ROOT CATEGORIES AND SIMPLE LIE ALGEBRAS

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1. Introduction

Let A be a finite dimensional associative hereditary algebra over some base field k which is representation-finite. In case k is algebraically closed, Gabriel [4] showed that the isomorphism classes of indecomposable A-modules correspond bijectively to the positive roots of the corresponding semisimple complex Lie algebra $\mathbf{g}$, and this result was extended to arbitrary base field by Dlab-Ringel [2]. It is natural to ask whether it is possible to recover a Lie algebra structure in terms of the representation theory of A. This was successfully attained by Ringel ([10],[11]). Let $K(\operatorname{mod} A)$ be the free abelian group with a basis indexed by the set of isomorphism classes of indecomposable A-modules. He showed that $K(\operatorname{mod} A)$ is the Chevalley $\mathbf{Z}$-form of $\mathbf{n}_+$ and hence $K(\operatorname{mod} A)\otimes_{\mathbf{Z}}\mathbf{C}$ and $\mathbf{n}_+$ are isomorphic Lie algebra, where $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ is a triangular decomposition of $!g$.

Ringel considered the case A over finite fields and filtrations of finite A-modules. He proved the existance of Hall polynomials for the filtrations and got an associative algebra which is called the Hall algebra of A, denoted by $\mathcal{H}_q(A)$, by using as structure constants the number of filtrations or evaluations of Hall polynomials at various values. $\mathcal{H}_1(A) \otimes \mathbf{C}$ is just the universal enveloping algebra of $K(\operatorname{mod} A) \otimes \mathbf{C}$ and $\mathcal{H}_1(A)$ is quite analogous to the Kostant $\mathbf{Z}$-form for a simple complex Lie algebra. Similar investigations were also given by Schofield [15] and Riedtmann [9] to consider the Euler characteristic of corresponding varieties instead of Hall polynomials.

Another advance in the representation theory of finite dimensional algebras is the theory of derived category due to Happel ([5],[6]). The derived category of a finite dimensional algebra B, denoted by $D^b(B)$, is a triangulated category with the translation functor T. In particular, the category $D^b(B)$ was well described when B is a tilted algebra.Happel pointed out that the isomorphism classes of indecomposable objects in $D^b(A)/T^2$ correspond bijectively to the root system of $\mathbf{g}$ (not only positive roots). For this reason, $\mathcal{R} = D^b(A)/T^2$ is called the root category. In [12], Ringel asked whether it is possible to use representations of algebras to recover the whole Lie algebra $\mathbf{g}$ and not only $\mathbf{n}_+$. The present article gives an affirmative answer for this question by using the root category $\mathcal{R}$. Schofield has constructed the complete

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Lie algebra $\mathbf{g}$ too, in a different way [15]. But we deal with the simply-laced and non-simply-laced ! cases in a unified way and our met hod is computable.

Unfortunately we can not find an associative algebra related to $\mathcal{R}$ analoguous to the Hall algebra of A.We consider the triangles in $\mathcal{R}$ which are heritaged from $D^b(A)$, investigate the orbit space of all triangles of fixed type under the actions of certain automorphism groups, prove that the Hall polynomials exist for the cardinality of this orbit spaces, and use the evaluations of Hall polynomials as structrue constants to get a Lie product. Particular interesting to us is that the Cartan subalgebra $\mathbf{h}$ is realized by the weighted Grothendieck group of $\mathcal{R}$. We prove that $\mathbf{h}\oplus K(\mathcal{R})$ is a $\mathbf{Z}$-form of $\mathbf{g}$ and hence $(\mathbf{h}\oplus K(\mathcal{R}))\otimes_{\mathbf{Z}}\mathbb{C}\simeq\mathbf{g}$, where $K(\mathcal{R})$ is the free abelian group with a basis indexed by the set of isomorphism classes of indecomposable objects in $\mathcal{R}$. The isomorphism classes of indecomposable objects in $\mathcal{R}$ constitute a Chevalley basis of $\mathbf{g}$. In! some sense, our description for $\mathbf{g}$ may be considered as a global version of semisimple complex Lie algebras.

Although the definition of Lie product is given only in terms of the root category $\mathcal{R}$, if we compute $[u_M, u_N]$ for indecomposables M and N, except the case $N\simeq TM$ and then $[u_M,u_N]=H_M$, it coincides with that of M and N as modules of a hereditary algebra, which is defined by Ringel. In [10] Ringel gave a complete list of all nontrivial Hall polynomials which occur for a representation-finite hereditary algebra A. So the structure constants for the Lie product of $\mathbf{n}_+$ can be easily calculated. Our investigation implies that the calculation of Ringel actually provide all possibilities of Hall polynomials for $\mathcal{R}$ and of structure constants for the whole $\mathbf{g}$.

For unexplained notions concerning the representation theory of algebras we refer to [13], for those concerning the theory of Lie algebras to [7], and for the derived category to [6]

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2. Preliminaries

1. Let A be a finite dimensional hereditary k-algebra and let $D^b(A)$ be the derived category of mod A, which is a triangulated category with the translation functor T. Denote by $\mathcal{R}$ the orbirt category $D^b(A)/T^2$ and let $F: D^b(A)\longrightarrow\mathcal{R}$ be the canonical covering functor.

Lemma 2.1. For any indecomposible objects $M,N\in D^b(A)$, there exist $i,j\in\mathbf{Z}$ such that F induces the k-isomorphisms

$$\mathrm{Hom}_{D^b(A)}(T^iM,N)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{R}}(FM,FN)$$

and

$$\mathrm{Hom}_{D^b(A)}(M,T^jN)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{R}}(FM,FN).$$

Proof. Since F is the covering functor, F induces the k-isomorphism

$$\bigoplus_{i\in\mathbf{Z}} \mathrm{Hom}_{D^b(A)}(T^{2i}M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{R}}(FM, FN).$$

However we know that $\mathrm{Hom}_{D^b(A)}(T^{2i}M, N) = 0$ for almost all $i \in \mathbf{Z}$ but at most one number in $\mathbf{Z}$. So the first assertion holds. Similarly we have the second one. $\square$

Recall that a short cycle in a category is a sequence $M \xrightarrow{f} N \xrightarrow{g} M$ where M and N are indecomposable objects and f and g are nonzero and non-isomorphic morphisms.

Lemma 2.2. If A is representation-finite, then $\mathcal{R}$ has no short cycle.

Proof. There is a short cycle in $D^b(A)/T^2$ if and only if there is a chain of non zero morphisms in $D^b(A)$ of form $M \xrightarrow{f} N \xrightarrow{g} T^2M$ with $M \not\simeq N$ and both indecomposible in $D^b(A)$. This implies that $N \prec TM$ and $T^2M \prec TN$, a contradiction. $\square$

Under the covering functor F, the isomorphism T of $D^b(A)$ induces an isomorphism of $\mathcal{R}$, denoted still by T. Then $T^2 = 1$ in $\mathcal{R}$. Although we don't know if $\mathcal{R}$ may become a triangulated category, we will point out that it has enough good triangles for our aim. Let us introduce the following notation.

Let $\mathcal{C}$ be an additive category together with a translation functor T and a set Ω of sextuples in $\mathcal{C}$. We call $(\mathcal{C}, T, \Omega)$ an 'almost' triangulated category and the elements of Ω are then called triangles if the following axioms are satisfied:

(Tr1) Every sextuple isomorphic to a triangle is a triangle. The sextuple $X \xrightarrow{1_X} X \xrightarrow{0} 0 \xrightarrow{0} TX$ is a triangle, where 1_X is the identity morphism.

(Tr2) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is a triangle,then $Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$ is a triangle.

(Tr3) Given two triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} TX'$ and morphisms $f : X \to X'$, $g : Y \to Y'$ such that $fu' = ug$, then there exists a morphism $h : Z \to Z'$ such that (f, g, h) induces a morphism from the first triangle to the second.

(Tr4) (The octahedral axiom) Suppose given triangles:

$$X \xrightarrow{u} Y \xrightarrow{i} Z' \xrightarrow{i'} TX$$

$$Y \xrightarrow{v} Z \xrightarrow{j} X' \xrightarrow{j'} TY$$

$$X \xrightarrow{uv} Z \xrightarrow{k} Y' \xrightarrow{k'} TX.$$

Then there exist morphisms $f : Z' \to Y'$, $g : Y' \to X'$ such that the following diagram commutes and the third row is a triangle.

$$\begin{array}{ccccccccc}
T^{-1}Y' & \xrightarrow{T^{-1}k'} & X & \xrightarrow{1_X} & X & & & & \\
\downarrow T^{-1}g & & \downarrow u & & \downarrow uv & & & & \\
T^{-1}X' & \xrightarrow{T^{-1}j'} & Y & \xrightarrow{v} & Z & \xrightarrow{j} & X' & \xrightarrow{j'} & TY \\
 & & \downarrow i & & \downarrow k & & \downarrow 1_{X'} & & \downarrow Ti \\
 & & Z' & \xrightarrow{f} & Y' & \xrightarrow{g} & X' & \xrightarrow{j'Ti} & TZ' \\
 & & \downarrow i' & & \downarrow k' & & & & \\
 & & TX & \xrightarrow{1_{TX}} & TX & & & &
\end{array}$$

Now we define a set in $\mathcal{R}$ consisting of all sxtuples isomorphic to

$$FM \xrightarrow{Ff} FN \xrightarrow{Fg} FL \xrightarrow{Fh} TFM$$

with

$$M \xrightarrow{f} N \xrightarrow{g} L \xrightarrow{h} TM$$

being a triangle in $D^b(A)$. It is not difficult to check that $\mathcal{R}$ together with T and Ω is an $'$almost$'$ triangulated category.

2. In the remaining part of this section, we assume that $(\mathcal{C}, T, \Omega)$ is an $'$almost$'$ triangulated category and there exists no short cycle in $\mathcal{C}$. We need the following easy lemmas.

Lemma 2.3. Assume that $M \xrightarrow{f} N \xrightarrow{g} L \xrightarrow{h} TM$ and $M \xrightarrow{f'} N \xrightarrow{g'} L \xrightarrow{h'} TM$ are two triangles in $\mathcal{C}$.

(i). If $f' = f\alpha$ with $\alpha \in \operatorname{Aut} N$, then there exists $\beta \in \operatorname{Aut} L$ such that $\alpha g' = g\beta$ and $\beta^{-1}h' = h$;

(ii). If $\beta h' = h$ with $\beta \in \operatorname{Aut} L$, then there exists $\alpha \in \operatorname{Aut} N$ such that $f'\alpha = f$ and $\alpha^{-1}g' = g\beta$.

Proof. Similar with the proof in [6, 1.2]. $\square$

Lemma 2.4. Given a triangle of the form

$$M \xrightarrow{(f_1, f_2)} N_1 \oplus N_2 \xrightarrow{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} L \xrightarrow{h} TM$$

in $\mathcal{C}$. If $f_2 = 0$, then it is isomorphic to the triangle of the form

$$M \xrightarrow{(f_1, f_2)} N_1 \oplus N_2 \xrightarrow{\begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}} L_1 \oplus L_2 \xrightarrow{\begin{pmatrix} h_1 \\ 0 \end{pmatrix}} TM,$$

where $g_{22} : N_2 \to L_2$ is an isomorphism and

$$M \xrightarrow{f_1} N_1 \xrightarrow{g_{11}} L_1 \xrightarrow{h_1} TM$$

is a triangle. In particular, if $h \neq 0$ and both M, L are indecomposible, then for any nonzero direct summand N' of N we have $\mathrm{Hom}_{\mathcal{C}}(M, N') \neq 0$ and $\mathrm{Hom}_{\mathcal{C}}(N', L) \neq 0$.

Proof. Since $f_2 = 0$, g_2 is a monomorphism. According to the axioms we have the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & N_2 & \xrightarrow{1} & N_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow g_2 & & \downarrow \\
M & \xrightarrow{(f_1,f_2)} & N_1 \oplus N_2 & \xrightarrow{(g_1,g_2)^t} & H & \xrightarrow{h} & TM \\
\downarrow & & \downarrow & & \downarrow l & & \downarrow \\
0 & \longrightarrow & N_2 & \xrightarrow{1} & N_2 & \longrightarrow & 0
\end{array}$$

where $N_2 \to N_1 \oplus N_2$ is canonical embedding and $N_1 \oplus N_2 \to N_2$ canonical surjection. So $g_2 l = 1$, then $H = H_1 \oplus H_2$, where $H_2 = g_2(N_2)$, and

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

It is easy to see that $g_{21} = 0, h_2 = 0$, and $g_{12} = g_{11}s$ where $s : H_1 \to H_2$. Hence

$$\begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix},$$

but

$$\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \in Aut(H_1 \oplus H_2).$$

We have proved that the given triangle is equivalent the required form with g_{22} to be an isomorphism.By (Tr4), the corresponding sextuple

$$M \xrightarrow{f_1} N_1 \xrightarrow{g_{11}} H_1 \xrightarrow{h_1} TM$$

is a triangle. $\square$

Given $X, Y, Z \in \mathcal{C}$, put

$$\begin{aligned} W(X, Y; Z) = \ & \{(f, g, h) \in \mathrm{Hom}_{\mathcal{C}}(X, Z) \times \mathrm{Hom}_{\mathcal{C}}(Z, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, TX) \\ & : X \xrightarrow{f} Z \xrightarrow{g} Y \xrightarrow{h} TX \text{ is a triangle } \}. \end{aligned}$$

If $W(X, Y; L) \times W(Z, L; M)$ is non-empty for $X, Y, Z, L, M \in \mathcal{C}$, we define an action of Aut L on it as follows

$$\alpha((f, g, h), (l, m, k)) = ((f\alpha, \alpha^{-1}g, h), (l, m\alpha, \alpha^{-1}k)) \tag{2.1}$$

for any $\alpha \in \mathrm{Aut}\, L$ and $((f, g, h), (l, m, k)) \in W(X, Y; L) \times W(Z, L; M)$. The orbit space $W(X, Y; L) \times W(Z, L; M)/\,\mathrm{Aut}\, L$ is denoted by $(W(X, Y; L) \times W(Z, L; M))^\wedge$ and the orbit of $((f, g, h), (l, m, k))$ by $((f, g, h), (l, m, k))^\wedge$.

Proposition 2.1. Given $X, Y, Z, M \in \operatorname{ind}\mathcal{C}$ and suppose that $Y \not\simeq TX, Y \not\simeq TZ, X \not\simeq TZ$. Then we have the following bijection.

$$\bigcup_{[L],\ L\in\mathcal{C}} (W(X,Y;L)\times W(Z,L;M))^{\wedge} \longrightarrow \bigcup_{[L'],\ L'\in\mathcal{C}} (W(Z,X;L')\times W(L',Y;M))^{\wedge}$$

$$((f,g,h),(l,m,k))^{\wedge} \longmapsto ((l',m',k'),(f',g',h'))^{\wedge},$$

where the morphisms fit into the following commutative diagram

$$\begin{array}{ccccccc}
 & & Z & = & Z & & \\
 & & \downarrow l' & & \downarrow l & & \\
T^{-1}Y & \xrightarrow{-Th'} & L' & \xrightarrow{f'} & M & \xrightarrow{g'} & Y \\
\| & & \downarrow m' & & \downarrow m & & \| \\
T^{-1}Y & \xrightarrow{-Th} & X & \xrightarrow{f} & L & \xrightarrow{g} & Y \\
 & & \downarrow k' & & \downarrow k & & \\
 & & TZ & = & TZ & &
\end{array}$$

with all rows and columns being triangles.

Proof. Given $((f,g,h),(l,m,k)) \in W(X,Y;L)\times W(Z,L;M)$, the axiom (Tr4) implies that there exists $L' \in \mathcal{C}$ and $((l',m',k'),(f',g',h')) \in W(Z,X;L')\times W(L',Y;M)$ such that the above diagram commutes which all rows and columns are triangles. If there exist another $L'' \in \mathcal{C}$ and $((l'',m'',k''),(f'',g'',h'')) \in W(Z,X;L'') \times W(L'',Y;M)$ such that the following diagram commutes which all rows and columns are triangles:

$$\begin{array}{ccccccc}
 & & Z & = & Z & & \\
 & & \downarrow l'' & & \downarrow l & & \\
T^{-1}Y & \xrightarrow{-Th''} & L'' & \xrightarrow{f''} & M & \xrightarrow{g''} & Y \\
\| & & \downarrow m'' & & \downarrow m & & \|, \\
T^{-1}Y & \xrightarrow{-Th} & X & \xrightarrow{f} & L & \xrightarrow{g} & Y \\
 & & \downarrow k'' & & \downarrow k & & \\
 & & TZ & = & TZ & &
\end{array}$$

then $g'' = g,\ k'' = k$ and so we must have $L'' \simeq L'$. Without loss of generality we assume that $L'' = L'$. It follows from Lemma 2.3 that there exist $\alpha, \beta \in \operatorname{Aut} L'$ such that $l'' = l'\alpha,\ m'' = \alpha^{-1}m',\ Th'' = T(h')\beta$ and $f'' = \beta^{-1}f'$. We claim that $l'' = l'\beta,\ m'' = \beta^{-1}m'$ or $Th'' = T(h')\alpha,\ f'' = \alpha^{-1}f'$.

In fact, if $l'' \neq l'\beta$ and $Th'' \neq T(h')\alpha$, then

$$(l'' - l'\beta)f'' = l - l'\beta f'' = l - l'f' = l - l = 0,$$

$$(Th'' - T(h')\alpha)m'' = Th - T(h')\alpha\alpha^{-1}m' = Th - Th = 0.$$

So there exist $0 \neq s \in \operatorname{Hom}_{\mathcal{C}}(Z, T^{-1}Y)$ and $0 \neq t \in \operatorname{Hom}_{\mathcal{C}}(T^{-1}Y, Z)$ such that $l'' - l'\beta = sTh''$ and $Th'' - T(h')\alpha = tl''$. We have $T^{-1}Y \simeq Z$ as no short cycle in

$\mathcal{C}$, a contradiction. Therefore we must have $l'' = l'\beta$ or $Th'' = T(h')\alpha$. Now we may assume that $Th'' = T(h')\alpha$ and try to show that $f'' = \alpha^{-1}f'$.

If $k'' = 0$, then l'' is a section and m'' a retraction. So $L'' \simeq Z \oplus X$. We see $g'' \neq 0$, otherwise $L'' \simeq T^{-1}Y \oplus M$ and hence $T^{-1}Y \simeq Z$ or $T^{-1}Y \simeq X$, a contradiction. In view of Lemma 2.4 we particularly have $\mathrm{Hom}_{\mathcal{C}}(X, M) \neq 0$ and $\mathrm{Hom}_{\mathcal{C}}(Z, M) \neq 0$. Observe that $T^{-1}Y \xrightarrow{-T(h')\alpha} L' \xrightarrow{\alpha^{-1}f'} M \xrightarrow{g'} Y$ is also a triangle. Since $Th'' = T(h')\alpha$, by Lemma 2.3 there is some $\gamma \in \mathrm{Aut}\, M$ such that $\alpha^{-1}f' = f''\gamma$ and $\gamma^{-1}g'' = g'$. But $g'' = g'$ and so $(1-\gamma^{-1})g' = 0$. Note that $\mathrm{End}_{\mathcal{C}}\, M$ is a division ring since M indecomposible and no short cycle in $\mathcal{C}$ and that $g' = g'' \neq 0$ as shown. Therefore $1 - \gamma^{-1} = 0$ and so $\gamma = 1$. Hence we have $f'' = \alpha^{-1}f'$ as required.

If $k'' \neq 0$, according to Lemma 2.4, l'' induces a nonzero morphism from Z to any nonzero direct summand of L'' since X indecomposible. Suppose that $f'' \neq \alpha^{-1}f'$. Since

$$(f'' - \alpha^{-1}f')m = f''m - \alpha^{-1}f'm = m''f - \alpha^{-1}m'f = \alpha^{-1}m'f - \alpha^{-1}m'f = 0,$$

we deduce $\mathrm{Hom}_{\mathcal{C}}(L', Z) \neq 0$. Note that no short cycle is in $\mathcal{C}$. So l'', in fact, induces an isomorphism from Z to some direct summand of L'', that is, l'' is a section and so $k'' = 0$, a contradiction. Hence we have $f'' = \alpha^{-1}f'$. That is, we have obtained

$$l'' = l'\alpha,\ m'' = \alpha^{-1}m',\ Th'' = T(h')\alpha,\ f'' = \alpha^{-1}f'.$$

If we assume that $l'' = l'\beta$. Similarly we deduce $m'' = \beta^{-1}m'$. So we obtain

$$l'' = l'\beta,\ m'' = \beta^{-1}m',\ Th'' = T(h')\beta,\ f'' = \beta^{-1}f'.$$

Anyway, we always have $((l'', m'', k''), (f'', g'', h'')) \in ((l', m', k'), (f', g', h'))^{\wedge}$. So the following map is well-defined:

$$\bigcup_{[L],\ L\in\mathcal{C}} (W(X,Y;L) \times W(Z,L;M)) \longrightarrow \bigcup_{[L'],\ L'\in\mathcal{C}} (W(Z,X;L') \times W(L',Y;M))^{\wedge}$$

$$((f,g,h),(l,m,k)) \longmapsto ((l',m',k'),(f',g',k'))^{\wedge},$$

where $((f,g,h),(l,m,k)), ((l',m',k'),(f',g',h'))$ fit into the commutative diagram in Proposition 2.1. Obviously the above mapping is invariant under the action of $\mathrm{Aut}\, L$ on $(W(X,Y;L) \times W(Z,L;M))$. Therefore the map in Proposition 2.1 is also well-defined. Note the symmetricity of the diagram in Proposition 2.1. We can naturally define a map from

$$\bigcup_{[L'],\ L'\in\mathcal{C}} (W(Z,X;L') \times W(L',Y;M))^{\wedge}$$

to

$$\bigcup_{[L],\ L\in\mathcal{C}} (W(X,Y;L) \times W(Z,L;M))^{\wedge}$$

which is the inverse one. So the proof is completed. $\square$

Let us introduce a new action of $\mathrm{Aut}X \times \mathrm{Aut}Y$ on $W(X, Y; L)$.

$$(\alpha, \beta)(f, g, h) = (\alpha f, g\beta, \beta^{-1}h(T\alpha)^{-1}) \tag{2.2}$$

for any $(\alpha, \beta) \in \mathrm{Aut}X \times \mathrm{Aut}Y$ and $(f, g, h) \in W(X, Y; L)$. For any $(f, g, h) \in W(X, Y; L)$ we define

$$(f, g, h)^\wedge = \{\ (\alpha f, g\beta, \beta^{-1}h(T\alpha)^{-1}) \mid (\alpha, \beta) \in \mathrm{Aut}X \times \mathrm{Aut}Y\ \}$$

Note that it is easy to see that $(\alpha f, g\beta, \beta^{-1}h(T\alpha)^{-1}) \in W(X, Y; L)$ again. Put

$$V(f, g, h) = \{\ (f, g, h)^\wedge \mid (f, g, h) \in W(X, Y; L)\ \} = W(X, Y; L)/(\mathrm{Aut}\, X \times \mathrm{Aut}\, Y).$$

Lemma 2.5. Let $X, Y, Z, M \in \mathrm{ind}\mathcal{C}$ and $Y \not\simeq TX, Y \not\simeq TZ, X \not\simeq TZ$. Then the action (2.1) of $\mathrm{Aut}\, L$ on $W(X, Y; L) \times W(Z, L; M)$ is free.

Proof. Since $Y \not\simeq TX$, $L \in \mathcal{C}$ is nonzero. Given $\alpha \in \mathrm{Aut}\, L$ and $((f, g, h), (l, m, k)) \in W(X, Y; L) \times W(Z, L; M)$, if $f\alpha = f, \alpha^{-1}g = g, m\alpha = m, \alpha^{-1}k = k$. Then $f(\alpha - 1) = 0, (\alpha^{-1} - 1)g = 0, m(\alpha - 1) = 0, (\alpha^{-1} - 1)k = 0$. If $\alpha \neq 1$, there exist nonzero $g' : Y \to L, f' : L \to X, k' : TZ \to L, m' : L \to M$ such that $gg' = \alpha - 1, f'f = \alpha^{-1} - 1, kk' = \alpha - 1, m'm = \alpha^{-1} - 1$. Write $L = L_1 \oplus \cdots \oplus L_s$ with $L_i, 1 \leq i \leq s$, indecomposable, then $f = (f_1, \cdots, f_s)$ and$g = (g_1, \cdots, g_s)^t$. If $h \neq 0$, then no f_i or g_j is isomorphic. It also follows from Lemma 2.4 that no f_i or g_j is zero. Obviously we get a contradiction to the condition which states no short cycle in $\mathcal{C}$. If $l \neq 0$, a same contradiction arise. It remains to consider the case $h = 0$ and $l = 0$. Thus we have $L \simeq X \oplus Y \simeq M \oplus TZ$. It implies that $X \simeq TZ$ or $Y \simeq TZ$, contradicts to the hypothese again. $\alpha = 1$ is proved. □

Lemma 2.6. Let $X, Y \in \mathrm{ind}\mathcal{C}$ and $L \in \mathcal{C}$ nonzero, then the action (2.2) of $\mathrm{Aut}\, X \times \mathrm{Aut}\, Y$ on $W(X, Y; L)$ is free.

Proof. Let $\alpha \in \mathrm{Aut}\, X, \beta \in \mathrm{Aut}\, Y$ and $(f, g, h) \in W(X, Y; L)$ satisfy that $\alpha f = f, g\beta = g, \beta^{-1}hT\alpha^{-1} = h$. Then $(\alpha - 1)f = 0, g(\beta - 1) = 0$. $\mathrm{End}\, X$ is a division ring because of indecomposablity of X and no short cycle in $\mathcal{C}$. If $\alpha \neq 1$, $\alpha - 1$ is invertable, hence $f = 0$. We have $Y \simeq L \oplus TX$, a contradiction. Therefore $\alpha = 1$ is proved, so is $\beta = 1$. □

Lemma 2.7. Let $Z, M \in \mathrm{ind}\mathcal{C}$ and nonzero $L \in \mathcal{C}$ with $L \not\simeq M \oplus TZ$, then the action (2.2) of $\mathrm{Aut}\, Z \times \mathrm{Aut}\, L$ on $W(Z, L; M)$ is free.

Proof. For any $(\alpha, \beta) \in \mathrm{Aut}\, Z \times \mathrm{Aut}\, L$ and $(l, m.k) \in W(Z, L; M)$, if $\alpha l = l, m\beta = m, \beta^{-1}k(T\alpha)^{-1} = k$, then $(\alpha - 1)l = 0, m(\beta - 1) = 0$. $l \neq 0$ since $L \not\simeq M \oplus TZ$. Because of the indecomposablity of Z and no short cycle in $\mathcal{C}$, $\mathrm{End}\, Z$ is a division ring. Then $\alpha - 1 = 0, \alpha = 1$. By $m(\beta - 1) = 0$, we have a morphism $k' : TZ \to L$ such that $kk' = \beta - 1$. Let $L = L_1 \oplus \cdots \oplus L_s$ with L_i indecomposable, then we write $k = (k_1, \cdots, k_s)^t, k' = (k'_1, \cdots, k'_s)$. If some k_i is isomorphic, k is a retraction; this implies $L \simeq M \oplus TZ$, a contradicction. If some $k_i = 0$, according to the dual of Lemma 2.4, M is decomposable, a contradiction too. So every k_i is nonzero and non-isomorphic. If $\beta \neq 1$, then $k' \neq 0$. It follows that some L_i is on a short cycle, a contradiction again. $\square$

There exist also the natural actions of $\mathrm{Aut}\, L$ on $W(X,Y;L)$ as follows.

$$\alpha(f,g,h) = (f\alpha^{-1}, \alpha g, h) \tag{2.3}$$

for any $\alpha \in \mathrm{Aut}\, L, (f,g,h) \in W(X,Y;L)$. Obviously $(f\alpha^{-1}, \alpha g, h) \in W(X,Y;L)$. For arbitrary $(f,g,h) \in W(X,Y;L)$, write

$$(f,g,h)^{\wedge}_L = \{ \ (f\alpha^{-1}, \alpha g, h) \mid \alpha \in \mathrm{Aut}\, L \ \}$$

and

$$U(X,Y;L) = \{ \ (f,g,h)^{\wedge}_L \mid (f,g,h) \in W(X,Y;L) \ \} = W(X,Y;L)/ \mathrm{Aut}\, L$$

Lemma 2.8. Let $X, Y \in \mathrm{ind}\mathcal{C}$ and nonzero $L \in \mathcal{C}$ with $L \not\simeq X \oplus Y$, then the action (2.3) of $\mathrm{Aut}\, L$ on $W(X,Y;L)$ is free.

Proof. Given $\alpha \in \mathrm{Aut}\, L$, $(f,g,h) \in W(X,Y;L)$ satisfying $f\alpha^{-1} = f, \alpha g = g$. Then we have $f' : L \to X$ and $g' : Y \to L$ such that $f'f = \alpha - 1, gg' = \alpha^{-1} - 1$. We only need the condition $f'f = \alpha - 1$ in the following. Suppose $L = L_1 \oplus \cdots \oplus L_t$ with L_i indecomposable, then we write $f = (f_1, \cdots, f_t)$. If some $f_i = 0$, according to Lemma 2.4, Y is decomposable, a contradiction. If some f_i is isomorphic, then f is a retraction, $L \simeq X \oplus Y$, a contradiction too. Hence every f_i is nonzero and non-isomorphic. If $f' \neq 0$, it is easily seen that some L_i lie in a short cycle, a contradiction again. So $f' = 0$, then $\alpha = 1$ as required. $\square$

Also k^* acts on $U(X,Y;L)$ naturally. Define

$$\lambda(f,g,h)^{\wedge}_L = (\lambda f, g, h\lambda^{-1})^{\wedge}_L \tag{2.4}$$

for any $\lambda \in k^*$ and $(f,g,h)^{\wedge}_L \in U(X,Y;L)$. It is easy to see that the action is well-defined.

Lemma 2.9. If $X, Y \in \mathrm{ind}\mathcal{C}$ and nonzero $L \in \mathcal{C}$ with $L \not\simeq X \oplus Y$, then the action of k^* on $U(X,Y;L)$ is free.

Proof. Easy. $\square$

Let k be a finite finite field, $\mathcal{C}$ a $k-$category such that $\mathrm{Hom}_{\mathcal{C}}(M,N)$ is finite dimensional over k for any objects M, N in $\mathcal{C}$. We use $|\cdot|$ to denote the ordinary cardinality.

Lemma 2.10. Let $X, Y \in \mathrm{ind}\mathcal{C}$, then

$$|W(X, Y \oplus TX; Y)| = |\,\mathrm{Aut}(Y \oplus TX)|/|\,\mathrm{Hom}(TX, Y)|.$$

Proof. Given $(f,g,h) \in W(X, Y \oplus TX; Y)$, we have the triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Y \oplus TX \xrightarrow{h} TX.$$

Applying $\mathrm{Hom}_{\mathcal{C}}(X,\ )$ we obtain the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(X,X) \xrightarrow{f^*} \mathrm{Hom}(X,Y) \xrightarrow{g^*}$$
$$\mathrm{Hom}(X, Y \oplus TX) \xrightarrow{h^*} \mathrm{Hom}(X, TX) \longrightarrow \cdots .$$

Since $\dim_k \mathrm{Hom}(X, Y \oplus TX) = \dim_k \mathrm{Hom}(X,Y) + \dim_k \mathrm{Hom}(X,TX)$, by the exactness of the sequence, we have $f^* = 0$. Then $f = 0$ and $Tf = 0$. Because $|W(X, Y \oplus TX; Y)| = |W(Y, TX; Y \oplus TX)|$, it suffices to count the number of the triangles of the form

$$Y \xrightarrow{g} Y \oplus TX \xrightarrow{h} TX \xrightarrow{0} TY.$$

g is a section and h is a retraction. We get the orbit space $U(Y, TX; Y \oplus TX)$ by using the action (2.3) of $\mathrm{Aut}(Y \oplus TX)$ on $W(Y, TX; Y \oplus TX)$. But this action may not be free. It is necessary to calculate the isotropy groups of this action. Following [8] or [7] the isotropy group is isomorphic to $\mathrm{Hom}(TX, Y)$. Meanwhile it is easy to see that the orbit space $U(Y, TX; Y \oplus TX)$ is trivial. So we have the formula in Lemma 2.10. $\square$

Later we will need the following known fact.

Lemma 2.11. Let B be a finite-dimensional algebra, $X, Y \in \mathrm{mod}\, B$. We have the canonical isomorphism $\mathrm{Hom}_{D^b(B)}(Y, TX) \simeq \mathrm{Ext}_B(Y, X)$, moreover, the correspondence is given by: $w \in \mathrm{Hom}_{D^b(B)}(Y, TX)$ goes to the exact sequences $0 \to X \xrightarrow{u} L \xrightarrow{v} Y \to 0$ in $\mathrm{Ext}_B(Y, X)$ such that $X \xrightarrow{u} L \xrightarrow{v} Y \xrightarrow{w} TX$ is a triangle in $D^b(B)$; particularly L must belong to $\mathrm{mod}\, B$. $\square$

3. Hall polynomials of a derived category

Let k be a finite field, A be a finite dimensional hereditary k-algebra and $D^b(A)$ the derived category of $\mathrm{mod}\, A$, which is a triangulated category with the translation functor T. we always assume that A is finite representation type.

By a lemma in [6, 5.2], any indecomposable object in $D^b(A)$ is isomorphic to a stalk complex with indecomposable stalk. Let $A^E = A \otimes_k E$ for every field extension E of k, and $X^E = X \otimes_k E$ is well-defined for any indecomposable object X in $D^b(A)$ which is taken as a stalk complex. Then X^E is still a stalk complex in $D^b(A^E)$. Let M be any object in $D^b(A)$. Then $M = X_1 \oplus \cdots \oplus X_t$ with each X_i indecomposable. We define $M^E = X_1^E \oplus \cdots \oplus X_t^E$. We denote by $D^b(A)^E$ the full subcategory of $D^b(A^E)$ whose objects are all M^E with $M \in D^b(A)$.

If a field E is a finite extension of k. Remember from [14] that E is called conservative for $D^b(A)$ provided $(\mathrm{End}\, X / \mathrm{rad}\, \mathrm{End}\, X)^E$ is a field for any indecomposable object $X \in D^b(A)$. It is easily seen that E is conservative for $D^b(A)$ if and only if so is E for mod A. Let Ω be the set of isomorphism classes of finite extension of k which are conservative for $D^b(A)$. It follows that Ω is an infinite set from [14]. Following [14] we have the following definition.

For objects $M, N, L \in D^b(A)$, if there exists an integral polynomial $\varphi^L_{MN} \in \mathbf{Z}[q]$ such that

$$\varphi^L_{MN}(|E|) = F^{L^E}_{M^E N^E} = |V(M^E, N^E; L^E)|$$

holds for any $E \in \Omega$. Then we say that $D^b(A)$ has the Hall polynomial φ^L_{MN}.

Lemma 3.1. Let X, Y and L be objects in $\mathcal{R}$ with X and Y indecomposable. If there is a hereditary subcategory $\mathcal{B}$ which contains X, Y and L, then the Hall polynomial φ^L_{XY} used here is as same as that defined by Ringel[11].

Proof. By Lemma 2.6 we have the formula $|W(X, Y; L)| = |\operatorname{Aut} X||V(X, Y; L)||\operatorname{Aut} Y|$. According to Lemma 2.11, $W(X, Y; L)$ is just the set

$$\{(f, g) \in \operatorname{Hom}_{\mathcal{B}}(X, L) \times \operatorname{Hom}_{\mathcal{B}}(L, Y) \mid f \textit{ injective}, g \textit{ surjective and } fg = 0 \textit{ in } \mathcal{B}\}$$

Comparing with the formula in the Section 3 of [8] we get that $|V(X, Y; L)|$ equals the number of filtrations $L \supseteq U \supseteq 0$ of L such that $U \simeq X$ and $L/U \simeq Y$ as modules in $\mathcal{B}$. So the proof is completed. $\square$

Lemma 3.2. We have $D^b(A)^E = D^b(A^E)$ for any $E \in \Omega$.

Proof. It follows $(\operatorname{mod} A)^E = \operatorname{mod} A^E$ from [8]. Therefore every indecomposable object in $D^b(A^E)$ can be taken as a stalk complex which isomorphic to X^E with X being a stalk complex in $D^b(A)$. so we have $D^b(A)^E = D^b(A^E)$. $\square$

Let T' be the translation functor of $D^b(A^E)$ which is induced by the shifting functor. It is easy to see that $T'M^E \simeq (TM)^E$ for any $M \in D^b(A)$. We may still write $T' = T$. Then $D^b(A)^E/T^2$ is canonically isomorphic to $D^b(A^E)/T^2$.

Lemma 3.3. Let $M \in D^b(A)$ and $E \in \Omega$. If L is a direct summend of M^E in $D^b(A^E)$, then $L \simeq N^E$ with N being a direct summand of M in $D^b(A)$ and independent on E.

Proof. Write $M = M_1 \oplus \cdots \oplus M_t$ with M_i indecomposable. By the well-known Krull-Schmidt Theorem, $L \simeq M^E_{i_1} \oplus \cdots \oplus M^E_{i_s}$. Take $N \simeq M_{i_1} \oplus \cdots \oplus M_{i_s}$. Then $L \simeq N^E$. $\square$

Proposition 3.1. $D^b(A)$ has the polynomials φ^L_{MN} and φ^L_{NM} for any $M, N, L \in D^b(A)$ with N indecomposable.

Proof. We first show the existence of φ^L_{NM}. Without loss of generality, we may assume tht N is a simple projective object of a hereditary category $\mathcal{B}$ in $D^b(A)$. Decompose $L = L_1 \oplus L_2$ with L_1 in $\mathcal{B}$ and no indecomposable direct summand of L_2 belong to $\mathcal{B}$. Then $\mathrm{Hom}_{D^b(A)}(N, L_2) = 0$. For any $E \in \Omega$ and any triangle in $D^b(A^E)$ of form

$$N^E \xrightarrow{f} L^E \xrightarrow{g} M^E \xrightarrow{h} TN^E,$$

then $f = (f_1, 0) : N^E \to L_1^E \oplus L_2^E$. Let $p : L^E \to L_2^E$ be the canonical projection. Then the composition $fp = 0$ implies that there is some $q : M^E \to L_2^E$ such that $gq = p$. So g maps L_2^E onto a direct summand of M^E. The above triangle decompose into the following form

$$N^E \xrightarrow{(f_1,0)} L_1^E \oplus L_2^E \xrightarrow{\begin{pmatrix} g_1 & g_2 \\ 0 & g_3 \end{pmatrix}} M_1^E \oplus M_2^E \xrightarrow{\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}} TN^E$$

with g_3 an isomorphism from L_2^E to M_2^E, where the decomposition $M = M_1 \oplus M_2$ is independent on E, see Lemma 3.3. It is easy to see $h_2 = 0$. By Lemma 2.4 we have $F^{L^E}_{N^E M^E} = F^{L_1^E}_{N^E M_1^E}$. So we can concentrate to consider the case $L \in \mathcal{B}$. If M is also in $\mathcal{B}$, in view of Lemma 3.1, it is just what we need that there exist the Hall polynomials φ^L_{MN} and φ^L_{NM} in $\mathcal{B}$ which are asserted in [11]. Therefore we only need to consider that M has an indecomposable direct summand M_2 not belonging to $\mathcal{B}$. If for any $E \in \Omega$ there exists no triangle of form $N^E \to L^E \to M^E \to TN^E$, we obtain that $\varphi^L_{NM} = 0$. Suppose that there is some $E \in \Omega$ and the triangle in $D^b(A^E)$ of form

$$N^E \xrightarrow{f} L^E \xrightarrow{(g_1,g_2)} M_1^E \oplus M_2^E \xrightarrow{\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}} TN^E$$

with $M^E \simeq M_1^E \oplus M_2^E$. If $h_2 = 0$, then L^E contains a direct summand isomorphic to M_2^E and so L contains a direct summand isomorphic to M_2 by Lemma 3.2, a contradiction since $M_2 \notin \mathcal{B}$. Hence we have $h_2 \neq 0$. By the assumption that N is the simple projective object in $\mathcal{B}$ and $M_2 \notin \mathcal{B}$, we see that h_2 must be an isomorphism. Then $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ is a retraction. Therefore $f = 0$ and $L \simeq M_1, \quad M \simeq L \oplus M_2$. This means that $F^{L^E}_{N^E,(L^E \oplus M_2^E)} = 1$ for any $E \in \Omega$. In this case $\varphi^L_{NM} = 1$. So we have proved that $D^b(A)$ has the Hall polynomial φ^E_{NM} in all cases.

Dually we can prove that there exists φ^L_{MN} for $D^b(A)$. $\square$

Completely similar to the Proposition 3.1 we have

Proposition 3.2. $\mathcal{R}$ has the polynomials φ^L_{MN} and φ^L_{NM} for any $M, N, L \in \mathcal{R}$ with N indecomposable.$\square$

Lemma 3.4. Let $X, Y \in \operatorname{ind}\mathcal{R}$ and a nonzero decomposable object $L \in \mathcal{R}$ satisfy $L \not\simeq X \oplus Y$. Then the Hall polynomial φ^L_{XY} is divided by $q-1$.

Proof. According to Lemma 2.6 and 2.8, we have

$$\begin{aligned} |V(X^E, Y^E; L^E)| &= |W(X^E, Y^E; L^E)|/(|\operatorname{Aut} X^E||\operatorname{Aut} Y^E|) \\ &= |U(X^E, Y^E; L^E)||\operatorname{Aut} L^E|/(|\operatorname{Aut} X^E||\operatorname{Aut} Y^E|). \end{aligned}$$

Although the category $\mathcal{R}$ is not directing, because $\mathcal{R}$ has no short cycle, the same method used by Ringel implies that $|\operatorname{Aut} L^E|$ is divided by $(|E|-1)^2$ (see [11],p440), or as a consequence of a general formula given in [8]. By Lemma 2.9 $|E|-1$ divide $|U(X^E, Y^E; L^E)|$. It follows that φ^L_{XY} is divided by $q-1$. $\square$

4. Lie algebra arising in a root category

Let $\mathcal{C}$ be a $'$almost$'$ triangulated categry over a central field k. In general, we assume that any object in $\mathcal{C}$ can be decomposed into the direct sum of finitely many indecomposable ones and the endomorphism ring of any indecomposable object is a finite dimensional local algebra over k. For any indecomposable object X, we denote $d(X) = \dim_k(\operatorname{End}_{\mathcal{C}} X/\operatorname{rad}\operatorname{End}_{\mathcal{C}} X)$. Now we define the weighted Grothendick group of $\mathcal{C}$, denoted by $G_w(\mathcal{C})$, to be the quotient of a free ablian group with a basis $\{H_{[M]}\}_{[M]}$, indexed by the isomorphism classes of all objects in $\mathcal{C}$ subject to the relations

$$H_{[M\oplus N]} - H_{[M]} - H_{[N]}$$

and

$$\sum_{i=1}^{l} d(X_i)H_{[X_i]} - \sum_{i=1}^{m} d(Y_i)H_{[Y_i]} + \sum_{i=1}^{n} d(Z_i)H_{[Z_i]}$$

provided there exists a triangle of form

$$\oplus_{i=1}^{l} X_i \longrightarrow \oplus_{i=1}^{m} Y_i \longrightarrow \oplus_{i=1}^{n} Z_i \longrightarrow T(\oplus_{i=1}^{l} X_i).$$

with X_i, Y_i and Z_i being indecomposable. We still denote in $G_w(\mathcal{C})$ by $H_{[M]}$ the corresponding basis element in the free ablian group, and instead of $H_{[M]}$ we usually write H_M

We also will consider the Grothendieck group of $\mathcal{C}$ as usual, denoted by $G(\mathcal{C})$, it is the quotient of a free ablian group with a basis $\{[M] \mid M \in \mathcal{C}\}$ modulo the relations $[X]-[Y]+[Z]$ provided there exists a triangle of form $X \to Y \to Z \to TX$.

¿From now on, we assume that k is a finite field and A is a finite dimensional associative k-algebra with unitary element which is hereditary and representation-finite.

Put $\mathcal{R} = D^b(A)/T^2$ the root category of A. For $M, N \in \mathcal{R}$, natually $\operatorname{Hom}_{\mathcal{R}}(M,N)$ becomes $\operatorname{End}_{\mathcal{R}} M$-$\operatorname{End}_{\mathcal{R}} N$-bimodule. We denote by $l({}_{\operatorname{End}_{\mathcal{R}} M}\operatorname{Hom}_{\mathcal{R}}(M,N))$ and $l(\operatorname{Hom}_{\mathcal{R}}(M,N)_{\operatorname{End}_{\mathcal{R}} N})$ the lengths of $\operatorname{Hom}_{\mathcal{R}}(M,N)$ respectively as a left $\operatorname{End}_{\mathcal{R}} M$-module and as a right $\operatorname{End}_{\mathcal{R}} N$-module.

We define a bilinear function $<-|->$ on $G_w(\mathcal{R})\times G(\mathcal{R})$ as follows: For $X,Y\in\mathcal{R}$, if X is indecomposable, then we define

$$< H_X \mid [Y] >= l({}_{\mathrm{End}_{\mathcal{R}} X}\mathrm{Hom}_{\mathcal{R}}(X,Y)) - l({}_{\mathrm{End}_{\mathcal{R}} X}\mathrm{Hom}_{\mathcal{R}}(X,TY))$$
$$+l(\mathrm{Hom}_{\mathcal{R}}(Y,X)_{\mathrm{End}_{\mathcal{R}} X}) - l(\mathrm{Hom}_{\mathcal{R}}(Y,TX)_{\mathrm{End}_{\mathcal{R}} TX});$$

if $X=\oplus_{i=1}^m X_i$ with all X_i indecomposable, then we define

$$< H_X \mid [Y] >= \sum_{i=1}^m < H_{X_i} \mid [Y] > .$$

Then we extend bilinearly this definition onto the whole $G_w(\mathcal{R})\times G(\mathcal{R})$. It is not difficult to check that this is well-defined. Note that for any indecomposable objects $X,Y\in\mathcal{R}$, we have $d(X)< H_X \mid [Y] >= d(Y) < H_Y \mid [X] >$.

Now we denote by $\mathcal{H}=G_w(\mathcal{R})$ the weighted Grothendieck group of $\mathcal{R}$. Let $\mathcal{N}$ be a free ablian group with a basis $\{u_{[X]}\}_{[X]}$, indexed by the isomorphism classes of all indecomposable objects in $\mathcal{R}$. We also write u_X for $u_{[X]}$. Put $\mathcal{G}=\mathcal{H}\oplus\mathcal{N}$ a direct sum of $\mathbf{Z}$-modules. We want to define an operation on $\mathcal{G}$ using Hall polynomials and the above bilinear functuion such that $\mathcal{G}$ becomes a Lie algebra. For any $L,M,N\in\mathcal{R}$ with M or N indecomposable, we put $\gamma_{MN}^L=\varphi_{MN}^L(1)-\varphi_{NM}^L(1)$, where φ_{MN}^L and φ_{NM}^L are the Hall polynomials and they always exist as seen in Section 3. Now we define a bilinear operation $[-,-]$ on $\mathcal{G}$ as follows:

(a) For any two indecomposable objects $X,Y\in\mathcal{R}$,

$$[u_X,u_Y]=\begin{cases}\sum_{[L],\ L\in\mathrm{ind}\,\mathcal{R}}\gamma_{XY}^L u_{[L]} & \text{if } Y\not\simeq TX\\ H_X & \text{if } Y\simeq TX\end{cases}$$

(b) For any objects $X,Y\in\mathcal{R}$ with Y indecomposable,

$$[H_X,u_Y]=-< H_X \mid [Y] > u_Y \quad\text{and}\quad [u_Y,H_X]=-[H_X,u_Y].$$

(c) $[\mathcal{H},\mathcal{H}]=0$,
and we extend $[-,-]$ bilinearly onto the whole $\mathcal{G}$. This is a well-defined operation since $<-|->$ is the bilinear function on $G_w(\mathcal{R})\times G(\mathcal{R})$.

With these notations, we have the following

Theorem 4.1. $\mathcal{G}$ together with the operation $[-,-]$ is a Lie algebra.

To prove the theorem, It suffices to show the Jacobi identity. We need the following preparation.

Proposition 4.1. Let $X,Y,Z\in\mathrm{ind}\,\mathcal{R}$. If $Y\not\simeq TX$, $Y\not\simeq TZ$ and $X\not\simeq TZ$, then $[[u_X,u_Y],u_Z]=[[u_X,u_Z],u_Y]+[u_X,[u_Y,u_Z]]$.

We remove the proofs of Proposition 4.1 and Theorem 4.1 to Section 5.

Let $P(1), \cdots, P(n)$ be a complete set of representatives of the isomorphism classes of indecomposable projective $A-$modules. For an $A-$module X, its dimension vector is defined by $\underline{dim}\ X = (l({}_{\mathrm{End}_A P(i)}\mathrm{Hom}_A(P(i),X)))_{i=1}^n$. The map $X \to \underline{dim}\ X$ induces an isomorphism of $G_W(A)$ with $\mathbf{Z}^n$. This can be extended to $\mathcal{R}$. If $X \in \mathcal{R}$, then there is $M \in D^b(A)$ such that $FM = X$. We can write $M = (M^i, d^i)$, a complex over A, and obtain $\underline{dim}X = \underline{dim}M = \sum_{i\in\mathbf{Z}}(-1)^i\underline{dim}M^i$. Note that we use two different notations to represent a same thing, in fact, H_X and $\underline{dim}X$ are essentially same. As pointed out in [5], the map $\underline{dim}$ induces a bijection between the isomorphism classes of indecomposables in $\mathcal{R}$ and the root system $\Phi(\mathcal{R})$, where the Dynkin graph of $\Phi(\mathcal{R})$ is just the graph of any complete section of $\mathcal{R}$. Denote by $\mathbf{g}(\Phi(\mathcal{R}))$ the simple ! Lie algebra with the root system $\Phi(\mathcal{R})$. Our main result is the following.

Theorem 4.2. Let $A, \mathcal{R}, \mathcal{H}$, and $\mathcal{N}$ as above. Then $\mathcal{G} = \mathcal{H} \oplus \mathcal{N}$ together with the operation $[-,-]$ is the $\mathbf{Z}-$form of $\mathbf{g}(\Phi(\mathcal{R}))$. In particular, $\mathcal{G} \otimes_{\mathbf{Z}} \mathbf{C}$ is isomorphic to $\mathbf{g}(\Phi(\mathcal{R}))$.

Proof. Choosing arbitrary complete section of $\mathcal{R}$, we get a hereditary subcategory $\mathcal{B}$ of $\mathcal{R}$ corresponding to it. Let $S_1, \cdots, S_n$ be the complete list of all non-isomorphic simple objects in $\mathcal{B}$. It is well-known that the Cartan matrix $(a_{ij})_{n\times n}$ of $\phi(\mathcal{R})$ is given by $a_{ij} = -l({}_{\mathrm{End}\, S_i}\mathrm{Ext}_{\mathcal{B}}(S_i,S_j)) - l(\mathrm{Ext}_{\mathcal{B}}(S_j,S_i)_{EndS_i})$ for $i \neq j$ and $a_{ii} = 2$. Since $\mathrm{Ext}_{\mathcal{B}}(S_i,S_j) \simeq \mathrm{Hom}_{\mathcal{R}}(S_i,TS_j)$, we always have

$$\begin{aligned} a_{ij} = < H_{S_i} \mid [S_j] >= & l({}_{\mathrm{End}\, S_i}\mathrm{Hom}_{\mathcal{R}}(S_i,S_j)) - l({}_{\mathrm{End}\, S_i}\mathrm{Hom}_{\mathcal{R}}(S_i,TS_j)) \\ & + l({}_{\mathrm{End}\, S_i}\mathrm{Hom}_{\mathcal{R}}(S_j,S_i)) - l(\mathrm{Hom}_{\mathcal{R}}(S_j,TS_i)_{\mathrm{End}\, TS_i}). \end{aligned}$$

Put $e_i = u_{S_i}, f_i = -u_{TS_i}$ and $h_i = -H_{S_i}$. We check the following.

(1) $[h_i, h_j] = 0$ for all $1 \leq i, j \leq n$.

(2) $[e_i, f_i] = h_i$ and $[e_i, f_j] = 0$ if $i \neq j$.

We have $[e_i, f_i] = [u_{S_i}, -u_{TS_i}] = -H_{S_i} = h_i$. If $i \neq j$ then $\mathrm{Hom}(S_i,S_j) = \mathrm{Hom}(S_j,S_i) = 0$; any triangle of form: $S_i \xrightarrow{f} M \xrightarrow{g} TS_j \xrightarrow{h} TS_i$ must be split, i.e, $h = 0$ and $M \simeq S_i \oplus TS_j$. Therefore $[u_{S_i}, u_{S_j}] = 0$, then $[e_i, f_j] = 0$.

(3) $[h_i, e_j] = a_{ij}e_j, [h_i, f_j] = -a_{ij}f_j$.

Obtained from the definition directly.

(4) $(ade_i)^{-a_{ij}+1}e_j = 0$ and $(adf_i)^{-a_{ij}+1}f_j = 0$ for $i \neq j$.

This is the Serre relation. We see from Lemma 3.1 that our Lie operation $[u_M, u_N]$ coincides with one defined by Ringel if both M and N lie in a hereditary subcategory.It was shown by him that the generators $u_{S_i}, 1 \leq i \leq n$ satisfy the Serre relation [10]. So we have the formulae (4).

Then Serre's Theorem tells us that there exists a natural epimorphism from $\mathbf{g}(\Phi(\mathcal{R}))$ to $\mathcal{G} \otimes_{\mathbf{Z}} \mathbf{C}$, which maps $\mathbf{g}_\alpha, \alpha \in \Phi(\mathcal{R})$, to the subspace of $\mathcal{G}\otimes_{\mathbf{Z}}\mathbf{C}$ generated by u_X where X is indecomposable such that $\underline{dim}\ X = \alpha$, and maps h_i to $-H_{S_i}$ for $1 \leq i \leq n$. Of course it must be an isomorphism. $\square$

In the definition of the Lie algebra $\mathcal{G} = \mathcal{H} \oplus \mathcal{N}$, the structure constants are always integral. The set $\{ u_M \mid M \in \operatorname{ind} \mathcal{R} \}$ constitute a $\mathbf{Z}-$basis of $\mathbf{g}(\Phi)$. Choosing arbitrary complete section of $\mathcal{R}$, we get a hereditary subcategory $\mathcal{B}$ of $\mathcal{R}$ corresponding to it. Then $\Phi = \Phi^+ \cup \Phi^-$, disjoint union of positive roots and nective roots, according to $\mathcal{B}$. We define $\{ x_\alpha \mid \alpha \in \Phi \}$ as follows: $x_\alpha = u_X$ if $\alpha \in \Phi^+$, where $X \in \operatorname{ind} \mathcal{B}$ with $\underline{dim} X = \alpha$; $x_\alpha = -u_{TX}$ if $\alpha \in \Phi^-$, where $X \in \operatorname{ind} \mathcal{B}$ with $\underline{dim} X = -\alpha$.

Proposition 4.2. The set $\{ x_\alpha \mid \alpha \in \Phi(\mathcal{R}) \}$ is a Chevalley basis of $\mathcal{G}$.

Proof. Let $h_\alpha = [x_\alpha, x_{-\alpha}]$. If $\alpha, \beta, \alpha+\beta \in \Phi$, suppose that $[x_\alpha, x_\beta] = c_{\alpha,\beta} x_{\alpha+\beta}$. We need to prove $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$. If $\alpha, \beta, \alpha+\beta \in \Phi^+$, let X, Y and Z in $\operatorname{ind} \mathcal{B}$ such that $\underline{dim} X = \alpha, \underline{dim} Y = \beta, \underline{dim} Z = \alpha + \beta$. Then $[x_\alpha, x_\beta] = [u_X, u_Y] = c_{\alpha,\beta} u_Z$, $[x_{-\alpha}, x_{-\beta}] = [-u_{TX}, -u_{TY}] = c_{\alpha,\beta} u_{TZ} = -c_{\alpha,\beta} x_{-\alpha,-\beta}$. If $\alpha \in \Phi^+, \beta \in \Phi^-$ but $\alpha + \beta \in \Phi^+$, let X, Y and Z in $\operatorname{ind} \mathcal{B}$ such that $\underline{dim} X = \alpha, \underline{dim} Y = -\beta, \underline{dim} Z = \alpha + \beta$. Then $[x_\alpha, x_\beta] = [u_X, -u_{TY}] = c_{\alpha,\beta} u_Z$, $[x_{-\alpha}, x_{-\beta}] = [-u_{TX}, u_Y] = c_{\alpha,\beta} u_{TZ} = -c_{\alpha,\beta} x_{-\alpha,-\beta}$. So we can claim that $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$ for all $\alpha, \beta, \alpha+\beta \in \phi$. The proof is finished. $\square$

By the Chevalley's Theorem, we have

Corollary 4.2.1. If $X, Y, Z \in \operatorname{ind} \mathcal{R}$ with $Y \neq TX$ and $\underline{dim} Z = \underline{dim} X + \underline{dim} Y$, then $\gamma_{XY}^Z = \pm(r+1)$, where r is the greatest integer such that $\underline{dim} Y - r \underline{dim} X \in \Phi$.

Note 1. The automorphism σ of $\mathcal{G}$ defined by $\sigma(H_X) = H_{TX}$ and $\sigma(u_X) = u_{TX}$ is the Chevalley involution. 2. The $\pm$ signs in Corollary 4.2.1 is not difficult to be determined from the structure of $\mathcal{R}$; in fact, if $X < Y$ in a hereditary subcategory then $\gamma_{XY}^Z = r+1$; if $Y < X$ in a hereditary subcategory then $\gamma_{XY}^Z = -(r+1)$. 3. For the simply-laced cases ($A_n, D_n, E_i, i = 6,7,8$), $d(X) = 1$ for any indecomposable $X \in \mathcal{R}$. So the weighted Grothendieck group is just the ordinary Grothendieck group. For the non-simply-laced cases (B_n, C_n, F_4, G_2), the 'weighted' is really necessary.

5. Proofs of Proposition 4.1 and Theorem 4.1

Proof of Proposition 4.1 By the defination we have

$$[[u_X, u_Y], u_Z] = \sum_{[L],[M]:\ L,M \in \operatorname{ind} \mathcal{R}} \gamma_{XY}^L \gamma_{LZ}^M u_M,$$

$$[u_X, [u_Y, u_Z]] = \sum_{[L'],[M]:\ L',M \in \operatorname{ind} \mathcal{R}} \gamma_{YZ}^{L'} \gamma_{XL'}^M u_M$$

and

$$[[u_X, u_Z], u_Y] = \sum_{[L''],[M]:\ L'',M \in \operatorname{ind} \mathcal{R}} \gamma_{XZ}^{L''} \gamma_{L''Y}^M u_M.$$

So it is enough to show that for any $M \in \operatorname{ind} \mathcal{R}$ we have

$$\sum_{[L]:\ L \in \operatorname{ind} \mathcal{R}} \gamma_{XY}^L \gamma_{LZ}^M = \sum_{[L']:\ L' \in \operatorname{ind} \mathcal{R}} \gamma_{YZ}^{L'} \gamma_{XL'}^M + \sum_{[L'']:\ L'' \in \operatorname{ind} \mathcal{R}} \gamma_{XZ}^{L''} \gamma_{L''Y}^M.$$

this is equivalent to the following equation by Lemma 3.4 and the obvious fact $\gamma_{XY}^{X\oplus Y} = \varphi_{XY}^{X\oplus Y}(1) - \varphi_{YX}^{X\oplus Y}(1) = 0$.

$$\sum_{[L]:\ L\in\mathcal{R}} \gamma_{XY}^{L}\gamma_{LZ}^{M} = \sum_{[L']:\ L'\in\mathcal{R}} \gamma_{YZ}^{L'}\gamma_{XL'}^{M} + \sum_{[L'']:\ L''\in\mathcal{R}} \gamma_{XZ}^{L''}\gamma_{L''Y}^{M}.$$

That is,

$$\sum_{[L]:\ L\in\mathcal{R}} (\varphi_{XY}^{L}(1)\varphi_{LZ}^{M}(1) + \varphi_{YX}^{L}(1)\varphi_{ZL}^{M}(1) - \varphi_{XY}^{L}(1)\varphi_{ZL}^{M}(1) - \varphi_{YX}^{L}(1)\varphi_{LZ}^{M}(1))$$
$$= \sum_{[L']:\ L'\in\mathcal{R}} (\varphi_{YZ}^{L'}(1)\varphi_{XL'}^{M}(1) + \varphi_{ZY}^{L'}(1)\varphi_{L'X}^{M}(1) - \varphi_{ZY}^{L'}(1)\varphi_{XL'}^{M}(1) - \varphi_{YZ}^{L'}(1)\varphi_{L'X}^{M}(1))$$
$$+ \sum_{[L'']:\ L''\in\mathcal{R}} (\varphi_{XZ}^{L''}(1)\varphi_{L''Y}^{M}(1) + \varphi_{ZX}^{L''}(1)\varphi_{YL''}^{M}(1) - \varphi_{XZ}^{L''}(1)\varphi_{YL''}^{M}(1) - \varphi_{ZX}^{L''}(1)\varphi_{L''Y}^{M}(1)).$$

For any $E\in\Omega$, by the Proposition 2.1 we have

$$\sum_{[L],\ L\in\mathcal{R}} |(W(X^E,Y^E;L^E)\times W(Z^E,L^E;M^E))^\wedge|$$
$$= \sum_{[L'],\ L'\in\mathcal{R}} |(W(Z^E,X^E;L'^E)\times W(L'^E,Y^E;M^E))^\wedge|.$$

It follows from Lemma 2.5

$$\sum_{[L],\ L\in\mathcal{R}} \frac{|W(X^E,Y^E;L^E)|\ |W(Z^E,L^E;M^E)|}{|\operatorname{Aut} L^E|}$$
$$= \sum_{[L'],\ L'\in\mathcal{R}} \frac{|W(Z^E,X^E;L'^E)|\ |W(L'^E,Y^E;M^E)|}{|\operatorname{Aut} L'^E|}.$$

Hence by Lemma 2.6 and Lemma 2.7 we have

$$\sum_{[L],\ L\not\simeq M\oplus TZ} \frac{|V(X^E,Y^E;L^E)|\ |\operatorname{Aut} X^E|\ |\operatorname{Aut} Y^E|\ |V(Z^E,L^E;M^E)|\ |\operatorname{Aut} Z^E|\ |\operatorname{Aut} L^E|}{|\operatorname{Aut} L^E|}$$
$$+\frac{|W(X^E,Y^E;M^E\oplus TZ^E)|\ |W(Z^E,M^E\oplus TZ^E;M^E)|}{|\operatorname{Aut}(M^E\oplus TZ^E)|}$$
$$= \sum_{[L'],\ L'\not\simeq M\oplus T^{-1}Y} \frac{|V(Z^E,X^E;L'^E)|\ |\operatorname{Aut} Z^E|\ |\operatorname{Aut} X^E|\ |V(L'^E,Y^E;M^E)|\ |\operatorname{Aut} L'^E|\ |\operatorname{Aut} Y^E|}{|\operatorname{Aut} L'^E|}$$
$$+\frac{|W(Z^E,X^E;M^E\oplus T^{-1}Y^E)|\ |W(M^E\oplus T^{-1}Y^E,Y^E;M^E)|}{|\operatorname{Aut}(M^E\oplus T^{-1}Y^E)|}.$$

¿From Lemma 2.10 we obtain

$$\sum_{[L],\ L\not\simeq M\oplus TZ} |\operatorname{Aut} X^E|\ |\operatorname{Aut} Y^E|\ |\operatorname{Aut} Z^E|\ |V(X^E,Y^E;L^E)|\ |V(Z^E,L^E;M^E)|$$

$$+\frac{|W(X^E,Y^E;M^E\oplus TZ^E)|}{|\operatorname{Hom}(TZ^E,M^E)|}$$

$$=\sum_{[L'],\ L'\not\simeq M\oplus T^{-1}Y}|\operatorname{Aut}X^E|\,|\operatorname{Aut}Z^E|\,|\operatorname{Aut}Y^E|\,|V(Z^E,X^E;L'^E)|\,|V(L'^E,Y^E;M^E)|$$

$$+\frac{|W(Z^E,X^E;M^E\oplus T^{-1}Y^E)|}{|\operatorname{Hom}(M^E,T^{-1}Y^E)|}.$$

Similarly we have

$$\sum_{[L],\ L\not\simeq M\oplus TX}|\operatorname{Aut}X^E|\,|\operatorname{Aut}Y^E|\,|\operatorname{Aut}Z^E|\,|V(Y^E,Z^E;L^E)|\,|V(X^E,L^E;M^E)|$$

$$+\frac{|W(Y^E,Z^E;M^E\oplus TX^E)|}{|\operatorname{Hom}(TX^E,M^E)|}$$

$$=\sum_{[L'],\ L'\not\simeq M\oplus T^{-1}Z}|\operatorname{Aut}X^E|\,|\operatorname{Aut}Z^E|\,|\operatorname{Aut}Y^E|\,|V(X^E,Y^E;L'^E)|\,|V(L'^E,Z^E;M^E)|$$

$$+\frac{|W(X^E,Y^E;M^E\oplus T^{-1}Z^E)|}{|\operatorname{Hom}(M^E,T^{-1}Z^E)|},$$

$$\sum_{[L],\ L\not\simeq M\oplus TZ}|\operatorname{Aut}X^E|\,|\operatorname{Aut}Y^E|\,|\operatorname{Aut}Z^E|\,|V(Y^E,X^E;L^E)|\,|V(Z^E,L^E;M^E)|$$

$$+\frac{|W(Y^E,X^E;M^E\oplus TZ^E)|}{|\operatorname{Hom}(TZ^E,M^E)|}$$

$$=\sum_{[L'],\ L'\not\simeq M\oplus T^{-1}X}|\operatorname{Aut}X^E|\,|\operatorname{Aut}Z^E|\,|\operatorname{Aut}Y^E|\,|V(Z^E,Y^E;L'^E)|\,|V(L'^E,X^E;M^E)|$$

$$+\frac{|W(Z^E,Y^E;M^E\oplus T^{-1}X^E)|}{|\operatorname{Hom}(M^E,T^{-1}X^E)|},$$

$$\sum_{[L],\ L\not\simeq M\oplus TY}|\operatorname{Aut}X^E|\,|\operatorname{Aut}Y^E|\,|\operatorname{Aut}Z^E|\,|V(X^E,Z^E;L^E)|\,|V(Y^E,L^E;M^E)|$$

$$+\frac{|W(X^E,Z^E;M^E\oplus TY^E)|}{|\operatorname{Hom}(TY^E,M^E)|}$$

$$=\sum_{[L'],\ L'\not\simeq M\oplus T^{-1}Z}|\operatorname{Aut}X^E|\,|\operatorname{Aut}Z^E|\,|\operatorname{Aut}Y^E|\,|V(Y^E,X^E;L'^E)|\,|V(L'^E,Z^E;M^E)|$$

$$+\frac{|W(Y^E,X^E;M^E\oplus T^{-1}Z^E)|}{|\operatorname{Hom}(M^E,T^{-1}Z^E)|},$$

$$\sum_{[L],\ L\not\simeq M\oplus TX}|\operatorname{Aut}X^E|\,|\operatorname{Aut}Y^E|\,|\operatorname{Aut}Z^E|\,|V(Z^E,Y^E;L^E)|\,|V(X^E,L^E;M^E)|$$

$$+\frac{|W(Z^E,Y^E;M^E\oplus TX^E)|}{|\operatorname{Hom}(TX^E,M^E)|}$$

$$= \sum_{[L'],\ L'\not\simeq M\oplus T^{-1}Y} |\operatorname{Aut} X^E|\ |\operatorname{Aut} Z^E|\ |\operatorname{Aut} Y^E|\ |V(X^E, Z^E; L'^E)|\ |V(L'^E, Y^E; M^E)| + \frac{|W(X^E, Z^E; M^E \oplus T^{-1}Y^E)|}{|\operatorname{Hom}(M^E, T^{-1}Y^E)|}$$

and

$$\sum_{[L],\ L\not\simeq M\oplus TY} |\operatorname{Aut} X^E|\ |\operatorname{Aut} Y^E|\ |\operatorname{Aut} Z^E|\ |V(Z^E, X^E; L^E)|\ |V(Y^E, L^E; M^E)| + \frac{|W(Z^E, X^E; M^E \oplus TY^E)|}{|\operatorname{Hom}(TY^E, M^E)|}$$

$$= \sum_{[L'],\ L'\not\simeq M\oplus T^{-1}X} |\operatorname{Aut} X^E|\ |\operatorname{Aut} Z^E|\ |\operatorname{Aut} Y^E|\ |V(Y^E, Z^E; L'^E)|\ |V(L'^E, X^E; M^E)| + \frac{|W(Y^E, Z^E; M^E \oplus T^{-1}X^E)|}{|\operatorname{Hom}(M^E, T^{-1}X^E)|}.$$

If M is indecomposable, there exists a polynomial $g_M \in \mathbf{Z}[q]$ such that $g_M(|E|) = |\operatorname{Aut} M^E|$ for any $E \in \Omega$ and the degree of the factor $q-1$ in g_M is just 1. In addition, for any $M, N \in \mathcal{R}$, let f_{MN} be an integral polynomial which satisfies $f_{MN}(|E|) = |\operatorname{Hom}_{\mathcal{R}}(M^E, N^E)|$ for all $E \in \Omega$.

According to Lemma 2.6, for any $E \in \Omega$ we have

$$\sum_{[L]:\ L\not\simeq M\oplus TZ} (\varphi^L_{XY}(|E|)\varphi^M_{ZL}(|E|) + \frac{\varphi^{M\oplus TZ}_{XY}(|E|)}{|\operatorname{Aut} Z^E| f_{TZ\ M}(|E|)}$$

$$= \sum_{[L']:\ L'\not\simeq M\oplus T^{-1}Y} (\varphi^{L'}_{ZX}(|E|)\varphi^M_{L'Y}(|E|) + \frac{\varphi^{M\oplus T^{-1}Y}_{ZX}(|E|)}{|\operatorname{Aut} Y^E| f_{MT^{-1}Y}(|E|)},$$

$$\sum_{[L]:\ L\not\simeq M\oplus TX} (\varphi^L_{YZ}(|E|)\varphi^M_{XL}(|E|) + \frac{\varphi^{M\oplus TX}_{YZ}(|E|)}{|\operatorname{Aut} X^E| f_{TX\ M}(|E|)}$$

$$= \sum_{[L']:\ L'\not\simeq M\oplus T^{-1}Z} (\varphi^{L'}_{XY}(|E|)\varphi^M_{L'Z}(|E|) + \frac{\varphi^{M\oplus T^{-1}Z}_{XY}(|E|)}{|\operatorname{Aut} Z^E| f_{MT^{-1}Z}(|E|)},$$

$$\sum_{[L]:\ L\not\simeq M\oplus TZ} (\varphi^L_{YX}(|E|)\varphi^M_{ZL}(|E|) + \frac{\varphi^{M\oplus TZ}_{YX}(|E|)}{|\operatorname{Aut} Z^E| f_{TZ\ M}(|E|)}$$

$$= \sum_{[L']:\ L'\not\simeq M\oplus T^{-1}X} (\varphi^{L'}_{ZY}(|E|)\varphi^M_{L'X}(|E|) + \frac{\varphi^{M\oplus T^{-1}X}_{ZY}(|E|)}{|\operatorname{Aut} X^E| f_{MT^{-1}X}(|E|)},$$

$$\sum_{[L]:\ L\not\simeq M\oplus TY} (\varphi^L_{XZ}(|E|)\varphi^M_{YL}(|E|) + \frac{\varphi^{M\oplus TY}_{XZ}(|E|)}{|\operatorname{Aut} Y^E| f_{TY\ M}(|E|)}$$

$$= \sum_{[L']:\ L'\not\simeq M\oplus T^{-1}Z} (\varphi'^{L}_{YX}(|E|)\varphi^{M}_{L'Z}(|E|) + \frac{\varphi^{M\oplus T^{-1}Z}_{YX}(|E|)}{|\operatorname{Aut} Z^E| f_{MT^{-1}Z}(|E|)},$$

$$\sum_{[L]:\ L\not\simeq M\oplus TX} (\varphi^{L}_{ZY}(|E|)\varphi^{M}_{XL}(|E|) + \frac{\varphi^{M\oplus TX}_{ZY}(|E|)}{|\operatorname{Aut} X^E| f_{TX\ M}(|E|)}$$

$$= \sum_{[L']:\ L'\not\simeq M\oplus T^{-1}Y} (\varphi^{L'}_{XZ}(|E|)\varphi^{M}_{L'Y}(|E|) + \frac{\varphi^{M\oplus T^{-1}Y}_{XZ}(|E|)}{|\operatorname{Aut} Y^E| f_{MT^{-1}Y}(|E|)},$$

and

$$\sum_{[L]:\ L\not\simeq M\oplus TY} (\varphi^{L}_{ZX}(|E|)\varphi^{M}_{YL}(|E|) + \frac{\varphi^{M\oplus TY}_{ZX}(|E|)}{|\operatorname{Aut} Y^E| f_{TY\ M}(|E|)}$$

$$= \sum_{[L']:\ L'\not\simeq M\oplus T^{-1}X} (\varphi^{L'}_{YZ}(|E|)\varphi^{M}_{L'X}(|E|) + \frac{\varphi^{M\oplus T^{-1}X}_{YZ}(|E|)}{|\operatorname{Aut} X^E| f_{MT^{-1}X}(|E|)}.$$

We will use the following two facts: (1). $\varphi^{M\oplus TX}_{ZY}, \cdots, \varphi^{M\oplus TY}_{XZ}$ are divided by $q-1$ by Lemma 3.4; (2). for any $M, N \in \mathcal{R}$, f_{MN} equals to some power of q and hence $f_{MN}(1) = 1$. Since Ω is infinite, we have

$$\sum_{[L]}(\varphi^{L}_{XY}(1)\varphi^{M}_{ZL}(1)) + (g_Z^{-1}\varphi^{M\oplus TZ}_{XY})(1) = \sum_{[L]}(\varphi^{L}_{ZX}(1)\varphi^{M}_{LY}(1)) + (g_Y^{-1}\varphi^{M\oplus T^{-1}Y}_{ZX})(1),$$

$$\sum_{[L]}(\varphi^{L}_{YZ}(1)\varphi^{M}_{XL}(1)) + (g_X^{-1}\varphi^{M\oplus TX}_{YZ})(1) = \sum_{[L]}(\varphi^{L}_{XY}(1)\varphi^{M}_{LZ}(1)) + (g_Z^{-1}\varphi^{M\oplus T^{-1}Z}_{XY})(1),$$

$$\sum_{[L]}(\varphi^{L}_{YX}(1)\varphi^{M}_{ZL}(1)) + (g_Z^{-1}\varphi^{M\oplus TZ}_{YX})(1) = \sum_{[L]}(\varphi^{L}_{ZY}(1)\varphi^{M}_{LX}(1)) + (g_X^{-1}\varphi^{M\oplus T^{-1}X}_{ZY})(1),$$

$$\sum_{[L]}(\varphi^{L}_{XZ}(1)\varphi^{M}_{YL}(1)) + (g_Y^{-1}\varphi^{M\oplus TY}_{XZ})(1) = \sum_{[L]}(\varphi^{L}_{YX}(1)\varphi^{M}_{LZ}(1)) + (g_Z^{-1}\varphi^{M\oplus T^{-1}Z}_{YX})(1),$$

$$\sum_{[L]}(\varphi^{L}_{ZY}(1)\varphi^{M}_{XL}(1)) + (g_X^{-1}\varphi^{M\oplus TX}_{ZY})(1) = \sum_{[L]}(\varphi^{L}_{XZ}(1)\varphi^{M}_{LY}(1)) + (g_Y^{-1}\varphi^{M\oplus T^{-1}Y}_{XZ})(1)$$

and

$$\sum_{[L]}(\varphi^{L}_{ZX}(1)\varphi^{M}_{YL}(1)) + (g_Y^{-1}\varphi^{M\oplus TY}_{ZX})(1) = \sum_{[L]}(\varphi^{L}_{YZ}(1)\varphi^{M}_{LX}(1)) + (g_X^{-1}\varphi^{M\oplus T^{-1}X}_{YZ})(1).$$

Therefore,

$$\sum_{[L]}(\varphi^{L}_{XY}(1)\varphi^{M}_{LZ}(1) + \varphi^{L}_{YX}(1)\varphi^{M}_{ZL}(1) - \varphi^{L}_{XY}(1)\varphi^{M}_{ZL}(1) - \varphi^{L}_{YX}(1)\varphi^{M}_{LZ}(1))$$

$$+(g_Z^{-1}\varphi^{M\oplus T^{-1}Z}_{XY})(1) + (g_Z^{-1}\varphi^{M\oplus TZ}_{YX})(1) - (g_Z^{-1}\varphi^{M\oplus TZ}_{XY})(1) - (g_Z^{-1}\varphi^{M\oplus T^{-1}Z}_{YX})(1)$$

$$= \sum_{[L]}(\varphi^L_{YZ}(1)\varphi^M_{XL}(1) + \varphi^L_{ZY}(1)\varphi^M_{LX}(1) - \varphi^L_{ZX}(1)\varphi^M_{LY}(1) - \varphi^L_{XZ}(1)\varphi^M_{YL}(1))$$

$$+(g_X^{-1}\varphi^{M\oplus TX}_{YZ})(1) + (g_X^{-1}\varphi^{M\oplus T^{-1}X}_{ZY})(1) - (g_Y^{-1}\varphi^{M\oplus T^{-1}Y}_{ZX})(1) - (g_Y^{-1}\varphi^{M\oplus TY}_{XZ})(1)$$

$$+ \sum_{[L]}(\varphi^L_{ZY}(1)\varphi^M_{LX}(1) - \varphi^L_{XZ}(1)\varphi^M_{LY}(1) + \varphi^L_{YZ}(1)\varphi^M_{LX}(1) - \varphi^L_{ZX}(1)\varphi^M_{YL}(1))$$

$$+(g_Y^{-1}\varphi^{M\oplus T^{-1}Y}_{XZ})(1) - (g_X^{-1}\varphi^{M\oplus TX}_{ZY})(1) + (g_Y^{-1}\varphi^{M\oplus TY}_{ZX})(1) - (g_X^{-1}\varphi^{M\oplus T^{-1}X}_{YZ})(1).$$

However, we have $T^2 = 1$ in $\mathcal{R}$. So we obtain

$$\sum_{[L]}(\varphi^L_{XY}(1)\varphi^M_{LZ}(1) + \varphi^L_{YX}(1)\varphi^M_{ZL}(1) - \varphi^L_{XY}(1)\varphi^M_{ZL}(1) - \varphi^L_{YX}(1)\varphi^M_{LZ}(1))$$

$$= \sum_{[L]}(\varphi^L_{YZ}(1)\varphi^M_{XL}(1) + \varphi^L_{ZY}(1)\varphi^M_{LX}(1) - \varphi^L_{ZY}(1)\varphi^M_{XL}(1) - \varphi^L_{YZ}(1)\varphi^M_{LX}(1))$$

$$+ \sum_{[L]}(\varphi^L_{XZ}(1)\varphi^M_{LY}(1) + \varphi^L_{ZX}(1)\varphi^M_{YL}(1) - \varphi^L_{XZ}(1)\varphi^M_{YL}(1) - \varphi^L_{ZX}(1)\varphi^M_{LY}(1))$$

as required.□

Proof of Theorem 4.1. Let X, Y, Z be indecomposible objects in $\mathcal{R}$.At first we will check the Jacobi identity

$$[u_X, [u_Y, u_Z]] = [[u_X, u_Y], u_Z] + [u_Y, [u_X, u_Z]] \tag{5.1}$$

Suppose that there exists a hereditarty category $\mathcal{B}$ contains X, Y, Z. Consider any triangle of form $X \to W \to Y \to TX$ in $\mathcal{R}$. Then we must have $W \in \mathcal{B}$. In view of Lemma 3.1, the Hall polynomial φ^W_{YX} defined in $\mathcal{R}$ is equal to the ordinary one in $\mathcal{B}$ in the sense of Ringel. So the validity of the Jacobi identity for X, Y, Z in $\mathcal{R}$ is equivalent to that in $\mathcal{B}$. The latter is well-known from Ringel's investigation.

So we consider in the remainder of our proof the case that there is no hereditary category containing X, Y, Z at the same time. Note that for any indecomposable objects $M, N \in \mathcal{R}$, if there exsits no hereditary category containing both M and N, then we easily see that $M \simeq TN$ and $N \simeq TM$. So there exist at least two of X, Y, Z belonging to a hereditary category. Without loss of generality, we can assume that X, Z lie in a hereditary category $\mathcal{B}$ and hence $X \not\simeq TZ$.

If we also have $Y \not\simeq TX$ and $Y \not\simeq TZ$, then the Jocobi identity holds from Proposition 4.1. So from now on, we always assume, without loss of generality, that $Y = TX$.

If for any hereditary category $\mathcal{B}$ which contains X, Z, we have that X and Z are not compariable in $\mathcal{B}$, then there is a section $\mathcal{S}$ in $\mathcal{R}$ such that X and Z both are sources in $\mathcal{S}$. This implies that

$$\mathrm{Hom}_{\mathcal{R}}(X, Z) = 0 = \mathrm{Hom}_{\mathcal{R}}(Z, X)$$

and

$$\mathrm{Hom}_{\mathcal{R}}(X, TZ) = 0 = \mathrm{Hom}_{\mathcal{R}}(Z, TX).$$

It follows that $[u_X, u_Z] = 0$, $[u_Z, u_Y] = [u_Z, u_{TX}] = 0$ and $< H_X \mid [Z] >= 0$. Hence we also have

$$[[u_X, u_Y], u_Z] = [[u_X, u_{[TX]}], u_Z] = [H_X, u_Z] = - < H_X \mid [Z] > u_Z = 0.$$

Therefore the Jacobi identity holds.

So we may assume that there exists a hereditary category $\mathcal{B}$ such that $X, Z \in \mathcal{B}$ and $X < Z$ in $\mathcal{B}$. We may assume further that X is the unique simple projective module in $\mathcal{B}$. We will use the induction on the length of Z as a module in $\mathcal{B}$.

If $Z \simeq X$, then we have

$$[[u_X, u_Y], u_Z] = [[u_X, u_{[TX]}], u_X] = [H_X, u_X] = - < H_X \mid [X] > u_X = -2u_X,$$
$$[[u_X, u_Z], u_Y] = [[u_X, u_X], u_{[TX]}] = 0$$

and

$$[u_X, [u_Y, u_Z]] = [u_X, [u_{[TX]}, u_X]] = [u_X, -H_X] = [H_X, u_X] = -2u_X.$$

So the Jacobi identity holds.

Suppose that Z is a simple module in $\mathcal{B}$ and $Z \not\simeq X$. When $\mathrm{Hom}_{\mathcal{R}}(Z, TX) = 0$, then $[u_X, u_Z] = [u_Y, u_Z] = 0$, also $[[u_X, u_Y], u_Z] = [H_X, u_Z] = 0$. So we may assume that $\mathrm{Hom}_{\mathcal{R}}(Z, TX) \neq 0$. There exists a unique indecomposable object $L \in \mathcal{B}$ with a non-split exact sequence $0 \to X \to L \to Z \to 0$ in $\mathcal{B}$. The length of L is 2 in $\mathcal{B}$, then $\varphi^L_{XZ} = 1$. Since there is no short cycle in $\mathcal{R}$, we have

$$\begin{aligned}[][[u_X, u_Y], u_Z] &= [H_X, u_Z] = - < H_X \mid [Z] > \ u_Z \\ &= l({}_{\mathrm{End}_{\mathcal{R}} X}\mathrm{Hom}_{\mathcal{R}}(X, TZ))u_Z = \frac{\dim_k \mathrm{Hom}_{\mathcal{R}}(Z, TX)}{d(X)} u_Z\end{aligned}$$

and

$$[[u_X, u_Z], u_Y] = \varphi^L_{XZ}(1)[u_L, u_{TX}] = \varphi^L_{XZ}(1)\varphi^Z_{LTX}(1)u_Z.$$

It is easy to see that $[u_{TX}, u_Z] = 0$. So $[u_X, [u_Y, u_Z]] = [u_X, [u_{TX}, u_Z]] = 0$. The proof of Jacobi identity is reduced to prove

$$\varphi^Z_{LTX}(1) = \frac{\dim_k \mathrm{Hom}_{\mathcal{R}}(Z, TX)}{d(X)}.$$

By Lemma 2.11, we have a natural map from the set of non-split extensions $0 \to X \xrightarrow{u} L \xrightarrow{v} Z \to 0$ onto the set of non-zero elements in $\mathrm{Hom}_{\mathcal{R}}(Z, TX)$. The sequences

$0 \to X \xrightarrow{u} L \xrightarrow{v} Z \to 0$ and $0 \to X \xrightarrow{u'} L \xrightarrow{v'} Z \to 0$ go to a same element w in $\mathrm{Hom}_{\mathcal{R}}(Z, TX)$ if and only if the triangles $X \xrightarrow{u} L \xrightarrow{v} Z \xrightarrow{w} TX$ and $X \xrightarrow{u'} L \xrightarrow{v'} Z \xrightarrow{w} TX$ belong to an equivalent class in $U(X, Z; L)$. So $|U(X, Z; L)| = |\mathrm{Hom}_{\mathcal{R}}(Z, TX)| - 1$. According to Lemma 2.6,

$$\varphi^Z_{LTX}(|E|) = \frac{|W(L^E, TX^E; Z^E)|}{|\mathrm{Aut}\, L^E|\ |\mathrm{Aut}\, TX^E|} = \frac{|W(X^E, Z^E; L^E)|}{|\mathrm{Aut}\, L^E|\ |\mathrm{Aut}\, X^E|}$$
$$= \frac{|U(X^E, Z^E; L^E)|\ |\mathrm{Aut}\, L^E|}{|\mathrm{Aut}\, L^E|\ |\mathrm{Aut}\, X^E|} = \frac{|\mathrm{Hom}_{\mathcal{R}}(Z^E, TX^E)| - 1}{|!ndX^E| - 1}$$

Since $\dim_k \mathrm{Hom}(Z, TX) = \dim_E \mathrm{Hom}(Z^E, TX^E)$ and $\dim_k \mathrm{End}\, X = \dim_E \mathrm{End}\, X^E$,

$$\varphi^Z_{LTX}(|E|) = \frac{|E|^{\dim_k \mathrm{Hom}(Z,TX)} + |E|^{\dim_k \mathrm{Hom}(Z,TX)-1} + \cdots + 1}{|E|^{d(X)} + |E|^{d(X)-1} + \cdots + 1}$$

hence

$$\varphi^Z_{LTX}(1) = \frac{\dim_k \mathrm{Hom}(Z, TX)}{d(X)}$$

as required.

Assume that Z is non-simple in $\mathcal{B}$. By a theorem of Bongartz (see [1]), there are indecomposable objects Z_1, Z_2 in $\mathcal{B}$ and a non-split extention $0 \to Z_1 \to Z \to Z_2 \to 0$ in $\mathcal{B}$. By Ringel [10], $\varphi^Z_{Z_1 Z_2}(1) \neq 0$. Put $a = \varphi^Z_{Z_1 Z_2}(1)^{-1}$. Since no short cycle in $\mathcal{R}$, $u_Z = a[u_{Z_1}, u_{Z_2}]$. Because X is the simple projective in $\mathcal{B}$ and $X \not\simeq Z_2, Y \not\simeq TZ_2$. We claim that

$$[u_Y, [u_{Z_1}, u_{Z_2}]] = [[u_Y, u_{Z_1}], u_{Z_2}] + [u_{Z_1}, [u_Y, u_{Z_2}] \qquad (5.2).$$

If $Z_1 \simeq X$, since the length of Z_2 is less than the length of Z, by the induction assumption the formula (5.2) is right. If $Z_1 \not\simeq X$, then $Y \not\simeq TZ_2$ and $Y \not\simeq TZ_1$, obviously $Z_2 \not\simeq TZ_1$. By Proposition 4.1 we have our asseration (5.2)

We come back to finish the final step. By (5.2),

$$a[u_X, [u_Y, u_Z]] = [u_X, [[u_{TX}, u_{Z_2}], u_{Z_1}]] + [u_X, [u_{Z_2}, [u_{TX}, u_{Z_1}]]].$$

If $[u_X, u_{Z_2}] = 0$, it is trivial for the proof. So the only possibility for our concerns is that we have the triangle $Z_1 \xrightarrow{f} L \xrightarrow{g} TX \xrightarrow{h} TZ$ with L indecomposable and $h \neq 0$ in $\mathcal{R}$. Hence X, L and Z_2 belong to $\mathcal{B}$ simultaneously since $L \xrightarrow{g} TX$ is a non-isomorphic morphism. We therefore have

$$[u_X, [[u_{TX}, u_{Z_1}], u_{Z_2}]] = [[u_X, [u_{TX}, u_{Z_1}]], u_{Z_2}] + [[u_{TX}, u_{Z_1}], [u_X, u_{Z_2}]]$$

Similarly we have

$$[u_X, [u_{Z_1}, [u_{TX}, u_{Z_2}]]] = [[u_X, u_{Z_1}], [u_{TX}, u_{Z_2}]] + [u_{Z_1}, [u_X, [u_{TX}, u_{Z_2}]]]$$
$$[u_{TX}, [[u_X, u_{Z_1}], u_{Z_2}]] = [[u_{TX}, [u_X, u_{Z_1}]], u_{Z_2}] + [[u_X, u_{Z_1}], [u_{TX}, u_{Z_2}]]$$

and

$$[u_{TX}, [u_{Z_1}, [u_X, u_{Z_2}]]] = [[u_{TX}, u_{Z_1}], [u_X, u_{Z_2}]] + [u_{Z_1}, [u_{TX}, [u_X, u_{Z_2}]]].$$

By the induction assumption,

$$[u_X, [u_{TX}, u_{Z_1}]] = [[u_X, u_{TX}], u_{Z_1}] + [u_{TX}, [u_X, u_{Z_1}]]$$
$$= [H_X, u_{Z_1}] + [u_{TX}, [u_X, u_{Z_1}]] = - < H_X \mid [Z_1] > \ u_{Z_1} + [u_{TX}, [u_X, u_{Z_1}]].$$
$$[u_X, [u_{TX}, u_{Z_2}]] = - < H_X \mid [Z_2] > \ u_{Z_2} + [u_{TX}, [u_X, u_{Z_2}]].$$

Then

$$a[u_X, [u_Y, u_Z]] = [u_X, [u_{TX}, [u_{Z_1}, u_{Z_2}]]] = [u_X, [[u_{TX}, u_{Z_1}], u_{Z_2}]] + [u_X, [u_{Z_1}, [u_{TX}, u_{Z_2}]]]$$
$$= [[u_X, [u_{TX}, u_{Z_1}]], u_{Z_2}] + [[u_{TX}, u_{Z_1}], [u_X, u_{Z_2}]] + [[u_X, u_{Z_1}], [u_{TX}, u_{Z_2}]] + [u_{Z_1}, [u_X, [u_{TX}, u_{Z_2}]]]$$
$$= - < H_X \mid [Z_1] > [u_{Z_1}, u_{Z_2}] + [[u_{TX}, [u_X, u_{Z_1}]], u_{Z_2}] + [[u_{TX}, u_{Z_1}], [u_X, u_{Z_2}]]$$
$$- < H_X \mid [Z_2] > [u_{Z_1}, u_{Z_2}] + [u_{Z_1}, [u_{TX}, [u_X, u_{Z_2}]]] + [[u_X, u_{Z_1}], [u_{TX}, u_{Z_2}]]$$
$$= - < H_X \mid [Z] > [u_{Z_1}, u_{Z_2}] + [u_{TX}, [[u_X, u_{Z_1}], u_{Z_2}]] + [u_{TX}, [u_{Z_1}, [u_X, u_{Z_2}]]]$$
$$= a[[u_X, u_{TX}], u_Z] + [u_{TX}, [u_X, [u_{Z_1}, u_{Z_2}]]]$$
$$= a[[u_X, u_{TX}], u_Z] + a[u_{TX}, [u_X, u_Z]]$$

So we have proved that

$$[u_X, [u_Y, u_Z]] = [[u_X, u_Y], u_Z] + [u_Y, [u_X, u_Z]].$$

This is the Jacobi identity (5.1).

We also need to check the Jacobi identity:

$$[[H_X, u_Y], u_Z] = [[H_X, u_Z], u_Y] + [H_X, [u_Y, u_Z]] \tag{5.3}$$

for any indecomposables X, Y and Z in $\mathcal{R}$. In fact

$$[[H_X, u_Y], u_Z] = - < H_X \mid [Y] > \ [u_Y, u_Z]$$
$$[[H_X, u_Z], u_Y] = - < H_X \mid [Z] > \ [u_Z, u_Y] = < H_X \mid [Z] > \ [u_Y, u_Z]$$

If $[u_Y, u_Z] = 0$, we have (5.3) clearly. If there exist trangles of form $Y \xrightarrow{f} L \xrightarrow{g} Z \xrightarrow{h} TY$ or $Z \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} TZ$ with L indecomposable and $h \neq 0$, then

$$[H_X, [u_Y, u_Z]] = - < H_X \mid [L] > \ [u_Y, u_Z] = (- < H_X \mid [Y] > - < H_X \mid [Z] >)\, [u_Y, u_Z].$$

Therefore

$$[[H_X, u_Z], u_Y] + [H_X, [u_Y, u_Z]] = - < H_X \mid [Y] > \ [u_Y, u_Z] = [[H_X, u_Y], u_Z].$$

(5.3) is proved. It is quite easy to show

$$[[H_X, H_Y], u_Z] = [H_X, [H_Y, u_Z]] + [[H_X, u_Z], H_Y] \tag{5.4}$$

We have examined the all possible situations. □

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