ON SHIFTS OF CONVEX MEASURES

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According to a classical theorem, for every Gaussian measure γ on a locally convex space X and every vector $h \in X$, the measures γ and $\gamma(\,\cdot\,+h)$ are either equivalent or mutually singular. This property has a lot of applications in both the theory of random processes and infinite dimensional analysis. The collection of all vectors h, for which such measures are equivalent is said to be the Cameron – Martin space of γ. It is known that for every element h of this space the mapping $t \mapsto \gamma(\,\cdot\,+th)$ is infinitely differentiable. This fact lies in the foundations of the analysis on the Wiener space (see [4]). Naturally the question arises about other classes of measures for which similar assertions are true.

In view of diverse applications in the stochastic analysis (see [1], [2]), it is of interest to study the differential properties of convex measures. A Gaussian measure is a very special case of a convex measure. In this paper we study the behavior of convex Radon measures on a locally convex topological vector space under shifts. One of the main results we obtain is the estimate

$$\|\mu_h - \mu\| \geq 2 - 2e^{\frac{-\|d_h\mu\|}{2}},$$

for the finite dimensional projections of the measure μ, where $d_h\mu$ denotes the Skorohod derivative of μ along h. Making use of this estimate, we prove that for every direction h, which is not a direction of differentiability of the measure μ in the sense of Skorohod, the measures μ and μ_h are mutually singular. Therefore, the above mentioned dichotomy extends from Gaussian measures to convex ones in the following form: a convex measure is either differentiable along a vector in the sense of Skorohod or all its translates in this direction are mutually singular.

Let us recall some definitions. Let X be a locally convex topological vector space, and $\mathcal{B}(X)$ its Borel σ-field. The term "measure on the space X" means a Radon (possibly, signed) measure on $\mathcal{B}(X)$ (see, e.g., [10]). For any measure μ on X we

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denote by $|\mu|$ its variation and by $\|\mu\| := |\mu|(X)$ its variation norm. Continuous linear mappings $P : X \to R^n$ are called finite dimensional projections. Sets of the form $C = P^{-1}(B)$, where $P : X \to R^n$ is a finite dimensional projection and $B \in \mathcal{B}(R^n)$, are said to be cylindrical. The induced measure $\mu \circ P^{-1}$ on R^n is defined by $\mu \circ P^{-1}(B) = \mu(P^{-1}(B))$. Finally, put $\mu_h(B) = \mu(B + h)$.

A measure μ on X is said to be Skorohod differentiable along a vector h if there exists a (signed) Radon measure ν such that for all continuous bounded functions f on X the following equality holds true:

$$\lim_{t\to 0} \int \frac{f(x - th) - f(x)}{t}\, \mu(dx) = \int f(x)\, \nu(dx).$$

Such a measure ν (which is clearly unique) is denoted by $d_h\mu$ and called the Skorohod derivative of μ along h (see [3], [5] for more details on differentiable measures).

In the case $X = R^n$, a measure μ is Skorohod differentiable along a vector h if and only if its partial derivative $\partial_h\mu$ in the sense of generalized functions (see [9], [11]) is a bounded measure. A measure on R^n is Skorohod differentiable along n independed vectors if and only if it is of the form $\mu = f\, dx$, where f is an integrable function of the class $BV(R^n)$ (see [11]).

A probability measure μ on X is said to be convex (see [6]) if

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda},$$

for all sets A and B from the σ-field $\sigma(X)$, generated by the cylindrical sets (i.e., by the dual X^*).

This condition is equivalent to

$$\mu_*(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda},$$

for all $A,\, B \in \mathcal{B}(X)$, $\lambda \in [0, 1]$, where $\mu_*(A) = \sup\{\mu(K),\ K \subset A, K \text{ is compact}\}$. Clearly, in the case where X is a Banach (or Fréchet) space, the set $\lambda A + (1 - \lambda)B$ is μ-measurable (since μ has separable support), so that one can replace μ_* by μ.

It was proved in [6] that a measure μ is convex if and only if all its finite dimensional projections are convex. In the same work the following theorem was proved, describing all convex measures on finite dimensional spaces.

Theorem 0. *A probability measure μ on R^n is convex if and only if there exist an integer number k, $0 \leq k \leq n$, a probability measure ν on R^k, absolutely continuous with respect to the Lebesgue measure m_k, and an affine transformation $h : R^k \to R^n$ such that $\mu = \nu \circ h^{-1}$ and the function $-\log(d\nu/dm_k)$ is convex if $k > 0$ (where $\log 0 = -\infty$).*

In particular, any convex measure on R^n is concentrated on an affine subspace of R^n on which it has a density with respect to the corresponding Lebesgue measure.

According to [8], every absolutely continuous convex measure on R^n is Skorohod differentiable along all vectors. In particular, every such measure has a density of the class $BV(R^n)$. Obviously, unlike the Gaussian case, a non-singular translate of a general convex measure μ may fail to be equivalent to μ (consider, e.g., the measure μ on R^1 with density $I_{[0,1]}$, i.e., the Lebesgue measure on $[0,1]$ extended to R^1 as zero outside $[0,1]$). Therefore, one should take care of a right form of the Gaussian dichotomy mentioned above. A natural candidate is: either $\mu \perp \mu_{th}$ for all $t \neq 0$ or μ is continuous along h in the sense that $\|\mu_{th} - \mu\| \to 0$ as $t \to 0$. In fact, we prove even a stronger version of this dichotomy both in finite and infinite dimensions.

Now we turn to the formulation and proving of our main results.

Theorem 1. *Let μ be a probability convex measure on R^n, absolutely continuous with respect to the Lebesgue measure. Then for every $h \in R^n$ one has*

$$\|\mu_h - \mu\| \geq 2 - 2e^{\frac{-\|d_h\mu\|}{2}}.$$

Proof. Step 1. Let us consider the case $n = 1$. In this case $\mu = e^{-V(x)}dx$, where $V(x): (a,b) \to R^1$, $V(\alpha x + (1-\alpha)y) \leq \alpha V(x) + (1-\alpha)V(y)$, $x, y \in (a,b)$, $0 \leq \alpha \leq 1$. Moreover, V is an absolutely continuous function, $V'(x) \geq 0$. Therefore, there exist the limits $\lim_{x\to a+} V(x) = V(a)$, $\lim_{x\to b-} V(x) = V(b)$, and, in addition, if a or $b = \pm\infty$, then $\lim_{x\to\pm\infty} V(x) = \infty$ (see [7], [8]). Set $V(x) = \infty$, $e^{-V(x)} = 0$ outside the segment $[a,b]$.

For such a measure μ we shall calculate $\|d_h\mu\|$ and $\|\mu_h - \mu\|$. It is easy to see that

$$d_1\mu = -V'(x)e^{-V(x)}dx + e^{-V(a)}\delta(a) - e^{-V(b)}\delta(b),$$

whence

$$\begin{aligned}
\|d_1\mu\| &= \int_a^b |V'(x)|e^{-V(x)}dx + e^{-V(a)} + e^{-V(b)} \\
&= \int_a^{x_0} (-V'(x))e^{-V(x)}dx + \int_{x_0}^b V'(x)e^{-V(x)}dx + e^{-V(a)} + e^{-V(b)} \\
&= e^{-V(x_0)} - e^{-V(a)} + e^{-V(x_0)} - e^{-V(b)} + e^{-V(a)} + e^{-V(b)} = 2e^{-V(x_0)},
\end{aligned}$$

where x_0 is determined by the condition $V(x_0) = \min_{x\in R} V(x)$, $d_h\mu = hd_1\mu$, $h > 0$. In addition, $\|\mu_h - \mu\| = 2\|\mu\|$ if $h \geq b - a$. Let us calculate

$$\|\mu_h - \mu\| = \int_{-\infty}^{+\infty} |e^{-V(x-h)} - e^{-V(x)}|dx = \int_a^{b+h} |e^{-V(x-h)} - e^{-V(x)}|dx$$

in the case $h < b - a$. In this case the function $V(x) - V(x-h)$ increases on the interval $(a+h, b)$, because $V'(x) - V'(x-h) \geq 0$. If the function $V(x) - V(x-h)$

on the interval $(a+h,b)$ takes both positive and negative values, then there exists $y_0 \in (a+h,b)$ such that $V(y_0) = V(y_0-h)$ (by virtue of the mean value theorem and the continuity of the function V). If $V(x)-V(x-h) > 0$ on $(a+h,b)$, we set $y_0 = a+h$, and if $V(x)-V(x-h) < 0$ everywhere on $(a+h,b)$, we set $y_0 = b$. In either of these two cases we obtain $e^{-V(x-h)} \le e^{-V(x)}$, when $x \ge y_0$, and $e^{-V(x_h)} \ge e^{-V(x)}$, when $x \le y_0$. We have

$$\begin{aligned}
&\int_a^{b+h} \left|e^{-V(x-h)} - e^{-V(x)}\right| dx \\
&= \int_a^{y_0} \left(e^{-V(x)} - e^{-V(x-h)}\right) dx + \int_{y_0}^{b+h} \left(e^{-V(x-h)} - e^{-V(x)}\right) dx \\
&= -\int_a^{y_0-h} e^{-V(x)} dx + \int_a^{y_0} e^{-V(x)} dx - \int_{y_0}^{b} e^{-V(x)} dx + \int_{y_0-h}^{b} e^{-V(x)} dx \\
&= 2\int_{y_0-h}^{y_0} e^{-V(x)} dx = 2\max_{\alpha\in R} \mu[\alpha, \alpha+h],
\end{aligned}$$

since for every y with $y_0 - h \le y \le y_0$ and $x \notin [y_0-h, y_0]$ one has $V(y) \le V(x)$.

Having got such expressions for the norms of the shift and the derivative, let us consider the measure ν, the density of which is the nonincreasing rearrangement g (see [9]) of the density of the measure μ. More precisely, let $\lambda(\alpha) = m_1\{x : e^{-V(x)} > \alpha\}$ be the distribution function of μ, then $g(x) = \lambda^{-1}(\alpha) = e^{-W(x)}$, where W is also a convex function, $W(0) = \min V(x) = V(x_0)$, $W(x) \ge W(y)$ if $x > y > 0$ and $W = \infty$ if $x < 0$ or $x > b-a$.

The function W can be obtained as follows. Assume we have a convex closed function V, that is, the epigraph of the function V defined by $G = \{(x,y) \in R^2 : y \ge V(x)\}$ is a convex closed subset of R^2. We define the function $f(y)$ on $[V(x_0), \infty)$ by $f(y) = \max\limits_{(x,y)\in G} x - \min\limits_{(x,y)\in G} x$. The function $f(y)$ does not decrease and

$$f(\alpha y + (1-\alpha)z) \ge \alpha f(y) + (1-\alpha) f(z).$$

Let $W(x) = f^{-1}(y)$. The function W is convex and does not decrease on $[0,\infty)$, $W(0) = V(x_0)$. By the construction of the function W, one has

$$m_1\{x : W(x) < \beta\} = m_1\{x : V(x) < \beta\}$$

for every $\beta \in R^1$. Hence, e^{-W} does not increase, and

$$\begin{aligned}
&m_1\{x : e^{-V(x)} > \alpha\} = m_1\{x : V(x) < -\log\alpha\} \\
&= m_1\{x : W(x) < -\log\alpha\} = m_1\{x : e^{-W(x)} > \alpha\}.
\end{aligned}$$

Thus, the distribution functions of e^{-V} and e^{-W} coincide, hence, e^{-W} is the non-increasing rearrangement of e^{-V}. Let $\nu = e^{-W(\circ)}dx$. Then, by the calculations above,

$$\|d_1\nu\| = 2e^{-\min W} = 2e^{-\min V} = \|d_1\mu\|;$$

$$\begin{aligned}\|\nu_h - \nu\| &= 2\max_\alpha \nu[\alpha, \alpha+h] \\ &= 2\int_0^h e^{-W(x)}dx = 2\max_\alpha \mu[\alpha,\alpha+h] = \|\mu_h - \mu\|;\end{aligned}$$

$$\|\mu\| = \|\nu\|.$$

Therefore, it suffices to consider measures μ of the following special form: $\mu = e^{-V(\circ)}dx$, where V is a convex function that does not decrease on $[0,\infty)$. In this case, $\|d_h\mu\| = 2|h|e^{-V(0)}$ and

$$\|\mu_h - \mu\| = 2\int_0^h e^{-V(x)}dx, \qquad \|\mu\| = 1 = \int_0^\infty e^{-V(x)}dx.$$

To obtain the desired estimate, we fix h and $\|d_h\mu\|$ (that is, the value of $V(0)$). We shall find a function V_0, for which the value of the integral $\int_0^h e^{-V_0(x)}dx$ is minimal under the restriction $\|\mu\| = \int_0^\infty e^{-V} = 1$. Let us prove that this function V_0 is the linear function with the corresponding normalization, that is,

$$e^{-V_0(x)} = \frac{\|d_1\mu\|}{2}e^{\frac{-\|d_1\mu\|}{2}x}.$$

Indeed, let $V_0(x) = \frac{\|d_1\mu\|}{2}x - \log\big(\frac{\|d_1\mu\|}{2}\big)$ and let V be an arbitrary convex function satisfying the conditions specified above.

(i) If $V(h) = \infty$, then

$$\int_0^h e^{-V(x)}dx = 1 > \int_0^h e^{-V_0(x)}dx.$$

(ii) Let $V_0(h) < V(h) < \infty$. Then

$$V(h) = V(0) + \int_0^h V'(x)dx \le V(0) + V'(h)h,$$

because V' does not decrease on $[0,\infty)$ (here $V'(h)$ stands for $\sup_{x<h} V'(x)$). Therefore,

$$V'(h) \ge \frac{V(h) - V(0)}{h} > \frac{V_0(h) - V_0(0)}{h} = V_0'(h)$$

since $V(h) > V_0(h)$, $V(0) = V_0(0)$. Then

$$\int_h^\infty e^{-V(x)}dx \leq \int_h^\infty e^{-V'(h)(x-h)-V(h)}dx < \int_h^\infty e^{-V_0'(h)(x-h)-V_0(h)}dx = \int_h^\infty e^{-V_0(x)}dx,$$

and since $\int_0^\infty e^{-V(x)}dx = \int_0^\infty e^{-V_0(x)}dx = 1$, one has

$$\int_0^h e^{-V(x)}dx > \int_0^h e^{-V_0(x)}dx.$$

(iii) Let $V(h) \leq V_0(h)$. Then, for every $0 < x < h$, $x = \alpha h$, one has

$$V(x) \leq (1-\alpha)V(0) + \alpha V(h) \leq (1-\alpha)V_0(0) + \alpha V_0(h) = V_0(\alpha h) = V_0(x),$$

and, consequently,

$$\int_0^h e^{-V(x)}dx \geq \int_0^h e^{-V_0(x)}dx.$$

Hence, for any fixed value of $\|d_h\mu\|$, the minimum of the norm $\|\mu_h - \mu\|$ is attained at the measure μ with density e^{-V_0} and equals

$$2\int_0^h \frac{\|d_1\mu\|}{2} e^{-\frac{d_1\mu\|}{2}x}dx = 2e^{-\frac{\|d_1\mu\|}{2}x}\Big|_0^h = 2 - 2e^{-\frac{\|d_1\mu\|}{2}h} = 2 - 2e^{-\frac{\|d_h\mu\|}{2}}.$$

Therefore, for an arbitrary probability convex measure on R^1, which is absolutely continuous with respect to the Lebesgue measure, the following estimate holds true:

$$\|\mu_h - \mu\| \geq 2 - 2e^{-\frac{\|d_h\mu\|}{2}}.$$

Step 2. Now we extend this statement to measures defined on the space R^n. Let μ be a probability convex measure on R^n, absolutely continuous with respect to the Lebesgue measure m_n. Then $\mu = e^{-V(x_1,\dots,x_n)}dx_1\dots dx_n$, where V is a convex function (i.e., its epigraph is a convex set). Moreover, we can redefine the function V on a set of the Lebesgue measure zero in such a manner that its epigraph becomes closed. Hence, we may suppose the epigraph to be closed. Let $domV = C = \{(x_1,\dots,x_n) : V(x) < \infty\}$ be the domain of definition of V and let $G = \{(x_1,\dots,x_n,y) : \ y \geq V(x)\}$ be its epigraph. It has been proved in [8] that in this case $V(x) \to \infty$, when $|x| \to \infty$, that the measure μ is Skorohod differentiable along every direction and that its Skorohod derivative equals

$$d_h\mu = \left(-\frac{\partial V}{\partial h}(x)\right)e^{-V(x)}dx - e^{-V(x)}(n_x,h)\big|_{x\in\partial C}d\sigma,$$

where $d\sigma$ is the surface measure on ∂C, n_x is the outer normal vector to the surface ∂C. Therefore,

$$\|d_h\mu\| = \int\limits_C \left|\frac{\partial V}{\partial h}\right| e^{-V(x)}dx + \int\limits_{\partial C} e^{-V(x)}|(n_x,h)|d\sigma$$
$$= 2|h| \int\limits_{h^\perp} \exp\bigl(-\min_{t\in R} V(x+th)\bigr)m_{n-1}(dx).$$

Indeed, let $h = |h|e_1$. Denote by $c_1(x_1,\dots,x_n) = c_1$ and $c_2(x_1,\dots,x_n) = c_2$ the points on the surface ∂C. Then

$$\|d_h\mu\| = |h|\int\limits_C \left|\frac{\partial V}{\partial x_1}\right| e^{-V(x)}dx + |h|\int\limits_{\partial C} e^{-V(x)}|(n_x,h)|d\sigma$$
$$= |h| \int\limits_{R^{n-1}} \Bigl(\int\limits_{c_1}^{c_2}\left|\frac{\partial V}{\partial x_1}\right|dx_1\Bigr)dx_2\dots dx_n$$
$$+ |h| \int\limits_{R^{n-1}} \Bigl(e^{-V(c_2,x_2,\dots,x_n)} + e^{-V(c_1,x_2,\dots,x_n)}\Bigr)dx_2...dx_n$$
$$= 2|h| \int\limits_{R^{n-1}} e^{-\min_{x_1} V(x_1,\dots,x_n)}dx_2\dots dx_n.$$

For the shift of the measure μ in the direction $h = |h|e_1$ we have

$$\|\mu_h - \mu\| = \int\limits_{R^n} \bigl|e^{-V(x_1-h,x_2,\dots,x_n)} - e^{-V(x_1,\dots,x_n)}\bigr|dx_1\dots dx_n$$
$$= \int\limits_{R^{n-1}} \Bigl(\int\limits_{R^1} \bigl|e^{-V(x_1-h,x_2,\dots,x_n)} - e^{-V(x_1,\dots,x_n)}\bigr|dx_1\Bigr)dx_2\dots dx_n$$
$$= 2\int\limits_{R^{n-1}} \max_{\alpha\in R^1}\Bigl\{\int\limits_{\alpha}^{\alpha+h} e^{-V(x_1,x_2,\dots,x_n)}dx_1\Bigr\}dx_2\dots dx_n.$$

Now we construct the function $W(x_1,\dots,x_n)$ similarly to the one-dimensional case. Denote by

$$G = \{(x_1,\dots,x_n,y) \in R^{n+1} : y \ge V(x_1,\dots,x_n)\}$$

the epigraph of the function V (which is a convex closed subset in R^{n+1}). Let

$$G' = \{(x_2,\dots,x_n,y) \in R^n \bigm| \exists x_1 : V(x_1,\dots,x_n) \le y\}$$

be its projection to $R^n(x_2,\dots,x_n,y)$. Let, further,

$$F_1(x_2,\dots,x_n,y)=\max_{(x_1,\dots,x_n,y)\in G}x_1,$$
$$F_2(x_2,\dots,x_n,y)=\min_{(x_1,\dots,x_n,y)\in G}x_1,$$
$$F(x_2,\dots,x_n,y)=F_1-F_2.$$

The function F is defined on G', it is nonnegative and concave. Set

$$D=\{(x_1,x_2,\dots,x_n,y)\in R^{n+1}:0\le x_1\le F(x_2,\dots,x_n,y)\}.$$

Then D is a convex closed subset in R^{n+1}; it is the epigraph of some function $W(x_1,\dots,x_n)$. Indeed, if $(x_1^0,\dots,x_n^0,y^0)\in D$, then $(x_1^0,\dots,x_n^0,y)\in D$ for every $y>y^0$.

Let us consider the measure $\nu=e^{-W(\circ)}dx_1\dots dx_n$. For this measure we have

$$\|\nu_h-\nu\|=\|\mu_h-\mu\|,\quad \|d_h\nu\|=\|d_h\mu\|,\quad \|\nu\|=\|\mu\|.$$

It follows from the equality of the expressions

$$\max_{\alpha\in R^1}\int\limits_{\alpha}^{\alpha+|h|}e^{-V(x_1,x_2^0,\dots,x_n^0)}dx_1=\max_{\alpha\in R^1}\int\limits_{\alpha}^{\alpha+|h|}e^{-W}dx_1$$

and

$$\exp\bigl(-\min_{x_1}V(x_1,x_2^0,\dots,x_n^0)\bigr)=\exp\bigl(-\min_{x_1}W(x_1,x_2^0,\dots,x_n^0)\bigr)$$

that

$$\int\limits_{R^1}e^{-V(x_1,x_2^0,\dots,x_n^0)}dx_1=\int\limits_{R^1}e^{-W(x_1,x_2^0,\dots,x_n^0)}dx_1$$

for any fixed $(x_2^0, \dots, x_n^0)$, as it has been proved in the one-dimensional case. Thus,

$$
\begin{aligned}
\|\nu_h - \nu\| &= 2 \int\limits_{R^{n-1}} \Big(\int\limits_0^{|h|} e^{-W(x_1,\dots,x_n)} dx_1 \Big) dx_2 \dots dx_n \\
&= 2 \int\limits_0^{|h|} \Big(\int\limits_{R^{n-1}} e^{-W(x_1,\dots,x_n)} dx_2 \dots dx_n \Big) dx_1 = 2 \int\limits_0^{|h|} e^{-U(x_1)} dx_1, \\
\|d_h\nu\| &= 2|h| \int\limits_{R^{n-1}} e^{-W(0,x_2,\dots,x_n)} dx_2 \dots dx_n = 2|h| e^{-U(0)}, \\
\|\nu\| &= \int\limits_{R^n} e^{-W(x_1,\dots,x_n)} dx_1 \dots dx_n \\
&= \int\limits_{R^1} \Big(\int\limits_{R^{n-1}} e^{-W(x_1,\dots,x_n)} dx_2 \dots dx_n \Big) dx_1 = \int\limits_{R^1} e^{-U(x_1)} dx_1 \\
&= \int\limits_0^{\infty} e^{-U(x_1)} dx_1 = 1
\end{aligned}
$$

by the construction of the function W and the Fubini theorem. Here

$$
e^{-U(x_1)} = \int\limits_{R^{n-1}} e^{-W(x_1,x_2,\dots,x_n)} dx_2 \dots dx_n.
$$

It has been shown in [7] that the function $U(x_1)$ in this case is also convex. Moreover, the function $e^{-U(x_1)}$ does not increase on $[0,\infty)$, since all the functions $e^{-W(x_1,x_2^0,\dots,x_n^0)}$ do not increase for any fixed $(x_2^0, \dots, x_n^0)$. Therefore, we can consider the convex measure $\lambda = e^{-U(x_1)}dx_1$ on R^1, for which

$$
\|\lambda_{|h|} - \lambda\| = \|\nu_h - \nu\| = \|\mu_h - \mu\|,
$$

$$
\|d_{|h|}\lambda\| = \|d_h\nu\| = \|d_h\mu\|,
$$

$$
\|\lambda\| = \|\nu\| = \|\mu\| = 1.
$$

For the measure λ the desired assertion has already been proved, hence, it holds true for the measure μ as well. The proof is complete.

As a corollary we obtain the following theorem about the directions of differentiability of a convex measure on an infinite dimensional space.

Theorem 2. *Let μ be a convex probability Radon measure on a locally convex space X. Then for every direction h, which is not the direction of Skorohod differentiability of μ, the measures μ and μ_h are mutually singular, while for every direction of differentiability there holds the following inequality:*

$$\|\mu_h - \mu\| \geq 2 - 2e^{-\frac{\|d_h\mu\|}{2}}.$$

Proof. Note that for any Radon measure ν on X its variation norm coincides with the supremum of $\|\nu \circ P^{-1}\|$ over all finite dimensional projections P. Indeed, let $X = A^+ \cup A^-$ and $P(X) = A_P{}^+ \cup A_P^-$ be the sets from the Hahn decompositions for the measures ν and $\nu \circ P^{-1}$, respectively. Then

$$\|\nu \circ P^{-1}\| = \nu \circ P^{-1}(A_P^+) - \nu \circ P^{-1}(A_P^-) \leq \nu(A^+) - \nu(A^-) = \|\nu\|.$$

On the other hand, since ν is Radon, for each n there exists a set $A_n = P_n^{-1}B_n$, where $B_n \in \mathcal{B}(R^n)$ and P_n is a finite dimensional projection, such that $|\nu|(A^+ \triangle A_n) \leq 1/n$. In this case

$$|\nu|(A^- \triangle (X \backslash A_n)) = |\nu|(A^+ \triangle A_n) \leq 1/n.$$

Then

$$\begin{aligned}\|\nu \circ P_n^{-1}\| &\geq \nu \circ P_n^{-1}(B_n) - \nu \circ P_n^{-1}(R^n \backslash B_n) \\ &\geq \nu(A^+) - 1/n - \nu(A^-) - 1/n = \|\nu\| - 2/n.\end{aligned}$$

Therefore, $\|\nu \circ P_n^{-1}\| \to \|\nu\|$ as $n \to \infty$.

Let us consider the finite dimensional projections of the measure μ. The following two cases are now possible:

(i) $\sup_P \|d_{Ph}\mu \circ P^{-1}\| = \infty$. Assume first that $\|d_{Ph}\mu \circ P^{-1}\| = \infty$ for some finite dimensional projection P. In this case, according to Theorem 0 and Theorem 1, we get

$$2 = \|(\mu \circ P^{-1})_{Ph} - \mu \circ P^{-1}\| \leq \|\mu_h - \mu\|.$$

Thus, the measures μ and μ_h are mutually singular. Now let $\|d_{Ph}\mu \circ P^{-1}\| < \infty$ for every P. Then there exists a sequence $\{P^n\}$ such that $\|d_{h_n}\mu^n\| \to \infty$ as $n \to \infty$, where $h_n = P_n h$ and $\mu_n = \mu \circ P_n^{-1}$. In this case, according to Theorem 1, we have

$$\|\mu^n_{h_n} - \mu^n\| \geq 2 - 2e^{-\frac{\|d_{h_n}\mu^n\|}{2}} \to 2 \qquad \text{as} \quad n \to \infty.$$

Hence, $\|\mu_h - \mu\| = 2$, and the measures μ and μ_h are mutually singular again.

(ii) $\|d_{Ph}\mu \circ P^{-1}\| \leq C$, where C is some positive constant. Then

$$\|(\mu \circ P^{-1})_{tPh} - \mu \circ P^{-1}\| \leq C|t|$$

for all t and all finite dimensional projections P (see [3] or Corollary of Lemma 2.4 in [5]). Hence, $\|\mu_{th} - \mu\| \leq C|t|$ for all t, which by virtue of Theorem 1 in [3] (see also

Theorem 2.7 in [5]) implies the Skorohod differentiability of μ along h. Therefore, we obtain

$$\|\mu_h - \mu\| = \sup_P \|(\mu \circ P^{-1})_{Ph} - \mu \circ P^{-1}\|$$
$$\geq \sup_P \left(2 - 2\exp\left(-\frac{\|d_{Ph}\mu \circ P^{-1}\|}{2}\right)\right) = 2 - 2\exp\left(-\frac{\|d_h\mu\|}{2}\right).$$

Note that in the case where the measure μ is not differentiable along h, it is not differentiable along th for every nonzero t. Hence, the measures μ and μ_{th} are mutually singular. The proof is complete.

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