)}^{H(\bar{k}(2))} & \to & \mathrm{Hom}_{H(\bar{k}(2))}(W(\bar{t}(2)),V(\bar{t}(2))) & \to & Z_{\bar{t}(2)} & \to & 0
\end{array}$$

Here $Z_{\bar{t}}=M_\Theta(\bar{t})^{H(\bar{k})}/\dot{M}(\bar{t})_\Theta^{H(\bar{k})}$.

For the sake of shortness denote any invariant algebra $K[R(Q,\bar{t})]^{H(\bar{k})}$ by $J(Q,\bar{t})$.

Repeating word by word the proof of the proposition 1 from [Zub1] (and using the lemma 1.1 from [Zub4] as well) we see that all vertical arrows in the last diagram are epimorphisms. In particular, we get the following

Proposition 2.1 *([Don2]) The epimorphism* $p_{\bar{t}(1),\bar{t}(2)}: K[R(Q,\bar{t}(1))]\to K[R(Q,\bar{t}(2)]$ *induces the epimorphism* $\phi_{\bar{t}(1),\bar{t}(2)}: J(Q,\bar{t}(1))\to J(Q,\bar{t}(2))$.

We have an inverse spectrum of algebras:

$$\{J(Q,\bar{t}),\phi_{\bar{t}(1),\bar{t}(2)}\mid \bar{k}(1)\geq\bar{k}(2)\}$$

Moreover, because of epimorphisms $\phi_{\bar{t}(1),\bar{t}(2)}$ are homogeneous we have the countable set of spectrums:

$$\{J(Q,\bar{t})(r),\phi_{\bar{t}(1),\bar{t}(2)}\mid \bar{k}(1)\geq\bar{k}(2)\}, r=0,1,2,\ldots$$

The inverse limit of r-th spectrum denote by $J(Q)(r)$. It is clear that $J(Q)=\oplus_{r\geq 0}J(Q)(r)$ can be endowed with an algebra structure in obvious way.

Definition 3 *The algebra $J(Q)$ is said to be a free invariant algebra of "mixed" representations of the quiver Q.*

From the geometrical point of view we have a commutative diagram

$$\begin{array}{ccc} R(Q,\bar{t}(2)) & \to & R(Q,\bar{t}(2))/H(\bar{k}(2)) \\ \downarrow & & \downarrow \\ R(Q,\bar{t}(1)) & \to & R(Q,\bar{t}(1))/H(\bar{k}(1)) \end{array}$$

which is dual to

$$\begin{array}{ccc} K[R(Q,\bar{t}(2))] & \leftarrow & J(Q,\bar{t}(2)) \\ \uparrow & & \uparrow \\ K[R(Q,\bar{t}(1))] & \to & J(Q,\bar{t}(1)) \end{array}$$

In the first diagram horizontal arrows are categorical quotients with respect to the corresponding reductive groups actions and vertical ones are isomorphisms onto closed subvarieties. The algebra $J(Q)$ can be regarded as a coordinate algebra of an infinitely dimensional variety which is the direct limit of varieties $\mathrm{Spec}\,(J(Q,\bar{t})) = R(Q,\bar{t})/H(\bar{k})$ or as an invariant algebra $K[R(Q)]^{H(Q)}$, where $K[R(Q)]$ is the "homogeneous" inverse limit of the algebras $K[R(Q,\bar{t})]$ defined by the same way as above and $H(Q)$ is the direct limit of the groups $H(\bar{k})$.

It is clear that any $J(Q,\bar{t})$ is an epimorphic image of $J(Q)$. Denote the kernel of this epimorphism by $T(Q,\bar{t})$. Notice that $J(Q)(r)$ can be identified with $J(Q,\bar{t})(r)$ for sufficiently "large" $\bar{t}$. The proof can be copied from [Don2, Zub4] (see below).

In particular, there is no necessary to consider the algebra $J = J(Q)$ as the inverse limit over all compatible dimensional vectors $\bar{t}$. One can replace the set of all dimensional vectors by any cofinal subset.

For example, we can take $\{\bar{N} = (T_1,\ldots,T_n) \mid N \geq 2\}$, where $T_i = N$ iff $t_i = k_i$ otherwise $T_i = N^*$. If $N \geq r$ then $J(Q)(r) \cong J(Q,\bar{N})(r)$. From now on we suppose that $\bar{t}(1) = \bar{N}$ and $\bar{t}(2) = \bar{t}$, where the number N is sufficiently large, say $N \geq r$. Denote by $\bar{k}$ the underlying vector of $\bar{t}$.

Finally, let us denote the image of any $\phi \in \mathrm{Hom}_{H(\bar{k})}(W(\bar{t}),V(\bar{t}))$ in the homogeneous component $K[R(Q,\bar{t})](\bar{r})$ by $c(\phi,\bar{t})$. Repeating again the proof of the same proposition 1 from [Zub1] we get

Proposition 2.2 *The $\bar{r}$-homogeneous component of the ideal $T(Q,\bar{t})$ is generated as a vector space by the elements $c(\phi,\bar{N})$, where $\phi \in Hom_{H(\bar{N})}(W(\bar{N}),V(\bar{N})) \neq 0$, where $V(\bar{N}), W(\bar{N})$ run over all "superpartitions" $(\lambda_{A_1},\mu_{A_2},\gamma_{A_3})$ of multidegree $\bar{r}$. Besides, one have to require that either at least one of the modules $d(W(\bar{N})) = W(\bar{t})$ and $d(V(\bar{N})) = W(\bar{t})$ is equal to zero or $\phi\mid_{W(\bar{t})} = 0$.*

Corollary 2.1 *The algebra $J(Q,\bar{t})$ is generated by all $c(\phi)$ without any restrictions on $\phi \in Hom_{H(\bar{k})}(W(\bar{t}),V(\bar{t})) \neq 0$ or by the elements $p_{\bar{N},\bar{t}}(c(\phi)), \phi \in Hom_{H(\bar{N})}(W(\bar{N}), V(\bar{N})) \neq 0$.*

Now our aim is to compute any element $c(\phi,\bar{t})$. Almost the same calculations have been done in [Zub5].

It is clear that the group $H(\bar{N})$ contains the diagonal subgroup which is isomorphic to $GL(N)$. In particular, the requirement $\mathrm{Hom}_{H(\bar{N})}(W(\bar{N}),V(\bar{N}))\neq 0$ implies that $\mathrm{Hom}_{GL(N)}(W(\bar{N}),V(\bar{N}))\neq 0$. Thus the degrees of the polynomial $GL(N)$-modules (see [Green] for definitions) $\Lambda^{\lambda_{A_1}}\otimes\Lambda^{\mu_{A_2},\mu_{A_2}}$ and $\Lambda^{\lambda_{A_1}}(\triangleright)\otimes\Lambda^{\gamma_{A_3},\gamma_{A_3}}$ must be the same. In particular,

$$s=r_2=r_3=\mid\mu_{A_2}\mid=\sum_{a\in A_2}\mid\mu_a\mid=\mid\gamma_{A_3}\mid=\sum_{a\in A_3}\mid\gamma_a\mid$$

Let us denote $\mid\lambda_{A_1}\mid$ by t. Then $r=t+2s$.

Definition 4 *A vertex $i\in V$ is said to be ordinary iff one belongs to some V_u having cardinality one otherwise this vertex is called doubled.*

Denote the set consisting of all ordinary vertexes by V_{ord}. One can decompose this set into two subsets, say $V_{ord}=U\bigsqcup U^*$, where $U=\{i\in V_{ord}\mid t_i=k_i\}$ and $U^*=\{i\in V_{ord}\mid t_i=k_i^*\}$.

It is clear that $V\setminus V_{ord}=\bigsqcup_{q\in\Omega}V_q$, where each V_q is equal to $\{i_q,j_q\}, t_{i_q}=k_{i_q}, t_{j_q}=k^*_{j_q}, k_{i_q}=k_{j_q}=s_q$. In other words, $U_q=\{i_q\}, U^*_q=\{j_q\}$.

Any space $\mathrm{Hom}_{H(\bar{k})}(W(\bar{t}),V(\bar{t}))$ can be written as

$$\otimes_{i\in U}\mathrm{Hom}_{GL(k_i)}(\otimes_{a\in A,t(a)=i}\Lambda^{\chi_a}(E_i),\otimes_{a\in A,h(a)=i}\Lambda^{\chi_a}(E_i))$$
$$\otimes_{i\in U^*}\mathrm{Hom}_{GL(k_i)}(\otimes_{a\in A,h(a)=i}\Lambda^{\chi_a}(E_i),\otimes_{a\in A,t(a)=i}\Lambda^{\chi_a}(E_i))$$
$$\otimes_{q\in\Omega}\mathrm{Hom}_{GL(s_q)}((\otimes_{a\in A,t(a)=i_q}\Lambda^{\chi_a}(E_{i_q}))\otimes(\otimes_{a\in A,h(a)=j_q}\Lambda^{\chi_a}(E_{j_q})),$$
$$(\otimes_{a\in A,h(a)=i_q}\Lambda^{\chi_a}(E_{i_q}))\otimes(\otimes_{a\in A,t(a)=j_q}\Lambda^{\chi_a}(E_{j_q}))),$$

where χ_a is equal to λ_a,μ_a or γ_a iff $a\in A_1, a\in A_2$ or $a\in A_3$ respectively.

In particular, the space $\mathrm{Hom}_{H(\bar{k})}(W(\bar{t}),V(\bar{t}))$ does not equal zero iff the following conditions are realized :

1. $\forall i\in V_{ord}, \sum_{a\in A,t(a)=i}r_a=\sum_{a\in A,h(a)=i}r_a=p_a$
2. $\forall q\in\Omega, \sum_{a\in A,t(a)=i_q}r_a+\sum_{a\in A,h(a)=j_q}r_a=\sum_{a\in A,h(a)=i_q}r_a+\sum_{a\in A,t(a)=j_q}r_a=p_q$

As in [Zub4] we extend the set of variables $\{Y(a)\mid a\in A\}$ by the following way. Let us replace each $Y(a)$ by some new set of matrices having the same size as $Y(a)$. The cardinality of this set is equal to r_a. Simultaneously, we replace each arrow a by r_a new arrows with the same origin and end as a one-to-one corresponding to these new matrices. The quiver that we get by this way denote by $\hat{Q}$. The vertex set of $\hat{Q}$ coincides with V but the arrow set $\hat{A}$ can be different from A. Roughly speaking, this quiver is a "complete linearization" of Q with respect to $\bar{r}$.

Let us set any linear order on A. Denote this order by usual symbol $<$. Let us enumerate arrows of the quiver $\hat{Q}$ by numbers $1,\dots,r$. One can assume that for any $a\in A$ the corresponding set of new arrows are enumerated by the numbers from the segment $[\dot{a},a]=[\sum_{b<a}r_b+1,\sum_{b\leq a}r_b]$.

We obtain some "specialization" $f:[1,r]=\hat{A}\to A$ defined as $f(j)=a$ iff $j\in[\dot{a},a], a\in A$. By the same way one can define the specialization $Y(j)\to Y(a)$ iff $j\in[\dot{a},a], a\in A$. Denote the last specialization by the same symbol f.

Without loss of generality it can be assumed that $\forall a\in A_1, b\in A_2, c\in A_3, a<b<c$. Thus follows that $\hat{A}_1=[1,t], \hat{A}_2=[t+1,t+s], \hat{A}_3=[t+s+1,r]$.

Besides, $f([1,t])=A_1,\ f([t+1,s+t])=A_2$ and $f([s+t+1,r])=A_3$. It is clear that $h(j)=i$ or $t(j)=i$ iff $h(f(j))=i$ or $t(f(j))=i$ respectively, $j\in\hat{A}=[1,\dots,r], i\in V$.

Let us set

$$\forall i\in U, T(i)=\{j\in\hat{A}\mid t(j)=i\}, H(i)=\{j\in\hat{A}\mid h(j)=i\},$$
$$\forall i\in U^*, T(i)=\{j\in\hat{A}\mid h(j)=i\}, H(i)=\{j\in\hat{A}\mid t(j)=i\}$$

Analogously, let

$$\forall q\in\Omega, T(q)=\{j\in\hat{A}\mid t(j)=i_q \text{ or } h(j)=j_q\}, H(q)=\{j\in\hat{A}\mid h(j)=i_q \text{ or } t(j)=j_q\}$$

It is obvious that $p_i=\mid T(i)\mid=\mid H(i)\mid$ for any $i\in V_{ord}$ and $p_q=\mid T(q)\mid=\mid H(q)\mid$ for any $q\in\Omega$.

Let us remind that the space $\mathrm{Hom}_{H(\bar{k})}(W(\bar{t}),V(\bar{t}))$ equals

$$\mathrm{Hom}_{H(\bar{k})}((\otimes_{a\in A_1}\Lambda^{\lambda_a}(E_{t(a)}))\otimes(\otimes_{a\in A_3}\Lambda^{\gamma_a}(E_{t(a)}))\otimes(\otimes_{a\in A_3}\Lambda^{\gamma_a}(E_{h(a)})),$$
$$(\otimes_{a\in A_1}(\Lambda^{\lambda_a}(E_{h(a)}))\otimes(\otimes_{a\in A_2}\Lambda^{\mu_a}(E_{t(a)}))\otimes(\otimes_{a\in A_2}\Lambda^{\mu_a}(E_{h(a)})))$$

In particular, one can define an inclusion of this space into

$$\mathrm{Hom}_{H(\bar{k})}((\otimes_{a\in A_1}E_{t(a)}^{\otimes r_a})\otimes(\otimes_{a\in A_3}E_{t(a)}^{\otimes r_a})\otimes(\otimes_{a\in A_3}E_{h(a)}^{\otimes r_a}),$$
$$(\otimes_{a\in A_1}(E_{h(a)}^{\otimes r_a})\otimes(\otimes_{a\in A_2}E_{t(a)}^{\otimes r_a})\otimes(\otimes_{a\in A_2}E_{h(a)}^{\otimes r_a}))$$

by the rule $\phi\to i_{\lambda_{A_1},\mu_{A_2},\mu_{A_2}}\phi p_{\lambda_{A_1},\gamma_{A_3},\gamma_{A_3}}$, where $p_{\lambda_{A_1},\gamma_{A_3},\gamma_{A_3}}=(\otimes_{a\in A_1}p_{\lambda_a})\otimes(\otimes_{a\in A_3}p_{\gamma_a})^{\otimes 2}$ and $i_{\lambda_{A_1},\mu_{A_2},\mu_{A_2}}=(\otimes_{a\in A_1}i_{\lambda_a})\otimes(\otimes_{a\in A_2}i_{\mu_a})^{\otimes 2}$. Let us denote the last space by $\mathrm{Hom}\,(\bar{t})$ and this inclusion by $\Phi_{\lambda_{A_1},\mu_{A_2},\gamma_{A_3}}$.

Let us consider the multilinear component of degree r of the ring $J(\hat{Q},\bar{t})$. This component is isomorphic to

$$\mathrm{Hom}_{H(\bar{k})}((\otimes_{a\in\hat{A}_1}E_{t(a)})\otimes(\otimes_{a\in\hat{A}_3}E_{t(a)})\otimes(\otimes_{a\in\hat{A}_3}E_{h(a)}),$$
$$(\otimes_{a\in\hat{A}_1}(E_{h(a)})\otimes(\otimes_{a\in\hat{A}_2}E_{t(a)})\otimes(\otimes_{a\in\hat{A}_2}E_{h(a)}))$$

This space is equal to Hom $(\bar{t})$ up to some trivial identifications.

We obtain the following commutative diagram

$$\begin{array}{ccccc} 0 & \to & \mathrm{Hom}_{H(\bar{N})}(W(\bar{N}),V(\bar{N})) & \to & \mathrm{Hom}\,(\bar{N}) \\ & & \downarrow & & \downarrow \\ 0 & \to & \mathrm{Hom}_{H(\bar{k})}(W(\bar{t}),V(\bar{t})) & \to & \mathrm{Hom}\,(\bar{t}) \end{array}$$

The horizontal arrows are the inclusions defined above and the vertical ones are the surjective restriction maps. In particular, if both modules $W(\bar{t}), V(\bar{t})$ are not equal to zero and if we identify the space $\mathrm{Hom}_{GL(\bar{N})}(W(\bar{N}), V(\bar{N})$ with its image in $\mathrm{Hom}\,(\bar{N})$ then the kernel of the map

$$\mathrm{Hom}_{H(\bar{N})}(W(\bar{N}),V(\bar{N})) \to \mathrm{Hom}_{H(\bar{k})}(W(\bar{t}),V(\bar{t}))$$

is the intersection of $\mathrm{Hom}_{H(\bar{N})}(W(\bar{N}),V(\bar{N}))$ with the kernel of the epimorphism $\mathrm{Hom}\,(\bar{N}) \to \mathrm{Hom}\,(\bar{t})$.

The space $\mathrm{Hom}\,(\bar{t})$ can be identified with

$$\otimes_{i\in V_{ord}}\mathrm{End}_{GL(k_i)}(E_i^{\otimes p_i}) \otimes \otimes_{q\in\Omega}\mathrm{End}_{GL(s_q)}(E(q)^{\otimes p_q})$$

Here $E(q) \cong E_{i_q} \cong E_{j_q}$.

It is known that there is an epimorphism $K[S_d] \to \mathrm{End}_{GL(V)}(V^{\otimes d})$ defined by the rule: $\sigma \to \tilde{\sigma}, \sigma \in S_d$, where $\tilde{\sigma}(v_1 \otimes \ldots \otimes v_d) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(d)}$. We will omit the upper tilde if it does not lead to confusion.

The kernel I_{p+1} of this epimorphism is not equal to zero iff $d > p = \dim V$ and in the last case this kernel is generated (as a two-sided ideal) by the element $\sum_{\tau\in S_{p+1}}(-1)^\tau\tau$, where $S_{p+1} = S_{[1,p+1]}$ [Pr].

In particular, the algebra $\mathrm{Hom}\,(\bar{N})$ is isomorphic to $\otimes_{i\in V_{ord}}K[S_{p_i}] \otimes \otimes_{q\in\Omega}K[S_{p_q}]$ since we assumed that $N \geq r$. The kernel of the epimorphism $\mathrm{Hom}\,(\bar{N}) \to \mathrm{Hom}\,(\bar{t})$ equals

$$I_{\bar{k}+1} = \sum_{i\in V_{ord}, p_i>k_i} \ldots \otimes \underbrace{I_{k_i+1}}_{\text{the place of } K[S_{p_i}]} \otimes \ldots + \sum_{q\in\Omega, p_q>s_q} \ldots \otimes \underbrace{I_{s_q+1}}_{\text{the place of } K[S_{p_q}]} \otimes \ldots$$

Therefore, for all $N' \geq N \geq r$ the epimorphism $\mathrm{Hom}\,(\bar{N}') \to \mathrm{Hom}\,(\bar{N})$ is an isomorphism. Besides, the same is valid for all epimorphisms

$$\mathrm{Hom}_{H(\bar{N}')}(W(\bar{N}'),V(\bar{N}')) \to \mathrm{Hom}_{H(\bar{N})}(W(\bar{N}),V(\bar{N}))$$

In particular, the $\bar{r}$-homogeneous component of the algebra $J(Q,\bar{N})$ does not dependent on the number N and can be identified with the $\bar{r}$-homogeneous component of the free invariant algebra $J(Q)$ as we remarked earlier on.

The space

$$\mathrm{Hom}_K((\otimes_{a\in A_1}E_{t(a)}^{\otimes r_a})\otimes(\otimes_{a\in A_3}E_{t(a)}^{\otimes r_a})\otimes(\otimes_{a\in A_3}E_{h(a)}^{\otimes r_a}),$$

$$(\otimes_{a\in A_1}(E_{h(a)}^{\otimes r_a})\otimes(\otimes_{a\in A_2}E_{t(a)}^{\otimes r_a})\otimes(\otimes_{a\in A_2}E_{h(a)}^{\otimes r_a}))$$

denote by $B(\bar t)$. Let us consider the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_K(W(\bar t),V(\bar t)) & \to & K[R(Q,\bar t)](\bar r) \\ \downarrow & & \uparrow \\ B(\bar t) & \to & K[R(\hat Q,\bar t)](\underbrace{1,\dots,1}_{r}) \end{array}$$

The horizontal arrows and the left vertical arrow were defined above. The right vertical one is induced by the specialization $Y(j)\to Y(f(j))$.

Let $\phi\in\mathrm{Hom}_K(W(\bar t),V(\bar t))$. Let us suppose that we pass over this diagram along the left vertical, low horizontal and right vertical arrows consequentially.

What will happen with this element after passing along this way? What is the difference between this way result and another one – along the top arrow?

In order to answer for these questions we must fix some notations.

Let $e_1^i,\dots,e_{k_i}^i$ be a base of the space E_i, $i\in V$. The dual base of E_i^* is $(e_1^i)^*,\dots,(e_{k_i}^i)^*$. Let us decompose the interval $[1,r]$ into subintervals by the following rule:

$$[1,r]=(\bigsqcup_{a\in A_1}[\dot a,a])\bigsqcup(\bigsqcup_{a\in A_3}[\dot a-s,a-s])\bigsqcup(\bigsqcup_{a\in A_3}[\dot a,a]),$$

where $[\dot a-s,a-s]$ equals $[\sum_{b<a}r_b-s+1,\sum_{b\le a}r_b-s]$.

By the same way one can decompose the interval $[1,r]$ into another subintervals:

$$[1,r]=(\bigsqcup_{a\in A_1}[\dot a,a])\bigsqcup(\bigsqcup_{a\in A_2}[\dot a,a])\bigsqcup(\bigsqcup_{a\in A_2}[\dot a+s,a+s]),$$

where $[\dot a+s,a+s]$ equals $[\sum_{b<a}r_b+s+1,\sum_{b\le a}r_b+s]$.

Let $I,J:[1,r]\to[1,\max_{i\in V}k_i]$ be two maps such that the following conditions are realized:

1. $\forall a\in A_1, I([\dot a,a])\subseteq[1,k_{t(a)}]$ and $J([\dot a,a])\subseteq[1,k_{h(a)}]$;
2. $\forall a\in A_2, J([\dot a,a])\subseteq[1,k_{t(a)}]$ and $J([\dot a+s,a+s])\subseteq[1,k_{h(a)}]$;
3. $\forall a\in A_3, I([\dot a-s,a-s])\subseteq[1,k_{t(a)}]$ and $I([\dot a,a])\subseteq[1,k_{h(a)}]$.

It is necessary to define some Young subgroups of the group S_r. Let $\lambda_a=(\lambda_1^{(a)},\dots,\lambda_{u_a}^{(a)}),\mu_b=(\mu_1^{(b)},\dots,\mu_{v_b}^{(b)}),\gamma_c=(\gamma_1^{(c)},\dots,\gamma_{w_c}^{(c)}),a\in A_1,b\in A_2,c\in A_3$.

One can define two refine decompositions of the previous ones. The first one is

$$\forall a \in A_1, [\dot{a}, a] = \bigsqcup_{1\le l\le u_a} [\sum_{b<a} r_b + \sum_{1\le j\le l-1} \lambda_j^{(a)} + 1, \sum_{b<a} r_b + \sum_{1\le j\le l} \lambda_j^{(a)}],$$

$$\forall a \in A_3, [\dot{a} - s, a - s] = \bigsqcup_{1\le l\le w_a} [\sum_{b<a} r_b - s + \sum_{1\le j\le l-1} \gamma_j^{(a)} + 1, \sum_{b<a} r_b - s + \sum_{1\le j\le l} \gamma_j^{(a)}],$$

$$\forall a \in A_3, [\dot{a}, a] = \bigsqcup_{1\le l\le w_a} [\sum_{b<a} r_b + \sum_{1\le j\le l-1} \gamma_j^{(a)} + 1, \sum_{b<a} r_b + \sum_{1\le j\le l} \gamma_j^{(a)}]$$

The second one is the same on the segment $[1, t]$ but outside this is

$$\forall a \in A_2, [\dot{a}, a] = \bigsqcup_{1\le l\le v_a} [\sum_{b<a} r_b + \sum_{1\le j\le l-1} \mu_j^{(a)} + 1, \sum_{b<a} r_b + \sum_{1\le j\le l} \mu_j^{(a)}],$$

$$\forall a \in A_2, [\dot{a} + s, a + s] = \bigsqcup_{1\le l\le v_a} [s + \sum_{b<a} r_b + \sum_{1\le j\le l-1} \mu_j^{(a)} + 1, s + \sum_{b<a} r_b + \sum_{1\le j\le l} \mu_j^{(a)}]$$

Besides, one can define the additional refine decomposition of the original one $[1, r] = \bigsqcup_{a\in A}[\dot{a}, a]$ by the obvious way:

$$\forall a \in A_1, [\dot{a}, a] = \bigsqcup_{1\le l\le u_a} [\sum_{b<a} r_b + \sum_{1\le j\le l-1} \lambda_j^{(a)} + 1, \sum_{b<a} r_b + \sum_{1\le j\le l} \lambda_j^{(a)}],$$

$$\forall a \in A_2, [\dot{a}, a] = \bigsqcup_{1\le l\le v_a} [\sum_{b<a} r_b + \sum_{1\le j\le l-1} \mu_j^{(a)} + 1, \sum_{b<a} r_b + \sum_{1\le j\le l} \mu_j^{(a)}],$$

$$\forall a \in A_3, [\dot{a}, a] = \bigsqcup_{1\le l\le w_a} [\sum_{b<a} r_b + \sum_{1\le j\le l-1} \gamma_j^{(a)} + 1, \sum_{b<a} r_b + \sum_{1\le j\le l} \gamma_j^{(a)}]$$

Let $S_{\lambda_{A_1},\gamma_{A_3},\gamma_{A_3}}$, $S_{\lambda_{A_1},\mu_{A_2},\mu_{A_2}}$ and $S_{\lambda_{A_1},\mu_{A_2},\gamma_{A_3}}$ be an Young subgroups of S_r defined by the first, second and third refine decompositions respectively.

Let us suppose that the following conditions are realized – the restrictions of the maps I and J on all intervals of the first and second refine decompositions respectively are injective ones.

Then a typical base vector of $\mathrm{Hom}_K(W(\bar{t}), V(\bar{t}))$ is $p_{\lambda_{A_1},\gamma_{A_3},\gamma_{A_3}}(e_I^*)\otimes p_{\lambda_{A_1},\mu_{A_2},\mu_{A_2}}(e_J)$, where

$$e_I^* = (\otimes_{a\in A_1}(\otimes_{l\in[\dot{a},a]}(e_{I(l)}^{t(a)})^*)) \otimes (\otimes_{a\in A_3}(\otimes_{l\in[\dot{a}-s,a-s]}(e_{I(l)}^{t(a)})^*)) \otimes (\otimes_{a\in A_3}(\otimes_{l\in[\dot{a},a]}(e_{I(l)}^{h(a)})^*))$$

and

$$e_J = (\otimes_{a\in A_1}(\otimes_{l\in[\dot{a},a]} e_{J(l)}^{h(a)})) \otimes (\otimes_{a\in A_2}(\otimes_{l\in[\dot{a},a]} e_{J(l)}^{t(a)})) \otimes (\otimes_{a\in A_2}(\otimes_{l\in[\dot{a}+s,a+s]} e_{J(l)}^{h(a)}))$$

After passing over the first way mentioned above we obtain

$$\sum_{\sigma_1,\sigma_2\in S_{\lambda_{A_1}}}(-1)^{\sigma_1}(-1)^{\sigma_2}\prod_{1\le j\le t}y(f(j))_{I(\sigma_1(j)),J(\sigma_2(j))}\times$$

$$\sum_{\sigma_1,\sigma_2\in S_{\mu_{A_2}}}(-1)^{\sigma_1}(-1)^{\sigma_2}\prod_{t\le j\le t+s}y(f(j))_{J(\sigma_1(j)),J(\sigma_2(j+s))}\times$$

$$\sum_{\sigma_1,\sigma_2\in S_{\gamma_{A_3}}}(-1)^{\sigma_1}(-1)^{\sigma_2}\prod_{t+s\le j\le r}y(f(j))_{I(\sigma_1(j-s)),I(\sigma_2(j))}$$

Ordering the factors of these products with respect to their first coordinates we get

$$\mid S_{\lambda_{A_1},\mu_{A_2},\gamma_{A_3}}\mid\sum_{\sigma\in S_{\lambda_{A_1}}}(-1)^{\sigma}\prod_{1\le j\le t}y(f(j))_{I(j),J(\sigma(j))}\times$$

$$\sum_{\sigma\in S_{\lambda_{A_2}}}(-1)^{\sigma}\prod_{t\le j\le t+s}y(f(j))_{J(j),J(\sigma(j+s))}\times$$

$$\sum_{\sigma\in S_{\lambda_{A_3}}}(-1)^{\sigma}\prod_{t+s\le j\le r}y(f(j))_{I(j-s),I(\sigma(j))}$$

We denote the image of any element $\psi\in B(\bar t)$ in the $K[R(\hat Q,\bar t)](\underbrace{1,\dots,1}_{r})$ by $tr^*(\psi)$ and its specialization under f by $tr^*(\psi,f)$.

The computation given above show that after passing over our diagram by the first way we obtained the element $tr^*(\Phi_{\lambda_{A_1},\mu_{A_2},\gamma_{A_3}}(\phi),f)=\mid S_{\lambda_{A_1},\mu_{A_2},\gamma_{A_3}}\mid c(\phi)$, where $c(\phi)=c(\phi,\bar t)$ is the image of ϕ under the top map $\mathrm{Hom}_K(W(\bar t),V(\bar t))\to K[R(Q,\bar t)](\bar r)$.

The given operator σ from $\mathrm{Hom}(\bar N)\subseteq \mathrm{End}_{GL(N)}(E^{\otimes r})=K[S_r]$, where E is a N-dimensional space which is isomorphic to all E_i, can be written as

$$\sum_{1\le j_1,\dots,j_r\le N}(\otimes_{1\le k\le t}(e^{t(k)}_{j_k})^*)\otimes(\otimes_{t+s+1\le k\le r}(e^{t(k)}_{j_{k-s}})^*)\otimes(\otimes_{t+s+1\le k\le r}(e^{h(k)}_{j_k})^*)\otimes$$

$$(\otimes_{1\le k\le t}e^{h(k)}_{j_{\sigma^{-1}(k)}})\otimes(\otimes_{t+1\le k\le t+s}e^{t(k)}_{j_{\sigma^{-1}(k)}})\otimes(\otimes_{t+1\le k\le t+s}e^{h(k)}_{j_{\sigma^{-1}(k+s)}})$$

Then $tr^*(\sigma)$ equals

$$\sum_{1\le j_1,\dots,j_r\le N}(\prod_{1\le k\le t}y(k)_{j_k,j_{\sigma^{-1}(k)}})(\prod_{t+1\le k\le t+s}y(k)_{j_{\sigma^{-1}(k)},j_{\sigma^{-1}(k+s)}})(\prod_{t+s+1\le k\le r}y(k)_{j_{k-s},j_k})$$

In order to contract this sum into a product of ordinary traces one can use the following rule (see [Zub5]). Let us consider the formal "product" of pairs:

$$\prod_{1\le k\le t}(k,\sigma^{-1}(k))\prod_{t+1\le k\le t+s}(\sigma^{-1}(k),\sigma^{-1}(k+s))\prod_{t+s+1\le k\le r}(k-s,k)$$

The next step is to divide this "product" into "cycles". By definition, any "cycle" is a "subproduct" $\prod_{1\le f\le l}(a_f,b_f)$ such that $b_f=a_{f+1}, 1\le f\le l-1$ and $b_l=a_1$. If it is necessary one can change the previous order of "coordinates" of any pair. This subdivision is possible because of the following fact – each symbol k appears two times in the original "product".

Finally, each "cycle"$\prod_{1\le f\le l}(a_f,b_f)$ corresponds to a trace $tr(Z(j_1)\dots Z(j_l))$, where j_f is the number of the pair (a_f,b_f) in the original "product" and $Z(j_f)$ coincides with $Y(j_f)$ iff we did not change the original order of the coordinates of this pair otherwise $Z(j_f)=Y(j_f)^t, 1\le f\le l$. It is more convenience for our further computations to denote $Y(?)^t$ by $\overline{Y(?)}$.

For example, let $t=3, s=2, r=7, \sigma=(1726)(354)\in S_7$. The "product" of pairs corresponds to σ is $(16)(27)(34)(52)(31)(46)(57)$. Decomposing into "cycles" we get $(16)(64)(43)(31)\cdot(27)(75)(52)$. Therefore,

$$tr^*(\sigma)=tr(Y(1)\overline{Y(6)}\ \overline{Y(3)}Y(5))tr(Y(2)\overline{Y(7)}Y(4))$$

Notice that if $s=0$ then $tr^*(\sigma)=tr(\sigma)$, where $tr(\sigma)=tr(Y(a)\dots Y(b))\dots tr(Y(c)\dots Y(d))$ and $(a\dots b)\dots(c\dots d)$ is the cyclic decomposition of σ^{-1}.

Let us set

$$T(i_q)=\{j\mid t(j)=i_q\}, H(i_q)=\{j\mid h(j)=i_q\},$$
$$T(j_q)=\{h(j)=j_q\}, H(j_q)=\{j\mid t(j)=j_q\}$$

It is clear that $\forall q\in\Omega, T(q)=T(i_q)\bigcup T(j_q), H(q)=H(i_q)\bigcup H(j_q)$.

Let us define a "doubled" quiver $Q^{(d)}$ as follows. The vertex set $V^{(d)}$ of this quiver equals $V\bigsqcup V^*_{ord}$, where $V^*_{ord}=\{i^*\mid i\in V_{ord}\}$. Respectively, the arrow set $A^{(d)}$ of one equals $A\bigsqcup\overline{A}$, where $\overline{A}=\{\bar a\mid a\in A\}$. By definition, $A^{(d)}_i=A_i\bigsqcup\overline{A}_i$, where $\overline{A}_i=\{\bar a\mid a\in A_i\}, i=1,2,3$.

Further, if $h(a),t(a)\in V_{ord}$ then $h(\bar a)=t(a)^*, t(\bar a)=h(a)^*$ but if $h(a)$ or $t(a)$ lies in some $V_q, q\in\Omega$, then

$$h(\bar a)=\begin{cases} j_q, t(a)=i_q,\\ i_q, t(a)=j_q\end{cases}$$

and symmetrically

$$t(\bar a)=\begin{cases} j_q, h(a)=i_q,\\ i_q, h(a)=j_q\end{cases}$$

By the same way one can construct the "doubled" quiver $\hat Q^{(d)}$.

Let us remind that for any $a\in A^{(d)}(\hat A^{(d)})$ we suppose $Z(a)=Y(a)$ iff $a\in A$ otherwise $Z(a)=\overline{Y(a)}$.

Definition 5 *Any product $Z(a_m)\dots Z(a_1)$ is said to be admissible iff $a_m,\dots,a_1$ is a path in $Q^{(d)}$, i.e. if $t(a_i)=h(a_{i+1}), i=1,\dots,m-1$. It is said to be strong admissible iff this path is closed, i.e. if $t(a_m)=h(a_1)$.*

Lemma 2.1 *Any $\sigma \in S_r$ lies in Hom$(\bar{N})$ iff the following equations are fulfilled:*

1. $\forall i \in U, \sigma((T(i) \bigcap \hat{A}_1) \bigsqcup (T(i) \bigcap \hat{A}_3 - s)) = (H(i) \bigcap \hat{A}_1) \bigsqcup (H(i) \bigcap \hat{A}_2 + s)$;
2. $\forall i \in U^*, \sigma((T(i)) = H(i)$;
3. $\forall q \in \Omega, \sigma((T(i_q) \bigcap \hat{A}_1) \bigsqcup (T(i_q) \bigcap \hat{A}_3 - s) \bigsqcup T(j_q)) = (H(i_q) \bigcap \hat{A}_1) \bigsqcup (H(i_q) \bigcap \hat{A}_2 + s) \bigsqcup H(j_q)$.

Proof. Any $\sigma \in \mathrm{Hom}\,(\bar{N})$ has an original record with respect to the embedding $\mathrm{Hom}\,(\bar{N}) \subseteq \mathrm{End}_{GL(N)}(E^{\otimes r}) = K[S_r]$, say

$$\sum_{1 \le j_1, \ldots, j_r \le N} (\otimes_{1 \le k \le t} (e_{j_k}^{t(k)})^*) \otimes (\otimes_{t+s+1 \le k \le r} (e_{j_{k-s}}^{t(k)})^*) \otimes (\otimes_{t+s+1 \le k \le r} (e_{j_k}^{h(k)})^*) \otimes$$

$$(\otimes_{1 \le k \le t} e_{j_{\sigma^{-1}(k)}}^{h(k)}) \otimes (\otimes_{t+1 \le k \le t+s} e_{j_{\sigma^{-1}(k)}}^{t(k)}) \otimes (\otimes_{t+1 \le k \le t+s} e_{j_{\sigma^{-1}(k+s)}}^{h(k)}),$$

introduced above. One can rewrite it in a more refined way

$$\sum_{1 \le j_1, \ldots, j_r \le N} \otimes_{i \in U} ((\otimes_{1 \le k \le t, t(k)=i} (e_{j_k}^{i})^*) \otimes (\otimes_{t+1 \le k \le t+s, t(k+s)=i} (e_{j_k}^{i})^*) \otimes (\otimes_{1 \le k \le t, h(k)=i} e_{j_{\sigma^{-1}(k)}}^{i})$$

$$\otimes (\otimes_{t+1 \le k \le t+s, h(k)=i} e_{j_{\sigma^{-1}(k+s)}}^{i})) \otimes_{i \in U^*} ((\otimes_{t+s+1 \le k \le r, h(k)=i} (e_{j_k}^{i})^*) \otimes (\otimes_{t+1 \le k \le t+s, t(k)=i} e_{j_{\sigma^{-1}(k)}}^{i}))$$

$$\otimes_{q \in \Omega} ((\otimes_{1 \le k \le t, t(k)=i_q} (e_{j_k}^{i_q})^*) \otimes (\otimes_{t+1 \le k \le t+s, t(k+s)=i_q} (e_{j_k}^{i_q})^*) \otimes (\otimes_{t+s+1 \le k \le r, h(k)=j_q} (e_{j_k}^{j_q})^*)$$

$$\otimes ((\otimes_{1 \le k \le t, h(k)=i_q} e_{j_{\sigma^{-1}(k)}}^{i_q}) \otimes (\otimes_{t+1 \le k \le t+s, h(k)=i_q} e_{j_{\sigma^{-1}(k+s)}}^{i_q}) \otimes (\otimes_{t+1 \le k \le t+s, t(k)=j_q} (e_{j_{\sigma^{-1}(k)}}^{j_q}))$$

It remains to notice that any factor $e_{??}^{?}$ in any summand of this sum must appear on a "dual" side of the same summand, i.e. like $(e_{??}^{?})^*$. This concludes the proof.

The equations in this lemma generalize the ordinary quiver case ones(compare with [Zub5]). For the sake of convenience denote the right side sets of these equations by $\mathcal{H}(i), \mathcal{H}(q)$ and the left side sets, i.e. we mean arguments of the substitution σ, by $\mathcal{T}(i), \mathcal{T}(q)$ respectively. Then they can be rewritten as $\sigma(\mathcal{T}(?)) = \mathcal{H}(?)$.

Lemma 2.2 *Any $tr(Z(a_m) \ldots Z(a_1))$ occurs as a factor of some multilinear trace products from $J(\hat{Q}, \bar{t})(\underbrace{1, \ldots, 1}_{r})$ iff $Z(a_m) \ldots Z(a_1)$ is strong admissible.*

Proof. Fix some subproduct UVW of any cyclic permutation of the product $Z(a_m) \ldots Z(a_1)$ consisting of three factors. We will prove that both UV, VW are admissible. In fact, it is enough to prove this assertion for UV or VW only but we need to describe both cases.

Without loss of generality one can assume that $V = Y(j)$. In the converse case one can transpose the product $\ldots UVW \ldots$. The following list contains all admissible cases of occupying both places around V.

1. If $j \in \hat{A}_1 = [1, \dots, t]$ then U can be occupied by $Y(\sigma(j))$. It happens iff $\sigma(j) \in \hat{A}_1$. Let $t(j) = i$. Then we have either $j \in T(i) \bigcap \hat{A}_1$ or $i = i_q, j \in T(i_q) \bigcap \hat{A}_1$. In both cases $\sigma(j) \in H(i)$, i.e. the product $\overline{Y(\sigma(j))}Y(j)$ is admissible. The matrix U can be equal to $Y(j')$ or $\overline{Y(j')}$, where $j' \in \hat{A}_2 = [t+1, \dots, t+s]$. The case $U = Y(j')$ takes place iff $\sigma(j) = j' + s \in \hat{A}_3$. It means that either $j' \in H(i) \bigcap \hat{A}_2$ or $i = i_q, j' \in H(i_q) \bigcap \hat{A}_2$ and in both cases the product $Y(j')Y(j)$ is admissible too.

   Finally, let $U = \overline{Y(j')}$. It means that $\sigma(j) = j'$. In this case $i = i_q$ only and $j' \in H(j_q) \bigcap \hat{A}_2$, i.e. the product $\overline{Y(j')}Y(j)$ is admissible again. As for W the possibilities are the following: $Y(\sigma^{-1}(j)), Y(j')$ or $\overline{Y(j')}$, where $j' \in \hat{A}_3$. The admissibility can be proved by the same way as above. Briefly one can describe all these ways of occupying as $\underbrace{(\hat{A}_2, \bar{\hat{A}}_2, \hat{A}_1)}_{U} \underbrace{\hat{A}_1}_{V} \underbrace{(\hat{A}_3, \bar{\hat{A}}_3, \hat{A}_1)}_{W}$.

   Other cases for V are listed without any comments. The interested reader can check ones very easily.

2. If $j \in \hat{A}_2$ then either $U = \overline{Y(j')}, j' \in \hat{A}_1, t(j') = i_q, t(j) = j_q$ or $U = Y(j'), \overline{Y(j')}, j' \in \hat{A}_3$. In the last case either $t(j) = h(j')$ or $t(j) = j_q, t(j') = i_q$. For W we have the following opportunities: $W = Y(j'), j' \in \hat{A}_1$ – described in the first item, $W = Y(j'), \overline{Y(j')}, j' \in \hat{A}_3$ and either $t(j') = h(j)$ or $h(j') = j_q, h(j) = i_q$. Briefly, $\underbrace{(\hat{A}_3, \bar{\hat{A}}_3, \bar{\hat{A}}_1)}_{U} \underbrace{\hat{A}_2}_{V} \underbrace{(\hat{A}_3, \bar{\hat{A}}_3, \hat{A}_1)}_{W}$.

3. If $j \in \hat{A}_3$ then either $U = Y(j'), j' \in \hat{A}_1, t(j) = h(j')$ or $U = Y(j'), \overline{Y(j')}, j' \in \hat{A}_2$. The last case is considered in the second item up to some transposition. For W we have the following opportunities: $W = \overline{Y(j')}, j' \in \hat{A}_1$ – described in the first item up to some transposition, and $W = Y(j'), \overline{Y(j')}, j' \in \hat{A}_2$ – described in the second item too. Briefly, $\underbrace{(\hat{A}_2, \bar{\hat{A}}_2, \hat{A}_1)}_{U} \underbrace{\hat{A}_3}_{V} \underbrace{(\hat{A}_2, \bar{\hat{A}}_2, \bar{\hat{A}}_1)}_{W}$.

It is clear that in all cases listed above the products UV, VW are admissible. The lemma is proved.

Any trace product $u = tr(Z(a_r) \dots Z(a_k)) \dots tr(Z(a_m) \dots Z(a_1))$ can be rewritten by many ways. We will fix only one writing of each product as follows. In any factor $tr(Z(a) \dots Z(b))$ each matrix $Z(?)$ is equal either $Y(j)$ or $\overline{Y(j)}$. Let us ascribe to $Z(?)$ its number j. The record of $tr(Z(a) \dots Z(b))$ is called right iff the matrix with maximal number, say j, occupies the first place, i.e. it is $Z(a)$, and also $Z(a) = Y(j)$. Let us call this number j the number of $tr(Z(a) \dots Z(b))$. The record of u is called right iff all its factors are right and their numbers increase on passing by this product from the left to the right. Of course, we assume that u satisfies all conditions of admissibility.

Proposition 2.3 *The right trace products form a base of* $J(\hat{Q},\bar{N})(\underbrace{1,\ldots,1}_{r})$. *In particular, they span* $J(\hat{Q},\bar{t})(\underbrace{1,\ldots,1}_{r})$ *for any* $\bar{k}$.

Proof. The first assertion has been proved in [Zub5]. The second one is a trivial consequence of the proposition 2.1.

For the sake of convenience we will omit the symbol tr in the record of any multilinear invariant from $J(\hat{Q},\bar{N})(\underbrace{1,\ldots,1}_{r})$ excepting the cases when it is necessary. Besides, we will replace any matrix $Y(j)$ or its transposed $\overline{Y(j)}$ by the number j or its "transposed" $\bar{j}$, $1 \leq j \leq r$.

For example, the invariant $tr(Y(1)\overline{Y(6)}\,\overline{Y(3)}Y(5))tr(Y(2)\overline{Y(7)}Y(4))$ given above can be rewritten as $(1\bar{6}\bar{3}5)(2\bar{7}4)$. In particular, if $s=0$ then for any $\sigma \in S_r, tr^*(\sigma) = tr(\sigma) = \sigma$.

Finally, let us suppose by definition that $\bar{\bar{i}} = i, i = 1,\ldots,r$ and $[\bar{1},\bar{r}] = \{\bar{1},\ldots,\bar{r}\}$.

The contracting rules mentioned above can be described more precisely.

Namely, let $\sigma \in \mathrm{Hom}\,(\bar{N})$ and $tr^*(\sigma) = (a\ldots b)\ldots(c\ldots d)$, where $a,\ldots,b,c,\ldots,d \in [1,r]\bigsqcup[\bar{1},\bar{r}]$. All we need is to define exactly what is a right side neighbor of any symbol j in this cyclic record of $tr^*(\sigma)$? If j is an ordinary symbol, i.e. if $j \in [1,r]$, then we have

1. If $j \in \hat{A}_1$ then $(\ldots j?\ldots)$, where

$$? = \begin{cases} \sigma^{-1}(j),\ \sigma^{-1}(j) \in \hat{A}_1, \\ \sigma^{-1}(j)+s,\ \sigma^{-1}(j) \in \hat{A}_2, \\ \overline{\sigma^{-1}(j)},\ \sigma(j) \in \hat{A}_3 \end{cases}$$

2. If $j \in \hat{A}_2$ then $(\ldots j?\ldots)$, where

$$? = \begin{cases} \sigma^{-1}(j+s),\ \sigma^{-1}(j+s) \in \hat{A}_1, \\ \sigma^{-1}(j+s)+s,\ \sigma^{-1}(j+s) \in \hat{A}_2, \\ \overline{\sigma^{-1}(j+s)},\ \sigma^{-1}(j+s) \in \hat{A}_3 \end{cases}$$

3. If $j \in \hat{A}_3$ then $(\ldots j?\ldots)$, where

$$? = \begin{cases} \overline{\sigma(j)},\ \sigma(j) \in \hat{A}_1, \\ \sigma(j),\ \sigma(j) \in \hat{A}_2, \\ \overline{\sigma(j)-s},\ \sigma(j) \in \hat{A}_3 \end{cases}$$

If $j = \bar{l}$ then the corresponding rules are:

1. If $l \in \hat{A}_1$ then $(\ldots j? \ldots)$, where

$$? = \begin{cases} \overline{\sigma(l)}, \ \sigma(l) \in \hat{A}_1, \\ \sigma(l), \ \sigma(l) \in \hat{A}_2, \\ \overline{\sigma(l) - s}, \ \sigma(l) \in \hat{A}_3 \end{cases}$$

2. If $j \in \hat{A}_2$ then $(\ldots j? \ldots)$, where

$$? = \begin{cases} \sigma^{-1}(l), \ \sigma^{-1}(l) \in \hat{A}_1, \\ \sigma^{-1}(l) + s, \ \sigma^{-1}(l) \in \hat{A}_2, \\ \overline{\sigma^{-1}(l)}, \ \sigma^{-1}(l) \in \hat{A}_3 \end{cases}$$

3. If $j \in \hat{A}_3$ then $(\ldots j? \ldots)$, where

$$? = \begin{cases} \overline{\sigma(l-s)}, \ \sigma(l-s) \in \hat{A}_1, \\ \sigma(l-s), \ \sigma(l-s) \in \hat{A}_2, \\ \overline{\sigma(l-s) - s}, \ \sigma(l-s) \in \hat{A}_3 \end{cases}$$

In the next section we will prove that each space $\mathrm{Hom}_{H(\bar{N})}(W(\bar{N}), V(\bar{N}))$ as well as the kernel $I_{\bar{k}+1}$ have bases defined over Z. In other words, any vector of ones is a linear combination of operators $\sigma \in S_r$ (let us remind that we identify $\mathrm{Hom}_{H(\bar{N})}(W(\bar{N}), V(\bar{N}))$ with a subspace of $\mathrm{Hom}(\bar{N}) \subseteq K[S_r]$) with coefficients ± 1. Let ϕ be a basic vector of such type from $\mathrm{Hom}_{H(\bar{N})}(W(\bar{N}, V(\bar{N}))$. We will show that the element $c(\phi)$ can be computed in the ring $Z[R(Q, \bar{N})]$. Moreover, we see that $c(\phi) = \frac{1}{|S_{\lambda_{A_1}, \mu_{A_2}, \gamma_{A_3}}|} tr^*(\Phi_{\lambda_{A_1}, \mu_{A_2}, \gamma_{A_3}}(\phi), f)$. Therefore, one can assume that $K = Q$ unless otherwise stated. (see [Zub1, Zub4]).

3 Suitable generators

Fix some $\sigma_0 \in \mathrm{Hom}(\bar{N})$. Then we have $\mathrm{Hom}(\bar{N}) = \sigma_0 \cdot (\otimes_{i \in V_{ord}} K[S_{\mathcal{T}_i}]) \otimes (\otimes_{q \in \Omega} K[S_{\mathcal{T}_q}])$. Moreover, the ideal $I_{\bar{k}+1}$ is equal to

$$\sigma_0 \cdot (\sum_{i \in V_{ord}, p_i > k_i} \ldots \otimes \underbrace{I_{k_i+1}}_{\text{the place of } K[S_{\mathcal{T}_i}]} \otimes \ldots + \sum_{q \in \Omega, p_q > s_q} \ldots \otimes \underbrace{I_{s_q+1}}_{\text{the place of } K[S_{\mathcal{T}_q}]} \otimes \ldots)$$

Denote by S_t the group $(\times_{i \in V_{ord}} S_{\mathcal{T}_i}) \otimes (\otimes_{q \in \Omega} S_{\mathcal{T}_q})$.

The image of the space $\mathrm{Hom}_{GL(\bar{N})}(W(\bar{N}), V(\bar{N}))$ in the space $\mathrm{Hom}(\bar{N})$ is equal to

$$N_{\lambda_{A_1}, \mu_{A_2}, \gamma_{A_3}} = \{\phi \in \sigma_0 \cdot K[S_t] \mid \tau_1 \phi \tau_2 = (-1)^{\tau_1} (-1)^{\tau_1} \phi,$$
$$\forall \tau_1 \in S_{\lambda_{A_1}, \mu_{A_2}, \mu_{A_2}}, \forall \tau_2 \in S_{\lambda_{A_1}, \gamma_{A_3}, \gamma_{A_3}}\}$$

or to

$$\sigma_0 \cdot \{\phi \in K[S_t] \mid \sigma_0^{-1}\tau_1\sigma_0\phi\tau_2 = (-1)^{\tau_1}(-1)^{\tau_1}\phi, \forall\tau_1 \in S_{\lambda_{A_1},\mu_{A_2},\mu_{A_2}}, \forall\tau_2 \in S_{\lambda_{A_1},\gamma_{A_3},\gamma_{A_3}}\}$$

(see [Zub1, Zub4]).

The next computations of the generators of the space $N_{\lambda_{A_1},\mu_{A_2},\gamma_{A_3}}$ are almost the same as in [Zub4] and we will omit superfluous details referring the reader to this article.

Up to the end of this section we denote by E_k a vector space of dimension k with a fixed base $e_1,\ldots,e_k$.

Let us identify the group algebra $K[S_r]$ with some subspace of the homogeneous component $S^r(E_r\otimes E_r)$ by the rule $\sigma \longleftrightarrow \prod_{i=1}^{i=r} e_{\sigma(i)}\otimes e_i$. For any subset $T\subseteq\{1,\ldots,r\}$ denote by E_T the subspace of E_r generated by all vectors $e_j, j\in T$.

We consider the space $S^r(E_r\otimes E_r)$ as a $GL(r)\times GL(r)$-module. The group S_r acts on the space E_r by the rule $\sigma(e_i)=e_{\sigma(i)}, \sigma\in S_r, 1\le i\le r$. In other words, we identify the group S_r with a subgroup of the group of monomial matrices by the rule $\sigma\to\sum_{1\le i\le r} e_{\sigma(i),i}$, where e_{kl} is a matrix unit which has zero coefficients outside of k-th row and l-th column place and this place is occupied by 1, $1\le k,l\le r$. Denote the matrix $\sum_{1\le i\le r} e_{\sigma(i),i}$ by the same symbol σ.

The inclusion $K[S_r]\to S^r(E_r\otimes E_r)$ is a morphism of $S_r\times S_r$-modules. Finally, it can easily be checked that $K[S_r]$ coincides with the weight subspace $S^r(E_r\otimes E_r)^{(1^r)\times(1^r)}$ under the induced action of the standard torus $T(r)\times T(r)$.

Using the same arguments we see that $K[S_t]$ coincides with the subspace

$$((\otimes_{i\in V_{ord}} S^{p_i}(E_{\mathcal{T}_i}\otimes E_{\mathcal{T}_i}))\otimes(\otimes_{q\in\Omega} S^{p_q}(E_{\mathcal{T}_q}\otimes E_{\mathcal{T}_q})))^{(1^r)\times(1^r)}$$

Let $GL(\lambda_{A_1},\mu_{A_2},\mu_{A_2})$ $(GL(\lambda_{A_1},\gamma_{A_3},\gamma_{A_3}))$ be a subgroup of the group $GL(r)$ consisting of all block diagonal matrices those satisfy the following requirement: if we decompose the interval $[1,r]$ into sequential subintervals of the same lengths as the sizes of their blocks from the top to the bottom then we get the second refine decomposition (respectively – the first one). It is not hard to prove that $\sigma_0^{-1}\cdot N_{\lambda_{A_1},\mu_{A_2},\gamma_{A_3}}$ can be identified with

$$\{g\in(\otimes_{i\in V_{ord}} S^{p_i}(E_{\mathcal{T}_i}\otimes E_{\mathcal{T}_i}))\otimes(\otimes_{q\in\Omega} S^{p_q}(E_{\mathcal{T}_q}\otimes E_{\mathcal{T}_q}))\mid \forall x\in GL(\lambda_{A_1},\mu_{A_2},\mu_{A_2}),$$

$$\forall y\in GL(\lambda_{A_1},\gamma_{A_3},\gamma_{A_3}), g^{(\sigma_0^{-1}x\sigma_0,y)}=\det(x)\det(y)g\}$$

Let us construct some filtration in $K[S_t]$. First of all, we divide each $\mathcal{T}(?)$ into some sublayers by a "monotonic" way. In other words, let $\mathcal{T}(?)=\sqcup_{1\le j\le l_?}\bar{\beta}_{?j}$, where $\max\bar{\beta}_{?j_1}<\min\bar{\beta}_{?j_2}$ as soon as $j_1<j_2$, and $\max(\min)\bar{\beta}_{?j}$ means the maximal (minimal) number from one. Joining over all indexes ? we obtain a decomposition of the segment $[1,r]$. Denote by $S_{\bar{\beta}}$ the Young subgroup $\times_{i\in V_{ord},q\in\Omega}(\times_{1\le j\le l_i}S_{\bar{\beta}_{ij}})\times(\times_{1\le j\le l_q}S_{\bar{\beta}_{qj}})$.

We call this subgroup as in [Zub1] a base one.

By the definition, the space Hom $(\bar{N})$ is invariant under the left (right) compositions with the elements from $GL(\lambda_{A_1}, \mu_{A_2}, \mu_{A_2})$ (respectively – from $GL(\lambda_{A_1}, \gamma_{A_3}, \gamma_{A_3})$).

Denote by $\Lambda^{\bar{\beta}}$ the space $\otimes_{i\in V_{ord}, q\in\Omega}(\otimes_{1\leq j\leq l_i}\Lambda^{p_{ij}}(E_{\mathcal{T}(i)})) \otimes (\otimes_{1\leq j\leq l_q}\Lambda^{p_{qj}}(E_{\mathcal{T}(q)}))$, where $p_{?j} =\mid \bar{\beta}_{?j} \mid$. The restriction of the pairing map $\delta_{\bar{\beta}}$ on the space $\Lambda^{\bar{\beta}} \otimes \Lambda^{\bar{\beta}}$ denote by the same symbol.

Repeating all arguments concerning ABW-filtrations from the second section one can define some filtration $\{M_{\bar{\beta}}\}$ of the space $(\otimes_{i\in V_{ord}} S^{p_i}(E_{\mathcal{T}_i} \otimes E_{\mathcal{T}_i})) \otimes (\otimes_{q\in\Omega} S^{p_q}(E_{\mathcal{T}_q} \otimes E_{\mathcal{T}_q}))$.

For any $\bar{\beta}$ we have an explicit sequence of $(\times_{i\in V_{ord}, q\in\Omega} GL(E_{\mathcal{T}(i)}) \times GL(E_{\mathcal{T}(q)})) \times (\times_{i\in V_{ord}, q\in\Omega} GL(E_{\mathcal{T}(i)}) \times GL(E_{\mathcal{T}(q)}))$-modules

$$0 \to \ker \delta_{\bar{\beta}} \to \Lambda^{\bar{\beta}} \otimes \Lambda^{\bar{\beta}} \to M_{\bar{\beta}}/\dot{M}_{\bar{\beta}} \to 0$$

All these modules with GF again.

Notice that the first refine decomposition is a "subdecomposition" of $(\bigsqcup_{i\in V_{ord}} \mathcal{T}(i)) \bigsqcup (\bigsqcup_{q\in\Omega} \mathcal{T}(q))$ and the second one is a "subdecomposition" of $(\bigsqcup_{i\in V_{ord}} \mathcal{H}(i)) \bigsqcup (\bigsqcup_{q\in\Omega} \mathcal{H}(q))$. Thus both groups $\sigma_0^{-1} GL(\lambda_{A_1}, \mu_{A_2}, \mu_{A_2})\sigma_0$ and $GL(\lambda_{A_1}, \gamma_{A_3}, \gamma_{A_3})$ are Levi subgroups of the group $\times_{i\in V_{ord}, q\in\Omega} GL(E_{\mathcal{T}(i)}) \times GL(E_{\mathcal{T}(q)})$.

Finally, we have the filtration $M_{\bar{\beta}}^{(1^r)\times(1^r)}$ of the space $K[S_t]$. Besides, $\sigma_0^{-1} \cdot I_{\bar{k}+1}$ is an union of members of this filtration whose "indexes" $\bar{\beta}$ satisfy the following condition: there is some $i \in V_{ord}$ or $q \in \Omega$ such that at least one subset $\bar{\beta}_{ij}$ or $\bar{\beta}_{qj}$ has the cardinality $p_{ij} \geq k_i + 1$ or $p_{qj} \geq s_q + 1$ respectively.

Now, it is not hard to prove that $\sigma_0^{-1}(N_{\lambda_{A_1}, \mu_{A_2}, \gamma_{A_3}} \bigcap I_{\bar{k}+1})$ has the filtration $(M_{\bar{\beta}} \otimes D)^G$, where $D = \det^{-1} \otimes \det^{-1}$ and $G = \sigma_0^{-1} GL(\lambda_{A_1}, \mu_{A_2}, \mu_{A_2})\sigma_0 \times GL(\lambda_{A_1}, \gamma_{A_3}, \gamma_{A_3})$.

Using the lemma 1.4 from [Zub4] we obtain the following short exact sequence

$$0 \to (\ker \delta_{\bar{\beta}} \otimes D)^G \to (\Lambda^{\bar{\beta}} \otimes \Lambda^{\bar{\beta}} \otimes D)^G \to (M_{\bar{\beta}}/\dot{M}_{\bar{\beta}} \otimes D)^G \to 0$$

Therefore, all we need is to find $(1^r) \times (1^r)$-weight subspace of the space $(\Lambda^{\bar{\beta}} \otimes \Lambda^{\bar{\beta}} \otimes D)^G$ which equals

$$(\Lambda^{\bar{\beta}} \otimes \overset{-1}{\det})^{\sigma_0^{-1} GL(\lambda_{A_1}, \mu_{A_2}, \mu_{A_2})\sigma_0} \otimes (\Lambda^{\bar{\beta}} \otimes \overset{-1}{\det})^{GL(\lambda_{A_1}, \gamma_{A_3}, \gamma_{A_3})}$$

and then we must compute the image of one under the pairing $\delta_{\bar{\beta}}$.

It can easily be checked that this subspace consists of all vectors x from $(\Lambda^{\bar{\beta}} \otimes \Lambda^{\bar{\beta}})^{(1^r)\times(1^r)}$ such that $x^{(\sigma_0^{-1}\tau_1\sigma_0, \tau_2)} = (-1)^{\tau_1}(-1)^{\tau_2}x$, for all $\tau_1 \in S_{\lambda_{A_1}, \mu_{A_2}, \mu_{A_2}}, \forall \tau_2 \in S_{\lambda_{A_1}, \gamma_{A_3}, \gamma_{A_3}}$ [Zub1, Zub4]. Denote this subspace by $V_{\bar{\beta}}$.

One can represent the groups $S_{\lambda_{A_1}, \gamma_{A_3}, \gamma_{A_3}}$ and $S_{\lambda_{A_1}, \mu_{A_2}, \mu_{A_2}}$ as $S_{\lambda_{A_1}} \times S_{\gamma_{A_3}-s} \times S_{\gamma_{A_3}}$ and $S_{\lambda_{A_1}} \times S_{\mu_{A_2}} \times S_{\mu_{A_2}+s}$ respectively. Thus any element $\pi \in S_{\lambda_{A_1}, \gamma_{A_3}, \gamma_{A_3}}$ can be written as the product $\pi_1\pi_2\pi_3$, where $\pi_1 \in S_{\lambda_{A_1}}, \pi_2 \in S_{\gamma_{A_3}-s}, \pi_3 \in S_{\gamma_{A_3}}$. Analogously, any element $\pi \in S_{\lambda_{A_1}, \mu_{A_2}, \mu_{A_2}}$ can be represented as the product $\pi_1\pi_2\pi_3$, where $\pi_1 \in S_{\lambda_{A_1}}, \pi_2 \in S_{\mu_{A_2}}, \pi_3 \in S_{\mu_{A_2}+s}$.

Let $\pi \in S_{[t+1,t+s]}$ and $(a \dots b)\dots(c \dots d)$ its cyclic decomposition. Denote by $\pi+s$ the element $(a+s\dots b+s)\dots(c+s\dots d+s) \in S_{[t+s+1,r]}$. Analogously, any $\pi = (a\dots b)\dots(c\dots d) \in S_{[t+s+1,r]}$ has a "double" $\pi - s = (a-s\dots b-s)\dots(c-s\dots d-s)$.

The set S_1 consisting of all $\pi \in S_{\lambda_{A_1},\mu_{A_2},\mu_{A_2}}$ such that $\pi_2 + s = \pi_3$ is a subgroup of the group $S_{\lambda_{A_1},\mu_{A_2},\mu_{A_2}}$. In the same way, $S_2 = \{\pi \in S_{\lambda_{A_1},\gamma_{A_3},\gamma_{A_3}} \mid \pi_2 = \pi_3 - s\}$ is a subgroup of the group $S_{\lambda_{A_1},\mu_{A_2},\mu_{A_2}}$.

Denote the groups $S_{\lambda_{A_1},\mu_{A_2},\gamma_{A_3}}$ and $S_{[1,t]} \times S_{[t+1,t+s]} \times S_{[t+s+1,r]}$ by S and S_0 respectively and define two homomorphisms ρ_1,ρ_2 from S_0 into the group S_r. The first homomorphism is given by $\pi \to \pi_1\pi_2(\pi+s)$. The second one takes any π to $\pi_1(\pi_3 - s)\pi_2$. It is clear that $\rho_1(S) = S_1, \rho_2(S) = S_2$.

Following the same idea as in [Zub1, Zub4] we compute some "suitable" generators, i.e. the generators of the space

$$W_{\bar\beta} = \{x \in (\Lambda^{\bar\beta} \otimes \Lambda^{\bar\beta})^{(1^r)\times(1^r)} \mid \forall \tau \in S, x^{(\sigma_0^{-1}\rho_1(\tau)\sigma_0,\rho_2(\tau))} = x\}$$

It is clear that this space contains the space $V_{\bar\beta}$ mentioned above.

The canonical projection $\otimes_{i\in V_{ord},q\in\Omega}(E_{\mathcal{T}(i)})^{\otimes p_i}) \otimes (E^{\otimes p_q}_{\mathcal{T}(q)}) \to \Lambda^{\bar\beta}$ denote by p. The vectors $\bar e_\sigma = p(e_\sigma)$ form a base of the space $(\Lambda^{\bar\beta})^{(1^r)}$, where

$$e_\sigma = \otimes_{i\in V_{ord},q\in\Omega}(\otimes_{j\in\mathcal{T}(i)} e_{\sigma(j)}) \otimes (\otimes_{j\in\mathcal{T}(q)} e_{\sigma(j)}), \sigma \in S_t/S_{\bar\beta},$$

see [Zub4]. Thus the space $(\Lambda^{\bar\beta})^{(1^r)} \otimes (\Lambda^{\bar\beta})^{(1^r)}$ has a base $\{\bar e_{\sigma_1} \otimes \bar e_{\sigma_2} \mid \sigma_1,\sigma_2 \in S_t/S_{\bar\beta}\}$. This base is decomposed into orbits under the "diagonal" action of the group S by the rule $(\bar e_{\sigma_1} \otimes \bar e_{\sigma_2})^\tau = \bar e_{\overline{\sigma_0^{-1}\rho_1(\tau)\sigma_0\sigma_1}} \otimes \bar e_{\overline{\rho_2(\tau)\sigma_2}}, \tau \in S, \sigma_1,\sigma_2 \in S_t/S_{\bar\beta}$. Therefore, one can represent the previous base as

$$\bigsqcup_{(\sigma_1,\sigma_2)\in Y} \{\bar e_{\sigma_0^{-1}\rho_1(\tau)\sigma_0\sigma_1} \otimes \bar e_{\rho_2(\tau)\sigma_2} \mid \tau \in S/(\rho_1^{-1}(S_{\bar\beta}^{\sigma_0\sigma_1}) \bigcap \rho_2^{-1}(S_{\bar\beta}^{\sigma_2}) \bigcap S)\},$$

where Y is some fixed representative set of all S-orbits and $S^\pi_{\bar\beta} = \pi S_{\bar\beta}\pi^{-1}, \pi \in S_r$.

It is clear that $\forall \tau \in S, \sigma_0^{-1}\rho_1(\tau)\sigma_0\sigma_1 S_{\bar\beta} = \sigma_0^{-1}\rho_1(\bar\tau)\sigma_0\sigma_1 S_{\bar\beta}, \rho_2(\tau)\sigma_2 S_{\bar\beta} = \rho_2(\bar\tau)\sigma_2 S_{\bar\beta}$. In particular, for any $\tau \in S$ we have

$$\bar e_{\sigma_0^{-1}\rho_1(\tau)\sigma_0\sigma_1} \otimes \bar e_{\rho_2(\tau)\sigma_2} =$$
$$(-1)^{\sigma_0^{-1}\rho_1(\tau)\sigma_0\sigma_1(\sigma_0^{-1}\rho_1(\bar\tau)\sigma_0\sigma_1)^{-1}}(-1)^{\rho_2(\tau)\sigma_2(\rho_2(\bar\tau)\sigma_2)^{-1}} \bar e_{\sigma_0^{-1}\rho_1(\bar\tau)\sigma_0\sigma_1} \otimes \bar e_{\rho_2(\bar\tau)\sigma_2} =$$
$$(-1)^{\rho_1(\tau)\rho_1(\bar\tau)^{-1}}(-1)^{\rho_2(\tau)\rho_2(\bar\tau)^{-1}} \bar e_{\sigma_0^{-1}\rho_1(\bar\tau)\sigma_0\sigma_1} \otimes \bar e_{\rho_2(\bar\tau)\sigma_2} = \bar e_{\sigma_0^{-1}\rho_1(\bar\tau)\sigma_0\sigma_1} \otimes \bar e_{\rho_2(\bar\tau)\sigma_2}$$

Therefore, the vectors

$$\sum_{\tau \in S/(\rho_1^{-1}(S_{\bar\beta}^{\sigma_0\sigma_1}) \bigcap \rho_2^{-1}(S_{\bar\beta}^{\sigma_2}) \bigcap S)} \bar e_{\sigma_0^{-1}\rho_1(\tau)\sigma_0\sigma_1} \otimes \bar e_{\rho_2(\tau)\sigma_2}$$

form a base of the space $W_{\bar{\beta}}$. By the same arguments one can obtain a base of the space $V_{\bar{\beta}}$. We omit these computations and refer the reader to [Zub1, Zub4].

The most important thing is the both bases are defined over Z. In other words, even we replace the ground field K by the ring Z, i.e. if we consider the space $\Lambda^{\bar{\beta}} \otimes \Lambda^{\bar{\beta}}$ as a free Z-module we obtain the same free generators of the free Z-modules $V_{\bar{\beta}}$ and $W_{\bar{\beta}}$ too. In particular, the free generators of $V_{\bar{\beta}}$ can be expressed by "suitable" ones with integral coefficients [Zub1, Zub4].

Finally, mapping the generators of the space $W_{\bar{\beta}}$ into $K[S_t]$ we obtain the generators of the space $\sigma_0^{-1} \cdot (N_{\lambda_{A_1}, \mu_{A_2}, \gamma_{A_3}} \bigcap I_{\bar{k}+1})$ as

$$g_{\sigma_1,\sigma_2} = \sum_{\tau \in S_{\bar{\beta}}} \sum_{\pi \in S/(\rho_1^{-1}(S_{\bar{\beta}}^{\sigma_0\sigma_1}) \bigcap \rho_2^{-1}(S_{\bar{\beta}}^{\sigma_2}) \bigcap S)} (-1)^{\tau} \sigma_0^{-1} \rho_1(\pi) \sigma_0 \sigma_1 \tau \sigma_2^{-1} \rho_2(\pi)^{-1}$$

and multiplying by σ_0, the generators

$$h_{\sigma_1,\sigma_2} = \sum_{\tau \in S_{\bar{\beta}}} \sum_{\pi \in S/(\rho_1^{-1}(S_{\bar{\beta}}^{\sigma_0\sigma_1}) \bigcap \rho_2^{-1}(S_{\bar{\beta}}^{\sigma_2}) \bigcap S)} (-1)^{\tau} \rho_1(\pi) \sigma_0 \sigma_1 \tau \sigma_2^{-1} \rho_2(\pi)^{-1}$$

of the space $N_{\lambda_{A_1}, \mu_{A_2}, \gamma_{A_3}} \bigcap I_{\bar{k}+1}$.

Summarizing we get the following

Proposition 3.1 *The $\bar{r}$-component of $T(Q, \bar{t})$ is generated as a vector space by all elements $\frac{1}{|S|} tr^*(h_{\sigma_1,\sigma_2}, f)$, where $S_{\bar{\beta}}$ runs over all Young subgroups of S_t satisfying the condition on its layers formulated above. If we ignore this condition we get the generators of $J(Q)(\bar{r})$ and mapping ones into $J(Q, \bar{t})(\bar{r})$ – the generators of this last homogeneous component.*

Notice that from the rigorous point of view it is not obvious that all elements $\frac{1}{|S|} tr^*(h_{\sigma_1,\sigma_2}, f)$ relative to a suitable base group $S_{\bar{\beta}}$ lie in $T(Q, \bar{t})$. But as we will prove below all these elements can be computed over Z. In particular, it remains to prove that their complete linearizations lie in this ideal. The complete linearization of $\frac{1}{|S|} tr^*(h_{\sigma_1,\sigma_2}, f)$ equals $tr^*(h_{\sigma_1,\sigma_2})$ (see the lemma 3.5 below) and it can easily be checked that $h_{\sigma_1,\sigma_2} \in I_{\bar{k}+1}$.

Now, let us describe an alternative way to set the correspondence $\sigma \longleftrightarrow tr^*(\sigma)$. Let us consider the symmetric group $S_{[1,r] \bigsqcup [\bar{1}, \bar{r}]}$ acting on the "doubled" set $[1, r] \bigsqcup [\bar{1}, \bar{r}]$. Denote by R the following substitution:

$$\forall i \in [1, r], R(i) = \begin{cases} i, \ i \in \hat{A}_1, \\ i + s, \ i \in \hat{A}_2, \\ \bar{i}, \ i \in \hat{A}_3, \end{cases}$$

$$\forall i \in [1, r], R(\bar{i}) = \begin{cases} \bar{i}, \ i \in \hat{A}_1, \\ i, \ i \in \hat{A}_2, \\ \overline{i - s}, \ i \in \hat{A}_3 \end{cases}$$

It is clear that $R = \prod_{i\in \hat{A}_2}(i\ i+s\ \overline{i+s}\ \bar{i})$.

For any $\pi \in S_{[1,r]\bigsqcup[\bar{1},\bar{r}]}$ having a cyclic decomposition $(a\dots b)\dots(c\dots d)$ denote $(\bar{a}\dots\bar{b})\dots(\bar{c}\dots\bar{d})$ by $\bar{\pi}$. We have a bijection $\iota : \pi \to \bar{\pi}^{-1}$ on $S_{[1,r]\bigsqcup[\bar{1},\bar{r}]}$. It is clear that this bijection induces an involution on the group $S_{[1,r]\bigsqcup[\bar{1},\bar{r}]}$.

Lemma 3.1 *Let $\sigma \in S_r$ and $tr^*(\sigma) = u = (a\dots b)\dots(c\dots d)$, where $\{a,\dots,b,c,\dots,d\}$ is a subset of $[1,r]\bigsqcup[\bar{1},\bar{r}]$ having cardinality r. Then $R\sigma^{-1}\bar{\sigma}R = u\bar{u}^{-1} = u\iota(u)$.*

Proof. It can easily be checked that for any $j \in [1,r]\bigsqcup[\bar{1},\bar{r}]$ its right side neighbor in both $R\sigma^{-1}\bar{\sigma}R$ and u cyclic decomposition are the same. For example, let $j = \bar{l}, l \in \hat{A}_3$. Then we have the following chain of sequential equations:

$$R(\bar{l}) = \overline{l-s}, \bar{\sigma}(\overline{l-s}) = \overline{\sigma(l-s)}$$

and finally

$$R(\overline{\sigma(l-s)}) = \begin{cases} \overline{\sigma(l-s)},\ \sigma(l-s) \in \hat{A}_1, \\ \sigma(l-s),\ \sigma(l-s) \in \hat{A}_2, \\ \overline{\sigma(l-s)-s},\ \sigma(l-s) \in \hat{A}_3, \end{cases}$$

i.e. the result is the same as in the rules defining u. Other cases can be checked similarly. Thus follows that any cycle of $R\sigma^{-1}\bar{\sigma}R$ is a cycle of u or its transposed $\bar{u}^{-1}$. This concludes the proof.

Let us denote by a the substitution $\prod_{i\in[1,r]}(i\bar{i})$. It is clear that $a\pi a^{-1} = \bar{\pi}$. In particular, the involution ι can be defined as $\iota(\pi) = a\pi^{-1}a^{-1}$.

Lemma 3.2 *Let $\sigma \in S_r$ and $R\sigma^{-1}\bar{\sigma}R = u\iota(u)$. Suppose that the cyclic record of u, including trivial cycles too, contains two symbols i, j belonging to the same set $\hat{A}_l$ or $\bar{\hat{A}}_l, l = 1,2,3$. Then $(ij)u\iota((ij)u) = R\sigma'^{-1}\bar{\sigma}'R$, where either $\sigma' = (i',j')\sigma$ or $\sigma' = \sigma(i',j')$ and i', j' belong to the same $\hat{A}_f$ or $\bar{\hat{A}}_f$, $f = 1,2,3$. More precisely,*

$$i', j' = \begin{cases} i, j,\ i, j \in \hat{A}_1, \\ \bar{i}, \bar{j},\ i, j \in \bar{\hat{A}}_1, \end{cases}$$

$$i', j' = \begin{cases} i, j,\ i, j \in \hat{A}_2, \\ \overline{i+s}, \overline{j+s},\ i, j \in \bar{\hat{A}}_2, \end{cases}$$

$$i', j' = \begin{cases} i-s, j-s,\ i, j \in \hat{A}_3, \\ \bar{i}, \bar{j},\ i, j \in \bar{\hat{A}}_3 \end{cases}$$

In particular, σ and σ' have the different parities.

Proof. Notice that a decomposition $u\iota(u)$ is not uniquely defined. One can interchange any cycle $(a\dots b)$ from the cyclic record of u with its transposed $(\bar{b}\dots\bar{a})$ from the record of $\iota(u)$. Therefore, a left factor u can be defined as a part of cyclic decomposition of $R\sigma^{-1}\bar{\sigma}R$ depending of r symbols from $[1,r]\bigsqcup[\bar{1},\bar{r}]$ and satisfying all conditions of admissibility.

Let $i,j\in\bar{\hat{A}}_3$, say $i=\bar{m}, j=\bar{n},\ m,n\in\hat{A}_3$. We have

$$(ij)u\ \iota((ij)u)=(\bar{m}\bar{n})R\sigma^{-1}\bar{\sigma}R(mn)=RR^{-1}(\bar{m}\bar{n})R\sigma^{-1}\times\bar{\sigma}R(mn)R^{-1}R$$

Further, $R^{-1}(\bar{m}\bar{n})R=(\bar{m}^{R^{-1}}\bar{n}^{R^{-1}})=(mn)$. Thus $(ij)u\ \iota((ij)u)=R\sigma'^{-1}\bar{\sigma}'R$, where $\sigma'=\sigma(mn)$. By the same way all another cases can be checked.

It remains to prove that $(ij)u$ is right defined. Using the identity $(ij)(iC)(jD)=(iCjD)$, where C,D are some fragments of these cycles, we see that the sets of symbols involved in the records of u and $(ij)u$ correspondently are the same. So it is enough to prove that $(ij)u$ does not contain ι-invariant cycles.

Let us suppose that $u=(iCjD)\ldots$. Then $(ij)u=(iC)(jD)\ldots$. If $(iC)=\iota((iC))$ then in the cycle (iC) there are two sequential symbols like $\bar{z},z$. Then it is true for $(iCjD)$ excepting the case $C=C_1\bar{i}$. In the last case we have $(iCjD)=(iC_1\bar{i}jD)$. But both cases are forbidden because of i,j or $\bar{i},\bar{j}$ belongs to the same set $\hat{A}_l, l=1,2,3$, see the short conditions of admissibility above. The case $u=(iC)(jD)$ is symmetrical to the previous one. The lemma is proved.

Lemma 3.3 *([Zub5]) Let $\pi=\pi_1\pi_2\pi_3\in S_0,\sigma\in S_r$ and $tr^*(\sigma)=u$. Then $u^{\pi\times\bar{\pi}}=tr^*(\rho_1(\pi)\sigma\rho_2(\pi)^{-1})$.*

Proof. It is enough to prove this equation for $\pi=(ij)$, where i,j lie in $\hat{A}_1,\hat{A}_2$ or $\hat{A}_3$ simultaneously.

Let $i,j\in\hat{A}_2$. Then $\rho_1(\pi)=(ij)(i+s,j+s)$ and $\rho_2(\pi)=\mathrm{id}$. We have

$$R(\bar{i}\bar{j})(\overline{i+s}\,\overline{j+s})\sigma^{-1}\bar{\sigma}(ij)(i{+}s,j{+}s)R=((\bar{i}\bar{j})(\overline{i+s}\,\overline{j+s}))^{\tau}R\sigma^{-1}\bar{\sigma}R((ij)(i{+}s,j{+}s))^{\tau^{-1}},$$

where $\tau=(i,i+s,\overline{i+s},\bar{i})(j,j+s,\overline{j+s},\bar{j})$. It remains to notice that

$$((\bar{i}\bar{j})(\overline{i+s}\ \overline{j+s}))^{\tau}=(ij)(\bar{i}\bar{j}),((ij)(i+s,j+s))^{\tau}=(\bar{i}\bar{j})(ij)$$

The rest cases can be checked by the same way. The lemma is proved.

Let us define some "intermediate" collection of matrices $U(l), 1\leq l\leq m$, where m is equal to the number of all layers of the group $G=\rho_1^{-1}(S_{\bar{\beta}}^{\sigma_0\sigma_1})\bigcap\rho_2^{-1}(S_{\bar{\beta}}^{\sigma_2})\bigcap S$. One can define the new specialization g which takes any matrix $Y(j)$ to $U(l)$ iff j belongs to the l-th layer of the group G. It is clear that there is some specialization h such that $f=h\circ g$.

Lemma 3.4 *Any element $\frac{1}{|S|}tr^*(h_{\sigma_1,\sigma_2},f)$ is obtained with the help h from the element $\frac{1}{|G|}tr^*(\sum_{\tau\in S_{\bar{\beta}}}(-1)^{\tau}\sigma_0\sigma_1\tau\sigma_2^{-1},g)$.*

Proof. Using the previous lemma we see that

$$tr^*(\rho_1(\pi)\sigma_0\sigma_1\tau\sigma_2^{-1}\rho_2(\pi)^{-1}, f) = tr^*(\sigma_0\sigma_1\tau\sigma_2^{-1}, f)$$

because of $f \circ \pi = f$. The final computations are trivial.

In other words, we can assume that $S = G \leq \rho_1^{-1}(S_{\bar\beta}^{\sigma_0\sigma_1}) \bigcap \rho_2^{-1}(S_{\bar\beta}^{\sigma_2})$ up to some "gluing" of matrix variables (see [Zub1, Zub4]). Besides, without loss of generality one can suppose that $\sigma_1 = 1$.

Replacing the group $S_{\bar\beta}$ by the group $S_{\bar\beta}^{\sigma_2} \leq S_t$ and the element σ_0 by the element $\sigma_0\sigma_2^{-1}$ one can assume that $\sigma_2 = 1$ too.

Lemma 3.5 *The invariant* $\frac{1}{|G|}tr^*(\sum_{\tau\in S_{\bar\beta}}(-1)^\tau\sigma\tau, g)$ *is some partial linearization (briefly – PL) of the invariant* $\frac{1}{|\rho_1^{-1}(S_{\bar\beta})\bigcap\rho_2^{-1}(S_{\bar\beta})|}tr^*(\sum_{\tau\in S_{\bar\beta}}(-1)^\tau\sigma\tau, f')$*, where the specialization* f' *corresponds to the group* $\rho_1^{-1}(S_{\bar\beta})\bigcap\rho_2^{-1}(S_{\bar\beta})$.

Proof. By the definition, $S_{f'} = \rho_1^{-1}(S_{\bar\beta})\bigcap\rho_2^{-1}(S_{\bar\beta}) \leq S_0$.

For example, let us consider two layers α, β of the group G which are contained in some layer of the group $S_{f'}\bigcap S_{[t+1,t+s]}$. For the sake of simplicity let us assume that these layers have numbers $m-1, m$ correspondently.

Let us define the new specialization g' such $g'(j) = m-1$ iff $j \in \alpha\bigcup\beta$ otherwise $g'(j) = g(j)$.

Let $x \in S_r$ and $tr^*(x, g') = (g'(a)\ldots g'(b))\ldots(g'(c)\ldots g'(d))$, where $\{a,\ldots,b, \underline{c,\ldots}, d\}$ is a subset of $[1,r]\bigcup[\bar 1, \bar r]$ having cardinality r. By definition, $g'(\bar j) = \overline{g'(j)}, j \in [1,r]$.

Extracting the homogeneous summands of degrees $\mid \alpha \mid$ and $\mid \beta \mid$ in $U(m-1)$ and $U(m)$ respectively from $tr^*(x, g') \mid_{U(m-1)\to U(m-1)+U(m)}$ we get the sum

$$\sum_{\pi\in S_{\alpha\cup\beta}/S_\alpha\times S_\beta}(g(\pi(a))\ldots g(\pi(b)))\ldots(g(\pi(c))\ldots g(\pi(d)))$$

Using the previous lemma we see that

$$(g(\pi(a))\ldots g(\pi(b)))\ldots(g(\pi(c))\ldots g(\pi(d))) = tr^*(\rho_1(\pi)x, g)$$

Thus our PL of the element $\frac{1}{|S_{g'}|}tr^*(\sum_{\tau\in S_{\bar\beta}}(-1)^\tau\sigma\tau, g')$ equals

$$\frac{1}{\mid S_{g'}\mid}\sum_{\tau\in S_{\bar\beta}}\sum_{\pi\in S_{\alpha\cup\beta}/S_\alpha\times S_\beta}(-1)^\tau tr^*(\rho_1(\pi)\sigma\tau, g)$$

Further, $\rho_1(\pi) \in S_{\bar\beta}^\sigma$, i.e. $\rho_1(\pi) = \sigma y\sigma^{-1}, y \in S_{\bar\beta}$. In particular, we get

$$\sum_{\tau\in S_{\bar\beta}}(-1)^\tau tr^*(\rho_1(\pi)\sigma\tau, g) = \sum_{\tau\in S_{\bar\beta}}(-1)^\tau tr^*(\sigma y\tau, g) = \sum_{\tau\in S_{\bar\beta}}(-1)^\tau tr^*(\sigma\tau, g)$$

since $\rho_1(\pi)$ is an even element. Therefore, our PL equals the original invariant $\frac{1}{|G|}tr^*(\sum_{\tau\in S_{\bar\beta}}(-1)^\tau\sigma\tau, g)$. Repeating these arguments as many times as we need we pass from the group G to the group $S_{f'}$. This concludes the proof.

Summarizing we see that up to some rearrangings, gluings of matrix variables and PL-s the generators of $J(Q)$ ($J(Q,\bar t)$) as well as the generators of $T(Q,\bar t)$ are

$$c(\phi, f) = \frac{1}{\mid S_f \mid} tr^*(\sum_{\tau\in S_{\bar\beta}}(-1)^\tau\sigma\tau, f),$$

where $S_f = \rho_1^{-1}(S_{\bar\beta})\bigcap\rho_2^{-1}(S_{\bar\beta})$ and $\phi = \sum_{\tau\in S_{\bar\beta}}(-1)^\tau\sigma\tau$. Notice that if $s = 0$ then these elements are the same as the suitable generators from [Zub1, Zub4].

Let us remind some definitions from [Don2]. Let $S_g \leq S_r$ is an Young subgroup corresponding to a map $g : [1, r] \to [1, m]$. Any sequence $p = j_1 \dots j_s$ of symbols from $[1, m]$ is said to be a primitive cycle iff there is not any proper subsequence q of p such that p graphically coincides with $q^k = \underbrace{q \dots q}_{k}, kd = s, s > k > 1$. For any $\tau = (a \dots b) \dots (c \dots d) \in S_r$ we have

$$\tau^g = (g(a) \dots g(b)) \dots (g(c) \dots g(d)) = \prod_{1\leq j\leq s_1}(p_1^{k_{1j}}) \dots \prod_{1\leq j\leq s_l}(p_l^{k_{lj}}),$$

where each p_i is a primitive cycle which is uniquely defined up to some cyclic permutation of its symbols, $i = 1, \dots, l$.

Definition 6 *Two substitutions $\mu, \pi \in S_r$ are called S_g-equivalent iff there is a sequence $\mu = \tau_1, \dots, \tau_k = \pi$ such that for any pair $\tau_i, \tau_{i+1}, 1 \leq i \leq k-1$, either there is $x \in S_g$ such that $\tau_{i+1} = \tau_i^x$ or for two cycles of τ_i (τ_{i+1}), say $(a \dots b), (c \dots d)$, we have $(g(a) \dots g(b)) = (p^f), (g(c) \dots g(d)) = (p^d)$, where p is a primitive cycle and $\tau_{i+1} = (ac)\tau_i$ (respectively – $\tau_i = (ac)\tau_{i+1}$).*

It can easily be checked that this relation between elements of S_r is really an equivalence. Donkin calls any equivalence class an Young superclass.

It is clear that all substitutions from the same Young superclass D have the same sets of primitive cycles. We denote anyone from these sets by P_D.

Besides, for any Young superclass D the element $\frac{1}{|S_g|}\sum_{x\in D}(-1)^x tr(x, g)$ can be written as a sum with integer coefficients of products of the elements $\sigma_j(p)$, where $p \in P_D$ [Don2]. The same is true for any quiver case (see [Don1, Zub4]). We call invariants of ordinary representations of quivers as ordinary ones including "invariants" from corresponding free invariant algebras. In our notations these invariants correspond to the case $s = 0$.

Theorem 3.1 *The algebra $J(Q, \bar t)$ is generated by the elements $\sigma_j(Z(a_d) \dots Z(a_1))$, where $1 \leq j \leq \max\limits_{1\leq i\leq n}\{k_i\}$, $a_r, \dots, a_1$ is a closed path in the double quiver $Q^{(d)}$ defined above.*

Proof. Without loss of generality one can work in $J(Q)$ or in $J(\hat{Q})$ if it is necessary.

Let us consider any suitable generator

$$z = c(\phi, f) = \frac{1}{\mid S_f \mid} tr^*(\sum_{\tau \in S_{\bar{\beta}}} (-1)^\tau \sigma\tau, f)$$

Fix a summand $tr^*(\sigma\tau) = u$. One can interpret the element u as an ordinary invariant of $\hat{Q}^{(d)}$ depending of r matrix variables $Z(j_1), \ldots, Z(j_r), j_1, \ldots, j_r \in [1, r] \bigcup [\bar{1}, \bar{r}]$ or as a substitution from $S_{\{j_1,\ldots,j_r\}}$.

It is easy that $\{j_1, \ldots, j_r\} = T_1 \bigcup \overline{T}_2$, where T_1, T_2 are two subsets of $[1, r]$ such that $T_1 \bigsqcup T_2 = [1, r]$. Denote by S'_f the group S^π_f, where $\pi = \prod_{i\in T_2}(i\bar{i})$. It is clear that $S'_f = S_{f'}$, where $f' = (f \times f \circ a) \mid_{j_1,\ldots,j_r}$.

I claim that S'_f Young superclass containing u is a subset of $tr^*(\sigma S_{\bar{\beta}})$. Indeed, for any $v' \in S'_f$ we have $v' = v^\pi, v \in S_f$ and $u^{v'} = u^{v\times\bar{v}} = tr^*(\rho_1(v)\sigma\tau\rho_2(v)^{-1})$ by the lemma 3.3. It remains to notice that $\sigma^{-1}\rho_1(S_f)\sigma \leq S_{\bar{\beta}}$ and $\rho_2(S_f) \leq S_{\bar{\beta}}$.

Next, any element (ab), where $a, b \in g^{-1}(j)$ and j is a symbol of some primitive cycle belonging to u, has a form (ij) or $(\bar{i}\bar{j})$, $(ij) \in S_f$. Using the lemma 3.2 we see that $(ab)u$ equals $tr^*((i'j')\sigma\tau)$ or $tr^*(\sigma\tau(i'j'))$ and $(i'j')^{\sigma^{-1}} \in S_{\bar{\beta}}$ or $(i'j') \in S_{\bar{\beta}}$ respectively.

For example, if $(ab) = (\bar{i}\bar{j}), i, j \in \hat{A}_2$ then $(ab)u = tr^*((i + s, j + s)\sigma\tau)$. But any layer of $S_f \bigcap S_{[t+1,t+s]}$ has a form $\sigma(\beta_{??}) \bigcap \hat{A}_2 \bigcap (\sigma(\beta_{**}) \bigcap \hat{A}_3 - s)$. Thus $(i + s, j + s)^{\sigma^{-1}} \in S_{\bar{\beta}} \bigcap S_{[t+s+1,r]}$.

Finally, our generator z can be represented as a sum of elements

$$\pm\frac{1}{\mid S_f \mid}\sum_{x\in D}(-1)^x x^{f'} = \pm\frac{1}{\mid S_{f'} \mid}\sum_{x\in D}(-1)^x tr(x, f'),$$

where any summand $tr(x, f')$ is identified with $x^{f'}$ due our conventions and D runs over all superclasses contained in $tr^*(\sigma S_{\bar{\beta}})$. In particular, using Donkin's theorem mentioned above we get our theorem.

In fact, all we need to prove is the coincidence of signs. But for any element $u' = tr^*(\sigma\tau')$ from the Young superclass of given $u = tr^*(\sigma\tau)$ we have $(-1)^{\tau'} = (-1)^\tau \frac{(-1)^u}{(-1)^{u'}}$ by the lemma 3.2. This concludes the proof.

4 Relative problems

In this section we will describe some further generalization of "mixed" representations of quivers. In particular, this new class of representations contains so-called orthogonal (symplectic) representations of symmetric quivers introduced in [DW3].

We show that in fact it does not give nothing new from invariant theory point of view. Namely, invariant rings of these new representations of quivers are epimorphic images of invariant rings of "mixed" representations of some another quivers those correspond to original ones. A partial case of this correspondence was described in the first section. The general case will be explained below. Moreover, we prove that defining relations between invariants of these new type representations can be described with the help of defining relations between invariants of "mixed" ones.

Let $R(Q,\bar{t})$ be an "mixed" representation space of a quiver Q of dimension $\bar{t} = (t_1,\ldots,t_n)$ with respect to some division of V into disjoint subsets, say $V = \bigsqcup_{i=1}^{i=l} V_i$. By definition $\bar{t}$ is compatible with this partition.

Let us replace some factors of the group $H = H(\bar{k})$ by orthogonal or symplectic subgroups requiring additionally that the characteristic of the ground field does not equal 2 if at least one factor is replaced by an orthogonal group. Denote a subgroup of H obtained by this way as $G = \times_{1\le d\le l} G_d$.

Next, let us extract among all components $\mathrm{Hom}_K(W_{h(a)}, W_{t(a)}), a \in A$, those having property $h(a), t(a) \in V_d, d = 1,\ldots,l$. Let $h(a) = i, t(a) = j$. We have three cases – $G_d = GL(k_d)$, $G_d = O(k_d)$ or $G_d = Sp(k_d)$, $k_d = k_i = k_j$.

Let us consider the first case $G_d = GL(k_d)$. Let $t_i = k_i^*, t_j = k_j$ or $t_i = k_i, t_j = k_j^*$.

Identifying $\mathrm{Hom}_K(W_i, W_j)$ with $M(k_d)$ one can replace this space by its subspaces of symmetric or skew-symmetric matrices. In notations of [DW3] these subspaces can be identified with $S^2(V)(S^2(V^*))$ or $\Lambda^2(V)(\Lambda^2(V^*))$ respectively in obvious way as a $GL(k_d)$-modules, where $V = E_i = E_j$. If $(t_i,t_j) = (k_i,k_j)$ then $\mathrm{Hom}_K(W_i, W_j)$ remains the same.

In two rest cases it does not matter does (t_i,t_j) coincide with (k_i,k_j) or not. Indeed, $V \cong V^*$ as a $O(V)$ or $Sp(V)$-module. If $G_d = O(k_d)$ then one can replace the space $\mathrm{Hom}_K(W_i, W_j) = M(k_d)$ by its subspaces of symmetric or skew-symmetric matrices again.

In the case $G_d = Sp(k_d)$ one can replace the space $\mathrm{Hom}_K(W_i, W_j) = M(k_d)$ by its subspaces $\mathrm{Lie}\,(Sp(k_d)) = \{A \in M(k_d) \mid AJ \text{ is a symmetric matrix}\}$ or $\{A \in M(k_d) \mid AJ \text{ is a skew-symmetric matrix}\}$, where $J = J_{k_d}$.

Denote a subspace of $R(Q,\bar{t})$ obtained with the help of some replacements described above by S.

Definition 7 *A subspace S is said to be a "supermixed" representation space of the quiver Q with respect to the induced action of the group G.*

The very close definition was introduced in [DW3]. Omitting details, they associate with any generalized quiver of $O(n)$ ($Sp(n)$) orthogonal (symplectic) representations of so-called symmetric quiver. For example, typical components of orthogonal representations of a symmetric quiver are

$$\mathrm{Hom}_K(V_1,V_2), \mathrm{Hom}_K(V_1,V_2^*), \mathrm{Hom}_K(V_1^*,V_2), \Lambda^2(V) \subseteq \mathrm{Hom}_K(V^*,V),$$

$$\Lambda^2(V^*) \subseteq \mathrm{Hom}_K(V,V^*),$$

$$\mathrm{Hom}_K(V,W), \mathrm{Hom}_K(V^*,W), \mathrm{Hom}_K(W_1,W_2), \Lambda^2(W) \subseteq \mathrm{Hom}_K(W,W)$$

The spaces V, V_i, W, W_j are regarded as standard $GL(V), GL(V_i), O(W), O(W_j)$ modules respectively, $i = 1,2, j = 1,2$. These spaces are isotypical components of the space K^n with respect to the action of an abelian reductive subgroup D of $O(n)$. The centralizer $R = Z_{O(n)}(D)$ is a product of all these groups $GL(?), O(??)$.

In the symplectic case one has to replace the components $\Lambda^2(V), \Lambda^2(V^*), \Lambda^2(W)$ by $S^2(V), S^2(V^*), S^2(W)$ up to some identifications like $A \longleftrightarrow AJ$ mentioned above. Besides, in the last case the groups $O(W), O(W_1), O(W_2)$ must be replaced by $Sp(W), Sp(W_1), Sp(W_2)$ correspondently.

It is clear that our definition admits more general situation than Derksen-Weyman's one. For example, their definition does not include any action of some orthogonal (symplectic) group on symmetric (skew-symmetric – with respect to an identification mentioned above) matrix component.

From now on we fix some "supermixed" representation space S of Q and its automorphism group G.

Theorem 4.1 *There is a quiver Q' such that the invariant algebra $K[S]^G$ is an epimorphic image of an invariant algebra of a "mixed" representation space of Q' of suitable dimension $\bar{t}'$. Moreover, the definition of this epimorphism affords to describe the generators of its kernel exactly.*

Proof. We describe the construction of Q' step by step with respect to the all replacements which were used to get S and G.

For example, let us consider the case when $G_d = Sp(k_d)$ acts on some component $S_a \subseteq \mathrm{Hom}_K(V_i, V_j), a \in A, h(a) = i, t(a) = j, V_i = V_j = K^{k_d}, k_d = k_i = k_j$ and S_a can be identified with the subspace of symmetric matrices by the rule $A \to AJ, A \in S_a$. With respect to this identification the group $G_d = Sp(k_d)$ acts on S_a by the rule $A^g = gAg^t, g \in G_d$.

Repeating word by word the proof of the lemma 1.3 [Zub5] we have an epimorphism $K[S' \times M(k_d)]^G \to K[S]^G$, where S' is a product of all components of S excepting S_a and G_d acts on $M(k_d)$ by the same rule $A^g = gAg^t$.

To be precise, S is a closed G-subvariety of $S' \times M(k_d)$. Moreover, it is a complete intersection defined by the relations $x_{ij} - x_{ji} = 0, 1 \le i < j \le k_d$, where $X = (x_{ij})$ is the general matrix corresponding to the factor $M(k_d)$. The ideal I of S is generated by G-invariant subspace $E = \oplus_{1 \le i < j \le k_d} K \cdot z_{ij}$, where $z_{ij} = x_{ij} - x_{ji}, 1 \le i, j \le k_d$.

The algebra $S(E)$ is a $GL(k_d)$-module with GF with respect to the induced action $Z \to g^{-1}Z(g^t)^{-1}, Z = (x_{ij} - x_{ji})$. It follows immediately from [Kur1, Kur2]. In particular, it is G-module with GF [Zub3]. Thus we get the following exact sequences (see [Don7])

$$0 \to (\Lambda^r(E) \otimes R)^G \to \ldots \to (\Lambda^2(E) \otimes R)^G \to (E \otimes R)^G \to I^G \to 0$$

and

$$0 \to I^G \to R^G \to K[S]^G \to 0$$

Here $R = K[S' \times M(k_d)], r = \dim E = \frac{k_d(k_d-1)}{2}$.

The space $E \otimes R$ can be considered as a homogeneous component of an invariant ring of a "supermixed" representation space of some new quiver Q' having the same set of vertexes as Q but the additional arrow a' such that $h(a') = i, t(a') = j$. Besides, in comparison with the previous representation space the new one has the additional component $S_{a'}$ which is a subspace of $M(k_d)$ consisting of all skew-symmetric matrices with respect to the same action of G.

If we introduce a new general matrix $X(a')$ corresponding to a' then $(E \otimes R)^G$ is the degree one in $X(a')$ homogeneous component of the invariant algebra of this new representation space.

The next step is using transfer principle reciprocity to replace the group $G_d = Sp(k_d)$ by $GL(k_d)$ as in the first section.

Let us remind that we have to add to the variety $S' \times M(k_d)$ the new factor $GL(k_d)/Sp(k_d)$ identified with a closed subvariety of $M(k_d)^2$ consisting of all pairs of matrices (x, y) such that $xy = I_{k_d}$ and both x and y are skew-symmetric. This subvariety is a complete intersection again.

This step was described in [Zub5] and by this reason we omit all details but briefly describe what we get in this case.

The algebra R^G is an epimorphic image of the algebra $R'^{G'}$, where $R' = K[S' \times M(k_d) \times M(k_d)^2]$, $G' = \times_{f, f\neq d} G_f \times GL(k_d)$ and $GL(k_d)$ acts on the additional factor $M(k_d)^2$ by the rule $(x, y)^g = (gxg^t, (g^t)^{-1} y g^{-1}), x, y \in M(k_d), g \in GL(k_d)$. It means that we add to our quiver Q one vertex, say with the number $n+1$, and two arrows b, c such that $h(b) = t(c) = i, t(b) = h(c) = n+1$.

Further, the vertex $n+1$ is occupied by the space $E_{n+1} = K^{k_d} = V$ as well as the vertex i is occupied by V^*. Our epimorphism is just the specialization $X(b) \to J, X(c) \to -J = J^{-1}$.

As above we have the following exact sequences

$$0 \to (\Lambda^{r'}(E') \otimes R')^{G'} \to \ldots \to (E' \otimes R')^{G'} \to I'^{G'} \to 0$$

and

$$0 \to I'^{G'} \to R'^{G'} \to R^G \to 0$$

Here $r' = \dim E' = k_d^2 + \frac{k_d(k_d-1)}{2} + k_d = \frac{k_d(3k_d+1)}{2}$. The ideal I' is generated by the G'-invariant subspace

$$E' = (\oplus_{1 \leq i < j \leq k_d} K \cdot z_{ij}) \oplus (\oplus_{1 \leq i \leq k_d} K \cdot z_i) \oplus (\oplus_{1 \leq i, j \leq k_d} K \cdot t_{ij}),$$

where $z_{ij} = x_{ij}(b) + x_{ji}(b), z_i = x_{ii}(b), 1 \leq i \neq j \leq k_d, t_{ij} = \sum_{1 \leq k \leq k_d} x_{ik}(b) x_{kj}(c) - \delta_{ij}, 1 \leq i, j \leq k_d$.

Finally, to compute the generators of the ideal $I'^{G'}$ one has to introduce a new representation space just adding to the variety $S' \times M(k_d) \times M(k_d)^2$ two new factors – the subspace of skew-symmetric matrices of $M(k_d)$ and $M(k_d)$ with respect to the actions $A^g = gAg^t$ and $A^g = gAg^{-1}$ correspondently.

In other words, we add to our quiver one arrow from i to j and one loop incidenced to j. Introducing two general matrices corresponding to these new factors, say U, V, where U is skew-symmetric, we see that $I'^{G'}$ is an epimorphic image of degree one in both U and V homogeneous component of the invariant algebra of the last representation space. This epimorphism is the specialization $U \to Z = (z_{ij}), V \to X(b)X(c) - I_{k_d}$.

All another cases can be considered by the same way as above. This concludes the proof.

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