Odd primary string cohomology

Iver Ottosen *

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld
Fax number: +49 521 106 4743
E-mail address: ottosen@mathematik.uni-bielefeld.de
26 September 2000

Abstract

Let X be a connected space and let $K = H^*(X; \mathbb{F}_p)$ where p is an odd prime. We construct functors ω and ℓ which approximate the cohomology of the free loop space ΛX as follows: There are morphisms $\omega(K) \to H^*(\Lambda X; \mathbb{F}_p)$ and $\ell(K) \to H^*_{S^1}(\Lambda X; \mathbb{F}_p)$. These are isomorphisms when X is a product of Eilenberg-MacLane spaces of type $K(\mathbb{F}_p, n)$ for $n \geq 1$.

MSC: 55N91; 55P35; 55R12

1 Introduction

The string cohomology of a topological space X with coefficients in a ring R is defined as follows:

$$H^*_{st}(X; R) := H^*_{S^1}(\Lambda X; R) = H^*(ES^1 \times_{S^1} \Lambda X; R)$$

where ΛX denotes the free loop space of X. These cohomology groups together with the cohomology of the free loop space itself $H^*(\Lambda X; R)$ plays a central role in geometry and topology. It is however not know how to compute these in general.

When $R = \mathbb{F}_2 = \mathbb{Z}/2$, M. Bökstedt and I found computable functors of $H^*(X; \mathbb{F}_2)$ which approximate these cohomology groups [2]. The purpose of this paper is to generalize these functors to the case $R = \mathbb{F}_p = \mathbb{Z}/p$ where p is any of the odd primes. Certain algebra generators in string cohomology are more difficult to construct in the odd primary case. Hence method and strategy differs from [2] at various places.

2 Notation

Fix an odd prime p. We use $\mathbb{F}_p$-coefficients everywhere unless otherwise is specified. $\mathcal{A}$ denotes the mod p Steenrod algebra, $\mathcal{U}$ the category of unstable $\mathcal{A}$-modules and $\mathcal{K}$ the category of unstable $\mathcal{A}$-algebras. We write $\mathcal{K}_0$ for the full

*The author was supported by the European Union TMR network ERB FMRX CT-97-0107: Algebraic K-theory, Linear Algebraic Groups and Related Structures.

subcategory of $\mathcal{K}$ with the connected unstable $\mathcal{A}$-algebras as its objects. The category of differential graded algebras is denoted DGA.

For $K \in \mathcal{K}$ we define $\lambda : K \to K$ as follows

$$\lambda x = \begin{cases} P^{\frac{|x|-1}{2}} x & , |x| \text{ odd} \\ 0 & , |x| \text{ even} \end{cases}$$

Note that $|\lambda x| = p(|x| - 1) + 1$. Note also that λ is a derivation over Frobenius by the Cartan formula

$$\lambda(xy) = \lambda(x)y^p + x^p\lambda(y).$$

In any graded $\mathbb{F}_p$-algebra K we define $\sigma : K \to \mathbb{F}_p$ by $\sigma(x) = 1$ for $|x|$ odd and $\sigma(x) = 0$ for $|x|$ even. We also define $\hat{\sigma} : K \to \mathbb{F}_p$ by $\hat{\sigma}(x) = 1 - \sigma(x)$.

3 The approximation functor ω

Let Y be a connected S^1-space with action map $\eta : S^1 \times Y \to Y$. We have $\eta \circ \gamma = id$ where $\gamma(y) = (1, y)$. Recall that $H^*(S^1) = \Lambda(v)$ where $|v| = 1$.

Definition 3.1. Define the map $d : H^*(Y) \to H^{*-1}(Y)$ by

$$\eta^*(y) = 1 \otimes y + v \otimes dy.$$

Proposition 3.2. *The map d satisfies the following:*

$$d \circ d = 0, \tag{1}$$
$$d(x + y) = dx + dy, \tag{2}$$
$$d(xy) = d(x)y + (-1)^{|x|}xd(y), \tag{3}$$
$$P^i(dx) = d(P^i x) \text{ for } i \geq 0, \tag{4}$$
$$\beta(dx) = -d(\beta x), \tag{5}$$
$$d(\lambda x) = (dx)^p, \tag{6}$$
$$d(\beta\lambda x) = 0. \tag{7}$$

Proof. Similar to the $p = 2$ case proved in [2] Proposition 3.2. □

Definition 3.3. For $K \in \mathcal{K}_0$ we define $\omega(K)$ as the symmetric product of K with the free graded commutative algebra on generators dx of degree $|x| - 1$ for $x \in K$ divided by the ideal generated by the elements

$$d(x + y) - dx - dy, \tag{8}$$
$$d(xy) - d(x)y - (-1)^{|x|}xd(y), \tag{9}$$
$$d(\lambda x) - (dx)^p, \tag{10}$$
$$d(\beta\lambda x). \tag{11}$$

There is a degree -1 differential d on $\omega(K)$ defined by $d(x) = dx$ for $x \in K$ as a derivation over K. Hence $(\omega(K), d) \in DGA$.

Proposition 3.4. *For $K \in \mathcal{K}$ we can define an $\mathcal{A}$-action on $\omega(K)$ by $\theta(x) = \theta x$ and $\theta(dx) = (-1)^{|\theta|}d(\theta x)$ for $x \in K$ and $\theta \in \mathcal{A}$. In this way ω becomes a functor $\omega : \mathcal{K}_0 \to \mathcal{K}$. Note that the differential d on $\omega(K)$ is graded $\mathcal{A}$-linear.*

Proof. See the appendix. □

Definition 3.5. For a connected space X we define the map

$$e : \omega(H^*X) \to H^*(\Lambda X)$$

by $x \mapsto ev_0^*(x)$ and $dx \mapsto dev_0^*(x)$ where $ev_0 : \Lambda X \to X; f \mapsto f(1)$.

Remark 3.6. The map e is a morphism in $\mathcal{K}$ as well as a morphism in DGA. It is also natural in X. We view $\omega(H^*X)$ as an approximation to $H^*(\Lambda X)$ via the morphism e.

Proposition 3.7. *Let $K, L \in \mathcal{K}_0$ and let $i : K \to K \otimes L$ and $j : L \to K \otimes L$ be the inclusions given by $i(x) = x \otimes 1$ and $j(y) = 1 \otimes y$. The composite*

$$\kappa : \omega(K) \otimes \omega(L) \xrightarrow{\omega(i)\otimes\omega(j)} \omega(K \otimes L) \otimes \omega(K \otimes L) \xrightarrow{\mu} \omega(K \otimes L),$$

where μ denotes the product, is an isomorphism in both DGA and $\mathcal{K}$. For connected spaces X and Y with homology of finite type the following diagram commutes.

$$\begin{array}{ccc} \omega(H^*X) \otimes \omega(H^*Y) & \xrightarrow{\kappa} & \omega(H^*X \otimes H^*Y) \\ \downarrow{\scriptstyle e\otimes e} & & \downarrow{\scriptstyle e} \\ H^*(\Lambda X) \otimes H^*(\Lambda Y) & \xrightarrow{\cong} & H^*(\Lambda(X \times Y)) \end{array}$$

Proof. The map κ is a morphism in both DGA and K since it is a composite of maps which are morphisms in both categories. For $a \in K$ and $b \in L$ we have that $\kappa(a \otimes b) = a \otimes b$ and $\kappa(d_\otimes(a \otimes b)) = d(a \otimes b)$. We verify that there is a well defined $\mathbb{F}_p$-algebra map γ in the opposite direction of κ with $\gamma(a \otimes b) = a \otimes b$ and $\gamma(d(a \otimes b)) = d_\otimes(a \otimes b)$ such that κ is an isomorphism.

Elements of the form (8) are mapped to zero by definition. By direct computation one sees that the elements of the form (9) are also mapped to zero. Assuming that $\sigma(a) = 1$ and $\sigma(b) = 0$ we find

$$\begin{aligned} d(\lambda(a \otimes b)) - (d(a \otimes b))^p =& d(\lambda(a) \otimes b^p) - d(a)^p \otimes b^p \mapsto \\ & d(\lambda a) \otimes b^p - d(a)^p \otimes b^p = 0, \\ d(\beta(\lambda(a \otimes b))) =& d(\beta\lambda(a) \otimes b^p) \mapsto \\ & d(\beta\lambda(a)) \otimes b^p = 0 \end{aligned}$$

and if $\sigma(a) = \sigma(b)$ the elements are already zero in $\omega(K \otimes L)$. The above description of κ on generators shows that the diagram commutes. □

4 The ω approximation for certain Eilenberg-MacLane spaces

For an Abelian group A and positive integer n we let B^nA denote the Eilenberg-MacLane space $K(A, n)$. In this section we prove the following result:

Theorem 4.1. *The map $e : \omega(H^*B^n\mathbb{F}_p) \to H^*(\Lambda B^n\mathbb{F}_p)$ is an isomorphism in $\mathcal{K}$ and DGA for each $n \geq 1$.*

We first consider the case $n \geq 2$. The Whitehead theorem together with the long exact sequence of homotopy groups and the five lemma proves the following:

Proposition 4.2. *Let G be a connected topological group. Then there is a commutative diagram as follows*

$$\begin{array}{ccccc}
\Omega G & \xrightarrow{i_2} & G \times \Omega G & \xrightarrow{pr_1} & G \\
{\scriptstyle id}\downarrow & & {\scriptstyle m}\downarrow & & {\scriptstyle id}\downarrow \\
\Omega G & \longrightarrow & \Lambda G & \xrightarrow{ev_0} & G
\end{array}$$

where $m(f,x) = (z \mapsto f(z) \cdot x)$. The map m is a weak homotopy equivalence.

Proposition 4.3. *For $n \geq 2$ there is an isomorphism*

$$m^* : H^*(\Lambda B^n \mathbb{F}_p) \to H^*(B^n \mathbb{F}_p) \otimes H^*(B^{n-1}\mathbb{F}_p)$$

with the property $m^* \circ e(\iota_n) = \iota_n \otimes 1$ and $m^* \circ e(d\iota_n) = c_n 1 \otimes \iota_{n-1}$ for some nonzero constant $c_n \in \mathbb{F}_p$.

Proof. The space $B^n\mathbb{F}_p$ is a topological Abelian group since $\mathbb{F}_p$ is an Abelian group. Proposition 4.2 with $G = B^n\mathbb{F}_p$ gives that m^* is an isomorphism and that $m^* \circ ev_0^*(\iota_n) = 1 \otimes \iota_n$ as stated. We have $m^*(dev_0^* \iota_n) = c_n 1 \otimes \iota_{n-1}$ for some $c_n \in \mathbb{F}_p$ since $1 \otimes \iota_{n-1}$ is the only class in degree $n-1$ of the right hand side. Let h denote the composite

$$h : S^1 \times B^n\mathbb{F}_p \times B^{n-1}\mathbb{F}_p \xrightarrow{m} S^1 \times \Lambda B^n \mathbb{F}_p \xrightarrow{\eta} \Lambda B^n\mathbb{F}_p \xrightarrow{ev_0} B^n\mathbb{F}_p$$

where η is the action map. Then $h^*(\iota_n) = 1 \otimes \iota_n \otimes 1 + c_n v \otimes 1 \otimes \iota_{n-1}$. If $c_n = 0$ then $ev_0 \circ \eta$ is homotopic to the composite

$$k : S^1 \times \Lambda B^n\mathbb{F}_p \xrightarrow{pr_2} \Lambda B^n\mathbb{F}_p \xrightarrow{ev_0} B^n\mathbb{F}_p$$

since $B^n\mathbb{F}_p$ classifies mod p cohomology in degree n. But we have $ev_0 \circ \eta(z,f) = f(z)$ so its adjoint is the identity on $\Lambda B^n\mathbb{F}_p$. The adjoint of k is the map $\Lambda B^n\mathbb{F}_p \to \Lambda B^n\mathbb{F}_p$ which sends a loop f to the constant loop with value $f(1)$. The maps k and $ev_0 \circ \eta$ cannot be homotopic since their adjoints are not. □

Proposition 4.4. *The map $e : \omega(H^*B^n\mathbb{F}_p) \to H^*(\Lambda B^n\mathbb{F}_p)$ is an isomorphism for each $n \geq 2$.*

Proof. Since the cohomology of the spaces $B^m\mathbb{F}_p$ are free objects in $\mathcal{K}$ we can define a morphism in $\mathcal{K}$ as follows:

$$\begin{aligned}
f : H^*(B^n\mathbb{F}_p) \otimes H^*(B^{n-1}\mathbb{F}_p) \to \omega(H^*B^n\mathbb{F}_p) \quad &, \quad \iota_n \otimes 1 \mapsto \iota_n \\
&, \quad 1 \otimes \iota_{n-1} \mapsto c_n^{-1} d\iota_n.
\end{aligned}$$

We have $m^* \circ e \circ f = id$ and $f \circ m^* \circ e = id$ since these equalities hold on generators. □

The case $n = 1$ is interesting since here $\Lambda B\mathbb{F}_p$ splits in p components. We will use the following result to see that $\omega(H^*B\mathbb{F}_p)$ split accordingly.

Lemma 4.5. *There is an isomorphism of rings as follows*

$$\alpha : \mathbb{F}_p[x]/(x^p - x) \to (\mathbb{F}_p)^p \quad ; \quad x \mapsto (0, 1, 2, \ldots, p-1)$$

where $\mathbb{F}_p[x]$ is the polynomial ring in one variable x of degree zero and $(\mathbb{F}_p)^p$ is the p-fold Cartesian product of $\mathbb{F}_p$ by itself.

Proof. Use the factorization $x^p - x = \prod_{n \in \mathbb{F}_p}(x - n)$ and the Chinese remainder theorem. $\square$

Remark 4.6. Let $e_n = \alpha^{-1}(0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 on the nth place for $n \in \mathbb{F}_p$. Clearly $e_n e_m = 0$ for $n \neq m$, $e_n^2 = e_n$ and $\sum e_n = 1$. Also $xe_n = ne_n$. Finding eigenvectors for $xf(x) = nf(x)$ and normalizing one gets

$$e_0 = 1 - x^{p-1}$$
$$e_m = -\sum_{i=1}^{p-1} (\frac{x}{m})^i \quad , \quad m \neq 0$$

Definition 4.7. For $n \in \mathbb{F}_p$ define the following action map

$$f_n : \mathbb{Z} \times \mathbb{F}_p \to \mathbb{F}_p \quad ; \quad (r, [s]) \mapsto [nr + s].$$

We let $B\mathbb{F}_p(n)$ denote $B\mathbb{F}_p$ equipped with S^1-action Bf_n and write d_n for the corresponding action differential on $H^*B\mathbb{F}_p(n)$.

Proposition 4.8. *We have $H^*B\mathbb{F}_p(n) = \Lambda(v_n) \otimes \mathbb{F}_p[\beta v_n]$ where $|v_n| = 1$. The differential d_n on this algebra is given by $d_n v_n = n$ and $d_n \beta v_n = 0$ for each $n \in \mathbb{F}_p$.*

Proof. We must show that $(Bf_n)^*(v_n) = 1 \otimes v_n + nv \otimes 1$. This follows from $H_1(Bf_n) = \pi_1(Bf_n) = f_n$ by taking duals. Since $\lambda v_n = \iota_n$ the class βv_n is mapped to zero. $\square$

Proposition 4.9. *The map $e : \omega(H^*B\mathbb{F}_p) \to H^*(\Lambda B\mathbb{F}_p)$ is an isomorphism.*

Proof. From [1] Lemma 7.11 we have $\Lambda B\mathbb{F}_p \simeq \sqcup_{n \in \mathbb{F}_p} B\mathbb{F}_p$. Define maps $j_n : B\mathbb{F}_p(n) \to \Lambda B\mathbb{F}_p$ by $x \mapsto (z \mapsto Bf_n(z, x))$ for $n \in \mathbb{F}_p$. These are S^1-maps. Let $(\Lambda B\mathbb{F}_p)(n)$ denote the component of $\Lambda B\mathbb{F}_p$ containing the image of j_n. Then the restriction $j_n| : B\mathbb{F}_p(n) \to (\Lambda B\mathbb{F}_p)(n)$ is an S^1-map and a homotopy equivalence. Especially the induced in cohomology $(j_n|)^*$ is an isomorphism of differential graded algebras. By Proposition 4.8 we see that $(\Lambda B\mathbb{F}_p)(n) \neq (\Lambda B\mathbb{F}_p)(m)$ for $n \neq m$ such that $\sqcup j_n : \sqcup B\mathbb{F}_p(n) \to \Lambda B\mathbb{F}_p$ is an S^1-map and a homotopy equivalence. Especially $(\sqcup j_n)^* = (j_0^*, \ldots, j_{p-1}^*)$ is an isomorphism. So it suffices to show that $g = (j_0^*, \ldots, j_{p-1}^*) \circ e$ is an isomorphism. We have

$$(j_n^* \circ e)(x) = j_n^* \circ ev_0^*(x) = (ev_0 \circ j_n)^*(x) = x, \tag{12}$$
$$(j_n^* \circ e)(dx) = j_n^*(dev_0^*(x)) = d_n \circ j_n^* \circ ev_0^*(x) = d_n x \tag{13}$$

for $x \in H^*B\mathbb{F}_p$ which describes the map g on generators. By definition we have

$$\omega(H^*B\mathbb{F}_p) = \Lambda(\iota_1) \otimes \mathbb{F}_p[\beta\iota_1] \otimes \big(\mathbb{F}_p[d\iota_1]/((d\iota_1)^p - d\iota_1)\big).$$

From Lemma 4.5 and (13) we see that g is an isomorphism in degree zero. By (12) we conclude that it is an isomorphism in all degrees. $\square$

5 Steenrod diagonal elements

In this and the following four sections we describe algebra related to certain classes in string cohomology. The motivation comes later in Theorem 11.3. In the following K denotes an unstable $\mathcal{A}$-algebra. The polynomial algebra $\mathbb{F}_p[u]$ where $|u| = 2$ is an object in $\mathcal{K}$ by the isomorphism $\mathbb{F}_p[u] \cong H^*(BS^1)$.

Definition 5.1. For $x \in K$ and $\epsilon = 0, 1$ we define $St_\epsilon(x) \in \mathbb{F}_p[u] \otimes K$ by

$$St_\epsilon(x) = u^{-\epsilon\hat{\sigma}(x)} \sum_{i \geq 0} (-u^{p-1})^{[|x|/2]-i} \otimes \beta^\epsilon P^i x.$$

Note that the terms where the total exponent of u is negative has $\beta^\epsilon P^i x = 0$. Let $R(K) \subseteq \mathbb{F}_p[u] \otimes K$ be the sub-$\mathbb{F}_p$-algebra generated by $u \otimes 1$ and $St_\epsilon(x)$ for all $x \in K$ and $\epsilon = 0, 1$.

Theorem 5.2. *For each $\theta \in \mathcal{A}$ one has $\theta R(K) \subseteq R(K)$. Thus R is a functor $R : \mathcal{K} \to \mathcal{K}$. The explicit formulas are as follows where $n = [|x|/2]$ and $\epsilon = 0, 1$:*

$$\begin{aligned} P^i St_\epsilon(x) =& \sum_t \binom{(p-1)(n-t) + \epsilon\sigma(x)}{i - pt} u^{(p-1)(i-pt)} St_\epsilon(P^t x) \\ -\epsilon(-1)^{\sigma(x)} & \sum_t \binom{(p-1)(n-t) - 1 + \sigma(x)}{i - pt - 1} u^{(p-1)(i-pt)-1+(2-p)\sigma(x)} St_0(\beta P^t x), \\ \beta St_\epsilon(x) =& (1-\epsilon) u^{\hat{\sigma}(x)} St_1(x). \end{aligned}$$

Proof. The formula for the Bockstein operation follows directly by the definition of $St_\epsilon(x)$. We use results from [3] to prove the other formula. By [8] we have that $\mathbb{F}_p[u, u^{-1}]$ is an $\mathcal{A}$-algebra with $\beta = 0$ and

$$P^i u^j = \binom{j}{i} u^{j+i(p-1)} \quad ; \quad i, j \in \mathbb{Z} \quad ; \quad i \geq 0.$$

Here the following extended definition of binomial coefficients is used where $r \in \mathbb{R}$ and $k \in \mathbb{Z}$.

$$\binom{r}{k} = \begin{cases} \frac{r(r-1)\dots(r-k+1)}{k!} & , k > 0 \\ 1 & , k = 0 \\ 0 & , k < 0 \end{cases}$$

Let $\Delta = \Lambda(a) \otimes \mathbb{F}_p[b, b^{-1}]$ with $|a| = 2p - 3$, $|b| = 2p - 2$ be the $\mathcal{A}$-algebra introduced in [3] (2.6). That is $\beta a = b$ and

$$\begin{aligned} P^i(b^j) &= (-1)^i \binom{(p-1)j}{i} b^{i+j}, \\ P^i(ab^{j-1}) &= (-1)^i \binom{(p-1)j - 1}{i} ab^{i+j-1}. \end{aligned}$$

Note that we have changed the names of the generators. In [3] they were named u and v instead of a and b. We define an additive transfer map as follows:

$$\tau : \Delta \to \mathbb{F}_p[u, u^{-1}] \quad ; \quad b^j \mapsto 0 \quad ; \quad ab^{j-1} \mapsto (-u^{p-1})^j u^{-1}.$$

Note that $|\tau| = -1$. A direct verification shows that τ is $\mathcal{A}$-linear.

A functor R_+ from the category of graded $\mathcal{A}$-modules to itself is constructed in [3]. In the case of an unstable $\mathcal{A}$-algebra K it comes with an $\mathcal{A}$-linear map $f : R_+K \to \Sigma\Delta \otimes K$ defined by [3] (3.1), (3.2). The composite

$$R_+K \xrightarrow{f} \sigma\Delta \otimes K \xrightarrow{\Sigma\tau\otimes 1} \Sigma\mathbb{F}_p[u,u^{-1}] \otimes K$$

is given by

$$\begin{aligned} sb^k \otimes x &\mapsto -s\sum_j (-u^{p-1})^{k-j}u^{-1} \otimes \beta P^j x, \\ sab^{k-1} \otimes x &\mapsto s\sum_j (-u^{p-1})^{k-j}u^{-1} \otimes P^j x. \end{aligned}$$

Especially $sb^n \otimes x \mapsto -su^{\sigma(x)}St_1(x)$ and $sab^{n-1} \otimes x \mapsto su^{-1}St_0(x)$ where $n = [|x|/2]$. The formulas [3] (3.4), (3.5) for the $\mathcal{A}$-action on R_+M gives the following formulas for the $\mathcal{A}$-action on $u^{\sigma(x)}St_1(x)$ and $u^{-1}St_0(x)$:

$$\begin{aligned} P^i(u^{\sigma(x)}St_1(x)) &= \sum_t \binom{(p-1)(n-t)}{i-pt} u^{(p-1)(i-pt)-\sigma(x)}St_1(P^t x) \\ &\quad - \sum_t (-1)^{\sigma(x)} \binom{(p-1)(n-t)-1}{i-pt-1} u^{(p-1)(i-pt-\sigma(x))-1}St_0(\beta P^t x), \\ P^i(u^{-1}St_0(x)) &= \sum_t \binom{(p-1)(n-t)-1}{i-pt} u^{(p-1)(i-pt)-1}St_0(P^t x). \end{aligned}$$

This proves the result directly for $\sigma(x) = 0$ and $\epsilon = 1$. By the Cartan formula applied to $uu^{-1}St_\epsilon(x)$ we have that $P^iSt_\epsilon(x) = uP^i(u^{-1}St_\epsilon(x)) + u^pP^{i-1}(u^{-1}St_\epsilon(x))$. By combining this with the formulas above we get the result in the other cases. □

6 The functor ℓ

In this section we describe the functor which approximates string cohomology. We also define maps which relate this functor to the functors R and ω.

Definition 6.1. For $K \in \mathcal{K}_0$ we define $\ell(K)$ as the graded commutative $\mathbb{F}_p$-algebra generated by the classes

$$\begin{aligned} &\phi(x) \text{ of degree } p|x| - \sigma(x)(p-1), \\ &\delta(x) \text{ of degree } |x| - 1, \\ &q(x) \text{ of degree } p|x| - 1 - \sigma(x)(p-3) \end{aligned}$$

for all homogeneous $x \in K$ and a class u of degree 2; modulo the ideal generated by

$$\phi(x+y) - \phi(x) - \phi(y) + \sigma(x)\sum_{i=0}^{p-2}(-1)^i\delta(x)^i\delta(y)^{p-2-i}\delta(xy), \tag{14}$$

$$\delta(x+y) - \delta(x) - \delta(y), \tag{15}$$

$$q(x+y) - q(x) - q(y) + \hat{\sigma}(x)\sum_{i=1}^{p-1}(-1)^i\frac{1}{i}\delta(x^iy^{p-i}), \tag{16}$$

$$(-1)^{\sigma(a)\hat{\sigma}(c)}\delta(a)\delta(bc)+(-1)^{\sigma(b)\hat{\sigma}(a)}\delta(b)\delta(ca)+(-1)^{\sigma(c)\hat{\sigma}(b)}\delta(c)\delta(ab), \tag{17}$$
$$\phi(ab)-(-u^{p-1})^{\sigma(a)\sigma(b)}\phi(a)\phi(b), \tag{18}$$
$$q(ab)-(-u^{p-1})^{\sigma(a)\sigma(b)}(u^{\sigma(b)}q(a)\phi(b)+(-u)^{\sigma(a)}\phi(a)q(b)), \tag{19}$$
$$q(x)^p-u^{p-1}q(\lambda x)-\phi(\beta\lambda x), \tag{20}$$
$$\delta(a)\phi(b)-\delta(ab^p)-\delta(a\lambda b)+\delta(ab)\delta(b)^{p-1}, \tag{21}$$
$$\delta(a)q(b)-\delta(ab^{p-1})\delta(b)-\delta(a\beta\lambda b), \tag{22}$$
$$\delta(x)u, \tag{23}$$
$$q(\beta\lambda x), \tag{24}$$
$$\delta(\beta\lambda x) \tag{25}$$

where $a, b, c, x, y \in K$ and $|x| = |y|$. It is understood that $\delta(1) = q(1) = 0$.

Remark 6.2. We have some immediate consequences of these relations:

- By (14)-(16) we have $\phi(0) = q(0) = \delta(0) = 0$.
- By (18), (19) and (21) the algebra $\ell(K)$ is unital with unit $\phi(1)$.
- By (18) and (19) we have $q(a^n) = n\phi(a)^{n-1}q(a)$ such that $q(a^p) = 0$.
- By (22) we have $\delta(b^p) = 0$ and also the important relation $\delta(\lambda b) = \delta(b)^p$.

Lemma 6.3. *For any $K \in \mathcal{K}_0$ the following relations hold in $\omega(K)$:*

$$\sum_{i=1}^{p-1}(-1)^{i+1}\frac{1}{i}d(x^iy^{p-i}) = (x+y)^{p-1}d(x+y)-x^{p-1}dx-y^{p-1}dy, \tag{26}$$
$$\sum_{j=0}^{p-2}(-1)^{j+1}(dx)^j(dy)^{p-2-j}d(xy) = (d(x+y))^{p-1}(x+y)-(dx)^{p-1}x-(dy)^{p-1}y. \tag{27}$$

Here $|x| = |y|$ is assumed to be even in (26) and odd in (27).

Proof. We verify (26) and omit the proof of (27) which is similar. Since d is a derivation we have

$$\sum_{i=1}^{p-1}(-1)^{i+1}\frac{1}{i}d(x^iy^{p-i}) = \sum_{i=1}^{p-1}(-1)^{i+1}(x^{i-1}y^{p-1}dx-x^iy^{p-i-1}dy).$$

By splitting the sum in two at the minus sign and substituting $j = i-1$ in the first of the resulting sums we see that the above equals the following:

$$\sum_{j=0}^{p-2}(-1)^jx^jy^{p-j-1}dx+\sum_{i=1}^{p-1}(-1)^ix^iy^{p-i-1}dy =$$
$$\sum_{t=0}^{p-1}(-1)^tx^ty^{p-t-1}(dx+dy)-x^{p-1}dx-y^{p-1}dy.$$

For $0 \leq t \leq p-1$ we have that $t!$ is invertible in $\mathbb{F}_p$ and also

$$\binom{p-1}{t} t! = (p-1)(p-2)\dots(p-t) = (-1)^t t! \bmod p.$$

Thus we have $\binom{p-1}{t} = (-1)^t$. Substituting this in the above and using the binomial formula the result follows. □

Proposition 6.4. *For any $K \in \mathcal{K}_0$ there is a natural homomorphism of $\mathbb{F}_p$-algebras which we call the de Rham map*

$$\begin{aligned} DR : \ell(K) \to \omega(K); \quad & \phi(x) \mapsto x^p + \lambda x - x(dx)^{p-1}, \\ & q(x) \mapsto x^{p-1}dx + \beta\lambda x, \\ & \delta(x) \mapsto dx, \quad u \mapsto 0. \end{aligned}$$

and we have $\mathrm{Im}(DR) \subseteq \ker(d : \omega(K) \to \omega(K))$. There is also a natural homomorphism of $\mathbb{F}_p$-algebras which we call the Steenrod map

$$\begin{aligned} St : \ell(K) \to \mathbb{F}_p[u] \otimes K; \quad & \phi(x) \mapsto St_0(x), \quad q(x) \mapsto St_1(x), \\ & \delta(x) \mapsto 0, \quad u \mapsto u \otimes 1. \end{aligned}$$

The image of this map is $\mathrm{Im}(St) = R(K)$. There is a commutative diagram of $\mathbb{F}_p$-algebras as follows.

$$\begin{tikzcd} \ell(K) \arrow[r, "St"] \arrow[d, "DR"'] & \mathbb{F}_p[u] \otimes K \arrow[d, "p_1"'] \\ \omega(K) \arrow[r, "p_2"] & K \end{tikzcd}$$

where the algebra maps p_1 and p_2 are given by $p_1(u) = 0$ and $p_2(dx) = 0$, $p_2(x) = x$ for each $x \in K$.

Proof. We check that the formulas for DR maps the relations (14)-(25) to zero. Formula (27) and the additivity of $x \mapsto x^p$ shows that (14) is mapped to zero. It is trivial that (15) is mapped to zero. By (26) and the additivity of $x \mapsto \beta\lambda x$ it follows that (16) is mapped to zero.

Taking the derivative of products and permuting factors we find the following equations:

$$\begin{aligned} & d(a)d(bc) = d(a)d(b)c + (-1)^{\sigma(b)} d(a)bd(c), \\ & d(b)d(ca) = (-1)^{\sigma(a)(\hat{\sigma}(b)+\hat{\sigma}(c))} ad(b)d(c) + (-1)^{\sigma(c)+\hat{\sigma}(a)(\hat{\sigma}(b)+\sigma(c))} d(a)d(b)c, \\ & d(c)d(ab) = (-1)^{\hat{\sigma}(c)(\hat{\sigma}(a)+\sigma(b))} d(a)bd(c) + (-1)^{\sigma(a)+\hat{\sigma}(c)(\sigma(a)+\hat{\sigma}(b))} ad(c)d(b). \end{aligned}$$

After some reductions (17) follows from these.

One easily checks that (18) and (19) are mapped to zero in each of the cases $\sigma(a) = \sigma(b) = 0$, $\sigma(a) = \sigma(b) = 1$ and $\sigma(a) = \hat{\sigma}(b) = 1$. It also follows by small direct computations that (20)-(25) are mapped to zero.

We check that (14)-(25) are mapped to zero by the formulas defining St. Since $\delta(x)$ is mapped to zero this is trivial for all elements except (18), (19), (20) and (24).

By the Cartan formula and $[\frac{|ab|}{2}] = [\frac{|a|}{2}] + [\frac{|b|}{2}] + \sigma(a)\sigma(b)$ one verifies that

$$St_0(ab) = (-u^{p-1})^{\sigma(a)\sigma(b)} St_0(a)St_0(b),$$
$$St_1(ab) = (-u^{p-1})^{\sigma(a)\sigma(b)}(u^{\sigma(b)}St_1(a)St_0(b) + (-u)^{\sigma(a)}St_0(a)St_1(b))$$

such that (18) and (19) are mapped to zero. Lemma 13.1 implies that (20) and (24) are mapped to zero. The diagram commutes by direct verification. □

7 The de Rham map and cohomology of $\omega(K)$

In this section K denotes a connected unstable $\mathcal{A}$-algebra. The de Rham map gives a map from $\ell(K)/(u)$ to the cycles in $\omega(K)$. We give an important criterion which ensures that this map is an isomorphism. The material corresponds to section 8 in [2].

Definition 7.1. Let $I_\delta \subseteq \ell(K)$ denote the ideal $I_\delta = (\delta(x) | x \in K)$.

Proposition 7.2. *There is an $\mathbb{F}_p$-linear map as follows*

$$\Psi : \omega(K) \to \ell(K) \quad ; \quad a_0 da_1 \dots da_n \mapsto \delta(a_0)\delta(a_1)\dots\delta(a_n)$$

where $a_0, \dots, a_n \in K$. Its image is the ideal $\mathrm{Im}(\Psi) = I_\delta$.

Proof. We must show that Ψ is well defined. The relations arising from (8), (9) and (11) are respected since we have the same relations in $\ell(K)$ with d replaced by δ. We must verify that the following relation is respected:

$$\begin{aligned} &a_0 da_1 \dots da_{i-1} d(a_i a_{i+1}) da_{i+2} \dots da_n = \\ &(-1)^{(k+\hat{\sigma}(a_i))\sigma(a_{i+1})} a_0 a_{i+1} da_1 \dots da_i da_{i+2} \dots da_n \\ &+(-1)^{(k+1)\sigma(a_i)} a_0 a_i da_1 \dots da_{i-1} da_{i+1} \dots da_n \end{aligned}$$

where $k = |da_1 \dots da_{i-1}|$. This follows if the relation

$$xd(yz) = (-1)^{\hat{\sigma}(y)\sigma(z)} xzd(y) + (-1)^{\sigma(y)} zyd(z)$$

is respected. By relation (17) one sees that it is after some work with the signs. □

Definition 7.3. Define the $\mathbb{F}_p$-algebra $\tilde{\omega}(K)$ as the quotient of $\ell(K)$ by the ideal $I_\delta + (u)$. Since $DR(I_\delta) \subseteq d\omega(K)$ we may define an $\mathbb{F}_p$-algebra map Φ by the following diagram where P denotes the canonical projection:

$$\begin{array}{ccc} \ell(K)/(u) & \xrightarrow{P} & \tilde{\omega}(K) \\ {\scriptstyle DR}\downarrow & & {\scriptstyle \Phi}\downarrow \\ \omega(K) & \longrightarrow & \omega(K)/d\omega(K) \end{array}$$

Since $d \circ DR = 0$ we have in fact defined a morphism $\Phi : \tilde{\omega}(K) \to H^*(\omega(K))$.

Remark 7.4. We have a filtration of $\ell(K)$ by ideals as follows:

$$\ell(K) \supseteq (u) \supseteq (u^2) \supseteq \dots \supseteq (u^i) \supseteq \dots$$

and $(u^i)/(u^{i+1}) = u^i \tilde{\omega}(K)$ for $i \geq 1$.

Proposition 7.5. *The composite $\Psi \circ d : \omega(K) \to \ell(K)$ is trivial, so we can define Ψ as a map on $\omega(K)/d\omega(K)$. This allows us to consider the composite $\Psi \circ \Phi : \tilde{\omega}(K) \to \ell(K)$. This composite is zero.*

Proof. By definition of Ψ we have $\Psi \circ d = 0$. The following rules hold for $b \in K$ and $z = a_0 da_1 \dots da_n \in \tilde{\omega}(K)$:

$$\Psi(\Phi(\phi(b))z) = (-1)^{|\phi(b)|}\phi(b)\Psi(z) \quad , \quad \Psi(\Phi(q(b))z) = (-1)^{|q(b)|}q(b)\Psi(z).$$

In fact the first rule follows from (21) and the second from (22) as one sees by direct verification. By these rules and the observation $\Psi(1) = 0$ the result follows. □

Remark 7.6. We can collect the information we have gathered so far in a commutative diagram:

$$\begin{tikzcd}
\tilde{\omega}(K) \arrow[r, "\Phi"] & \omega(K)/d\omega(K) \arrow[r, "\Psi"] & \ell(K)/(u) \arrow[r, "P"] \arrow[d, "DR"'] & \tilde{\omega}(K) \arrow[d, "\Phi"'] \\
& \omega(K) \arrow[u] \arrow[r, "d"] & \omega(K) \arrow[r] & \omega(K)/d\omega(K)
\end{tikzcd}$$

where the composite $\Psi \circ \Phi$ vanishes and $\ker(P) = \mathrm{Im}(\Psi)$.

Theorem 7.7. *Assume that the map $\Phi : \tilde{\omega}(K) \to H^*(\omega(K))$ is an isomorphism. Then so is*

$$DR : \ell(K)/(u) \to \ker(d : \omega(K) \to \omega(K)).$$

Proof. The diagram is formally the same as the one above Theorem 8.5 of [2]. So the same diagram chase as in the proof of Theorem 8.5 in [2] gives the result. □

Proposition 7.8. *For any pair $K, L \in \mathcal{K}_0$ the following composite is an isomorphism*

$$\tilde{\kappa} : \tilde{\omega}(K) \otimes \tilde{\omega}(L) \xrightarrow{\tilde{\omega}(i)\otimes\tilde{\omega}(j)} \tilde{\omega}(K \otimes L) \otimes \tilde{\omega}(K \otimes L) \xrightarrow{\mu} \tilde{\omega}(K \otimes L)$$

where $i : K \to K \otimes L$; $i(a) = a \otimes 1$ and $j : L \to K \otimes L$; $j(b) = 1 \otimes b$ are the canonical inclusions and μ denotes the product homomorphism. The following diagram commutes.

$$\begin{tikzcd}
\tilde{\omega}(K \otimes L) \arrow[r, "\Phi_{K\otimes L}"] & H^*(\omega(K \otimes L)) \\
\tilde{\omega}(K) \otimes \tilde{\omega}(L) \arrow[u, "\tilde{\kappa}"] \arrow[r, "\Phi_K \otimes \Phi_L"] & H^*(\omega(K)) \otimes H^*(\omega(L)) \arrow[u, "\kappa^*"']
\end{tikzcd}$$

Proof. We verify that $\tilde{\kappa}$ is an isomorphism. For $a \in K$ and $b \in L$ we have that $a \otimes b = (a \otimes 1)(1 \otimes b)$ giving the following relations in $\tilde{\omega}(K \otimes L)$:

$$\begin{aligned}
&\phi(a \otimes b) = (1 - \sigma(a)\sigma(b))\phi(a \otimes 1)\phi(1 \otimes b), \\
&q(a \otimes b) = \hat{\sigma}(b)q(a \otimes 1)\phi(1 \otimes b) + \hat{\sigma}(a)\phi(a \otimes 1)q(1 \otimes b).
\end{aligned}$$

Since $\phi(a) \otimes 1 \mapsto \phi(a \otimes 1)$, $q(a) \otimes 1 \mapsto q(a \otimes 1)$, $1 \otimes \phi(b) \mapsto \phi(1 \otimes b)$ and $1 \otimes q(b) \mapsto q(1 \otimes b)$ we see that $\tilde{\kappa}$ is surjective. One checks that the following gives a well defined map $\tilde{\gamma}$ in the opposite direction of $\tilde{\kappa}$:

$$\begin{aligned}
&\phi(a \otimes b) \mapsto (1 - \sigma(a)\sigma(b))\phi(a) \otimes \phi(b), \\
&q(a \otimes b) \mapsto \hat{\sigma}(b) q(a) \otimes \phi(b) + \hat{\sigma}(a)\phi(a) \otimes q(b).
\end{aligned}$$

By checking on generators we see that $\tilde{\gamma} \circ \tilde{\kappa} = id$ so $\tilde{\kappa}$ is injective as well and hence an isomorphism.

The isomorphisms κ and $\tilde{\kappa}$ have corresponding factorizations. The diagram splits in two commuting squares accordingly. $\square$

8 Frobenius algebras

Definition 8.1. A *Frobenius algebra* is a graded commutative $\mathbb{F}_p$-algebra K equipped with two $\mathbb{F}_p$-linear maps $\beta, \lambda : K \to K$ satisfying the following conditions:

- K is connected ($K^q = 0$ for $q < 0$ and $K^0 = \mathbb{F}_p$) and finite dimensional in each degree.
- $|\beta| = 1$, $\beta \circ \beta = 0$ and $\beta(xy) = \beta(x)y + (-1)^{|x|} x\beta(y)$ for all $x, y \in K$.
- $|\lambda x| = p(|x| - 1) + 1$, $\lambda x = 0$ when $|x|$ is even, $\lambda x = x$ when $|x| = 1$ and $\lambda(xy) = \lambda(x)y^p + x^p\lambda(y)$ for all $x, y \in K$.

A morphism of Frobenius algebras $f : (K, \beta, \lambda) \to (K', \beta', \lambda')$ is an $\mathbb{F}_p$-algebra map $f : K \to K'$ of degree zero such that $f \circ \lambda = \lambda' \circ f$ and $f \circ \beta = \beta' \circ f$. The category of Frobenius algebras is denoted $\mathcal{F}$.

Remark 8.2. There is a forgetful functor $\mathcal{K}_0 \to \mathcal{F}$.

Definition 8.3. Let $v\mathbb{F}_p$ denote the category of positively graded $\mathbb{F}_p$-vector spaces which are finite dimensional in each degree. Let $I : \mathcal{F} \to v\mathbb{F}_p$ denote the functor which takes a Frobenius algebra to its augmentation ideal (that is $I(K) = K^{*>0}$). Define the functor $S_{\mathcal{F}} : v\mathbb{F}_p \to \mathcal{F}$ as the left adjoint of I. For $V \in v\mathbb{F}_p$ we call $S_{\mathcal{F}}(V)$ the *free Frobenius algebra* on V.

Remark 8.4. We have $S_{\mathcal{F}}(V \oplus W) = S_{\mathcal{F}}(V) \otimes S_{\mathcal{F}}(W)$. Furthermore there is an explicit description as follows

$$S_{\mathcal{F}}(V) = S_{CA}\big(V \oplus \beta V \oplus \bigoplus_{i \geq 1, \nu \in \{0,1\}} \beta^\nu \lambda^i (\beta V^{even} \oplus V^{odd, *>1})\big)$$

where S_{CA} denotes the free graded commutative algebra functor.

Recall that a sequence of integers $I = (\epsilon_1, s_1, \epsilon_2, s_2, \ldots, \epsilon_k, s_k, \epsilon_{k+1})$ with $s_i \geq 0$ and $\epsilon_i \in \{0, 1\}$ is called admissible if $s_i \geq ps_{i+1} + \epsilon_{i+1}$ and $s_k \geq 1$ or if $k = 0$ when $I = (\epsilon)$. The degree of I is defined as $|I| = \sum \epsilon_j + \sum 2s_j(p - 1)$ and the excess is defined recursively by $e((\epsilon, s), J) = 2s + \epsilon - |J|$. We use the following notation $P^I = \beta^{\epsilon_1} P^{s_1} \beta^{\epsilon_2} P^{s_2} \ldots \beta^{\epsilon_k} P^{s_k} \beta^{\epsilon_{k+1}}$.

Lemma 8.5. *If $n \geq 2$ then $H^*B^n\mathbb{F}_p$ is the free Frobenius algebra on the vector space generated by the following set:*

$$\{P^I\iota_n | I \text{ is admissible }, e(I) \leq n-2, \epsilon_1 = 0\}.$$

Furthermore $H^*B\mathbb{F}_p$ is the free Frobenius algebra on the vector space generated by the class ι_1.

Proof. The case $n = 1$ is trivial. Assume that $n \geq 2$ and define the set

$$A(n) = \{I | I \text{ is admisseble }, e(I) \leq n-1, |I| + n \text{ is odd }\}.$$

Remark that if $I \in A(n)$ then $(0, (|I| + n - 1)/2, I) \in A(n)$. To see this write $I \in A(n)$ as $I = (\epsilon, s, I')$. Then $e(I) = 2s + \epsilon - |I'| \leq n-1$ or equivalently $2sp + 2\epsilon - |I| \leq n-1$ such that the sequence $(0, (|I| + n - 1)/2, I)$ is admissible. Its excess is $n-1$ and its degree plus n is odd since $p-1$ is even.

By Cartan's computation (a special case of [5], Theorem 10.3) we have that $H^*B^n\mathbb{F}_p$ is the free graded commutative algebra on the set

$$B = \{P^J\iota_n | J \text{ is admissible }, e(J) < n \text{ or } (e(J) = n \text{ and } \epsilon_1 = 1)\}.$$

Assume that $P^I\iota_n$ belongs to the set in the statement of the lemma. Then $P^I\iota_n$ and $\beta P^I\iota_n$ belongs to B. By the remark we see that if $|I| + n$ is even then $\beta^\epsilon\lambda^i\beta P^I\iota_n \in B$ and if $|I| + n$ is odd then $\beta^\epsilon\lambda^i P^I\iota_n \in B$ for $\epsilon = 0, 1$ and $i \geq 1$.

Conversely, assume that $P^J\iota_n \in B$. If $e(J) \leq n-2$ or $e(J) = n-1$ and $\epsilon_1 = 1$ it is clearly one of the generators described in the lemma. It suffices to handle the case $e(J) = n-1, \epsilon_1 = 0$ since the case $e(J) = n, \epsilon_1 = 1$ then follows. Write J as $J = (0, s, J')$ where $e(J) = 2s - |J|' = n-1$. Then $2s = n + |J'| - 1$ such that $P^J\iota_n = \lambda P^{J'}\iota_n$ and $e(J) \leq e(J')$. We can continue this process until the next ϵ equals one or the excess drops below $n-1$. $\square$

Theorem 8.6. *The map $\Phi : \tilde{\omega}(K) \to H^*(\omega(K))$ is an isomorphism when K is a free Frobenius algebra.*

Proof. It suffices to show this when K is a free Frobenius algebra on a one dimensional vector space. Let v be the generator of this vector space. We first check the case $|v| = 1$ where $K = \Lambda(v) \otimes \mathbb{F}_p[\beta v]$. The idempotents from Remark 4.6 (with $x = dv$) gives the following splitting:

$$\omega(K) = \bigoplus_{i \in \mathbb{F}_p} e_i\omega(K).$$

For each i we have $de_i = 0$ and $(dv)e_i = ie_i$. Also $\lambda v = v$ such that $d\beta v = 0$. From this we see that $d(v^\epsilon(\beta v)^r e_i) = \epsilon i(\beta v)^r e_i$. It follows that $H^*(e_i\omega(K)) = 0$ for $i \neq 0$ and $H^*(e_0\omega(K)) = K$. So $H^*(\omega(K)) = K$ and since $\Phi(\phi(v)) = ve_0$ and $\Phi(q(v)) = \beta v$ we see that Φ is surjective. The relations $\phi(\beta v) = q(v)^p$ and $q(\beta v) = 0$ shows that $\phi(v)$ and $q(v)$ generate $\tilde{\omega}(K)$ so Φ is also injective.

Assume that $|v|$ is even. In the following we write $[-]$ for the functor which takes a set to the vector space it generates. We have

$$K = S_{CA}[v, \beta v, \lambda^i\beta v, \beta\lambda^i\beta v | i \geq 1]$$

and we find that $\omega(K) = K \otimes S_{CA}[dv, d\beta v]$. We change basis such that the differential becomes easier to describe:

$$\begin{aligned}\omega(K) =& S_{CA}[v, dv] \otimes S_{CA}[\beta v, d\beta v] \otimes \\ & S_{CA}[\lambda^i \beta v - (d\lambda^{i-1}\beta v)^{p-1}\lambda^{i-1}\beta v, \beta\lambda^i\beta v | i \geq 1].\end{aligned}$$

By the Künneth formula we find that $H^*(\omega(K))$ equals

$$S_{CA}[v^p, v^{p-1}dv] \otimes S_{CA}[\lambda^i\beta v - (d\lambda^{i-1}\beta v)^{p-1}\lambda^{i-1}\beta v, \beta\lambda^i\beta v | i \geq 1].$$

The algebra $\tilde{\omega}(K)$ is generated by the classes $\phi(v)$, $\phi(\lambda^i\beta v)$, $q(v)$ and $q(\lambda^i\beta v)$ where $i \geq 0$. We see that Φ maps these generators to the free generators for the cohomology of $\omega(K)$. Hence Φ is an isomorphism. The case where $|v|$ is odd is similar. □

9 A pullback description of the functor ℓ

Proposition 9.1. *Let (n_i) be a sequence of positive integers such that the set $\{i | n_i = N\}$ is finite for each N. In particular (n_i) may be a finite sequence. If we let $K = H^*(\prod B^{n_i}\mathbb{F}_p)$ then* $\ker(St) = I_\delta$.

Proof. We must show that $St : \ell(K)/I_\delta \to \mathbb{F}_p[u] \otimes K$ is injective. The algebra $\ell(K)/I_\delta$ has generators $\phi(x)$, $q(x)$ for $x \in K$ together with u and the relations are that ϕ and q are additive and that (18), (19), (20) and (24) equals zero.

For a free Frobenius algebra $K = S_{\mathcal{F}}(V)$ we have listed a set of algebra generators for $\ell(K)/I_\delta$ below and we have written how they are mapped by St modulo elements in the ideal (u^{p-1}). Here $z \in V^1$, $v \in V^{odd,*>1}$ and $w \in V^{even}$ runs through a basis for V and $i \geq 0$:

$$\begin{aligned}
\phi(z) &\mapsto 1 \otimes z, & q(z) &\mapsto 1 \otimes \beta z,\\
\phi(\beta v) &\mapsto 1 \otimes (\beta v)^p, & q(\beta v) &\mapsto -u^{p-2} \otimes \beta P^{(|v|-1)/2}\beta v,\\
\phi(\lambda^i v) &\mapsto 1 \otimes \lambda^{i+1} v, & q(\lambda^i v) &\mapsto 1 \otimes \beta\lambda^{i+1} v,\\
\phi(w) &\mapsto 1 \otimes w^p, & q(w) &\mapsto -u^{p-2} \otimes \beta P^{|w|/2-1} w,\\
\phi(\lambda^i\beta w) &\mapsto 1 \otimes \lambda^{i+1}\beta w, & q(\lambda^i\beta w) &\mapsto 1 \otimes \beta\lambda^{i+1}\beta w,\\
u &\mapsto u.
\end{aligned}$$

We claim that these generators are mapped to algebraically independent elements in $\mathbb{F}_p[u] \otimes K$ when K is the cohomology of a product of Eilenberg-MacLane spaces as stated. It suffices to show this in the case of one single Eilenberg-MacLane space.

The claim is trivial for $K = H^*B\mathbb{F}_p$. Let $K = H^*B^n\mathbb{F}_p$ where $n \geq 2$. We can then cancel the first line in the list above. By Lemma 8.5 we have that K is the free Frobenius algebra on the vector space V with basis $P^I\iota_n$ where I is admissible, $e(I) \leq n-2$ and $\epsilon_1 = 0$.

If $|I| + n$ is odd we must look closer at $\beta P^{(|I|+n-1)/2}\beta P^I\iota_n$. Write I as $I = (0, s, I')$. We have $e(I) = 2s - |I'| \leq n-2$ which implies that $(0, (|I|+n-1)/2, 1, s, I')$ is admissible. Its excess equals $n-2$ and we see that $P^{(|I|+n-1)/2}\beta P^I\iota_n$ is a basis element in V^{even}.

If $|I|+n$ is even we must look at $\beta P^{(|I|+n-2)/2}P^I\iota_n$. As in the odd case we see that $P^{(|I|+n-2)/2}P^I\iota_n$ is a basis element in V^{even}. However there is no β between the first two P-operations from the left.

By the above list of the lowest terms and the description of K as a free Frobenius algebra we now see that the generators are mapped to algebraically independent elements. Especially they are free generators. □

Lemma 9.2. *Let $a_1,\dots,a_p \in K$ be elements of odd degree and define the following element in I_δ:*

$$\Delta(a_1,\dots,a_p) = \sum_{i=2}^{p} \delta(a_1a_i)\delta(a_2)\dots\widehat{\delta(a_i)}\dots\delta(a_p).$$

where the hat means that the factor is left out. Then for any permutation $\tau\in\Sigma_p$ one has $\Delta(a_1,\dots,a_p)=\Delta(a_{\tau(1)},\dots,a_{\tau(p)})$. Further the element is mapped as follows by the de Rham map:

$$DR(\Delta(a_1,\dots,a_p)) = \sum_{i=1}^{p} a_i da_1\dots\widehat{da_i}\dots da_p.$$

Proof. We first show the invariance under permutation. Since the degree of $\delta(a_i)$ is even it suffices to show that $\Delta(a_1,a_2,\dots,a_p)=\Delta(a_2,\dots,a_p,a_1)$. We prove the following more general statement for $n\geq 3$:

$$\sum_{i=2}^{n}\delta(a_1a_i)\delta(a_2)\dots\widehat{\delta(a_i)}\dots\delta(a_n) =$$
$$\sum_{j=3}^{n}\delta(a_2a_j)\delta(a_1)\delta(a_3)\dots\widehat{\delta(a_j)}\dots\delta(a_n) - (n-1)\delta(a_2a_1)\delta(a_3)\dots\delta(a_n).$$

The proof is by induction on n. For $n=3$ we have

$$\begin{aligned}\delta(a_1a_2)\delta(a_3)+\delta(a_1a_3)\delta(a_2) &= \delta(a_1a_2)\delta(a_3)-\delta(a_3a_1)\delta(a_2)\\ &= 2\delta(a_1a_2)\delta(a_3)+\delta(a_2a_3)\delta(a_1)\\ &= -2\delta(a_2a_1)\delta(a_3)+\delta(a_2a_3)\delta(a_1)\end{aligned}$$

where we used (17) at the second equality sign. Assume that the formula holds for $n-1$. Then we have

$$\sum_{i=2}^{n}\delta(a_1a_i)\delta(a_2)\dots\widehat{\delta(a_i)}\dots\delta(a_n) =$$
$$\Big(\sum_{i=2}^{n-1}\delta(a_1a_i)\delta(a_2)\dots\widehat{\delta(a_i)}\dots\delta(a_{n-1})\Big)\delta(a_n)+\delta(a_1a_n)\delta(a_2)\dots\delta(a_{n-1}) =$$
$$\Big(\sum_{j=3}^{n-1}\delta(a_2a_j)\delta(a_1)\delta(a_3)\dots\widehat{\delta(a_j)}\dots\delta(a_{n-1})\Big)\delta(a_n)$$
$$-(n-2)\delta(a_2a_1)\delta(a_3)\dots\delta(a_{n-1})\delta(a_n)+\delta(a_1a_n)\delta(a_2)\dots\delta(a_{n-1}).$$

We have that $\delta(a_1a_n)\delta(a_2)+\delta(a_2a_1)\delta(a_n)=\delta(a_2a_n)\delta(a_1)$ by relation (17) such that the sum of the last two terms above equals

$$-(n-1)\delta(a_2a_1)\delta(a_3)\dots\delta(a_n)+\delta(a_2a_n)\delta(a_1)\dots\delta(a_{n-1})$$

so we recover the formula for n.

We use that $d(a_1a_i) = a_i da_1 - a_1 da_i$ to compute the image under the de Rham map:

$$\begin{aligned} DR(\Delta(a_1,\dots,a_p)) &= \sum_{i=2}^{p} d(a_1a_i)da_2\dots\widehat{da_i}\dots da_p \\ &= \sum_{i=2}^{p} a_i da_1\dots\widehat{da_i}\dots da_p - (p-1)a_1da_2\dots da_p \end{aligned}$$

which modulo p gives the stated result. □

Definition 9.3. For any non negative integer n we let $B(n)$ denote the following set:

$$B(n) = \{(\beta_1,\dots,\beta_p) \in \mathbb{Z}^p | \forall i : \beta_i \geq 0, \beta_1 + \dots + \beta_p = n, \exists i,j : \beta_i \neq \beta_j\}.$$

The cyclic group on p elements C_p act on $B(n)$ by cyclic permutation of coordinates. For $x \in K$ we define the following elements in I_δ:

$$\begin{aligned} D_0^n(x) =& -\sigma(x)\sum \Delta(P^{\beta_1}(x), P^{\beta_2}(x),\dots,P^{\beta_p}(x)), \\ D_1^n(x) =& \hat{\sigma}(x)\sum \delta(P^{\beta_1}(x)P^{\beta_2}(x)\dots P^{\beta_p}(x)) \end{aligned}$$

where both sums are taken over $\beta \in B(n)/C_p$. Note that $D_0(x)$ is well defined by Lemma 9.2

Lemma 9.4. *For any $x \in K$ the following formulas hold in $\omega(K)$:*

$$P^i \circ DR(\phi(x)) = DR(\phi(P^{i/p}x) + D_0^i(x)), \tag{28}$$

$$P^i \circ DR(q(x)) = DR(q(P^{i/p}x) + D_1^i(x)) \tag{29}$$

where by convention $P^t = 0$ when t is a rational number which is not a non negative integer.

Proof. We first prove (28). Recall that $DR(\phi(x)) = x^p + \lambda x - x(dx)^{p-1}$. We have $P^i\lambda x = \lambda P^{i/p}x$ by Lemma 13.1 and also $P^i(x^p) = (P^{i/p}x)^p$ so it suffices to prove the following for $|x|$ odd:

$$P^i(x(dx)^{p-1}) = (P^{i/p}x)(dP^{i/p}x)^{p-1} - DR(D_0^i(x)).$$

By the Cartan formula we have

$$P^i(x(dx)^{p-1}) = \sum P^{\beta_1}(x)dP^{\beta_2}(x)\dots dP^{\beta_p}(x)$$

where we sum over the tuples $(\beta_1,\dots,\beta_p)$ with $\sum \beta_j = i$. The cyclic group C_p acts on the set of such tuples and an orbit has length 1 or p. Arranging the terms according to this the result follows by the definition of $D_0^i(x)$ and Lemma 9.2.

For the proof of (29) recall that $DR(q(x)) = x^{p-1}dx + \beta\lambda x$. We have $P^i(\beta\lambda x) = \beta\lambda(P^{i/p}x) + (\beta P^{(i-1)/p}x)^p$ by Lemma 13.1 so when $|x|$ is odd we are done. For $|x|$ even we must show that

$$P^i(x^{p-1}dx) = (P^{i/p}x)^{p-1}dP^{i/p}x + DR(D_1^i(x)).$$

This follows by the Cartan formula and a similar argument on orbits as the above. □

Theorem 9.5. *For any $K \in \mathcal{K}_0$ there is an $\mathcal{A}$-module structure on $\ell(K)$ such that ℓ becomes a functor $\ell : \mathcal{K}_0 \to \mathcal{K}$. The explicit formulas for the action are as follows where $x \in K$, $n = [|x|/2]$ and $i \geq 0$. Firstly, the action on $\phi(x)$ is given by:*

$$\begin{aligned} P^i\phi(x) =& D_0^i(x) + \sum_t \binom{(p-1)(n-t)}{i-pt} u^{(p-1)(i-pt)}\phi(P^t x), \\ \beta\phi(x) =& u^{\hat{\sigma}(x)}(q(x) - \delta(x)^{p-2}\delta(x\beta x)). \end{aligned}$$

Secondly, the action on $q(x)$ is given by:

$$\begin{aligned} P^i q(x) =& D_1^i(x) + \sum_t \binom{(p-1)(n-t)+\sigma(x)}{i-pt} u^{(p-1)(i-pt)} q(P^t x) \\ -(-1)^{\sigma(x)} & \sum_t \binom{(p-1)(n-t)-1+\sigma(x)}{i-pt-1} u^{(p-1)(i-pt)-1+(2-p)\sigma(x)}\phi(\beta P^t x), \\ \beta q(x) =& -\delta(x^{p-1}\beta x). \end{aligned}$$

Thirdly, the actions on $\delta(x)$ and u are as follows:

$$P^i\delta(x) = \delta(P^i x), \quad \beta\delta(x) = -\delta(\beta x), \quad P^1 u = u^p, \quad \beta u = 0.$$

The maps DR and St becomes $\mathcal{A}$-linear such that we have a commutative diagram in $\mathcal{K}$:

$$\begin{tikzcd} \ell(K) \arrow[r, "St"] \arrow[d, "DR"'] & R(K) \arrow[d, "p_1"] \\ \ker(d) \arrow[r, "p_2"] & K \end{tikzcd}$$

If $K = H^*(\prod B^{n_i}\mathbb{F}_p)$ where (n_i) is a (possible finite) sequence of positive integers satisfying that the set $\{i | n_i = N\}$ is finite for each N, then the diagram is a pullback square.

Proof. We first prove that we have a pullback when K is a product of Eilenberg-MacLane spaces as stated. By Proposition 7.8 and Theorem 8.6 the map Φ is an isomorphism. So by Theorem 7.7 the kernel of DR is the ideal $(u) \subseteq \ell(K)$. The kernel of p_1 is the ideal $(u \otimes 1) \subseteq R(K)$ and it suffices to show that the restriction of the Steenrod map to these kernels $St| : (u) \to (u \otimes 1)$ is an isomorphism. It is surjective since St is surjective and $St(u) = u \otimes 1$. By Proposition 9.1 we have $\ker(St) = I_\delta$ such that $\ker(St|) = (u) \cap I_\delta$ which is trivial because of the relation $\delta(x)u = 0$ in $\ell(K)$. Hence $St|$ is also injective.

In the Eilenberg-MacLane case the pullback defines an $\mathcal{A}$-module structure on $\ell(K)$. By Theorem 5.2 and Lemma 9.4 we see that the stated formulas describe this action. A standard argument using the fact that $B^n\mathbb{F}_p$ classifies degree n cohomology together with naturality takes care of the statements for general $K \in \mathcal{K}$. □

10 Homotopy orbits of S^1-spaces

In this section we list some results on homotopy orbits of S^1-spaces. They are all similar to results for $p = 2$ considered in [2] and we often refer to the proofs given there. In the entire section Y denotes an S^1-space. We write C_n for the cyclic group of order n. We let u of degree $|u| = 2$ and v of degree $|v| = 1$ denote algebra generators as follows: $H^*S^1 = \Lambda(v)$, $H^*BS^1 = \mathbb{F}_p[u]$ and $H^*BC_{p^n} = \Lambda(v) \otimes \mathbb{F}_p[u]$.

Proposition 10.1. *The fibration $Y \to ES^1 \times_{S^1} Y \to BS^1$ has the following Leray-Serre spectral sequence:*

$$E_2^{**} = H^*(BS^1) \otimes H^*(Y) \Rightarrow H^*_{S^1}(Y).$$

The differential in the E_2-term is given by

$$d_2 : H^*(Y) \to uH^*(Y) \quad ; \quad d_2(y) = ud(y)$$

where d is the action differential.

Proof. Similar to the proof of [2] Proposition 3.3. □

Definition 10.2. Define the spaces E_nY for $n = 0, 1, 2, \ldots, \infty$ by

$$\begin{aligned} E_nY &= ES^1 \times_{C_{p^n}} Y \quad , \quad n < \infty \\ E_\infty Y &= ES^1 \times_{S^1} Y. \end{aligned}$$

For nonnegative integers n and m with $m > n$ define the maps

$$q^n_m : H^*E_mY \to H^*E_nY \quad , \quad \tau^m_n : H^*E_nY \to H^*E_mY$$

by letting q^n_m be the map induced by the quotient map and τ^m_n be the transfer map. Also define $q^n_\infty : H^*E_\infty Y \to H^*E_nY$ as the map induced by the quotient.

The following theorem is inspired by a result of Tom Goodwillie which can be found in [4] p. 279. We use it to give a convenient definition of the S^1-transfer.

Theorem 10.3. *There is a commutative diagram as follows for any $m \geq 1$.*

$$\begin{array}{ccc} E_mY & \xrightarrow{Q} & E_\infty Y \\ {\scriptstyle pr_1}\downarrow & & {\scriptstyle pr_1}\downarrow \\ BC_{p^m} & \xrightarrow{Bj} & BS^1 \end{array} \tag{30}$$

Here Q denotes the quotient map and $j : C_{p^m} \hookrightarrow S^1$ the inclusion. The diagram gives rise to an isomorphism.

$$\Theta : H^*(BC_{p^m}) \otimes_{H^*(BS^1)} H^*(E_\infty Y) \cong H^*(E_mY) \quad ; \quad x \otimes y \mapsto pr_1^*(x)q^m_\infty(y)$$

The transfer map $\tau^{m+1}_m : H^*E_mY \to H^*E_{m+1}Y$ is zero on elements of the form $\Theta(1 \otimes y)$ and the identity on elements of the form $\Theta(v \otimes y)$. We get an isomorphism

$$\mathrm{colim} H^*E_mY = vH^*E_\infty Y \cong \widetilde{H}^*(\Sigma(E_\infty Y)_+).$$

Proof. Similar to the proof of [2] Theorem 4.2. □

Definition 10.4. For any non negative integer n the S^1-transfer

$$\tau_n^\infty : H^*E_nY \to H^*E_\infty Y$$

is defined as the composite

$$H^*E_nY \longrightarrow \operatorname{colim} H^*E_mY \xrightarrow{v^{-1}} H^*E_\infty Y$$

where the direct limit is over the transfer maps τ_n^m. Note that $|\tau_n^\infty| = -1$.

Definition 10.5. Let θ_0 denote the S^1-equivariant map

$$\theta_0 : S^1 \times Y_0 \to ES^1 \times Y \quad ; \quad (z,y) \mapsto (ze, zy)$$

and let θ_n for $n = 1, 2, \ldots, \infty$ be the maps one obtains by passing to the quotients

$$\begin{aligned} &\theta_n : S^1/C_{p^n} \times Y \to E_mY \quad , \quad m < \infty \\ &\theta_\infty : * \times Y \to E_\infty Y. \end{aligned}$$

Proposition 10.6. *For non negative integers m and n with $n < m$ the following squares commutes.*

$$\begin{tikzcd} H^*E_nY \arrow[r, "\theta_n^*"] & H^*(S^1 \times Y) \\ H^*E_mY \arrow[u, "q_m^n"] \arrow[r, "\theta_m^*"] & H^*(S^1 \times Y) \arrow[u, "q_m^n \otimes 1"'] \end{tikzcd} \qquad \begin{tikzcd} H^*E_nY \arrow[r, "\theta_n^*"] \arrow[d, "\tau_n^m"'] & H^*(S^1 \times Y) \arrow[d, "\tau_n^m \otimes 1"'] \\ H^*E_mY \arrow[r, "\theta_m^*"] & H^*(S^1 \times Y) \end{tikzcd}$$

There are also commutative squares

$$\begin{tikzcd} H^*E_nY \arrow[r, "\theta_n^*"] & H^*(S^1 \times Y) \\ H^*E_\infty Y \arrow[u, "q_\infty^n"] \arrow[r, "\theta_\infty^*"] & H^*Y \arrow[u, "pr_2^*"'] \end{tikzcd} \qquad \begin{tikzcd} H^*E_nY \arrow[r, "\theta_n^*"] \arrow[d, "\tau_n^\infty"'] & H^*(S^1 \times Y) \arrow[d, "\tau_n^\infty"'] \\ H^*E_\infty Y \arrow[r, "\theta_\infty^*"] & H^*Y \end{tikzcd}$$

where $\theta_\infty^* = q_\infty^0$ is the map induced by the inclusion of the fiber, and the transfer on the right hand side is given by $1 \otimes y \mapsto 0$ and $v \otimes y \mapsto y$.

Proof. Similar to the proof of [2] Proposition 4.6. □

Proposition 10.7. *Frobenius reciprocity holds for any $n \geq 0$:*

$$\tau_n^\infty(q_\infty^n(x)y) = (-1)^{|x|} x \tau_n^\infty(y).$$

Furthermore the following composition formulas hold.

$$\tau_0^\infty \circ q_\infty^0 = 0 \quad , \quad q_\infty^0 \circ \tau_0^\infty = d.$$

Proof. Similar to the proof of [2] Proposition 4.7, Proposition 4.8. □

Proposition 10.8. *There is always an inclusion $\operatorname{Im}(q_\infty^0) \subseteq \ker(d)$. If we have equality $\operatorname{Im}(q_\infty^0) = \ker(d)$ then the Leray-Serre spectral sequence of the fibration $Y \to ES^1 \times_{S^1} Y \to BS^1$ collapses at the E_3-term.*

Proof. By Proposition 10.7 we have $d \circ q^0_\infty = q^0_\infty \circ \tau^\infty_0 \circ q^0_\infty = 0$. The collapse statement follows by Proposition 10.1. $\square$

Definition 10.9. Put $\zeta_p = exp(2\pi i/p)$ and define the map

$$\begin{array}{rcl} f'_Y : S^1 \times Y & \to & ES^1 \times Y^p \\ (z,y) & \mapsto & (ze, zy, \zeta_p zy, \zeta_p^2 zy, \dots, \zeta_p^{p-1} zy). \end{array}$$

We let C_p act on the space to the left by $\zeta_p \cdot (z,y) = (\zeta_p z, y)$ and on the space to the right by $\zeta_p \cdot (e, y_1, \dots, y_p) = (\zeta_p e, y_2, \dots, y_p, y_1)$. Then the above map is C_p-equivariant. Passing to the quotients we get a map

$$f_Y : S^1/C_p \times Y \to ES^1 \times_{C_p} Y^p.$$

Note that this map is natural in Y with respect to C_p-equivariant maps.

Recall the followings facts on the order p cyclic construction [6], [5] and [7]. For any space X with homology of finite type there is a natural isomorphism

$$H^*(ES^1 \times_{C_p} X^p) \cong H^*(C_p; H^*(X)^{\otimes p})$$

where C_p acts on $H^*(X)^{\otimes p}$ by cyclic permutation with the usual sign convention. For a homogeneous element $y \in H^*X$ the C_p invariant $y^{\otimes p}$ defines an element $1 \otimes y^{\otimes p}$ in the zeroth cohomology group of C_p. Let $N = 1 + \zeta_p + \zeta_p^2 + \dots + \zeta_p^{p-1}$ be the norm element in the group ring $\mathbb{F}_p[C_p]$. If $x_1, \dots, x_p \in H^*X$ are homogeneous elements, which are not all equal, then the invariant $Nx_1 \otimes \dots \otimes x_p$ also defines an element $1 \otimes Nx_1 \otimes \dots \otimes x_p$ in the zeroth cohomology group of C_p.

Theorem 10.10. *The following formula holds where $\delta_{i,j}$ denotes the Kronecker delta:*

$$f_Y^*(1 \otimes y^{\otimes p}) = 1 \otimes y^p + v \otimes y^{p-1} dy + \delta_{p,3} v \otimes \beta\lambda y.$$

Proof. We write Y_0 for the space Y with trivial S^1-action. We first prove the theorem in the special case $Y = Y_0$. Here the differential is zero. There is a factorization

$$f_{Y_0} : S^1/C_p \times Y_0 \xrightarrow{i \times 1} ES^1/C_p \times Y_0 \xrightarrow{1 \times \Delta} ES^1 \times_{C_p} Y_0^p.$$

By this and the formula for the Steenrod diagonal, [7] p. 119 & Errata, the result follows.

Next we prove the following formula for a general S^1-space:

$$f_Y^*(1 \otimes Nx_1 \otimes \dots \otimes x_p) = v \otimes d(x_1 \dots x_p). \tag{31}$$

There is a commutative diagram as follows:

$$\begin{array}{ccc} H^*(S^1/C_p \times Y) & \xleftarrow{f_Y^*} & H^*(ES^1 \times_{C_p} Y^p) \\ {\scriptstyle \tau_0^1 \otimes 1}\uparrow & & {\scriptstyle \tau_0^1}\uparrow \\ H^*(S^1 \times Y) & \xleftarrow{f_Y'^*} & H^*(ES^1 \times Y^p) \end{array}$$

The lower horizontal map is given by

$$f_Y'^*(1 \otimes x_1 \otimes \cdots \otimes x_p) = \prod_{i=1}^{p}(1 \otimes x_i + v \otimes dx_i)$$

as seen by the factorization

$$\begin{aligned} f_Y' : S^1 \times Y \ &\xrightarrow{\Delta_2}\ (S^1 \times Y)^2 \ \xrightarrow{pr_1 \times \Delta_p}\ S^1 \times (S^1 \times Y)^p \\ &\xrightarrow{i \times \eta^p}\ ES^1 \times Y^p \ \xrightarrow{1 \times 1 \times \zeta_p \times \cdots \times \zeta_p^{p-1}}\ ES^1 \times Y^p. \end{aligned}$$

The norm class is hit by the transfer and by finding the coefficient to v in the above formula (31) follows.

Finally we prove the Theorem in the general case. Because of the degrees $f_{S^1}^*(1 \otimes v^{\otimes p}) = 0$. The two projection maps $pr_1 : S^1 \times Y_0 \to S^1$ and $pr_2 : S^1 \times Y_0 \to Y_0$ are S^1-equivariant. Thus we can use naturality together with the case $Y = Y_0$ and the above equation to find the equations below

$$\begin{aligned} &f_{S^1 \times Y_0}^*(1 \otimes (1 \otimes y)^{\otimes p}) = 1 \otimes 1 \otimes y^p + \delta_{p,3} v \otimes 1 \otimes \beta\lambda y, \\ &f_{S^1 \times Y_0}^*(1 \otimes (v \otimes 1)^{\otimes p}) = f_{S^1 \times Y_0}^*(1 \otimes (v \otimes dy)^{\otimes p}) = 0. \end{aligned}$$

The action map $\eta : S^1 \times Y_0 \to Y$ is also an S^1-equivariant map, hence by naturality we have a commutative diagram

$$\begin{array}{ccc} S^1/C_p \times (S^1 \times Y_0) & \xrightarrow{f_{S^1 \times Y_0}} & ES^1 \times_{C_p} (S^1 \times Y_0)^p \\ {\scriptstyle 1 \times \eta}\downarrow & & {\scriptstyle 1 \times \eta^p}\downarrow \\ S^1/C_p \times Y & \xrightarrow{f_Y} & ES^1 \times_{C_p} Y^p \end{array}$$

We compute the pull back of the class $1 \otimes y^{\otimes p}$ to the cohomology of the upper left corner. First we find

$$\begin{aligned} &(1 \times \eta^p)^*(1 \otimes y^{\otimes p}) = 1 \otimes (1 \otimes y + v \otimes dy)^{\otimes p} = \\ &1 \otimes (1 \otimes y)^{\otimes p} + 1 \otimes (v \otimes dy)^{\otimes p} + \sum_{i=1}^{p-1} 1 \otimes N(1 \otimes y)^{\otimes i} \otimes (v \otimes dy)^{\otimes (p-i)}. \end{aligned}$$

By (31) we can compute $f_{S^1 \times Y_0}^*$ applied to the norm element terms. Only the $i = p-1$ term contributes.

$$\begin{aligned} &f_{S^1 \times Y_0}^*(1 \otimes N(1 \otimes y)^{\otimes (p-1)} \otimes (v \otimes dy)) = v \otimes d_{S^1 \times Y_0}(v \otimes y^{p-1} dy) \\ &= v \otimes (d_{S^1}(v) \otimes y^{p-1} dy + v \otimes d_{Y_0}(y^{p-1} dy)) \\ &= v \otimes 1 \otimes y^{p-1} dy \end{aligned}$$

Altogether we have

$$\begin{aligned} (1 \otimes \eta^*) \circ f_Y^*(1 \otimes y^{\otimes p}) &= f_{S^1 \times Y_0}^* \circ (1 \times \eta^p)^*(1 \otimes y^{\otimes p}) \\ &= f_{S^1 \times Y_0}^*(1 \otimes (1 \otimes y)^{\otimes p}) + v \otimes 1 \otimes y^{p-1} dy. \end{aligned}$$

We now apply $1 \otimes \gamma^*$ on both sides (for the map γ, see the beginning of §3) and the result follows. □

11 Construction of classes in string cohomology from classes in ordinary cohomology

In this section X denotes a connected space.

Definition 11.1. Put $\zeta_p = \exp(2\pi i/p)$ and define evaluation maps as follows:

$$\begin{aligned} &ev_0 : \Lambda X \to X \quad ; \quad \gamma \mapsto \gamma(1) \quad , \\ &ev_1 : ES^1 \times_{C_p} \Lambda X \to ES^1 \times_{C_p} X^p \quad ; \quad [e,\gamma] \mapsto [e, \gamma(1), \gamma(\zeta_p), \dots, \gamma(\zeta_p^{p-1})]. \end{aligned}$$

Definition 11.2. The classes $u, f(x), g(x), \delta(x) \in H^*_{S^1}(\Lambda X)$ for $x \in H^*X$ are defined by

$$\begin{aligned} f(x) &= \tau_1^\infty \circ ev_1^*(v \otimes x^{\otimes p}), & g(x) &= \tau_1^\infty \circ ev_1^*(1 \otimes x^{\otimes p}), \\ \delta(x) &= \tau_0^\infty \circ ev_0^*(x), & u &= \tau_1^\infty \circ ev_1^*(vu \otimes 1^{\otimes p}). \end{aligned}$$

Theorem 11.3. *Let $i_0 : X \hookrightarrow \Lambda X$ denote the constant loop inclusion and let i_∞ be the corresponding map of S^1-homotopy orbits. There is a commutative diagram as follows*

$$\begin{array}{ccc} H^*(ES^1 \times_{S^1} \Lambda X) & \xrightarrow{i_\infty^*} & H^*(BS^1 \times X) \\ {\scriptstyle q_\infty^0}\downarrow & & \downarrow \\ H^*(\Lambda X) & \xrightarrow{i_0^*} & H^*(X) \end{array} \tag{32}$$

and an inclusion $\mathrm{Im}(q_\infty^0) \subseteq \ker(d : H^*(\Lambda X) \to H^*(\Lambda X))$. *The constructed classes are mapped as follows under i_∞^*.*

$$\begin{aligned} &i_\infty^*(f(x)) = \hat{\sigma}(x) St_0(x) + \sigma(x)(-1)^m m! u^m St_0(x), \\ &i_\infty^*(g(x)) = \hat{\sigma}(x) St_1(x) + \sigma(x)(-1)^m m! u^{m-1} St_1(x), \\ &i_\infty^*(\delta(x)) = 0 \quad \text{and} \quad i_\infty^*(u) = u \otimes 1. \end{aligned}$$

Here $m = (p-1)/2$. Under q_∞^0 the images of the classes are as follows.

$$\begin{aligned} &q_\infty^0(f(x)) = \hat{\sigma}(x) e(x^p), \\ &q_\infty^0(g(x)) = \hat{\sigma}(x) e(x^{p-1}dx) + \sigma(x)\delta_{p,3} e(\beta\lambda x), \\ &q_\infty^0(\delta(x)) = e(dx) \quad \text{and} \quad q_\infty^0(u) = 0. \end{aligned}$$

Here $\delta_{p,3} = 1$ for $p = 3$ and zero otherwise.

Proof. A commutative diagram of spaces gives the diagram (32) and Proposition 10.7 gives the stated inclusion.

We check the formulas involving i_∞^*. There is a commutative diagram as follows where $\Delta : X \to X^p$ is the diagonal and i_1 is the map of C_p-homotopy orbits induced by i_0.

$$\begin{tikzcd}
H^*(X) \arrow[r, "ev_0^*"] & H^*(\Lambda X) & \\
H^*(X^p) \arrow[u, "\Delta^*"] \arrow[r] \arrow[d, "Tr_0^1"'] & H^*(\Lambda X) \arrow[u, "id"'] \arrow[d, "\tau_0^1"'] & \\
H^*(ES^1 \times_{C_p} X^p) \arrow[r, "ev_1^*"] & H^*(ES^1 \times_{C_p} \Lambda X) \arrow[r, "i_1^*"] \arrow[d, "\tau_1^\infty"'] & H^*(BC_p \times X) \arrow[d, "\tau_1^\infty \otimes 1"'] \\
& H^*(ES^1 \times_{S^1} \Lambda X) \arrow[r, "i_\infty^*"] & H^*(BS^1 \times X)
\end{tikzcd}$$

The horizontal map with no label is the induced in cohomology of the map $\gamma \mapsto (\gamma(1), \gamma(\zeta_p), \ldots, \gamma(\zeta_p^{p-1}))$. A homotopy commutative square of spaces shows that the upper square commutes and it is obvious that the other two are commutative. We see that $i_\infty^*(u) = u \otimes 1$ as stated.

The composite $ev_1 \circ i_1$ is the diagonal Δ_1. Its induced in cohomology is the Steenrod diagonal Δ_1^* given by the following ([7] p. 119 & Errata):

$$\nu(q)\Delta_1^*(1 \otimes x^{\otimes p}) = \sum_i (-1)^i u^{m(q-2i)} \otimes P^i x + \sum_i (-1)^i v u^{m(q-2i)-1} \otimes \beta P^i x$$

where $q = |x|$ and $\nu(q) = (m!)^q(-1)^{m(q^2+q)/2}$. From this formula and the lower part of the diagram we see that

$$\nu(q) i_\infty^*(f(x)) = \sum_i (-1)^i u^{m(q-2i)} \otimes P^i x = (-1)^{[q/2]} u^{\sigma(x)m} St_0(x),$$
$$\nu(q) i_\infty^*(g(x)) = \sum_i (-1)^i u^{m(q-2i)-1} \otimes \beta P^i x = (-1)^{[q/2]} u^{\sigma(x)(m-1)} St_1(x).$$

By [7] Lemma 6.3 one has $(m!)^2 = (-1)^{m+1} \bmod p$ and from this one sees that $\nu(q)^{-1}(-1)^{[q/2]} = 1$ for q even and $\nu(q)^{-1}(-1)^{[q/2]} = (-1)^m m!$ for q odd. Hence we have verified the formulas for $i_\infty^*(f(x))$ and $i_\infty^*(g(x))$.

By the left part of the diagram we see that

$$\delta(x) = \tau_1^\infty \circ ev_1^* \circ Tr_0^1(x \otimes 1 \otimes \cdots \otimes 1).$$

The composite $\Delta_1^* \circ Tr_0^1$ is zero by [7] Lemma 4.1 so $i_\infty^*(\delta(x)) = 0$.

Finally we check the formulas for q_∞^0. It follows directly from Proposition 10.7 that $\delta(x)$ is mapped as stated and clearly u is mapped to zero. For the classes $f(x)$ and $g(x)$ we use Proposition 10.6:

$$q_\infty^0 \circ \tau_1^\infty \circ ev_1^* = \tau_1^\infty \circ \theta_1^* \circ ev_1^* = \tau_1^\infty \circ (ev_1 \circ \theta_1)^*.$$

Note that $ev_1 \circ \theta_1$ equals the composite

$$S^1/C_p \times \Lambda X \xrightarrow{f_{\Lambda X}} ES^1 \times_{C_p} (\Lambda X)^p \xrightarrow{1 \times ev_0^p} ES^1 \times_{C_p} X^p$$

where $f_{\Lambda X}$ is the map from Definition 10.9. Thus we have

$$q_\infty^0 \circ \tau_1^\infty \circ ev_1^* = \tau_1^\infty \circ f_{\Lambda X}^* \circ (1 \times ev_0^p)^*.$$

From this and Theorem 10.10 we get the stated results. $\square$

Proposition 11.4. *The following diagram is a pullback square.*

$$\begin{array}{ccc} H^*(ES^1 \times_{S^1} \Lambda B\mathbb{F}_p) & \xrightarrow{i_\infty^*} & \mathbb{F}_p[u] \otimes H^*B\mathbb{F}_p \\ {\scriptstyle q_\infty^0}\downarrow & & \downarrow \\ \ker(d) & \xrightarrow{i_0^*} & H^*BF_p \end{array}$$

Proof. In the proof of Proposition 4.9 we saw that $\sqcup j_n : \sqcup BF_p(n) \to \Lambda B\mathbb{F}_p$ was both an S^1-map and a homotopy equivalence. So the induced map of S^1-homotopy orbits $(\sqcup j_n)_\infty$ is a weak homotopy equivalence. The maps in the diagram have nice descriptions in terms of this equivalence since there are commutative diagrams as follows where Q denotes quotient maps.

$$\begin{array}{ccc} \sqcup ES^1 \times_{S^1} B\mathbb{F}_p(n) & \xrightarrow{(\sqcup j_n)_\infty} & ES^1 \times_{S^1} \Lambda B\mathbb{F}_p \\ {\scriptstyle \sqcup Q}\uparrow & & {\scriptstyle Q}\uparrow \\ \sqcup ES^1 \times B\mathbb{F}_p(n) & \xrightarrow{\sqcup j_n} & ES^1 \times \Lambda B\mathbb{F}_p \end{array}$$

$$\begin{array}{ccc} \sqcup ES^1 \times_{S^1} B\mathbb{F}_p(n) & \xrightarrow{(\sqcup j_n)_\infty} & ES^1 \times_{S^1} \Lambda B\mathbb{F}_p \\ \uparrow & & {\scriptstyle i_\infty}\uparrow \\ ES^1 \times B\mathbb{F}_p(0) & \xrightarrow{\cong} & BS^1 \times B\mathbb{F}_p \end{array}$$

Hence it suffices to show that the following diagram is a pullback where $d_{(n)}$ denotes the differential on $H^*B\mathbb{F}_p(n)$.

$$\begin{array}{ccc} \oplus H^*(ES^1 \times_{S^1} B\mathbb{F}_p(n)) & \xrightarrow{pr_0} & H^*(BS^1 \times B\mathbb{F}_p) \\ {\scriptstyle \oplus Q^*}\downarrow & & \downarrow \\ \oplus \ker(d_{(n)}) & \longrightarrow & H^*(B\mathbb{F}_p) \end{array}$$

We have $H^*(ES^1 \times_{S^1} B\mathbb{F}_p(n)) \cong \ker(d_{(n)})$ for $n \neq 0$ since here the Leray-Serre spectral sequence has $E_3^{i,*} = 0$ for $i \geq 1$. The result follows. $\square$

As indicated by Theorem 11.3 above it turns out that when $|x|$ is odd then both $f(x)$ and $g(x)$ can be written as a product of some power of u with another class. This was not the case for $p = 2$ as described in [2]. We construct new classes to get around this difficulty.

Theorem 11.5. *Let $x \in H^*X$ be a cohomology class of odd degree. Then there exists classes $\phi(x), q(x) \in H^*(ES^1 \times_{S^1} \Lambda X)$ with $|\phi(x)| = p(|x| - 1) + 1$ and $|q(x)| = p(|x| - 1) + 2$ such that*

$$i_\infty^*(\phi(x)) = St_0(x), \quad q_\infty^0(\phi(x)) = \lambda x - x(dx)^{p-1},$$
$$i_\infty^*(q(x)) = St_1(x), \quad q_\infty^0(q(x)) = \beta\lambda x.$$

Proof. It suffices to prove the theorem when $X = B^n\mathbb{F}_p$ for odd $n \geq 1$. The general case then follows by defining $\phi(x) = (1 \times_{S^1} \Lambda h)^*\phi(\iota_n)$ and $q(x) = (1 \times_{S^1} \Lambda h)^*q(\iota_n)$ where $|x| = n$ and $h : X \to B^n\mathbb{F}_p$ has $h^*(\iota_n) = x$.

For $n = 1$ we have $St_0(\iota_1) = 1 \otimes \iota_1$ and $St_1(\iota_1) = 1 \otimes \beta\iota_1$ so here the result follows from Proposition 11.4.

Assume that $n = 2r+1$ where $r \geq 1$. By Proposition 10.1, Theorem 4.1 and Theorem 8.6 the E_3-term of the Leray-Serre spectral sequence for the fibration $\Lambda B^n\mathbb{F}_p \to ES^1 \times_{S^1} \Lambda B^n\mathbb{F}_p \to BS^1$ has the following form:

$$E_3 \cong \mathrm{Im}(d) \oplus (\mathbb{F}_p[u] \otimes \tilde{\omega}(K))$$

where $K = H^*B^n\mathbb{F}_p$. Here u has bidegree $(2,0)$ and an element y in $\mathrm{Im}(d)$ or $\tilde{\omega}(K)$ has bidegree $(0,|y|)$. Define $s : BS^1 \to ES^1 \times_{S^1} \Lambda B^n\mathbb{F}_p$ such that $pr_1 \circ s = id$ by choosing a constant loop. By s^* we see that the vertical line $(*,0)$ survives to E_∞.

Up to dimension $2rp + 2p - 1$ the only nonzero vertical lines are $(*,0)$, $(*, 2rp+1)$, $(*, 2rp+2)$ and $(*, 2rp+2p-1)$ corresponding to the classes u, $\phi(\iota_n)$, $q(\iota_n)$ and $q(\beta\iota_n)$ respectively. Hence we can define $\phi(\iota_n)$ and $q(\iota_n)$ by

$$\begin{aligned} &q_0^\infty(\phi(\iota_n)) = \lambda\iota_n - \iota_n(d\iota_n)^{p-1}, \\ &q_0^\infty(q(\iota_n)) = \beta\lambda\iota_n \text{ and } s^*(q(\iota_n)) = 0. \end{aligned}$$

Since $|f(\iota_n)| = 2rp+p$ and $|g(\iota_n)| = 2rp+p-1$ we see that $f(\iota_n) = C_1u^m\phi(\iota_n)$ and $g(\iota_n) = C_2u^{m-1}q(\iota_n)$ where $C_1, C_2 \in \mathbb{F}_p$ and $m = (p-1)/2$ as before. By Theorem 11.3 we conclude that

$$\begin{aligned} &C_1u^m i_\infty^*(\phi(\iota_n)) = (-1)^m m! u^m St_0(\iota_n), \\ &C_2u^{m-1} i_\infty^*(q(\iota_n)) = (-1)^m m! u^{m-1} St_1(\iota_n) \end{aligned}$$

and the result follows. □

Definition 11.6. For $x \in H^*X$ of even degree we define $\phi(x) = f(x)$ and $q(x) = g(x)$.

12 String cohomology and the functor ℓ

In this section we prove the main result of this paper.

Theorem 12.1. *Let X be a connected space which has finite dimensional mod p homology in each degree. Then there is a morphism of unstable $\mathcal{A}$-algebras*

$$\psi : \ell(H^*X) \to H^*(ES^1 \times_{S^1} \Lambda X)$$

which sends $\phi(x)$, $q(x)$, $\delta(x)$ for $x \in H^*X$ and u to the constructed classes with the same names. The morphism is natural in X. If both of the maps

$$e : \omega(H^*X) \to H^*(\Lambda X) \quad , \quad \Phi : \tilde{\omega}(H^*X) \to H^*(\omega(H^*X))$$

are isomorphisms then so is ψ. In particular, when (n_i) is a sequence (possibly finite) of positive integers such that the set $\{i|n_i = N\}$ is finite for each N and $X = \prod B^{n_i}\mathbb{F}_p$, then ψ is an isomorphism.

Proof. Assume that both e and Φ are isomorphisms and put $K = H^*X$. By Theorem 7.7 we have that DR surjects $\ker(d)$ and from the results in Section 11 we see that $\mathrm{Im}(DR) \subseteq \mathrm{Im}(q_\infty^0)$. Hence $\mathrm{Im}(q_\infty^0) = \ker(d)$. It then follows from Proposition 10.8 that the Leray-Serre spectral sequence associated to the fibration $\Lambda X \to ES^1 \times_{S^1} \Lambda X \to BS^1$ collapses at the E_3-term:

$$E_\infty = E_3 \cong \ker(d) \oplus u\tilde{\omega}(K) \oplus u^2\tilde{\omega}(K) \oplus \dots \tag{33}$$

By Remark 7.4 there is a filtration of $\ell(K)$ which associated graded object is also (33). If we fix a dimension the filtration is finite and we conclude that $\ell(K)$ and $H^*(ES^1 \times_{S^1} \Lambda X)$ have the same dimension in each degree. Hence it suffices to show that the map ψ in the statement is a well defined morphism which is surjective.

The constructed classes are algebra generators for $H^*(ES^1 \times_{S^1} \Lambda X)$ by the collapse, and the formulas for their images under i_∞^* given in Section 11 shows that $\mathrm{Im}(i_\infty^*) = R(K)$. Hence we have a commutative diagram as follows:

$$\begin{array}{ccc} H^*(ES^1 \times_{S^1} \Lambda X) & \xrightarrow{i_\infty^*} & R(K) \\ {\scriptstyle q_\infty^0}\downarrow & & \downarrow{\scriptstyle p_1} \\ \ker(d) & \xrightarrow{p_2} & K \end{array}$$

The kernel of p_1 is the ideal $(u \otimes 1)$ and $i_\infty^*(u) = u \otimes 1$. Since $u \in \ker(q_\infty^0)$ and i_∞^* is surjective we conclude that the restriction $i_\infty^*| : \ker(q_\infty^0) \to \ker(p_1)$ is surjective. Hence we have a surjection into the pullback.

We now restrict to the case where X is a product of Eilenberg-MacLane spaces as in the last part of the statement. Here e is an isomorphism by Proposition 3.7 and Theorem 4.1 and Φ is an isomorphism by Proposition 7.8 and Theorem 8.6.

The above surjection into the pullback together with Theorem 9.5 gives us a surjective morphism $\psi' : H^*(ES^1 \times_{S^1} \Lambda X) \to \ell(K)$ which is then an isomorphism. By definition it has inverse ψ.

By the fact that $B^n\mathbb{F}_p$ classifies degree n cohomology and naturality of the constructed classes, we can now conclude that the defining relations for $\ell(K)$ are universal for the constructed classes. Hence ψ is a well defined morphism in general.

In the case where e and Φ are isomorphisms, the collapse ensures that ψ is surjective and hence an isomorphism. $\square$

Corollary 12.2. *Let X be a connected space and assume that H^*X is a polynomial algebra on a set of even dimensional generators. Assume also that H_iX is finite for each i. Then ψ is an isomorphism.*

Proof. If K is zero in odd degrees then $\omega(K)$ is the ordinary de Rham complex $\Omega(K|\mathbb{F}_p)$. Furthermore, $\tilde{\omega}(K)$ is the de Rham complex $\Omega(\bar{K}|\mathbb{F}_p)$ where $\bar{K}$ is the algebra defined by $\bar{K}^{np} = K^n$ and $\bar{K}^m = 0$ for $m \neq 0 \bmod p$. The map Φ is the Cartier map.

The Eilenberg-Moore spectral sequence for $H^*(\Lambda X)$ has Hochschild homology of H^*X as its E_2-term and it collapses since the algebra generators sit in $E_2^{0,*}$ and $E_2^{-1,*}$. By the Hochschild-Konstant-Rosenberger theorem Hochschild homology is isomorphic to the de Rham complex and one concludes that e is an isomorphism. The Cartier map Φ is also an isomorphism. $\square$

Remark 12.3. Theorem 12.1 respects products in the following sense. Assume that X and Y are connected spaces with mod p homology of finite type and that e_X, Φ_X, e_Y, Φ_Y are isomorphisms. Then $e_{X\times Y}$ and $\Phi_{X\times Y}$ are isomorphisms and $\psi_{X\times Y}$ is an isomorphism.

13 Appendix: Unstable $\mathcal{A}$-algebras

In this appendix we prove Proposition 3.4. We shall need the following result.

Lemma 13.1. *For any unstable $\mathcal{A}$-algebra K and $x \in K$ the following equations hold.*

$$P^i\lambda x = \begin{cases} \lambda(P^{\frac{i}{p}}x) & , \quad i = 0 \ mod \ p \\ 0 & , \quad otherwise \end{cases} \tag{34}$$

$$P^i\beta\lambda x = \begin{cases} \beta\lambda(P^{\frac{i}{p}}x) & , \quad i = 0 \ mod \ p \\ (\beta P^{\frac{i-1}{p}}x)^p & , \quad i = 1 \ mod \ p \\ 0 & , \quad otherwise \end{cases} \tag{35}$$

Proof. We just prove (34) since the proof of (35) is similar. When $|x|$ is even both sides in the equation are zero. Assume that $|x|$ is odd. By the instability condition $P^i\lambda x = 0$ when $2i > p(|x|-1)+1$. When i is divisible by p this inequality implies $2i \geq p(|x|-1)+p$ or $\frac{2i}{p} \geq |x|$ and since $|x|$ is odd $\frac{2i}{p} > |x|$. So $P^{i/p}x = 0$ and the equation holds in this case. If $2i = p(|x|-1)$ then $P^i\lambda x = \lambda^2 x = \lambda(P^{i/p}x)$.

Finally assume that $2i < p(|x|-1)$. Then we can apply the Adem relation:

$$P^iP^{\frac{|x|-1}{2}}x = \sum_{t=0}^{[\frac{i}{p}]}(-1)^{i+t}\binom{(p-1)(\frac{|x|-1}{2}-t)-1}{i-pt}P^{i+\frac{|x|-1}{2}-t}P^tx.$$

The instability condition shows that $P^{i+\frac{|x|-1}{2}-t}P^tx = 0$ unless $i \leq pt$. But the binomial coefficient is zero when $i < pt$. So we get zero when $i \neq 0$ mod p and the term corresponding to $t = i/p$ when $i = 0$ mod p. $\square$

We now prove Proposition 3.4.

Proof. Let dK denote the graded $\mathbb{F}_p$-vector space given by $(dK)^n = K^{n+1}$ and $(dK)^{-1} = 0$. We write dx for the element in dK corresponding to x in K hence $d(x+y) = dx + dy$. We define an $\mathcal{A}$-algebra structure on dK by $P^idx = dP^ix$ and $\beta dx = -d\beta x$. Let $F(dK)$ denote the free graded commutative algebra on the $\mathbb{F}_p$-vector space dK. By the Cartan formula $F(dK)$ is an $\mathcal{A}$-algebra and the graded symmetric product $K \odot F(dK)$ is an $\mathcal{A}$-algebra. By definition $\omega(K) = K \odot F(dK)/I$ where I is the ideal generated by

$$1 \odot d(xy) - d(x) \odot y - (-1)^{|x|}x \odot d(y), \tag{36}$$

$$1 \odot (d(\lambda x) - (dx)^p), \tag{37}$$

$$1 \odot d(\beta\lambda x). \tag{38}$$

We verify that $\mathcal{A} \cdot I \subseteq I$ such that $\omega(K)$ is an $\mathcal{A}$-algebra. We have

$$P^n(1 \odot d(xy) - dx \odot y - (-1)^{|x|} x \odot dy) = \\ \sum_{i+j=n} (1 \odot d(P^i(x)P^j(y)) - dP^i x \odot P^j y - (-1)^{|x|} P^i x \odot dP^j y)$$

which is in I by (36) since the degree of P^i is even. Further

$$\beta(1 \odot d(xy) - dx \odot y - (-1)^{|x|} x \odot dy) = \\ -(1 \odot d(\beta(x)y) - d\beta x \odot y - (-1)^{|\beta x|} \beta x \odot dy) \\ -(-1)^{|x|}(1 \odot d(x\beta y) - dx \odot \beta y - (-1)^{|x|} x \odot d\beta y)$$

which is also in I by (36).

In any $\mathcal{A}$-algebra one has $P^i(a^p) = (P^{i/p}a)^p$ when $i = 0 \bmod p$ and zero otherwise, since this fact is a consequence of the Cartan formula alone. So by Lemma 13.1 we have the following relation in $F(dK)$ when $i = 0 \bmod p$:

$$P^i(d(\lambda x) - (dx)^p) = d(P^i \lambda x) - (P^{\frac{i}{p}} dx)^p = d(\lambda P^{\frac{i}{p}} x) - (dP^{\frac{i}{p}} x)^p.$$

For $i \neq 0 \bmod p$ we get zero. So P^i applied to an element of the form (37) lies in I. If we apply β to such an element we also land in I by (38). Finally Lemma 13.1 shows that $P^i(1 \odot d(\beta\lambda x)) \in I$ and trivially $\beta(1 \odot d(\beta\lambda x)) \in I$.

We verify that $\omega(K) \in \mathcal{U}$. We must show that $P^i dx = 0$ if $2i > |x| - 1$. This holds if $2i > |x|$ since $K \in \mathcal{U}$. If $2i = |x|$ we have $P^i dx = dP^i x = d(x^p) = 0$. We must also show that $\beta P^i dx = 0$ when $2i + 1 > |x| - 1$. This holds if $2i + 1 > |x|$ since $K \in \mathcal{U}$ and if $2i + 1 = |x|$ we have $\beta P^i dx = -d\beta P^i x = -d\beta\lambda x = 0$. Since the action on products are by the Cartan formula we have shown that $\omega(K) \in \mathcal{U}$.

Finally we check that $\omega(K) \in \mathcal{K}$. The Cartan formula holds by definition. For $|x|$ odd we have $P^{\frac{|dx|}{2}}(dx) = d\lambda x = (dx)^p$ and the result follows. □

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