# Compactness properties of locally compact groups

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For discrete groups there are finiteness properties $FP_n$ and $F_n$: A group $\Gamma$ is said to be of type $FP_n$ if there is a projective resolution over $\boldsymbol{Z}\Gamma$ of the trivial $\boldsymbol{Z}\Gamma$–module $\boldsymbol{Z}$ with finitely generated modules in dimension $\leq n$. If there is an Eilenberg–MacLane space $K(\Gamma, 1)$ with finite $n$–skeleton, then $\Gamma$ is said to be of type $F_n$. The properties $F_1$ and $FP_1$ are both equivalent with finite generation; $F_2$ is equivalent with finite presentability and implies $FP_2$. Whether $F_2$ and $FP_2$ are in fact equivalent is still an open question. A group is of type $F_n$ if and only if it is of type $FP_n$ and (for $n \geq 2$) of type $F_2$ (see e.g. [Se], [Wa 1], [Wa 2]).

In this paper we introduce compactness properties $C_n$ and $CP_n$ for locally compact groups, which generalize these finiteness properties.

In section 1 we give the basic definitions. Starting point is a theorem of K. Brown ([Br 2, thm 2.2]) giving a necessary and sufficient condition for a group to be of type $FP_n$.
Section 2 is devoted to the investigation of the property $C_n$ for $n \leq 2$. We prove that there is (as in the discrete case) a group–theoretical interpretation: The property $C_1$ is equivalent with compact generation, $C_2$ is equivalent with compact presentability.
In section 3 we prove some basic results about compactness properties. In some sense this section shows that the compactness properties are reasonably defined. Specifically it is proved that the compactness properties remain unchanged by passing to a cocompact subgroup or by taking the quotient by a compact normal subgroup and that they are preserved under direct products.

In a different publication the second named author will apply compactness properties to $S$–arithmetic groups and prove a local–global principle for the finiteness properties of these groups.

## 1 ind–objects

In this chapter we give the definition of the compactness properties $C_n$ and $CP_n$. To do this we use the notion of an ind–object in a given category (following Grothendieck) as well as some familiar constructions in the category of simplicial sets, which we recall. It will be an immediate consequence of [Br 2, thm. 2.2] that for discrete groups finiteness properties and compactness properties coincide.

## 1.1 A Hurewicz theorem for ind–spaces

Let $\mathcal{C}$ be any category. Then the category of ind–objects in $\mathcal{C}$, denoted by $\underline{\text{ind}}-\mathcal{C}$, is given by the following definitions (see also [Gr, 8.2]):

An object of $\underline{\text{ind}}-\mathcal{C}$ is a functor $\underline{A} : I \to \mathcal{C}$, where $I$ is a filtered set regarded as a category. We often write $A_i$ instead of $\underline{A}(i)$ and $\{A_i\}_{i\in I}$ instead of $\underline{A}$. In the terminology of [Gr] all these ind–objects are "essentially small".

A morphism in this category, however, is *not* a natural transformation of functors or some version of it, which takes into account the fact that the Category $I$ is variable, but

$$\mathrm{Hom}_{\underline{\text{ind}}-\mathcal{C}}(\{A_i\}_{i\in I}, \{B_j\}_{j\in J}) = \lim_i \ \mathrm{colim}_j \ \mathrm{Hom}_{\mathcal{C}}(A_i, B_j).$$

In more detail, a morphism in $\underline{\text{ind}}-\mathcal{C}$, say from $\underline{A} : I \to \mathcal{C}$ to $\underline{B} : J \to \mathcal{C}$, is given by

(i) An (arbitrary) map $\alpha : I \to J$

(ii) A collection of $\mathcal{C}$–morphisms $\{\phi_i\}_{i\in I}$, where $\phi_i : A_i \to B_{\alpha(i)}$

For these data to give rise to a morphism in $\underline{\text{ind}}-\mathcal{C}$ we require: For all $i$, $i'$ $\in I$ with $i < i'$ there is a $j \in J$ with $\alpha(i) < j > \alpha(i')$, such that the diagram

$$\begin{tikzcd}
A_i \arrow[r, "\phi_i"] \arrow[d] & B_{\alpha(i)} \arrow[dr] & \\
A_{i'} \arrow[r, "\phi_{i'}"] & B_{\alpha(i')} \arrow[r] & B_j
\end{tikzcd}$$

commutes, where the unmarked arrows are the structure maps of $I$ and $J$.

Two such data $\Phi = (\alpha, \{\phi_i\}_{i\in I})$ and $\Psi = (\beta, \{\psi_i\}_{i\in I})$ give rise to the same morphism, if $\phi_i$ and $\psi_i$ finally coincide. More precisely we have $\Phi = \Psi$, if and only if for each $i \in I$ there is a $j \in J$ with $\alpha(i) < j > \beta(i)$ such that the following diagram commutes:

$$\begin{tikzcd}
& B_{\alpha(i)} \arrow[dr] & \\
A_i \arrow[ur, "\phi_i"] \arrow[dr, "\psi_i"'] & & B_j \\
& B_{\beta(i)} \arrow[ur] &
\end{tikzcd}$$

If $\mathcal{C}$ is a pointed category, then so is $\underline{\mathrm{ind}} - \mathcal{C}$. The functor $\underline{A} : I \to \mathcal{C}$ is isomorphic to the nullobject if and only if for every $i \in I$ there is a $j \geq i$ such that the morphism $A_i \to A_j$ is trivial.

The notion of an ind–object can be seen as a refinement of the "colimit" notion. In the sequel we will be regarding ind–objects in the category of groups, and for most of these $\underline{\mathrm{ind}}$–groups $\{A_i\}_{i\in I}$ we know, that $\mathrm{colim}_{i\in I} A_i$ is isomorphic to the trivial group. What we have to investigate, however, is whether or not $\{A_i\}_{i\in I}$ is trivial as an $\underline{\mathrm{ind}}$–group.

For the following let the category $\mathcal{D}$ be either *Top* (topological spaces having the homotopy type of a CW–Complex) or $\mathcal{S}$ (simplicial sets) and let $\mathcal{C}$ be either $Top^*$ or $\mathcal{S}^*$, where "$*$" means "with basepoint". We can define homotopy groups for ind–objects in $\mathcal{C}$ as follows:

**Definition 1.1.1** *Let $\underline{A} : I \to \mathcal{C}$ be an ind–object and $n \geq 1$. The n-th homotopy group of $\underline{A}$, denoted by $\pi_n(\underline{A})$, is defined as the ind–object of groups $\{\pi_n(A_i)\}_{i\in I}$. The 0–th homotopy set $\pi_0(\underline{A})$ is defined as $\{\pi_0(A_i)\}_{i\in I} \in \underline{\mathrm{ind}}$–$Set^*$.*

Analogously we can define homology groups:

**Definition 1.1.2** *Let $\underline{B} : I \to \mathcal{D}$ be an ind–object. The homology groups of $\underline{B}$ are given by*

$$H_n(\underline{B}) := \{H_n(B_i)\}_{i\in I} \in \underline{\mathrm{ind}}\text{–}Gps$$

We are mainly interested in the question whether or not these groups are trivial for a given $\underline{\mathrm{ind}}$–space. Therefore it is useful to have an ind version of the Hurewicz–theorem. We start with a lemma:

**Lemma 1.1.3** *Let $\underline{A} : I \to Top^*$ be an $\underline{\mathrm{ind}}$–space and let $n \in \boldsymbol{N}$ be such that $\pi_k(\underline{A}) = 0$ for all $k \leq n$. Then for every $i \in I$ there is a $j \geq i$ and an $n$–connected space $B$ together with maps $\varphi : A_i \to B$, $\psi : B \to A_j$, such that the diagramm*

$$\begin{tikzcd} & B \arrow[dr, "\psi"] & \\ A_i \arrow[ur, "\varphi"] \arrow[rr, "\alpha_{ij}"] & & A_j \end{tikzcd}$$

*commutes, where $\alpha_{ij}$ is the structure map of $\underline{A}$.*
**Proof:** By induction on $n$:

$n = 0$ : Suppose $x_0$ is the basepoint and let $\Sigma$ be the set of connected components of $A_i$. For $K \in \Sigma$ choose $f_K : S^0 \cong \{0,1\} \to A_i$ such that $f_K(0) = x_0$ and $f_K(1) \in K$. Define

$$B := colim \left( \begin{tikzcd} \coprod_{K\in\Sigma} S^0 \arrow[r] \arrow[d, "\coprod f_K"] & \coprod_{K\in\Sigma} [0,1] \\ A_i & \end{tikzcd} \right)$$

Let $\varphi : A_i \to B$ be the canonical inclusion. $B$ is connected by construction. If $j \geq i$ is such that $\alpha_{ij}(A_i)$ is contained in the connected component of $A_j$, then $\alpha_{ij}$ can be extended to a map $\psi : B \to A_j$. Now such a $j$ always exists, since $\pi_0(\underline{A}) = 0$.

$n \to n+1$ : Let $i \in I$ and choose $j < k < l$ such that the structure maps $\alpha_{ij}$, $\alpha_{jk}$ and $\alpha_{kl}$ induce the zero map on the first $n+1$ homotopy groups. Suppose $B$ and $B'$ are two $n$–connected spaces (existing according to the induction hypothesis) such that the diagramm

$$\begin{tikzcd} & B \arrow[dr, "\psi"] & & & B' \arrow[dr, "\psi'"] & \\ A_i \arrow[ur, "\varphi"] \arrow[rr, "\alpha_{ij}"] & & A_j \arrow[r, "\alpha_{jk}"] & A_k \arrow[ur, "\varphi'"] \arrow[rr, "\alpha_{kl}"] & & A_l \end{tikzcd}$$

commutes.

The map $\rho := \varphi' \circ \alpha_{jk} \circ \psi : B \to B'$ induces the zero map on $\pi_{n+1}$. In analogy with the case $n = 0$ we can construct an $n+1$–connected space $C$ by attaching $n+2$–cells to $B$, such that $\rho$ and hence $\alpha_{il} : A_i \to A_l$ factor through $C$. •

We can now prove the Hurewicz–theorem for $\underline{\text{ind}}$–spaces:

**Theorem 1.1.4** *Suppose $\underline{A} : I \to Top^*$ is an $\underline{\text{ind}}$–space, $n \geq 1$ and $\pi_k(\underline{A}) = 0$ for $k \leq n$. Then $\pi_{n+1}(\underline{A}) \cong H_{n+1}(\underline{A})$ as $\underline{\text{ind}}$–groups.*
**Proof:** Lemma 1.1.3 shows, that $\underline{A}$ is isomorphic to and $\underline{\text{ind}}$–space consisting of $n$–connected spaces. But for those $\underline{\text{ind}}$–spaces the theorem follows immediately from the usual Hurewicz theorem. •

**Remark 1.1.5** *Because the geometric realization of a simplicial set is always a CW–complex, theorem 1.1.4 is also valid for $\underline{\text{ind}}$–$S^*$.*

## 1.2 Filtrations

In the following $\mathcal{C}$ denotes either *Top* or $S$.

**Definition 1.2.1** *Let $X \in \mathcal{C}$ and $\{X_\alpha\}_{\alpha\in I}$ a set of subspaces of $X$, such that*
*(i) For $\alpha, \beta \in I$ there is a $\gamma \in I$ such that $X_\alpha \cup X_\beta \subseteq X_\gamma$*
*(ii) $\bigcup_{\alpha\in I} X_\alpha = X$*
*Then $\{X_\alpha\}_{\alpha\in I}$ is called filtration of $X$.*

We can associate an ind–object in $\mathcal{C}$ to a filtration $\{X_\alpha\}_{\alpha\in I}$ in a natural way: Order $I$ by defining $\alpha \le \beta$ iff $X_\alpha \subseteq X_\beta$ and define $\underline{A} : I \to \mathcal{C}$ by $\underline{A}(\alpha) := X_\alpha$. Two filtrations $\{X_\alpha\}_{\alpha\in I}$ and $\{Y_\beta\}_{\beta\in J}$ of $X$ are called cofinal, if for every $\alpha \in I$ there is a $\beta \in J$ such that $X_\alpha \subseteq Y_\beta$ and vice versa. In this case the associated ind–objects are isomorphic.

We are now going to describe two filtrations, which are of basic importance for the rest of the paper:
For a set $X$ let $EX$ be the free simplicial set on $X$, i.e. the simplicial set given by

$$[n] \mapsto X^{n+1},$$

with the usual face- and degeneracy maps given by removing resp. doubling one component. It is easy to see that $EX$ is contractible. If $G$ is a group acting on $X$, then $G$ also acts on $(EX)_n = X^{n+1}$ by the diagonal action and the simplicial structure maps commute with this action. Hence $EX$ can be regarded as a simplicial $G$–set.

Remark: In case $X = G$ there is another description of $EG$ which is also commonly used in the literature. There $G$ acts on $G^{n+1}$ by multiplication on the first factor, the face maps are given by internal multiplication and the degeneracy maps by inserting the neutral element. Both descriptions are $G$–isomorphic, the isomorphism from the second one to the first one is in dimension $n$ given by

$$(g_0, \ldots, g_n) \mapsto (g_0, g_0 g_1, \ldots, g_0 \cdot \ldots \cdot g_n)$$

Now suppose $G$ and $X$ are locally compact and $G$ acts continuously on $X$. Nevertheless $EX$ is still regarded as a simplicial *set*, not as a simplicial space. We have the following two filtrations of $EX$:

*$C_{/G}$–filtration:* $C_{/G}$ stands for compact modulo $G$. Let $C$ be a compact subset of $X$, then $EC$ is a subcomplex of $EX$. Let $G\cdot EC$ denote the smallest $G$–invariant subcomplex of $EX$ containing $EC$. Then $(x_o, \ldots, x_n)$ is a simplex of $G\cdot EC$ if and only if $\{x_0, \ldots, x_n\} \subseteq g\cdot C$ for some $g \in G$. Now let $\{C_i\}_{i\in I}$

be the set of compact subsets of $X$. Then $I$ is filtered in a natural way and $\{G{\cdot}EC_i\}_{i\in I}$ is called the $C_{/G}$–filtration of $EX$.

*Filtration by $G$–convenient subcomplexes:* Let $G$ and $X$ be as above and suppose $A$ is a $G$–invariant subcomplex of $EX$. The set of $n$–simplices $A_n$ is then a $G$–invariant subset of $X^{n+1}$, hence when equipped with the subspace topology may be regarded as a $G$–space. $A$ is called $G$–convenient, if for every $n \in \boldsymbol{N}$ the orbit space $A_n/G$ is compact. The set of all $G$–convenient subcomplexes of $EX$, $\{A_j\}_{j\in J}$ is then a filtration of $EX$.

**Lemma 1.2.2** *The two filtrations described above have isomorphic homotopy and homology groups.*
**Proof:** It is sufficient to show that the induced filtrations of the $n$–skeleton of $EX$ are cofinal. Denote by $sk_n(-)$ the $n$–skeleton of the given simplicial set.
The complexes of the $C_{/G}$–filtrations are obviously $G$–convenient. Now fix a $G$–convenient subcomplex $A$, we have to show that there is a compact set $L \subseteq X$ such that we have the inclusion $sk_n(A) \subseteq sk_n(GEL)$ for the $n$–skeleta. For $k \in \boldsymbol{N}$ the set of $k$–simplices of $A$, denoted by $A_k$, is contained in the locally compact $G$–space $X^{k+1}$, and the orbit space $A_k/G$ is compact. Then there is a compact set $C_k \subseteq X^{k+1}$ such that $G{\cdot}C_k \supseteq A_k$. Now choose a compact $L \subseteq X$ such that $C_k \subseteq L^{k+1}$ for all $k \leq n$, then $sk_n(A) \subseteq sk_n(G{\cdot}EL)$. •

## 1.3 Brown's criterion

To begin with we briefly summarize the first part of the paper of K. Brown already mentioned in the introduction ([Br 2]).

**Definition 1.3.1** *Let $\Gamma$ be a discrete group and $X$ a CW–complex on which $\Gamma$ acts by permuting the cells. $X$ is called $n$–good for $\Gamma$ if*
*(i)* $H_k(X) = 0$ *for* $k < n$
*(ii) The stabilizer* $\Gamma_\sigma$ *of a p-cell* $\sigma$ *is of type* $FP_{n-p}$

A subcomplex $Y \subseteq X$ is called of finite $n$–type, if $Y$ is $\Gamma$–invariant and has finite $n$–skeleton modulo $\Gamma$. The following theorems hold:

**Theorem 1.3.2** *let $X$ be an $n$–good $\Gamma$–complex and $\underline{Y} = \{Y_\alpha\}_{\alpha\in I}$ the filtration of $X$ by the complexes of finite $n$–type. Then $\Gamma$ is of type $FP_n$ if and only if $H_i(\underline{Y}) = 0$ for all $i < n$.*

**Theorem 1.3.3** *Under the same hypothesis as above $\Gamma$ is of type $F_n$ if and only if $H_i(\underline{Y}) = 0$ for all $i < n$ and (for $n \geq 2$) $\pi_1(\underline{Y}) = 0$.*

With these theorems in mind we can now define compactness properties :

**Definition 1.3.4** *Let $G$ be a locally compact group, $EG$ the free simplicial set on $G$ and $\underline{A}$ the $C_{/G}$–filtration of $EG$.*
*$G$ is called of type $CP_n$ if $H_i(\underline{A}) = 0$ for all $i < n$.*
*$G$ is called of type $C_n$ if $\pi_i(\underline{A}) = 0$ for all $i < n$.*

As claimed before, for discrete groups the properties $C_n$ and $F_n$ resp. $CP_n$ and $FP_n$ coincide: The free simplicial set $EG$ is contractible and has a free $G$–action, so it is $n$–good for each $n \in \boldsymbol{N}$. If $G$ is discrete, then the filtration of the $n$–skeleton of $EG$ by $n$–good complexes is the same as the filtration by $G$–convenient complexes, therefore the equivalence of $CP_n$ and $FP_n$ follows directly from lemma 1.2.2. For the equivalence of the properties $F_n$ and $C_n$ one additionally has to apply theorem 1.1.4.

It is also immediate from theorem 1.1.4 that a group is of type $C_n$ if and only if it is of type $CP_n$ and (for $n \geq 2$) of type $C_2$.

# 2 Compact generation, compact presentation

## 2.1 Preliminaries

In this section we investigate the compactness properties $C_n$ in low dimensions, i.e. for $n \leq 2$. We will show that there are group theoretical interpretations analogously to the discrete case: The property $C_1$ is equivalent with compact generation, $C_2$ is equivalent with compact presentability.

Most of the proofs in this (and the next) section are based on the inspection of certain simplicial sets, simplicial maps and homotopy classes of simplicial maps. So before starting we make some remarks on this subject:
All the simplicial sets we are considering in this paper have a special property,which we will call "vertex determined". Namely, they can be embedded in a natural way in the free simplicial set on their vertices. This means the same as saying that each simplex $\sigma$ is uniquely determined by the ordered(!) tupel $(x_0, \ldots, x_n)$ of its vertices. A simplicial map from a vertex determined simplicial set $S$ is uniquely determined by its values on the vertices of $S$, so we can identify the map with its restriction to the 0–skeleton. This fact is quite often explicitly or implicitly used in the arguments of this section.

By a simplicial $n$–sphere, $n$–ball, circle, disc etc. we mean a simplicial set homeomorphic to the corresponding object.
In this section we often have to represent homotopy classes of continuous maps between the realizations of simplicial sets by simplicial maps. It follows from the simplicial approximation theorem (e.g. [Cu, thm.

11.6]), that this is always possible, more precisely that the following lemmata hold true:

**Lemma 2.1.1** *Let $(S, x_0)$ be a pointed simplicial set and let $\varphi \in \pi_1(S, x_0)$.*

*Then there is a pointed simplicial circle $C$ and a pointed simplicial map $f : C \to S$, which represents $\varphi$.*

**Lemma 2.1.2** *Let $S$ be a simplicial set, $C$ a simplicial circle and $f : C \to S$ a map which is nulllhomotopic. Then there is a simplicial disc $D$, a weak equivalence $b : \partial D \xrightarrow{\sim} S$ and a simplicial map $g : D \to S$, such that $g_{|\partial D} = f \circ b$.*

One last point: We often have to regard products in non abelian groups. For abbreviation we write

$$\prod_{i=0}^{n} g_i \text{ instead of } g_0 \cdot \ldots \cdot g_n$$

## 2.2 Compact generation

In the following $G$ always denotes a locally compact group. For subsets $C \subseteq G$ there is a connection between connectivity properties of $G \cdot EC$, the smallest $G$–invariant subcomplex of $EG$ containing $EC$, and group theoretical properties of $C$. For example we have:

**Lemma 2.2.1** *Let $C \subseteq G$ be compact and $X := G{\cdot}EC$. Then $X$ is connected if and only if $G$ is generated by $C^{-1}C$.*
**Proof:** Denote by $X_n$ the set of $n$–simplices of $X$. Then $X_0 = G$ and $X_1 = G{\cdot}(C, C)$. Hence $\sigma = (g_1, g_2)$ is a 1–simplex of $X$ if and only if $g_1^{-1} g_2 \in C^{-1}C$.
Now suppose $G$ is connected and $g \in G$. Then there is a sequence $(w_0, \ldots, w_n)$ of 1–simplices of $X$ which connects $g$ and the neutral element $e$. We may assume that $d_1 w_0 = e$, $d_0 w_n = g$ and
$d_0 w_i = d_1 w_{i+1}$ for $0 \leq i \leq n$. Let $g_i := d_0 w_i$ and $g_{-1} := e$, then we have $w_i = (g_{i-1}, g_i) \in X_1$, hence $g_{i-1}^{-1} g_i \in C^{-1}C$ and therefore

$$g = \prod_{i=0}^{n} g_{i-1}^{-1} g_i \in (C^{-1}C)^n.$$

Conversely let $C^{-1}C$ be a set of generators of $G$ and let $g \in G$. Then we can write $g$ as

$$g = \prod_{i=0}^{n} c_i l_i \text{ with } c_i \in C^{-1},\ l_i \in C.$$

Now let $g_i := \prod_{j\leq i} c_j l_j$ and $w_i := (g_{i-1}, g_i)$. Then $(w_0, \ldots, w_n)$ is a simplicial path from $e$ to $g$. Each of the egdes $w_i$ is a 1–simplex of $X$,
because $g_{i-1}^{-1} g_i = c_i l_i \in C^{-1}C$. Therefore the simplicial path is contained in $X$. •

**Theorem 2.2.2** *A locally compact group $G$ is of type $C_1$ if and only if it is compactly generated.*
**Proof:**
Let $C := \{e\}$ and choose a compact $L \subseteq G$ such that the map $G{\cdot}EC \to G{\cdot}EL$ induces the zero map on $\pi_0$. Because of $sk_0(G{\cdot}EC) = sk_0(G{\cdot}EL)$ it follows that $G{\cdot}EL$
is connected, hence $L^{-1}L$ generates $G$ by lemma 2.2.1.
Now let $G$ be compactly generated, $C$ be a compact generating set for $G$ and let $L$ be another compact subset of $G$. Define $M := K \cup L \cup \{e\}$. Then $G{\cdot}EL \subseteq G{\cdot}EM$ and
$G{\cdot}EM$ is connected by lemma 2.2.1. •

## 2.3 Compact presentability

First of all we have to explain what we mean by compact presentability: Let $C$ be a generating set of $G$, let $F(C)$ be the free group on the set $C$ and $R(C)$ the kernel of the map $F(C) \to G$. The elements of $R(C)$ are called relations in $C$ (with respect to the group $G$). Let $R_p(C)$ be the normal subgroup of $F(C)$ generated by the relation of length $\leq p$.

**Definition 2.3.1** *A locally compact group $G$ is called compactly presentable if there is a compact generating system $C$ of $G$ and a number $p \in \boldsymbol{N}$ such that $R_p(C) = R(C)$.*
For more details about this notion see [Ab, chapter 1].

There is a 1–1 correspondance between relations in $G$ and pointed loops in $EG$: Let $S$ be a pointed and oriented simplicial circle and $\varphi : S \to EG$ a pointed map (i.e. such that the basepoint maps onto the neutral element). Then $\varphi$ is called a loop in $EG$. Let $s_0$ denote the basepoint, $s_1, \ldots, s_n$ the other vertices of $S$ in the order given by the orientation and define $s_{n+1} := s_0$. Let $r_i \in G$ be given by $r_i = \varphi(s_i)^{-1}\varphi(s_{i+1})$.
Then the relation associated with $\varphi$ is the element of $F(G)$ given by

$$R_\varphi := \prod_{i=0}^{n} r_i \ ,$$

where the $r_i$ are viewed as elements of $F(G)$ and the product is taken in $F(G)$. Up to inversion and conjugation $R_\varphi$ does not depend on the basepoint and the orientation of $S$.
If $\sigma = (g_0, g_1, g_2)$ is a 2–simplex of $EG$, then we can similarly associate with $\sigma$ a relation $R_\sigma$ in $F(G)$: Define
$s_0 := g_0^{-1} g_1,\ s_1 := g_1^{-1} g_2$ and $s_2 := g_2^{-1} g_0$. Now regard the $s_i$ as elements of $F(G)$ and let

$$R_\sigma := \prod_{i=0}^{2} s_i \ .$$

As above, permuting the vertices of $\sigma$ changes $R_\sigma$ only up to conjugation and inversion.

**Lemma 2.3.2** *Let $D$ be a simplicial disc and $S := \partial D$. Choose a basepoint $x_0$ and an orientation on $S$ and let $\varphi : S \to EG$ be a pointed map. Then the relation associated to $\varphi|_S$ is contained in the normal subgroup generated by the relations $\{R_{\varphi(\sigma)} : \sigma \in D_2\}$.*
**Proof:**
By induction on $\# D_2$.
It is always possible to find a 2–simplex $\mu \in D$, such that the closure of $D \backslash \mu$ is again a simplicial disc; call it $D'$. Let $S' := \partial D'$ and $R'$ be the relation associated with $\varphi|_{S'}$. It is easy to see that $R$ lies in the normal closure of $R'$ and $R_{\varphi(\mu)}$, and by the inductive hypothesis $R'$ is contained in
the normal closure of $\{R_{\varphi(\sigma)} : \sigma \in D_2 \backslash \mu\}$. •

If vice versa $R = \prod_{i=0}^{n} g_i$ is a relation in $G$, we can associate with $R$ a loop in $EG$ given by the following data: Let $S$ be the simplicial circle with vertices $x_0, \ldots, x_n$ and non degenerated 1–simplices $(x_i, x_{i+1})$ and $(x_n, x_0)$. Let $x_0$ be the basepoint, choose the orientation from $x_0$ in direction $x_1$ and define $\varphi_R : S \to EG$ by

$$\varphi_R(x_i) = \prod_{j=0}^{i-1} g_j.$$

If $R$ is a relation in a subset $C$ of $G$ containing the neutral element $e$, then the image of $\varphi_R$ is contained in $G \cdot EC$. We are now ready to prove

**Theorem 2.3.3** *A locally compact group $G$ is of type $C_2$ if and only if it is compactly presentable.*
**Proof:** First of all some remarks: Because of theorem 2.2.2 we can assume that there is a compact generating system $C$ of $G$. Choose $C$ symmetric, i. e. such that $C = C^{-1}$, then $\bigcup_{n \in N} C^n = G$ and it follows from the Baire category theorem ([He, lemma 3.1]) that there is a number $n_0 \in \boldsymbol{N}$ such that $C^{n_0}$ containes an inner point. Therefore the filtration of $G$ by its compact subsets is cofinally isomorphic to the filtration by the subsets $\{C^n : n \in \boldsymbol{N}\}$.

Suppose first that $G$ is of type $C_2$. Let $C$ be a compact symmetric generating system with $e \in C$, then there is a $p \in \boldsymbol{N}$ such that

$$\iota : \pi_1(G \cdot EC) \to \pi_1(G \cdot EC^p)$$

is the zero map. Define $X := G \cdot EC$ and $Y := G \cdot EC^p$. Let $R$ be a relation in $C$ and $\varphi : S \to X$ the corresponding loop in $X$. By hypothesis $\varphi$ is nullhomotopic in $Y$, hence we can find by lemma 2.1.2 a simplicial disc $D$, a weak equivalence $b : \partial D \xrightarrow{\sim} S$ and a map $\psi : D \to Y$, such that $\psi|_{\partial D} = \varphi \circ b$. If we choose basepoint and orientation on the simplicial circle $\partial D$ compatible with $S$, then the relation associated with $\psi|_{\partial D}$ is equal to $R$ (maybe it is in non–reduced form, but this is of no importance). By lemma 2.3.2 we know that then $R$ is contained in the normal subgroup generated by the relations $R_\sigma$, where $\sigma$ is a 2–simplex of $Y$. But these are relations of length $\leq 6p$ in $C$.

Now let $G$ be compactly presentable and $C$ be a compact symmetric generating system of $G$ with $R_p(C) = R(C)$. We may assume $p = 3$ by [Ab, prop. 1.1.1]. It is easy to see that for every $q \in \boldsymbol{N}$ we have $R(C^q) = R_3(C^q)$. So it suffices to show that for a compact generating system $C$ with $R(C) = R_3(C)$ the map

$$\iota : \pi_1(G \cdot EC) \to \pi_1(G \cdot EC^2)$$

is zero. Let $X := G \cdot EC$, $Y := G \cdot EC^2$ and $\mu \in \pi_1(X)$. It follows from lemma 2.1.1 that $\mu$ can be represented by a map of the form $\varphi_R : S \to X$, where $R = \prod r_i$ is a relation in $C^2$. So $R$ can be written as

$$R = \prod_{j=1}^{m} \alpha_j R_j \alpha_j^{-1},$$

where $\alpha_j \in F(C^2)$ and $R_j$ is a relation of length 3 in $C^2$. Now every relation of length 3 in $C^2$ is of the form $R_\sigma$ for a 2–simplex $\sigma$ of $Y$. It follows that the loops belonging to the relations $\alpha_j R_j \alpha_j^{-1}$ are nullhomotopic in $Y$, therefore the same is true for $\varphi$. •

The proofs of the theorems 2.2.2 and 2.3.3 show, that for $n \leq 2$ the condition "$\pi_i(\underline{A}) = 0$ for all $i < n$" on the $C_{/G}$–filtration $\underline{A}$ of $EG$ can be weakened. In fact it is sufficient to go to zero once. This means:
Start with an arbitrary compact subset $K_0 \subseteq G$, e.g. $K_0 = \{e\}$. For $G$ to be of type $C_1$ it is sufficient to find another compact subset $K_1$ of $G$ with $K_0 \subseteq K_1$, such that

$$i : G{\cdot}EK_0 \to G{\cdot}EK_1$$

induces the trivial map on $\pi_0$.
For $G$ to be of type $C_2$ it is sufficient to find a third compact subset $K_2$ with $K_1 \subseteq K_2$, such that

$$i : G{\cdot}EK_1 \to G{\cdot}EK_2$$

induces zero on $\pi_1$.

One may ask, if this generalizes. It does so for discrete groups, but in general it is not known. If it was true this would imply that the "if" part of the question 3.9 of [AH] can be answered with "yes".

# 3 Elementary properties

## 3.1 Direct products

As one expects, a finite direct product of groups has a certain compactness property if and only if each of the factors has. Nevertheless the proof of this fact is not so easy that we could omit it.

**Lemma 3.1.1** *Let $F$ and $G$ be locally compact groups. Then $F \times G$ is of type $CP_n$ (resp. $C_n$), if and only if $F$ and $G$ are so.*
**Proof:** Let $H := F \times G$ and denote the projections from $H$ onto $F$ and $G$ by $p_F$ and $p_G$ and the inclusions into $H$ by $i_F$ and $i_G$. For $K \subseteq H$ let $K_F := p_F(K)$ and $K_G := p_G(H)$, then we always have $K \subseteq K_F \times K_G$. Consequently the filtration of $H$ by compact subsets is cofinal to the filtration by products of compact subsets of $F$ and $G$.
Now suppose $EF$, $EG$ and $EH$ are the free simplicial sets on $F$, $G$ and $H$ respectively. The map $\varphi : EF \times EG \to EH$ is a simplicial isomorphism which is compatible with every reasonable group action. Let $K \subseteq H$ be compact, we may assume that $K = i_F(K) \cdot i_G(K)$. Let $\psi$ be the restriction of $\varphi$ to $F{\cdot}EK_F \times G{\cdot}EK_G$, then $\psi$ is an isomorphism onto its image $H{\cdot}EK$.

The map $\psi$ induces an isomorphism of the corresponding homotopy groups. ¿From this the lemma follows immediately for the property $C_n$, because homotopy groups commute with direct products.

For the property $CP_n$ the proof is a bit more complicated: Let $K \subseteq L$ be two compact subsets of $H$, then the maps $i_F$ and $p_F$ induce a commutative diagram

$$\begin{array}{ccc} F\cdot EK_F & \longrightarrow & F\cdot EL_F \\ \downarrow{\scriptstyle i_F} & & \uparrow{\scriptstyle p_F} \\ H\cdot EK & \longrightarrow & H\cdot EL \end{array}$$

from which we get a commutative diagramm of homology groups. So if $H$ is of type $CP_n$, then $F$ is, too, and so is $G$ by the same reasoning.

The property $CP_1$ is equivalent with compact generation, so we can assume that the subcomplexes of the filtration of $EH$ are connected. Then the "only if" part is a consequence of the following lemma:

**Lemma 3.1.2** *Suppose $X_0 \subseteq X_1 \subseteq X_2$ and $Y_0 \subseteq Y_1 \subseteq Y_2$ are connected topological spaces and for all $i \leq n$ the inclusions induce the zero map on the $i$–th homology group. Then $H_n(X_0 \times Y_0) \to H_n(X_2 \times Y_2)$ is the zero map.*
**Proof:** For the homology of a direct product there is a natural exact sequence, the so called Kuenneth–formula (e.g [HW, Cor. 5.7.19])

$$0 \to \sum_{p+q=n} H_p(X) \otimes H_q(Y) \to H_n(X \times Y) \to \sum_{p+q=n-1} H_p(X) * H_q(Y) \to 0$$

The inclusions $X_0 \times Y_0 \subseteq X_1 \times Y_1 \subseteq X_2 \times Y_2$ induce maps of the corresponding Kuenneth–formulas, so we get a commutative diagram with exact rows of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_0 & \longrightarrow & B_0 & \longrightarrow & C_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \end{array}$$

where the vertical maps on the left and on the right are zero. It follows by diagram chasing that then the map $B_0 \to B_2$ is zero, too. •

## 3.2 Variation of the free simplicial set

In order to compare compactness properties of different groups it is necessary to be flexible in the choice of the "test space" $EG$ in definition 1.3.4. Some more flexibility is given by theorem 3.2.2. As a consequence we will see that the compactness properties remain unchanged by passing to a cocompact subgroup or by dividing out a compact normal subgroup. As in section 2 all the simplicial sets will be vertex determined, so we can (and sometimes will) identify a map with its restriction to the set of vertices. Moreover we will make use of the following

**Lemma 3.2.1** *Let $S$ and $T$ be vertex determined simplicial sets and suppose $f$ and $g$ are maps from $S$ to $T$. If for every simplex $\sigma = (x_0, \dots, x_n) \in S$ and for every $0 \leq i \leq n$ the tupel $(f(x_0), \dots, f(x_i), g(x_i), \dots, g(x_n))$ is a simplex in $T$, then $f$ and $g$ are homotopic.*
**Proof:** The simplicial set $S \times \Delta(1)$ is vertex determined. Define the homotopy $H : S \times \Delta(1) \to T$ from $f$ to $g$ by

$$H((s,0)) := f(s), \quad H((s,1)) := g(s) \text{ for } s \in S$$

It follows from the assumptions that $H$ is well defined. •

**Theorem 3.2.2** *Let $G$ be a locally compact group and $T$ be a locally compact space, on which $G$ acts properly. Let $ET$ be the free simplicial set on $T$ and $\underline{A} = \{G \cdot EK : K \subseteq T \text{ compact}\}$ its $C_{/G}$–filtration. Then $G$ is of type $CP_n$ if and only if $H_i(\underline{A}) = 0$ for all $i < n$. It is of type $C_n$ if and only if $\pi_i(\underline{A}) = 0$ for all $i < n$.*
**Proof:** First we need some notation: Let $EG$ be the free simplicial set on $G$, $\Sigma$ the set of compact subsets of $G$ and $\Lambda$ the set of compact subsets of $T$. For $K \in \Sigma$ let $X_K := G \cdot EK$, for $L \in \Lambda$ let $Y_L := G \cdot EL$ and $\mathrm{Ret}(L) := \{g \in G : gL \cap L \neq \emptyset\}$. The set $\mathrm{Ret}(L)$ is compact, because $G$ acts properly on $T$.

The main idea of the proof is the following: For $L \in \Lambda$ we construct a diagram:

(D1)

$$\begin{tikzcd}
& X_K \arrow[rr, "i_3"] \arrow[dr, "p_K"] & & X_{K'} \arrow[dr, "p_{K'}"] & \\
Y_L \arrow[ur, "s_L"] \arrow[rr, "i_1"] & & Y_{L'} \arrow[rr, "i_2"] & & Y_{L''}
\end{tikzcd}$$

in which $p_K \circ s_L \simeq i_1$ and $p_{K'} i_3 = i_2 p_K$. Now if $i_3$ induces the zero map on $H_i(X_K)$ (or $\pi_i(X_K)$), then so does $i_2 i_1$ on $H_i(X_K)$ (resp. $\pi_i(X_K)$).

Conversely for $K \in \Sigma$ we construct a diagram:

$$\text{(D2)}\qquad \begin{tikzcd} X_K \arrow[rr, "i_1"] \arrow[dr, "p_K"'] & & X_{K'} \arrow[rr, "i_2"] & & X_{K''} \\ & Y_L \arrow[ur, "s_L"'] \arrow[rr, "i_3"] & & Y_{L'} \arrow[ur, "s_{L'}"'] & \end{tikzcd}$$

in which then $s_L \circ p_K \simeq i_1$ and $s_{L'} i_3 = i_2 s_L$.

The crucial part is the definition of the maps $s_L$ and $p_K$ and the calculation of their images. Notice that all simplicial sets are vertex determined, because they are subsets of free simplicial sets.

(1) Definition of $p_K$:
Let $K \in \Sigma$, fix $x_0 \in T$ and let $L \supseteq K x_0$. Define $p_K : X_K \to Y_L$ by $p_K(g) := g x_0$. If $\sigma$ is a simplex of $X_K$, then $p_K(\sigma)$ is a simplex of $Y_L$, hence $p_K$ is well defined.
So if $K' \in \Sigma$ with $K \subseteq K'$ we have for every $L' \in \Lambda$ with $L' \supseteq K' x_o \cup L$ a commutative diagram

$$\begin{tikzcd} X_K \arrow[r] \arrow[d, "p_K"'] & X_{K'} \arrow[d, "p_{K'}"'] \\ Y_L \arrow[r] & Y_{L'} \end{tikzcd}$$

(2) Definition of $s_L$:
Let Ł $\in \Lambda$ and $\{O_i\}_{i \in I}$ the set of $G$–orbits of $G \cdot L$. For each $i \in I$ choose $x_i \in O_i \cap L$, let $p_i : G \to O_i$ be the projection given by $g \mapsto g x_i$ and let $s_i : O_i \to G$ be a section of $p_i$ (not necessarily continous). These sections define a map $s_L : Y_L \to EG$. Let $H := \mathrm{Ret}(L)$. We will show, that the image of $s_L$ is contained in $X_{H^2}$:
Let $g \in G$ and $l \in L$. Then $l = h x_i$ for suitable $i \in I$ and $h \in H$. Now we have

$$\begin{array}{ll} & g h x_i = p_i s_i(g h x_i) = s_i(g h x_i) \cdot x_i, \\ \text{hence} & (gh)^{-1} \cdot s_i(g h x_i) \subseteq \{g \in G : g x_i = x_i\} \subseteq H, \\ \text{hence} & s_i(gl) = s_i(g h x_i) \subseteq g h H \subseteq g H^2. \end{array}$$

So $s_i(gL) \subseteq gH^2$ for every $g \in G$. Now if $\sigma$ is a simplex of $Y_L$, then $s_i(\sigma)$ is a simplex of $X_{H^2}$.
This shows that $s_L$ can be regarded as a map to $X_K$ for every $K \in \Sigma$ with $K \supseteq H^2$. If $L' \in \Lambda$ with $L \subseteq L'$, then we can choose the set of representatives of the $G$–orbits of $G{\cdot}L'$ consistent with the choice made for $G{\cdot}L$. Define $H' :=$

$\mathrm{Ret}(L')$, then we get for every $K' \in \Sigma$ with $K' \supseteq (H')^2 \cup K$ a commutative diagram

$$\begin{tikzcd} X_K \arrow[r] & X_{K'} \\ Y_L \arrow[u, "s_L"] \arrow[r] & Y_{L'} \arrow[u, "s_{L'}"'] \end{tikzcd}$$

Now suppose $G$ is of type $F_n$, say, and let $L \in \Lambda$. Let (as usual) $H := \mathrm{Ret}(L)$ and $K := H^2$. Choose $x_0 \in L$ and let $L' := Kx_0 \cup L$. Define $p_K$ and $s_L$ as above. Then the diagram

$$\begin{tikzcd} & X_K \arrow[dr, "p_K"] & \\ Y_L \arrow[ur, "s_L"] \arrow[rr, "i"] & & Y_{L'} \end{tikzcd}$$

commutes up to homotopy: It is an immediate consequence of lemma 3.2.1 that $p_k \circ s_L : Y_L \to Y_{L'}$ is homotopic to the inclusion, if $p_K \circ s_L(gL) \subseteq gL'$ for every $g \in G$. But

$$p_K(s_L(gL)) \subseteq p_K(gH^2) = gH^2x_0 \subseteq gL',$$

the first inclusion holding by the calculations made in (2). So we can construct a diagram (D1), in which $i_3$ induces the zero map on $\pi_i$ for all $i < n$, hence the same is true for $i_2 i_1$.

Conversely let $K \in \Sigma$, choose $x_0 \in T$ and define $L := Kx_0$. Let $H := \mathrm{Ret}(L)$, $K' := H^2$ and $s_L : Y_L \to X_{K'}$ as above. Then the diagram

$$\begin{tikzcd} & Y_L \arrow[dr, "p_K"] & \\ X_K \arrow[ur, "s_L"] \arrow[rr, "i"] & & X_{K'} \end{tikzcd}$$

commutes up to homotopy. For, by the same reason as above, it suffices to show that $s_L \circ p_K(gK) \subseteq gK'$ for all $g \in G$. And indeed

$$s_L(p_K(gK)) = s_L(gL) \subseteq gH^2 = gK',$$

again the inclusion holding by (2). So if, for example, $\pi_i(\underline{A}) = 0$ for an $i \in \boldsymbol{N}$, then we can construct a diagram (D2), in which $\pi_i(i_3)$ is the zero map, hence $\pi_i(i_2 i_1)$ is zero, too. •

This proof looks like magic, because we can conclude a topological fact from the existence of a non continuous section, used in the definition of $s_L$. But note that the only information about the topology of $G$ encoded in our construction is its filtering set of compact subsets. So only a very coarse sort of "continuity" is needed, namely a map sending compact sets to relatively compact sets. This is where we used the fact that the action of $G$ on $T$ is proper.

As claimed in the beginning of this subsection we can now prove that the compactness properties are invariant "up to compactness":

**Corollary 3.2.3** *Let $G$ be a locally compact group and let $H$ be a closed subgroup of $G$ such that $G/H$ is compact. Then $G$ and $H$ have the same compactness properties.*
**Proof:** $G$ and $H$ act properly on $G$. By theorem 3.2.2 it suffices to show that the $C_{/G}$–filtration of $EG$ with respect to $G$ and the $C_{/G}$–filtration of $EG$ with respect to $H$ are cofinal.
Because $G/H$ is compact, there is a compact subset $R \subseteq G$ such that $H \cdot R = G$. Then we have for all compact subsets $C \subseteq G$:

$$H \cdot EC \subseteq G \cdot EC \subseteq H \cdot E(RC).$$

•

**Corollary 3.2.4** *Let $G$ be a locally compact group and $K$ a compact normal subgroup of $G$. Then $G$ and $G/K$ have the same compactness properties.*
**Proof:** Let $H := G/K$. Then $G$ and $H$ both act properly on $H$. Again by theorem 3.2.2 we only have to show that the two $C_{/G}$–filtrations of $EH$ (with respect to $G$ and with respect to $H$) are cofinal. But they are even equal, because for every subset $C \subseteq H$ we have $G \cdot EC = H \cdot EC$. •

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