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A COMBINATORIAL APPROACH TO COMPLEXITY THEORY VIA ORDINAL HIERARCHIES

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§ 0 Introduction

For many mathematicians the most noble activity lies in proving theorems. It must have come as a blow for them when Gödel [Gö 31] showed that there are unprovable theorems. At the beginning they still could find some consolation in hoping that such culprits might occur in Peano arithmetics through esoteric diagonalization arguments only. Nowadays there is a wealth of most natural valid theorems which can be stated in the language of finite combinatorics but are not provable within that system.

Mathematicians understand to a certain extent how to find unprovable theorems and how to prove their unprovability within a formal system. In that sense we are relying on the classical work by Gentzen [Ge 36], Kreisel [Kr 52] and Wainer [Wa 72]. Moreover we shall apply their beautiful ideas to something which seems to be well understood, viz to well quasi orderings. This is an old concept found in Gordan [Go 1885], and Kruskal [Kr 72] correctly pointed out that it was "a frequently discovered concept". That is why we are not reinventing it and are well aware that any sequence (s_i) of specialists starting with the author must contain an arbitrary long subsequence of experts knowing more than s_0, a fact, which gives a nice theme for this paper. Leeb was one of the first dealing with structural problems of $wqo's$, which are related to this paper [Le 73]. Beautiful ideas of P. Erdös are valuable for the analysis of such phenomena occuring in all well quasi orders. For related combinatorial questions we would also like to draw the reader's attention to the beautiful paper of Nešetřil and Loebl cf [LN 91].

§ 1 How to use complexity theory

We are interested in first order statements $\forall x \exists y A(x,y)$ in the language of Peano arithmetics where A is primitive recursive. Let $g(x)$ be the smallest y satisfying $A(x,y)$. We are interested in the question of whether g is defined for every x . Let us anticipate the answer, which has been known for a long time: If g grows fast enough then the statement " g is defined everywhere" is not provable within Peano arithmetics.

In order to specify growth rates in complexity theory define a hierarchy of reference functions. There are various hierarchies available and depending on the combinatorial problems and personal taste one can make a choice. Here we concentrate on the Wainer-Grzegorczyk hierarchy, cf [Gr 53] and [Wa 72].

The first few functions are defined as follows

$$\begin{aligned} f_0(n) &= n+1 \\ f_{i+1}(n) &= f_i \circ \ldots \circ f_i(n), \text{ where the iteration is } n \text{ fold, and finally} \\ f_\omega(n) &= f_n(n) \text{ is the Ackermann function defined by diagonalization.} \end{aligned}$$

The first few levels are well known: f_0 grows like the identity, f_1 linearly, f_2 exponentially, f_3 is the tower function; f_4 is sometimes called the "wow"-function [GRS 90], ...

Using the Cantor Normal form to define fundamental sequences representing ordinals there is no difficulty extending the hierarchy up to f_{ε_0} , for instance

$$\begin{aligned} f_{\omega+i+1}(n) &:= f_{\omega+i} \circ \cdots \circ f_{\omega+i}(n) \quad n\text{-times} \\ f_{\omega+\omega}(n) &:= f_{\omega+n}(n) \quad \text{diagonalization} \\ f_{\omega^\omega}(n) &:= f_{\omega^n}(n) \quad \text{diagonalization} \\ f_{\varepsilon_0}(n) &:= f_{\omega^{\cdot^{\cdot^{\omega}}}}(n) \quad \text{diagonalization with the } \omega- \text{ tower of height } n. \end{aligned}$$

For details cf [Wa 72].

One can measure complexity with respect to these reference functions by defining

$$(*) \begin{cases} g > h & \text{iff } \lim\limits_{n\to\infty} \dfrac{h(n)}{g(n)} = 0 \\ \text{and } g \sim f_\alpha & \text{iff } \alpha \text{ is the smallest ordinal with } f_{\alpha+1} > g. \end{cases}$$

One should be a aware that this complexity measure is fairly insensitive to small changes but, as we shall see, it will allow rather clean-cut statements on combinatorial complexity.

Theorem (Kreisel). *Let $A(x,y)$ be a primitive recursive formula in the language of Peano arithmetics and $g(x)$ be the smallest witness y for $A(x,y)$. If $g > f_{\varepsilon_0}$ or $g \sim f_{\varepsilon_0}$.* then " g is defined for all x " is not provable in Peano arithmetic.

This theorem demonstrates that it might be useful to understand complexity theory with respect to such hierarchies. From the point of view of nonprovability in Peano arithmetic only certain reference functions like f_{ε_0} are of interest but we shall see that the other levels of complexity occur in rather natural contexts too. Here we concentrate on surveying some of these results and give examples for combinatorial problems which correspond to various levels.

§ 2 Regressive sequences in wqo 's

Recall that a *well quasi ordering* is a poset $(A, \leq)$ that contains no infinite antichains and no infinite strictly descending sequence; thus any infinite sequence of elements of A must contain an infinite weakly ascending subsequence.

Let $(A, \leq)$ be a wqo -set with an obvious ranking r defined by successively taking minimal elements. Call a sequence $(a_0, a_1, \dots)$ **regressive** iff $r(a_i) \leq i$ for every $i \in \omega$.

Theorem 2.1. *Let $(A, \leq)$ be ωqo . Then there exists a function $H_{(A,\leq)} : \omega \to \omega$ such that every regressive sequence $(a_0, \dots, a_{H(n)})$ contains a weakly ascending subsequence with n terms.*

Harzheim proved this for $(\mathbb{N}, \leq)$ cf [Ha 67] and $(\mathbb{N}^d, \leq)$ cf [Ha 82]. The general version might be folklore. The following proof should be known to all specialists. It came to the authors mind when teaching on fixed point theorems in compact spaces.

Proof. Consider the space $\boldsymbol{S}$ of regressive ω -sequences over A . Finite sequences should be filled up with minimal elements. Thus with

$$\boldsymbol{R}_i = \{x \in A | r(x) \leq i\} \text{ one has } \boldsymbol{S} = \Pi_\omega \boldsymbol{R}_i.$$

As a product of the finite sets $\boldsymbol{R}_i$ the space $\boldsymbol{S}$ is compact in the Tychonoff topology, and as a metric space it is also sequentially compact.

Assuming that the theorem fails, pick a wqo -set $(A, \leq)$ and an $n \in \omega$ such that for every $h \in \omega$ there exists a regressive "bad" sequence $\boldsymbol{a}(h) = (a_0^{(h)}, \dots, a_h^{(h)})$ i.e. a h -term sequence not containing any n term ascending subsequence. Thus the sequence $(\boldsymbol{a}^{(h)})_{h \in \omega}$ has an accumulation point $\boldsymbol{a} \in \boldsymbol{S}$. As A is wqo it follows that $\boldsymbol{a}$ must contain an infinite weakly ascending subsequence. So it contains a weakly ascending subsequence $\boldsymbol{a}'$ with n terms. Of course $\boldsymbol{a}'$ is contained in an initial segment of $\boldsymbol{a}$, the accumulation point. Thus it is contained in an initial segment of some $\boldsymbol{a}^{(h)}$, yielding the desired contradiction. $\square$

Of course one could also use König's infinity lemma for a proof. We do not know whether the theorem can be generalized. To start with finite sets R_i in order to have a compact S does not seem to be the most general idea, cf [NR 90].

In this paper we are going to explore the complexity of $H_{(A,\leq)}$ for various posets $(A,\leq)$. For some of the most natural and commonly occuring wqo's the Wainer-Grzegorczyk hierarchy seems to be quite adequate for neat results.

§ 3 Low complexity levels, product of chains

Harzheim [Ha 67] established the following

Theorem 3.1. $H_{(\mathbb{N},\leq)}(n) = 2^{n-1}$. *Thus* $H_{(\mathbb{N},\leq)} \sim f_1$.

Proof. In order to establish the result we proceed in the framework of complexity theory and show

i) the upper bound $H(n) \leq 2^{n-1}$

ii) the lower bound $H(n) \geq 2^{n-1}$.

For (i) we make use of beautiful ideas of [ES 35].

Let (a_i) $i = 1,\ldots,H(n)$ be a regressive sequence of positive integers. So far $H(n)$ is unknown and we want to show $2^{n-1} \geq H(n)$. Define a mapping

$$\ell : \{1,\ldots,2^{n-1}\} \to \{1,\ldots,2^{n-1}\} \quad \text{by}$$

$$\ell(i) = \begin{cases} \text{length of a weakly ascending sequence} \\ \text{of maximal length with first element } a_i. \end{cases}$$

Case α . There is an i with $\ell(i) \geq n$.

Obviously this shows that an ascending subsequence of length n exists.

Case β . $\ell(i) < n$ for all i.

Thus ℓ may be viewed as a coloring of $\{1,\ldots,2^{n-1}\}$ with at most $n-1$ colors.

Definition. A subset X of $\mathbb{N}$ is called **large** iff $|X| > \min X$.

By the pigeon hole principle there is a large subset $X = \{i_1,\ldots,i_{i_1+1}\} \subseteq \{1,\ldots,2^{n-1}\}$ which is monochromatic for a certain ℓ . By definition each of the following elements

$$a_{i_1},\ldots,a_{i_{i_1}+1} \qquad (\,*\,)$$

is the starting point for a weakly ascending subsequence of maximal length ℓ . Therefore (*) has to be a strongly descending sequence. (In order to see $a_{i_1} > a_{i_2}$ suppose that, on the contrary $a_{i_1} \leq a_{i_2}$. Then a longest sequence starting at a_{i_2} could be extended by a_{i_1} yielding a longest ascending sequence of length $\ell + 1$). The length of $(*)$ is $i_1 + 1$ and its first element has rank $\leq i_1$, a contradiction.

It remains to be shown that 2^{n-1} is such that it allows the application of the pigeon hole principle. Color $\{1,\ldots,a-1\}$ with $n-1$ colors in such a way that no large subset occurs monochromatically. Observe that $f : \{1, \ldots, 7\} \to \{1, 2, 3\}$ defined by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ f_1 & f_2 & f_2 & f_3 & f_3 & f_3 & f_3 \end{pmatrix}$$

is a coloring such that every extension to $8 = 2^3$ would yield either a large set or need a new color. It is easy to see that any coloring of $\{1, \ldots, 7\}$ in which the colors do not occur successively either already contains a large set or can be rearranged to the above example, showing that the greedy strategy yields $a = 2^{n-1}$ as an upper bound for $H(n)$.

As for the lower bound (ii) we give an explicit regressive sequence $\boldsymbol{a}$ of length $2^{n-1} - 1$ without weakly ascending subsequence of length n : $\boldsymbol{a} = (0103210)$. $\square$

$n = 4$:

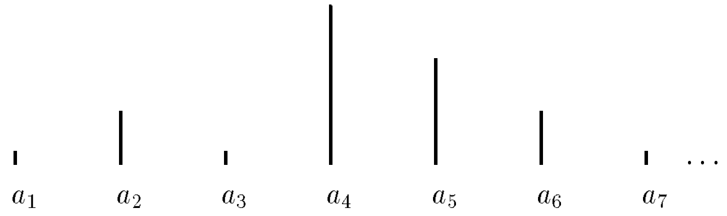


Figure 1.

Another possibility for establishing upper bounds which turned out to be useful in more complicated situations employs a tree argument, a beautiful idea occuring in [EHMR 84].

Given a regressive sequence $\boldsymbol{a}$ defined on the first few, say a, integers we recursively construct a sequence of binary trees $T_1, \ldots, T_j, \ldots, T_a$, in which

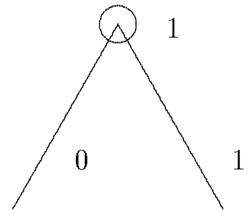
– the internal nodes are labelled by $1, \ldots, j$

– the leaves are unlabelled

– the pendant edges (those going into leaves) are labelled by $0, \ldots, j$

The construction is initiated by

$T_1 =$



(**)

Figure 2.

Given T_j with internal nodes $1, \ldots, j$ and pendant edges labelled $0, \ldots, j$, cf (**), define T_{j+1} as follows: As $\boldsymbol{a}$ is regressive we know that $\boldsymbol{a}(j+1) \leq j$.

Thus there is a pendant edge labelled with $\boldsymbol{a}(j+1)$. The corresponding leave in T_j now becomes an internal node of T_{j+1} labelled $j+1$. Moreover two new pendant edges are attached to it and labelled by $\boldsymbol{a}(j+1), \boldsymbol{a}(j+1)+1$. The other pendant edges of T_j are kept as such in T_{j+1} , those with labels $< \boldsymbol{a}(j+1)$ going unchanged and on those with labels $> \boldsymbol{a}(j+1)$ the labelling being increased by one. It is immediate from the construction that the function $\boldsymbol{a}$ is always increasing along the paths of internal nodes. If the size of the tree becomes larger than $2^{n-1}-1$ we cannot help avoiding increasing subsequences of size at least n which proves the upper bound. The example $\boldsymbol{a} = (0103210)$ of figure 1 leads to

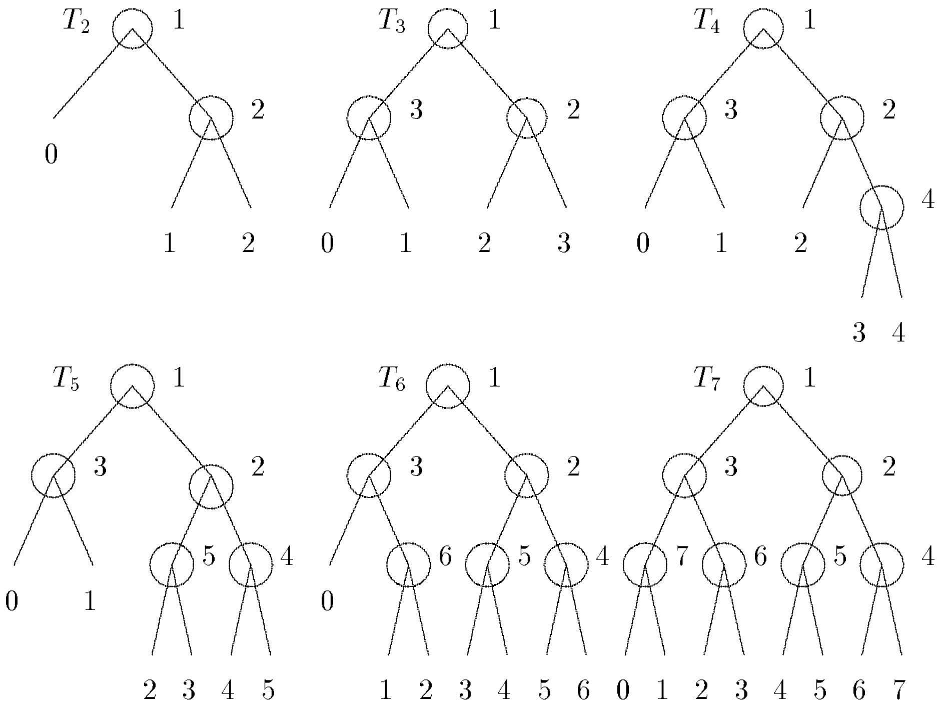


Figure 3.

The complete binary tree of depth $n-1$ may be obtained from the example for the lower bound which shows that $H(n) \geq 2^{n-1}$.

Needless to say, in the simple situation of Harzheim's result our efforts for proving upper and lower bounds by rather sophisticated looking methods may give an overloaded impression. To us it seems to be the simplest approach, cf [Th 89], and moreover it is generalizable to more tricky situations [cf § 5]. The general case for products of chains was given by [Ha 82].

Theorem 3.2. *$H_{\mathbb{N}^d} \sim f_{d-1}$ for $d \geq 2$.*

§ 4 The intermediate levels, Higman's theorem for finite alphabets

One of the classical results in wqo theory is Higman's theorem:

If $(X \leq)$ is wqo then $Hig(X, \leq)$, the set of finite words over X endowed with embeddability into subwords, is wqo.

We shall indicate the complexity of the corresponding H -functions. For doing so we observed a proof for Higman's theorem, which is as constructive as possible and astonishingly avoids minimal bad sequences. The proof makes use of some early observations of [Ju 68] on finite sets, but apart from it should be folklore to the specialists.

The crucial phenomenon is best observed by taking $\boldsymbol{t} = \{1 \leq, \ldots, \leq t\}$ to be a finite, linearly ordered alphabet. Let $\vec{a}, \vec{b} \in \mathrm{Hig}\ t$. If $\vec{a} \not\leq_{\mathrm{Hig}} \vec{b}$ then $\vec{b}$ has a certain structure imposed by $\vec{a}$. In order to appreciate the idea, let $t = 10, \vec{a} = (2, 10, 7)$ and $\vec{b} = (b_1, \ldots, b_n)$.

Case 1. $a_1 = 2 \not\leq b_i$ for all i (the embedding of $\vec{a}$ fails already in the first place). Then $\vec{b} \in$ Hig $(\boldsymbol{a}_1 - 1)$.

Case 2. Let i_1 be minimal with $b_{i_1} \geq 2$. Thus $\vec{b} = (b_1 \ldots b_{i_1 - 1}) * b_{i_1} * (b_{i_1+1} \ldots b_n)$ with $(b_1 \ldots b_{i_1-1}) \in$ Hig $(\boldsymbol{a}_1 - 1)$, $b_{i_1} \geq a_1$ and $(b_{i_1+1} \ldots b_n) \in$ Hig $\boldsymbol{t}$ is such that $(a_2, a_3) \not\leq_{\mathrm{Hig}} (b_{i_1+1}, \ldots, b_n)$.

By iteration one obtains the following general result:

Definition. Let X be a poset and $a \in X$. Then $[a)$ is the principal filter $\{z | a \leq z\}$ generated by a and $X \backslash [a)$ is the complement of the principal filter $[a)$.

Fact. *Let $(X \leq)$ be a poset, $\vec{a} = (a_1 \ldots a_n)$, $\vec{b} = (b_i \ldots b_m) \in$ Hig (X) and $\vec{a} \not\leq_{\mathrm{Hig}} \vec{b}$. Then there exist $\ell < n$, $\vec{b}_0 \in$ Hig $(X \backslash [a_1)), \ldots, \vec{b}_{\ell+1} \in$ Hig $(X \backslash [a_{\ell+1}))$ and elements $b_1^* \in [a_1), \ldots, b_\ell^* \in [a_\ell)$ with*

$$\vec{b} = \vec{b}_0 \times b_1^* \times \vec{b}_1 \times b_2^* \cdots \times \vec{b}_{\ell+1}.$$

Basically the fact says: If $\vec{a} \not\leq_{\mathrm{Hig}} \vec{b}$ then $\vec{b}$ is contained in a product whose factors are of the form Hig $(X \backslash [a_i))$ or principal filters. Moreover the length of all these products has an upper bound depending on the length of $\vec{a}$.

Theorem 4.1. (Higman) *If $(X \leq)$ is wqo then $Hig(X, \leq)$ is wqo.*

Proof. The theorem holds for $X = \varnothing$. So assume that the theorem holds for all complements of principal filters $X \backslash [a)$ of some X. We shall show that it holds for X . As such an induction works for wqo's, the theorem follows.

So let $\vec{a}_0, \vec{a}_1, \vec{a}_2 \ldots$ be a sequence of elements of Hig$X$. Assume $\vec{a}_0 \not\leq \vec{a}_i$ for all i (i.e. the greedy approach to show that the sequence is "good" fails), then each

$\vec{a}_i,\ i = 1, 2 \ldots$ has a structure as given by the fact. Thus there exists an ℓ such that for infinitely many i

$$\vec{a}_i \in \prod^{\ell+1} (\text{complement of principal ultrafilter}) \times \prod^{\ell} X.$$

The induction hypothesis and the product lemma [Hi 52] implies that there is a weakly increasing subsequence of $\vec{a}_i$'s. Thus X is wqo. $\square$

The proof looks quite constructive at first sight. A careful analysis reveals that for a general poset X the proof is not at all constructive. Nevertheless for special X 's it is strong enough that with some additional work [Th 93] one can obtain.

Theorem 4.2. *Let* $t < \omega$ *. Then* $H_{\mathrm{Hig}(t)} \simeq f_{\omega^{t-1}}$.

Remark. In the framework of regressive sequences the problem asking for $H_{\mathrm{Hig}(\omega)}$ seems to be ill–posed as the rank function according to our definition ($r(\mathbf{a}) = lgth(\mathbf{a})$) does not make sense. It is imaginable that for adequate definitions reasonable results could be obtained for $H_{Hig(\omega)}$ and beyond. For well posed but somewhat artificial modificati ons of this problem cf [Th 89].

§ 5 The upper levels, the ω - towers

Kanamori - McAloon [KM 87] gave a model theoretic proof for the unprovability of a theorem on regressive colorings of k -element sets. Here we shall analyze the corresponding complexity questions. By doing so we shall explain how "canonical Ramsey theory", "large sets" in the sense of Paris Harrington, cf [PH 77] and "tree arguments" can be applied in order to obtain sharp complexity results. These are related to the results of [EM 81].

Before generalizing the concept of regressive sequences we indicate as a combinatorial tool the Erdös-Rado canonization lemma cf [ER 50]. We need

Definition. Let $n, k \in \omega \cup \{\omega\}$. $\binom{\{1,\ldots,n\}}{k} = \binom{n}{k}$ denotes the set of all k element (rsp. infinite) subsets of n. Let $X = \{x_0, \ldots, x_{k-1}\}_<, Y = \{y_0, \ldots, y_{k-1}\}_<$ be $k-$ element subsets of M and I be a subset of $\{0, \ldots, \ (k-1)\}$. Let $X : I = \{x_i \in X / i \in I\}_<$; thus we have

$$X : I = Y : I \quad \text{iff} \quad x_i = y_i \quad \text{for all} \quad i \in I.$$

The countable version of the Erdös-Rado canonization lemma can be stated as follows:

Canonization lemma [ER 50]. *Let* $k \in \omega$ *be fixed and* $\Delta : \binom{\omega}{k} \to \omega$ *be a coloring into the natural numbers. Then there exist* $I \subseteq \{0, \ldots, (k-1)\}$ *and* *an infinite subset* $M \in \binom{\omega}{\omega}$ *of natural numbers such for all* $X, Y \in \binom{M}{k}$ *the relation*

$$X : I = Y : I \quad \textit{holds iff} \quad \Delta(X) = \Delta(Y).$$

Example: In the special case where $k = 2$ the theorem assures the existence of an infinite set such that the restricted coloring is

- constant

$$\Delta(X) = \Delta(Y) \quad \text{for all} \quad X, Y \in M \qquad (I = \varnothing),$$

- or injective

$$\Delta(X) = \Delta(Y) \quad \text{iff} \quad X = Y \quad (I = \{0, 1\}),$$

- or depends only on minimum elements

$$\Delta(X) = \Delta(Y) \quad \text{iff} \qquad \min\ (X) = \min\ (Y) \quad (I = \{0\}),$$

- or depends only on maximum elements

$$\Delta(X) = \Delta(Y) \quad \text{iff} \quad \max\ (X) = \max\ (Y) \quad (I = \{1\}).$$

In order to generalize regressive sequences we make use of the following definition

Definition. Fix n and k as above. A coloring $\Delta : \binom{n}{k} \to \omega$ is called **min-regressive** if $\Delta(x) < \min X$ for all $X \in \binom{n}{k}$. For $M \subseteq n$ we call a coloring $\Delta : \binom{M}{k} \to \omega$ **min homogeneous** if

$$\min X = \min Y \quad \text{implies} \quad \Delta(X) = \Delta(Y) \quad \text{for all} \quad X, Y \in \binom{M}{k}.$$

Note that for $k = 1$ we recover the notion of a regressive sequence.

A classical example is the van der Waerden coloring which assigns to every arithmetic progression of length k in $\{1, \ldots, n\}$ its first element diminished by one, and which assigns 0 to the other k tuples.

The following theorem is obvious to all those familiar with canonical Ramsey theory:

Theorem 5.1. *Let* $k \in \omega$ *be fixed. For every* m *there exists a smallest* $n = H_k(m)$ *with the following property:*

Let $\Delta : \binom{n}{k} \to \omega$ *be* min *-regressive. Then there exists an* m *-element subset* M *of* n *such that the restriction* $\Delta \restriction \binom{M}{k}$ *is* min *-homogenous.*

Proof. Work with the countable version of the Erdös-Rado canonization lemma. As the coloring Δ is min -regressive the pertinent canonical cases must be min - homogeneous. Finally apply compactness in order to obtain the existence of H_k . $\square$

A natural problem is the analysis of the complexity of H_k . Here we rely on [PV 89], [PTV 92] and [Th 89].

Theorem 5.2. *Let* $k \geq 2$. *Then*

$$H_k \sim f_{\omega^{\cdot^{\cdot^{\omega}}} \text{ tower of height } k-1}.$$

Here we shall give an explicit description of the arguments showing $H_2 \sim f_\omega$, the Ackermann function. As in §3 the proof consists in giving

(i) a lower bound $\sim f_\omega$

(ii) an upper bound $\sim f_\omega$.

For the lower bound we need:

Lemma 5.3. $H_2(Ram(2, m+3, k)) \geq f_k(m)$, *where* $Ram(2, m+3, k)$ *is the ordinary Ramsey number, arrowing* $(m+3)^2_k$.

Proof. Given m, k let $m^* = Ram(2, m+3, k)$, $n^* = H_2(m^*)$. Observe that for $x < y < \omega$ there exist unique $0 \leq k^* < k$ and $1 \leq \ell < x$ satisfying

$$f^{(\ell)}_{k^*}(x) := \underbrace{f_{k^*} \circ \cdots \circ f_{k^*}}_{l- \text{ times}}(x) \leq y < \underbrace{f_{k^*} \circ \cdots \circ f_{k^*}}_{l+1- \text{ times}}(x).$$

This is well defined as

$$f_{k^*}(x) < f_{k^*+1}(x) = f^{(x)}_{k^*}(x).$$

Define a regressive mapping by

$$\Delta(\{x, y\}) = \begin{cases} 0, & \text{if } f_k(x) \leq y, \\ \ell & \text{otherwise.} \end{cases}$$

Let $M^* \in \binom{n^*}{m^*}$ be such that Δ is min-homogeneous on M^*. We define a k-coloring $\Delta^* : \binom{M^*}{2} \to k$ by

$$\Delta^*(\{x,y\}) = \begin{cases} 0, & \text{if } f_k(x) \le y, \\ k^* & \text{otherwise.} \end{cases}$$

Let $M \in \binom{M^*}{m+3}$ be such that $\Delta^* \restriction \binom{M}{2}$ is a constant coloring and let $x < y < z$ be the three largest lements of M. Then $m \le x$ and as the function f_k is increasing it suffices to show that $f_k(x) \le z$. Assume to the contrary that $f_k(x) > z > y$. Hence also $f_k(y) > z$, as $f_k(x) \le f_k(y)$. Say, $\Delta(\{x,y\}) = \Delta(\{x,z\}) = \ell$ and $\Delta(\{x,y\}) = \Delta(\{x,z\}) = \Delta(\{x,z\}) = k^*$. Then $f_{k^*}^{\ell}(x) \le y < z < f_{k^*}^{\ell+1}(x)$. Apply f_{k^*} to this inequality. Then $z < f_{k^*}^{\ell+1}(x) \le f_{k^*}(y)$. But this contradicts that $f_{k^*}(y) \le z$. $\square$

Corollary 5.4. *The function $H_2(m)$ is not primitive recursive.*

Proof. As $Ram(2, m+3, k) \le k^{(m+3)\cdot k}$ it is primitive recursive.
But $H_2(Ram(2, m+3, m)) \ge f_m(m) = f_\omega(m)$ by the lemma. As the primitive recursive functions are closed under composition the assertion follows. $\square$

For the upper bounds we have:

Theorem 5.5. $H_2(m) < f_{m-1}(3) < f_\omega(m)$ *for all* $m \ge 3$.

Proof: In order to prove this theorem we consider trees as partially ordered sets, the smallest element being the root. Similarily as in §3 we use a **tree argument**: For a given regressive mapping $\Delta : \binom{n}{2} \to n$ define a tree $(T_\Delta, \le_T)$ on $\{2, \dots, n-1\}$ by

$$\ell <_T m \text{ iff } \Delta(\{k,\ell\}) = \Delta(\{k,m\}) \text{ for all } k \text{ with } k <_T \ell.$$

For example, the tree depicted in figure 4 corresponds to regressive mappings $\Delta : \binom{15}{2} \to 15$ such that:

$$\begin{aligned} \Delta(2,3) = &\Delta(2,4) = \Delta(2,5) = \Delta(2,6), \\ &\Delta(3,4) = \Delta(3,5) = \Delta(3,6), \\ \Delta(2,7) = &\Delta(2,8) = \cdots = \Delta(2,14), \\ &\Delta(7,8) = \cdots = \Delta(7,14). \end{aligned}$$

Nothing is asserted about the remaining pairs.

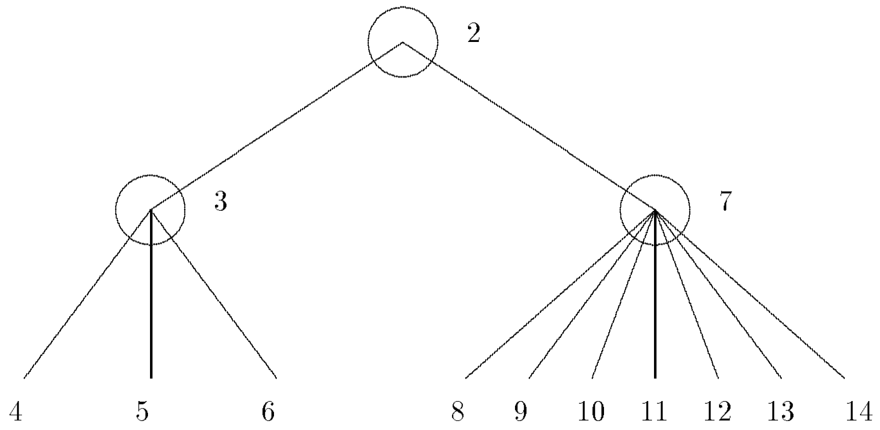


Figure 4.

Mills [Mi 80] called a tree **small branching** provided the successor-degree of each node i is at most i.

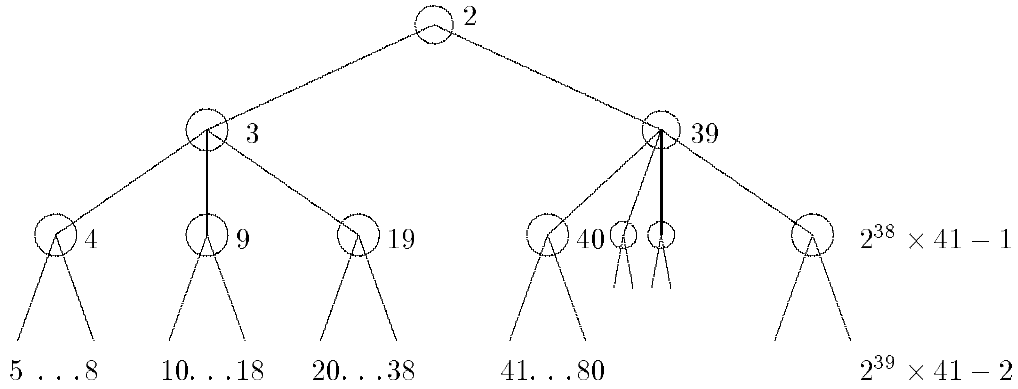


Figure 5.

The following observation is trivial but useful:

Observation 1. *Let* $\Delta : \binom{n}{2} \to n$ *be a regressive mapping and* $(T_\Delta, \leq_T)$ *be the associated tree. Then:*

(i) $k <_T \ell$ *implies that* $k < \ell$.

(ii) T_Δ *is small branching.*

(iii) Every chain is min-homogeneous.

For estimating $H_2(m)$ from above we ask how large n must at least be, such that every small branching tree T_Δ contains a chain of length m. Denote by $M(m)$ the smallest such n.

Figure 4. shows that $M(4) > 14$ and it is easy to see that, in fact, $M(4) = 15$.

Figure 5. indicates that $M(5) > 2^{39} \cdot 41 - 2$ and again it is not difficult to see that $M(5) = 2^{39} \cdot 41 - 1$. The idea behind Figures 4. and 5. is fairly obvious. To build a large small branching tree without chains of length m one fills in the branches from left to right by placing smaller numbers as far down on the tree as possible to save vertices higher up for larger numbers. These larger numbers then allow more immediate successors, thus making the tree as big as possible. Such trees are well known in computer science as **balanced preordered trees**.

Lemma 5.6. *Let $n = M(m) - 1$ and let T be a small branching tree defined on $\{2, \ldots, n\}$ without any m-element chains. Then T is a balanced preorder tree.*

The somewhat technical proof was given in [Mi 80] and [Th 89] and a beautifully illustrated version may be found in the highly recommended forthcoming book "Aspects of Ramsey Theory" [PV 93].

Let $M_m(k)$ be the smallest positive integer n such that every small branching balanced preorder tree on $[k, n]$ contains an m-element chain, thus $M(m) + 1 = M_m(2)$.

Observation 2.

(1) $M_2(k) = k + 1$,

(2) $M_{m+1}(k) \leq M_m^k(k+1)$, ($k$ *– fold iteration*) .

Proof. (1) is obvious and (2) follows immediately from the construction of small branching trees, cf Figure 5. □

By boolean combination of definitions obtain

Observation 3. $M_m(k) \leq f_{m-1}(k+1) - 1$ *for all* $k, m \geq 1$.

In order to obtain the upper bound f_ω for H_2 it suffices to combine Observations 1-3. □

Remark 1. It is possible to extend these arguments and obtain a proof of theorem 5.2 in general. For details cf [Th 89].

Remark 2. Erdös and Mills [EM 81] gave upper bounds for the Paris-Harrington function for coloring pairs with a fixed number of colors, the Ramsey case [Ra 30]. The above results cover the canonical min-homogeneous case for pairs and k-tuples in general.

§ 6 Outlook and problems

Here we concentrated on the levels of the Grzegorczyk-Wainer hierarchy up to $\varepsilon_\circ$. Of course one could and did go beyond. [S 87] gives an account on the finite miniaturisation of Kruskal's theorem for trees, another classic in wqo'theory. For the case of binary trees [Th 93] shows that for regressive sequences of binary trees $H_{Bin} \sim f_{\varepsilon_\circ}$, whereas [S 87] indicates that the general case for regressive sequences of arbitrary trees is far beyond $f_{\Gamma_\circ}$. Finally we would like to mention Leeb's *jungles* [Le], which unfortunately have not really been penetrable for us so far.

As a general problem and idea let us suggest to search for other "natural" combinatorial features which may be extended via compactness arguments and lead to fast growing functions and unprovability results.

Closer to the extension of Harzheim's result (cf Theorem 3.1) it would be interesting to find orders related to each level of the hierarchy. When stating Higman's theorem, we assumed the alphabet to be an antichain. Of course such alphabets may be partially or totally ordered. How does this order affect the growth of the corresponding $H-$ functions?

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