THE PARABOLIC SUBGROUPS OF EXCEPTIONAL ALGEBRAIC GROUPS WITH A FINITE NUMBER OF ORBITS ON THE UNIPOTENT RADICAL

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ABSTRACT. Let G be a simple algebraic group and P a parabolic subgroup of G. The group P acts on the Lie algebra $\mathfrak{p}_u$ of its unipotent radical P_u via the adjoint action. We classify all parabolic subgroups P of exceptional algebraic groups with a finite number of orbits on $\mathfrak{p}_u$. This is achieved by means of an algorithmic procedure. Combined with the solution of this problem for all classical instances from [5] this gives a complete classification of all such finite orbit cases.

1. Introduction

Throughout, G is a simple algebraic group over an algebraically closed field k, and $\operatorname{char} k$ is either zero or a good prime for G. Let P be a parabolic subgroup of G. We consider the adjoint action of P on the Lie algebra $\mathfrak{p}_u$ of its unipotent radical P_u. By saying that P is of a particular type, we mean the Dynkin type of a Levi subgroup of P. The class of nilpotency of P_u is denoted by $\ell(P_u)$.

The main result of this note is

Theorem 1.1. *Suppose G is of exceptional type. Then P acts on $\mathfrak{p}_u$ with a finite number of orbits if and only if one of the following holds:*

(i) $\ell(P_u) \leq 4$;
(ii) *G is of type E_6, $\ell(P_u) = 5$, and P is of type $A_1^2A_2$ or A_3;*
(iii) *G is of type E_7, $\ell(P_u) = 5$, and P is of type A_1A_4.*

Theorem 1.1 combined with the analogous result for classical groups [5] completes the classification of parabolic subgroups P of reductive groups with a finite number of orbits on $\mathfrak{p}_u$; this problem was first posed in [9].

These finiteness results may be viewed in context of the more general concept of the *modality of the action of P on $\mathfrak{p}_u$*, or simply the *modality of P*, denoted by $\operatorname{mod} P := \operatorname{mod}(P : \mathfrak{p}_u)$, which is the maximal number of parameters upon which a family of P-orbits on $\mathfrak{p}_u$ depends. Observe

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that $\operatorname{mod} P$ is zero precisely when P operates on $\mathfrak{p}_u$ with a finite number of orbits.

The basic machinery for investigating the modality of parabolic subgroups of reductive groups was introduced in [9]. Apart from the classification of the finite cases for classical groups from [5], the finite instances for Borel and semisimple rank one parabolic subgroups were classified in [8] and [9], respectively. Partial results for exceptional groups were obtained in [6], [9], and [10].

In the cases when the Dynkin diagram of G is simply laced the finiteness statements of Theorem 1.1 were obtained by means of an algorithmic procedure which allows one to determine an upper bound for the modality of the action of P on $\mathfrak{p}_u$. This program, referred to as MOP (Modality Of Parabolics), is available as a GAP share package, cf. [4]. For a description of this algorithm, its mathematical background, further applications, as well as examples, we refer to [7]. MOP's application is limited to the instances when the Dynkin diagram of G is simply laced. The computational results obtained in [6] are based on a precursor of MOP.

2. Preliminaries

We briefly recall the notion of the modality of a group action. Suppose that the connected algebraic group R acts morphically on the algebraic variety X. For x in X the R-orbit in X through x is denoted by $R \cdot x$. The *modality of the action of R on X* is defined as

$$\operatorname{mod}(R : X) := \max_{Z} \min_{z \in Z} \operatorname{codim}_Z R \cdot z,$$

where Z runs through all irreducible R-invariant subvarieties of X. In case X is an irreducible variety let $k(X)^R$ denote the field of R-invariant rational functions on X. By a result due to M. Rosenlicht $\min_{x \in X} \operatorname{codim}_X R \cdot x = \operatorname{trdeg} k(X)^R$. Therefore, $\operatorname{mod}(R : X)$ measures the maximal number of parameters upon which a family of R-orbits on X depends. The modality of the action of R on X is zero precisely when R admits only a finite number of orbits on X.

We require some basic facts concerning modality in our context; the first of which is elementary (cf. [9, Lem. 4.3], or [10, Lem. 2.8]):

Lemma 2.1. *Let $Q \subseteq P$ be parabolic subgroups of G. Then* $\operatorname{mod} P \leq \operatorname{mod} Q$.

This follows readily from the definition, since $\mathfrak{p}_u \subseteq \mathfrak{q}_u$ and any irreducible P-invariant subvariety of $\mathfrak{p}_u$ is also Q-invariant.

For an automorphism Θ of G we denote the set of fixed points by G^Θ, likewise for Θ-stable subgroups of G. We recall [11, Thm. 1.1] (cf. [9, Cor. 2.8]):

Lemma 2.2. *Suppose that Θ is a semisimple automorphism of G and that P is Θ-stable. Then* $\operatorname{mod} P^\Theta \leq \operatorname{mod} P$.

Since $\operatorname{char} k$ is assumed to be zero or a good prime for G, we have $\operatorname{mod}(P : P_u) = \operatorname{mod}(P : \mathfrak{p}_u)$, thanks to [11, Thm. 1.3]; thus we obtain the finiteness statement of Theorem 1.1 also for the action of P on P_u.

3. Proof of Theorem 1.1

We combine the exceptional cases from [10, Thm. 3.1] and [6, Lem. 3.13]:

Proposition 3.1. *Suppose G is of exceptional type. Then P acts on $\mathfrak{p}_u$ with an infinite number of orbits provided one of the following holds:*

(i) *G is of type E_8, F_4, or G_2 and $\ell(P_u) \geq 5$;*
(ii) *G is of type E_6 or E_7 and $\ell(P_u) \geq 6$;*
(iii) *G is of type E_6, $\ell(P_u) = 5$, and P is not of type $A_1^2A_2$ or A_3;*
(iv) *G is of type E_7, $\ell(P_u) = 5$, and P is not of type A_1A_4.*

Proof of Theorem 1.1. It follows from Proposition 3.1 that $\operatorname{mod} P > 0$ provided none of the conditions of Theorem 1.1 is satisfied.

In each of the cases of Theorem 1.1 when the Dynkin diagram of G is simply laced the desired finiteness statements were obtained directly using MOP. The classification of modality zero parabolics in G_2 already follows from [2, Table 2] and [9, Thm. 4.2].

Thus, only the instances of F_4 remain. Let G be of type E_6 and let τ be the graph automorphism of G of order 2. The fixed point subgroup G^τ is of type F_4. Let Q be a parabolic subgroup of G^τ. Then, $\operatorname{mod} Q > 0$ provided $\ell(Q_u) \geq 5$, by Proposition 3.1(i). In order to show the converse it suffices to prove that $\operatorname{mod} Q = 0$ provided Q is minimal with respect to satisfying $\ell(Q_u) \leq 4$, by Lemma 2.1. This leads to the three instances when Q is of type B_2, $A_1\widetilde{A}_2$, or $\widetilde{A}_1A_2$, where $\widetilde{A}_i$ represents a subsystem of type A_i consisting of short roots. Each such Q can be realized as the τ-fixed point subgroup of a parabolic subgroup P of G; see Figure 1 below. Each occurring P satisfies $\ell(P_u) \leq 4$ and thus $\operatorname{mod} P = 0$ by the finiteness result for E_6. The desired result for F_4 then follows by Lemma 2.2. □

Figure 1 presents the crucial F_4 cases from the proof of Theorem 1.1. The solid nodes indicate the Levi subgroup of P and P^τ, respectively.

Owing to the Chevalley commutator relations $\ell(P_u)$ is readily determined to be the sum of the coefficients of the simple roots α in the highest root of G such that $\mathfrak{g}_\alpha \subseteq \mathfrak{p}_u$, as indicated in Figure 1.

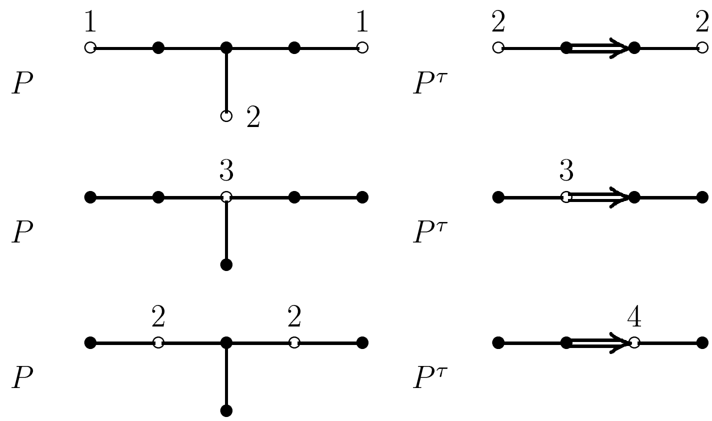

FIGURE 1

We conclude with two *a posteriori* consequences of the classification of modality zero parabolic groups.

Remark 3.2. Suppose P and Q are associated parabolic subgroups of G. Then $\operatorname{mod} P = 0$ if and only if $\operatorname{mod} Q = 0$. This follows from [5, Thm. 1.1] and Theorem 1.1 using the classification of the conjugacy classes of parabolic subsystems of Ψ (cf. [1, Prop. 6.3], [3, Thm. 5.4]).

This supports the conjecture that more generally $\operatorname{mod} P = \operatorname{mod} Q$ whenever P and Q are associated parabolic subgroups of G.

Remark 3.3. Suppose that $\operatorname{mod} P > 0$. Then there exists a connected simple regular subgroup H of G such that the parabolic $Q := H \cap P$ of H is the standard Borel subgroup of H and $\operatorname{mod} P \geq \operatorname{mod} Q > 0$. More specifically, H can always be chosen to be of type A_5, B_3, C_3, D_4, or G_2. This is a consequence of the inductive construction of all the cases when P is of positive modality, see [10, Thm. 3.1], [6, Lem. 3.13], and [5, Lem. 2.3].

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