STRONGLY HOMOTOPY-COMMUTATIVE MONOIDS REVISITED

MICHAEL BRINKMEIER

ABSTRACT. We prove that the delooping, i.e. the classifying space, of a grouplike monoid is an H-space if and only if its multiplication is a homotopy homomorphism, extending and clarifying a result of Sugawara. Furthermore it is shown that the Moore loop space functor and the construction of the classifying space induce an adjunction of the according homotopy categories.

INTRODUCTION

In [Sug60] Sugawara examined structures on topological monoids, which induce H-space multiplications on the classifying spaces. He introduced a form of coherently homotopy commutative monoids, which he called *strongly homotopy commutative*. His main result is that a countable CW-group G is strongly homotopy-commutative if and only if its classifying space BG is an H-space. The proof proceeds as follows. One first shows that the multiplication $G \times G \to G$ of a strongly homotopy commutative group is a homotopy homomorphism (Sugawara called such maps strongly homotopy multiplicative), i.e. a homomorphism up to coherent homotopies. Then one shows that this map induces an H-space structure on BG. The proof of the converse is very sketchy and far from convincing.

We start with an easy to handle reformulation of the notion of homotopy homomorphisms. The well-pointed and grouplike monoids (cmp. Def. 2.4) and homotoy classes of these homotopy homomorphisms form a category $\mathcal{H}\mathbf{Gr}_H$. If $\mathfrak{Top}^*_H$ is the category of well-pointed spaces and based homotopy classes of maps, then the classifying space and the Moore loop space functors induces functors $B_H : \mathcal{H}\mathbf{Gr}_H \to \mathfrak{Top}^*_H$ and $\Omega_H : \mathfrak{Top}^*_H \to \mathcal{H}\mathbf{Gr}_H$. We first prove the following strengthening of a result of Fuchs ([Fuc65]).

Theorem (3.7). *The functor B_H is left adjoint to Ω_H.*

The adjunction induces an equivalence of the full subcategories of monoids in $\mathcal{H}\mathbf{Gr}_H$ of the homotopy type of CW-complexes and of the

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full subcategory of $\mathfrak{Top}^*_H$ of connected spaces of the homotopy type of CW-complexes.

We then reexamine Sugawara's result starting with grouplike monoids whose multiplications are homotopy homomorphisms. They give rise to H-objects (i.e. Hopf objects) in the category $\mathcal{H}\mathbf{Gr}_H$. We obtain the following extension of Sugawara's theorem.

Theorem (3.8 and 4.2). *The classifying space of a grouplike and well-pointed monoid M is an H-space if and only if M is an H-object in $\mathcal{H}\mathbf{Gr}_H$.*

As mentioned above the multiplication of a strongly homotopy commutative monoid is a homotopy homomorphism. We were not able to prove the converse and consider it an open question.

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1. The W-construction

Let **Mon** be the category of well-pointed, topological monoids and continuous homomorphisms between them. Here well-pointed means, that the inclusion of the unit is a closed cofibration.

Remark 1.1. One can functorially replace any monoid M by well-pointed one by adding a whisker (cmp. [BV68], pg 1130f.). This does not change the (unbased) homotopy type of M.

Definition 1.2. Let M and N be topological monoids. A homotopy $H_t : M \to N$ is called a *homotopy through homomorphisms* if for each $t \in I$ the map $H_t : M \to N$ is a homomorphism.

Definition 1.3. (cmp. [BV73],[Vog73],[SV86]) We define a functor $W : \mathbf{Mon} \to \mathbf{Mon}$. For $M \in \mathfrak{ob}\,\mathbf{Mon}$ the monoid WM is the space

$$WM = \coprod_{n\in\mathbb{N}} M^{n+1} \times I^n / \sim$$

with the relation

$$(x_0, t_1, x_1, \ldots, t_n, x_n) = \begin{cases} (x_0, \ldots, t_{i-1}, x_{i-1}x_i, t_{i+1}, \ldots, x_n) & \text{for } t_i = 0 \\ (x_1, t_2, \ldots, x_n) & \text{for } x_0 = e \\ (x_0, \ldots, x_{i-1}, \max(t_i, t_{i+1}), x_{i+1}, \ldots, x_n) & \text{for } x_i = e \\ (x_0, \ldots, t_{n-1}, x_{n-1}) & \text{for } x_n = e. \end{cases}$$

The multiplication is given by

$$(x_0, \ldots, t_n, x_n) \cdot (y_0, s_1, \ldots, y_k) = (x_0, \ldots, t_n, x_n, 1, y_0, s_1, \ldots, y_k).$$

A continuous homomorphism $F : M \to N$ is mapped to $WF : WM \to WN$ with

$$WF(x_0, t_1, x_1 \dots, x_n) = \big(F(x_0), t_1, F(x_1), \dots, F(x_n))\big).$$

The *augmentation* $\varepsilon_M : WM \to M$ with $\varepsilon_M(x_0, \dots, x_n) = x_0 \cdot \dots \cdot x_n$ defines a natural transformation $\varepsilon : W \to \mathrm{id}$. If $i_M : M \to WM$ is the inclusion, which maps every element x of M to the chain (x), we get $\varepsilon_M \circ i_M = \mathrm{id}_M$ and a non-homomorphic homotopy $h_t : WM \to WM$ from $i_M \circ \varepsilon_M$ to id_M, given by

$$h_t(x_0, t_1, x_1, \dots, t_n, x_n) = (x_0, tt_1, x_1, \dots, tt_n, x_n).$$

Therefore ε_M is a homotopy equivalence and M a strong deformation retract of WM at *space level*, i.e. its homotopy inverse is no homomorphism.

One of the most important properties of the W-construction is the following lifting theorem, which is a slight variation of [SV86, 4.2] and is proven in the same way.

Theorem 1.4. *Given the following diagram in* **Mon** *with* $0 \leq n \leq \infty$ *such that*

$$\begin{tikzcd} WM \arrow[rr, dotted, "H"] \arrow[dr, "F"'] & & B \arrow[dl, "L"] \\ & N & \end{tikzcd}$$

1. *M is well-pointed and*
2. *L is a homotopy equivalence.*

Then there exists a homomorphism $H : WM \to B$ and a homotopy $K_t : WM \to N$ through homomorphisms from $L \circ H$ to F. Furthermore H is unique up to homotopy through homomorphisms.

2. Homotopy homomorphisms

Definition 2.1. Let M and N be two well-pointed monoids. A *homotopy homomorphism* F from M to N is a homomorphism $F : WM \to WN$. The map $f := \varepsilon_N \circ F \circ i_M : M \to N$ is the *underlying map* of F.

Let $\mathcal{H}$**Mon** be the category whose objects are well-pointed, topological monoids, and whose morphisms are homotopy homomorphisms.

Remark 2.2. Our homotopy homomorphisms are closely related to Sugawara's approach. If we compose a homotopy homomorphism with the augmentation, we obtain a map $WM \to N$ which is, up to the conditions for the unit, a strong homotopy multiplicative map in Sugawara's sense. Since ε_N is a homotopy equivalence, the resulting structures are equivalent, after passage to the homotopy category.

The Moore loop-space construction $\Omega_M X$ and the classifying space functor B define functors $\Omega_W : \mathfrak{Top}^* \to \mathcal{H}\mathbf{Mon}$ and $B_W : \mathcal{H}\mathbf{Mon} \to \mathfrak{Top}^*$ by $\Omega_W(X) = \Omega_M X$ and $B_W(M) = B(WM)$ on objects and $\Omega_W(f) = W\Omega_M f$ and $B_W(F) = BF$ on morphisms.

For a based map $f : X \to Y$ let $[f]_*$ denote its based homotopy class. For a homomorphism F of monoids let $[F]$ denote its homotopy class with respect to homotopies through homomorphisms.

Let $\mathfrak{Top}^*_H$ be the category of based, well-pointed spaces and based homotopy classes of based spaces and $\mathcal{H}\mathbf{Mon}_H$ the category of well-pointed monoids and homotopy classes of homotopy homomorphisms.

Remark 2.3. One can prove that the homotopy homomorphisms, which are homotopy equivalences on space level, represent isomorphisms in $\mathcal{H}\mathbf{Mon}_H$.

Since Ω_W and B_W preserve homotopies, they induce a pair of functors.

$$B_H : \mathfrak{Top}^*_H \rightleftarrows \mathcal{H}\mathbf{Mon}_H : \Omega_H$$

Definition 2.4. A monoid M with multiplication μ and unit e is called *grouplike*, if there a continuous map $i : M \to M$ such that the maps $x \mapsto \mu(x, i(x))$ and $x \mapsto \mu(i(x), x)$ are homotopic to the constant map on e.

Since the Moore loop-spaces are grouplike and since this notion is homotopy invariant, an additional restriction is necessary for Theorem 3.7 to be true. Let $\mathcal{H}\mathbf{Gr}$ be the full subcategory of $\mathcal{H}\mathbf{Mon}$, whose objects are grouplike, and let $\mathcal{H}\mathbf{Gr}_H$ be the corresponding homotopy category. Then B_H and Ω_H give rise to a pair of functors

$$B_H : \mathfrak{Top}^*_H \rightleftarrows \mathcal{H}\mathbf{Gr}_H : \Omega_H.$$

We make use of a construction from [SV86]. For an arbitrary monoid M let EM be the contractible space with right M-action such that $EM/M \simeq BM$. We define a monoid structure on the Moore path space

$$P(EM; e, M) :=$$
$$\left\{(\omega, l) \in EM^{\mathbb{R}_+} \times \mathbb{R}_+ : \omega(0) = e, \omega(l) \in M, \omega(t) = \omega(l) \text{ for } t \geq l\right\}.$$

The product of two paths (ω, l) and (ν, k) is given by $(\rho, l + k)$, with

$$\rho(t) = \begin{cases} \omega(t) & \text{if } 0 \leq t \leq l \\ \omega(l) \cdot \nu(t - l) & \text{if } l \leq t \leq l + k. \end{cases}$$

The end-point projection $\pi_M : P(EM; e, M) \to M, (\omega, l) \mapsto \omega(l)$ a continuous homomorphism. Since $P(EM; e, M)$ is the homotopy fiber of the inclusion $i : M \hookrightarrow EM$ and since EM is contractible, π_M is a homotopy equivalence.

By Theorem 1.4 there exists a homomorphism $\bar{T}_M : WM \to P(EWM; e, WM)$ such that the following diagram commutes up to homotopy through homomorphisms.

$$\begin{tikzcd} WM \arrow[rr, "\bar{T}_M"] \arrow[dr, equal] & & P(EWM; e, WM) \arrow[dl, "\pi_{WM}"] \\ & WM & \end{tikzcd}$$

Because π_{WM} is strictly natural in WM, $\bar{T}_M$ is natural up to homotopy through homomorphism.

Obviously we have $P(BWM, *, *) = \Omega_M BWM$. Hence the projection $p_{WM} : EWM \to BWM$ induces a natural homomorphism $P(p_{WM}) : P(EWM; e, WM) \to \Omega_M BWM$. Because WM is grouplike, $P(p_{WM})$ is a homotopy equivalence. Therefore we obtain a homomorphism $T_M : WM \to W\Omega_M BWM$, which is induced by Theorem 1.4 and the following diagram.

$$\begin{tikzcd} WM \arrow[r, "T_M"] \arrow[d, "\bar{T}_M"'] & W\Omega_M BWM \arrow[d, "\varepsilon_{\Omega_M BWM}"] \\ P(EWM; e, WM) \arrow[r, "P(p_M)"'] & \Omega_M BWM \end{tikzcd}$$

Since all morphisms are natural up to homotopy through homomorphisms, the T_M form a natural transformation $[T]$ from $\mathrm{id}_{\mathcal{H}\mathbf{Gr}_H}$ to $\Omega_H B_H$ and each T_M is a homotopy equivalence and hence an isomorphism in $\mathcal{H}\mathbf{Gr}_H$. Its inverse $[K_M]$ can be constructed by Theorem 1.4 and the following diagram.

$$\begin{tikzcd} W\Omega_M BWM \arrow[rr, dotted, "K_M"] \arrow[dr, equal] & & WM \arrow[dl, "T_M"] \\ & W\Omega_M BWM & \end{tikzcd}$$

For each well-pointed space X, we chose E_X to be the dotted arrow in the following diagram.

$$\begin{tikzcd} BW\Omega_M BW\Omega_M X \arrow[r, "BK_{\Omega_M X}"] \arrow[d, "B\varepsilon_{\Omega_M BW\Omega_M X}"'] & BW\Omega_M X \arrow[d, "B\varepsilon_{\Omega_M X}"] \\ B\Omega_M BW\Omega_M X \arrow[d, "e_{BW\Omega_M X}"'] & B\Omega_M X \arrow[d, "e_X"] \\ BW\Omega_M X \arrow[r, dotted, "E_X"'] & X \end{tikzcd}$$

Here the $e_\bullet$ are the maps described in Proposition 5.1. Since all solid arrows, except for e_X, are based homotopy equivalences the morphism E_X exists and is uniquely determined up to based homotopy. The naturality of E_X follows from the naturality up to homotopy of all other maps. Hence we have a natural transformation $[E]_*$ from $B_H\Omega_H$ to the identity on $\mathfrak{Top}^*_H$.

Theorem 2.5. *The functor $B_H : \mathcal{H}\mathbf{Gr}_H \to \mathfrak{Top}^*_H$ is left adjoint to Ω_H. The natural isomorphism $[T]$ is the unit, and the natural transformation $[E]_*$ the counit of this adjunction.*

Proof. The definition of E_{BWM} and the naturality of several morphisms imply

$$[E_{BWM} \circ BT_M \circ e_{BWM}]_* = [e_{BWM}]_*$$

and since e_{BWM} is a based homotopy equivalence by Proposition 5.1 this results in

$$[E_{B_H(M)}]_* \circ B_H[T_M] = [E_{BWM}]_* \circ [BT_M]_* = [\mathrm{id}_{BM}]_*.$$

The definition of E_X implies

$$[W\Omega_M E_X \circ W\Omega_M e_{BW\Omega_M X} \circ W\Omega_M B\varepsilon_{\Omega_M BW\Omega_M X} \circ W\Omega_M BT_{\Omega_M X}] = [W\Omega_M e_X \circ W\Omega_M B\varepsilon_{\Omega_M X}]$$

and the naturality of several maps leads to

$$[W\Omega_M E_X \circ W\Omega_M e_{BW\Omega_M X} \circ W\Omega_M B\varepsilon_{\Omega_M BW\Omega_M X} \circ W\Omega_M BT_{\Omega_M X}] = [W\Omega_M e_X \circ W\Omega_M B\varepsilon_{\Omega_M X} \circ W\Omega_M BW\Omega_M E_X \circ W\Omega_M BT_{\Omega_M X}] .$$

Since $\varepsilon_{\Omega_M X}$ and $\Omega_M e_X$ are homotopy equivalences the homomorphisms $W\Omega_M e_X$ and $W\Omega_B \varepsilon_{\Omega_M X}$ represent isomorphisms in $\mathcal{H}\mathbf{Gr}_H$. Therefore we have

$$[W\Omega_M BW\Omega_M E_X \circ W\Omega_M BT_{\Omega_M X}] = [\mathrm{id}_{W\Omega_M BW\Omega_M X}] .$$

The facts that $T_{\Omega_M X}$ is an isomorphism in $\mathcal{H}\mathbf{Gr}_H$ and that

$$[T_{\Omega_M X} \circ W\Omega_M E_X \circ T_{\Omega_M X}] = [W\Omega_M BW\Omega_M E_X \circ W\Omega_M BT_{\Omega_M X} \circ T_{\Omega_M X}]$$

imply

$$\Omega_H[E_X]_* \circ [T_{\Omega_H(X)}] = [W\Omega_M E_X \circ T_{\Omega_M X}] = [\mathrm{id}_{W\Omega_M X}] .$$

□

3. Hopf-objects

Definition 3.1. An *H-* or *Hopf-object* (X, μ, ρ) in a monoidal category[1] $(\mathcal{C}, \otimes, e)$ is a non-associative monoid, i.e. an object X of $\mathcal{C}$ together with morphisms $\mu : X \otimes X \to X$ and $\rho : e \to X$ such that the

[1]For a definition of monoidal categories see [McL71].

following diagram commutes.

$$\begin{tikzcd} e \otimes X \arrow[r, "\rho\otimes\mathrm{id}_X"] \arrow[dr, "\simeq"'] & X \otimes X \arrow[d, "\mu"] & X \otimes e \arrow[l, "\mathrm{id}_X\otimes\rho"'] \arrow[dl, "\simeq"] \\ & X. & \end{tikzcd}$$

A morphism of H-objects (or H-morphism) $f : X \to Y$ is a morphism such that $\mu_Y \circ (f \otimes f) = f \circ \mu_X$. The H-objects of $\mathcal{C}$ and the H-morphisms form a category $Hopf\mathcal{C}$.

Proposition 3.2. *Let $(\mathcal{C}, \odot, e_{\mathcal{C}})$ and $(\mathcal{D}, \otimes, e_{\mathcal{D}})$ be monoidal categories and*

$$(F, G, \eta, \varepsilon) : \mathcal{C} \to \mathcal{D}$$

an adjunction of monoidal functors[2] *such that the diagrams*

$$\begin{tikzcd} Y \odot Y \arrow[r, "\eta_Y\odot\eta_Y"'] \arrow[d, "\eta_{Y\odot Y}"] & GFY \odot GFY \arrow[d] \\ GF(Y \odot Y) & G(FY \otimes FY) \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} FGX \otimes FGX \arrow[r] \arrow[d, "\varepsilon_X\otimes\varepsilon_X"'] & F(GX \odot GX) \arrow[d] \\ X \otimes X & FG(X \otimes X) \arrow[l] \end{tikzcd}$$

commute for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, then there exists an adjoint pair of functors

$$HopfF : HopfH\mathcal{C} \leftrightarrows Hopf\mathcal{D} : HopfG.$$

Proof. $HopfF$ is given by

$$HopfF(X, \mu, \rho) = (FX, F\mu \circ \varphi, F\rho) \text{ and } HopfF(f) = Ff,$$

with $\varphi : FX \otimes FX \to F(X \odot X)$ the natural transformation. Its adjoint $HopfG$ is given analogously. The two commutative diagrams imply that the units η_X and the counits ε_Y of the adjunction are H-morphisms. Therefore they form the unit and counit of an adjunction. □

Example 3.3. $\mathfrak{Top}^*_H$ with its product is a monoidal category. The H-objects in $\mathfrak{Top}^*_H$ are precisely the H-spaces with the base point as unit. The homotopy class $[\mu]_*$ of the multiplication is called *H-space structure* of X. H-morphisms are the homotopy classes of H-space morphisms up to homotopy.

Example 3.4. $\mathcal{H}\mathbf{Gr}_H$ has a monoidal structure $\otimes$ given on objects by $M \otimes N = M \times N$. For morphisms $F : WM \to WM'$ and $G : WN \to WN'$ we define $F \otimes G : W(M \times N) \to W(M' \times N')$ as follows: Let $S_{M,N} = (W\mathrm{pr}_M, W\mathrm{pr}_N) : W(M \times N) \to WM \times WN$ be induced by

[2] For a definition of monoidal functors see [BFSV98]

the two projections. Then the diagram

$$\begin{tikzcd}
W(M \times N) \arrow[rr, "S_{M,N}"] \arrow[dr, "\varepsilon_{M\times N}"'] & & WM \times WN \arrow[dl, "\varepsilon_M \times \varepsilon_N"] \\
& M \times N. &
\end{tikzcd}$$

commutes. Obviously $S_{M,N}$ is a homotopy equivalence. By Theorem 1.4 the homotopy class of $S_{M,N}$ in $\mathcal{H}\mathbf{Mon}$ is uniquely determined.

For two homotopy homomorphisms $F : WM \to WM'$ and $G : WN \to WN'$, we define $F \otimes G : W(M \times N) \to W(M' \times N')$ to be the lifting in the following diagram.

$$\begin{tikzcd}
W(M \times N) \arrow[r, "F\otimes G"] \arrow[d, "S_{M,N}"'] & W(M' \times N') \arrow[d, "S_{M',N'}"] \\
WM \times WN \arrow[r, "F\times G"'] & WM' \times WN'.
\end{tikzcd}$$

This construction is compatible with the composition and we can define a functor $\otimes : \mathcal{H}\mathbf{Gr}_H \times \mathcal{H}\mathbf{Gr}_H \to \mathcal{H}\mathbf{Gr}_H$ with $M \otimes N = M \times N$ and $[F] \otimes [G] = [F \otimes G]$.

The projections $[P_M]$ and $[P_N]$ on $M \otimes N$ are given by $[p_i \circ S_{M,N}]$, where p_i is the according projection from $WM \times WN$. It is easy to check that $\otimes$ and these projections form a product in $\mathcal{H}\mathbf{Gr}_H$ and that the trivial monoid $*$ is a terminal and initial object of $\mathcal{H}\mathbf{Gr}_H$. Therefore $\mathcal{H}\mathbf{Gr}_H$ is monoidal and we have a notion of H-objects in $\mathcal{H}\mathbf{Gr}_H$.

The unit of an H-object in $\mathcal{H}\mathbf{Gr}_H$ is always the unit of the underlying monoid.

Lemma 3.5. *If $(M, [F])$ is a H-object in $\mathcal{H}\mathbf{Gr}_H$, then the underlying map f of F is homotopic to the multiplication μ of M.*

Proof. The homomorphism $\bar{F} = \varepsilon_M \circ F$ has the property $[\bar{F} \circ Wi_k] = [\varepsilon_M]$ for $k = 1, 2$. The homotopy $h_t : M \times M \to M$ with $h_t(x, y) = \bar{F}\big((x, e), t, (e, y)\big)$ runs from $f(x, y)$ to $f(x, e)f(e, y)$, and hence f and μ are based homotopic. □

Thus the multiplication μ of an H-object $(M, [F])$ in $\mathcal{H}\mathbf{Gr}_H$ is homotopic to the underlying map of F, and therefore homotopy-commutative with the commuting homotopy from xy to yx derived from $F\big((e, y), t, (x, e)\big)$. The relations in $W(M \times M)$ define higher homotopies so that the underlying monoid is homotopy commutative in a strong sense.

We now want to examine the structure on a monoid M, that leads to the existence of an H-space multiplication on its classifying space.

Proposition 3.6. *B_H and Ω_H are monoidal functors.*

Proof. For $M, N \in \mathcal{H}\mathbf{Gr}_H$ the morphism

$$s_{M,N} : BW(M \times N) \to BWM \times BWN$$

is given by the based homotopy equivalence (BWp_1, BWp_2), where $p_1, p_2 : M \times M \to M$ are the projections.

For $X, Y \in \mathfrak{Top}_H^*$ the morphism $\Omega_H(X \times Y) \simeq \Omega_H X \otimes \Omega_H Y$ is given by $W(\Omega_M p_1, \Omega_M p_2) : W\Omega_M(X \times Y) \to W(\Omega_M X \times \Omega_M Y)$. □

Theorem 3.2 now implies

Theorem 3.7. *B_H and Ω_H induce an adjunction*

$$HopfB_H : Hopf\mathcal{H}\mathbf{Gr}_H \leftrightarrows Hopf\mathfrak{Top}_H^* : Hopf\Omega_H$$

with

$$HopfB_H(M, [F]) = (BWM, [BF \circ s_{M,M}]_*)$$

and

$$Hopf\Omega_H(X, [\mu]_*) = (\Omega_M X, [W\Omega_M \mu \circ R_{X,X}]).$$

Theorem 3.8. *The classifying space BM of a grouplike and well-pointed monoid M is an H-space if and only if M is an H-object in $\mathcal{H}\mathbf{Gr}_H$.*

Proof. If M is an H-object, then BWM and thus BM are H-spaces.

Now let BM be an H-space. Then $\Omega_M BWM$ is an H-object in $Hopf\mathcal{H}\mathbf{Gr}_H$. Since $T_M : WM \to W\Omega_M BWM$ is a homotopy equivalence, M is an H-object, too. □

4. Extensions

A monoid in $Hopf\mathfrak{Top}_H^*$ is a homotopy-associative H-space (X, μ). A monoid $Hopf\mathcal{H}\mathbf{Gr}_H$ consists of a well-pointed and grouplike monoid together with homotopy homomorphisms $F_2 : W(M \times M) \to WM$ and $F_3 : W(M \times M \times M) \to WM$ such that $(M, [F_2])$ is an H-object and

$$[F_2 \circ (F_2 \otimes \mathrm{id})] = [F_3] = [F_2 \circ (\mathrm{id} \otimes F_2)].$$

We call the H-object $(M, [F_2])$ *associative.*

Since these structures are invariant under isomorphisms we obtain, similar to the non-associative case, the following

Theorem 4.1. *The classifying space BM of a well-pointed, grouplike monoid M is an homotopy associative H-space, if M is an associative H-object in $\mathcal{H}\mathbf{Gr}_H$.*

As we realized earlier, the morphism $e_X : B\Omega_M X \to X$ need not be a homotopy equivalence. But by Proposition 5.1 $\Omega_M e_X$ is a based homotopy equivalence. Hence, if we restrict to connected, based spaces of the homotopy type of CW-complexes, e_X is a homotopy equivalence.

This implies that the adjunction

$$B_H : \mathcal{H}\mathbf{Gr}_H \leftrightarrows \mathfrak{Top}_H^* : \Omega_H$$

induces an equivalence of categories, if we restrict to the full subcategories of based spaces of the homotopy type of connected CW-complexes and grouplike monoids of the homotopy type of CW-complexes.

Theorem 4.2. *The full subcategories $Hopf\mathcal{H}\mathbf{Gr}_H^{CW} \subset Hopf\mathcal{H}\mathbf{Gr}_H$ of H-objects of the homotopy type of CW-complexes, and $Hopf\mathfrak{Top}_H^{*,CW} \subset Hopf\mathfrak{Top}_H^*$ of connected H-spaces of the homotopy type of CW-complexes, are equivalent.*

5. Appendix: The evaluation map

This section is dedicated to the proof of the following theorem.

Proposition 5.1. *For each based space X there exists a natural map $e_X : B\Omega_M X \to X$ such that*

1. *$\Omega_M e_X$ is a homotopy equivalence for each based space X and*
2. *if M is a grouplike wellpointed monoid then e_{BM} is a homotopy equivalence.*

To prove this we will use based simplicial spaces. A *based simplicial space* is a functor from the dual of the category Δ of finite, ordered sets $[n] = \{0, 1, \ldots, n\}$ to $\mathfrak{Top}_*$. The *based standard simplices* $\nabla_*(n)$ are given by the quotient space $\nabla(n)/V_n$ with $\nabla(n)$ the n-th standard simplex and V_n its subspace of vertices. They induce a based cosimplicial space $\nabla_* : \Delta \to \mathfrak{Top}_*$.

We define the *based geometric realization* of a based simplicial space $\mathfrak{X}$ as

$$|\cdot|_* = \coprod_n \mathfrak{X}(n) \wedge \nabla_*(n)/\sim$$

with the relation $\sim$ generated by the same equalities as in the unbased case. This induces a functor $|\cdot|_*$ from the category of based simplicial spaces to $\mathfrak{Top}_*$.

Analogous to the unbased singular complex we can define the *based singular complex* $S_*X : \Delta^{op} \to \mathfrak{Top}_*$ of a based space X by

$$[n] \mapsto \mathfrak{Top}_*(\nabla_*(n), X).$$

S_* induces a functor from $\mathfrak{Top}_*$ to the category of based simplicial sets. As in the unbased case this right adjoint to the based realization $|\cdot|_*$. The unit $\tau_* : \mathrm{id} \to S_*|\cdot|_*$ is given by

$$\tau_{*,\mathfrak{X}}(x) = (t \mapsto (x,t)), \quad x \in \mathfrak{X}_n, t \in \nabla_*(n)$$

and the counit $\eta_* : |S_* \cdot|_* \to \mathrm{id}$ by

$$\eta_{*,X}(\omega, t) = \omega(t), \quad \omega \in S_*Y(n), t \in \nabla_*(n).$$

Definition 5.2. (cmp. [Seg74, A.4.]) A based simplicial space $\mathfrak{X}$ is *good* if for each n and $0 \le i \le n$ the inclusion $s_i(\mathfrak{X}_{n-1}) \hookrightarrow \mathfrak{X}_n$ is a closed cofibration.

Now observe that the based realization $|\mathfrak{X}|_*$ coincides with the unbased realization $|\mathfrak{X}|$ if the simplicial space $\mathfrak{X}$ has only one 0-simplex. Therefore we obtain the following lemma from well-known facts.

Lemma 5.3. *(cmp.* [Seg74, A.1]*) Let $\mathfrak{X}$ and $\mathfrak{Y}$ be good, based simplicial spaces with $\mathfrak{X}_0 = * = \mathfrak{Y}_0$ and let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a based simplicial map. If each map $\mathfrak{f}_n$ is a based homotopy equivalence, then the map*

$$|\mathfrak{f}|_* : |\mathfrak{X}|_* \to |\mathfrak{Y}|_*$$

is a based homotopy equivalence.

In the following we will show that the nerve $\Omega_M^\bullet X$ of the Moore loop space of an arbitrary wellpointed space X is homotopy equivalent to its based simplicial complex. There exists a based simplicial map $a : \Omega_M^\bullet X \to S_* X$, given by

$$a_n(\omega_1, \dots, \omega_n)(t_0, \dots, t_n) = (\omega_1 + \dots + \omega_n)\left(\sum_{i=1}^{n} \sum_{j=1}^{i} t_i l_j\right)$$

(l_j is the length of the loop ω_j and $+$ the loop addition). Let $\mathfrak{e}_j = (t_0, \dots, t_n)$ be the vertex of $\nabla(n)$ given by $t_j = 1, t_k = 0, k \neq j$. Then a maps the loop ω_j to the edge running from $\mathfrak{e}_{j-1}$ to $\mathfrak{e}_j$.

$E_n := \{(t_0, \dots, t_n) \in \nabla(n) : t_i + t_{i+1} = 1 \text{ for some } i\}$ is a strong deformation retract of $\nabla(n)$ and there exists a sequence of homotopy equivalences

$$\mathfrak{Top}_*(\nabla_*(n), X) \simeq \mathfrak{Top}_*(E_n, X) \simeq (\Omega X)^n \simeq (\Omega_M X)^n$$

such that the composition of a with these maps is the endomorphism of $(\Omega_M X)^n$ which changes the length of the loops to length 1. This map is homotopic to the identity, and hence a is a homotopy equivalence. Furthermore a is natural in X and defines a natural transformation from $\Omega_M^\bullet$ to S_*. If X and hence $\Omega_M X$ and $\mathfrak{Top}_*(\nabla_*(n), X)$ are wellpointed, then a_X is a based homotopy equivalence.

The map $e_X := \eta_{*,X} \circ |a_X|_* : |\Omega_M^\bullet X|_* \to X$ is natural in X and therefore induces a natural transformation from $|\Omega_M^\bullet \cdot|_*$ to id. Since $\Omega_M^\bullet$ is the nerve of a topological monoid, e is in fact a natural transformation from $B\Omega_M$ to $\mathrm{id}_{\mathfrak{Top}_*}$.

By [Seg74, 1.5] the canonical map $\tau_{\Omega_M X} : \Omega_M X \to \Omega B \Omega_M X$ with $\tau_{\Omega_M X}(\omega)(t) = (\omega; 1-t, t)$ is a homotopy equivalence because $\Omega_M X$ is grouplike. The composition $\Omega e_X \circ \tau_{\Omega_M X} : \Omega_M X \to \Omega X$ is the map normalizing the loops to length 1 and hence a homotopy equivalence. Therefore Ωe_X is a homotopy equivalence. Since the maps $\Omega_M X \to \Omega X$ are natural in X, this implies the first statement of Proposition 5.1.

Let M be a wellpointed grouplike monoid. Using the adjunction of the based realization and the based singular complex functors, we obtain a sequence

$$BM = |M^\bullet|_* \xrightarrow[|\tau_{*,M}|_*]{} |S_* BM|_* \xrightarrow[\eta_{*,BM}]{} |M^\bullet|_* = BM$$

The map $\eta_{*,BM} \circ |\tau_{*,M^\bullet}|_*$ is the identity. $S_*BM(1)$ is precisely the non-associative loop space ΩBM and, by [Seg74, 1.5], the map $\tau_{*,M^\bullet}$ is a homotopy equivalence on the 1-simplices. Furthermore $S_*BM(n)$ is based homotopy equivalent to $(\Omega_M BM)^n$ and $S_*BM(n)$ is special, i.e. it satisfies the conditions of [Seg74, 1.5]. Therefore $\tau_{*,M^\bullet}$ is a based homotopy equivalence in each dimension and thus $|\tau_{*,M^\bullet}|_*$ and $\eta_{*,BM}$. Since $|a_{BM}|_*$ is a based homotopy equivalence this implies the second statement of Proposition 5.1.

References

[BFSV98] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, and R. Vogt. Iterated monoidal categories II. preprint 98–035, Universität Bielefeld, 1998.

[BV68] J.M. Boardman and R.M. Vogt. Homotopy-everything H-spaces. *Bull. Amer. Math. Soc.*, 74:1117–1122, 1968.

[BV73] J.M. Boardman and R.M. Vogt. *Homotopy Invariant Algebraic Structures on Topological Spaces*, volume 347 of *Lecture Notes in Mathematics*. Springer, Berlin, 1973.

[Fuc65] M. Fuchs. Verallgemeinerte Homotopie-Homomorphismen und klassifizierende Räume. *Math. Annalen*, 161:197–230, 1965.

[McL71] S. McLane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer, Berlin, 1971.

[Seg74] G. Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

[Sug60] M. Sugawara. On the homotopy-commutativity of groups and loop spaces. *Mem. Coll. Sci. Univ. Kyoto, Ser. A*, 33:257–269, 1960.

[SV86] R. Schwänzl and R.M. Vogt. *Coherence in Homotopy Group Actions*, volume 1217 of *Lecture Notes in Mathematics*, pages 364–390. Springer, Berlin, 1986.

[Vog73] R.M. Vogt. Homotopy limits and colimits. *Math. Z.*, 134:11–52, 1973.