Combinatorial classification of finitely generated virtually free groups

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1. Introduction

For a finitely generated virtually free group Γ denote by m_Γ the least common multiple of the orders of the finite subgroups in Γ and let

$$b_\Gamma(\lambda) = \text{ number of free subgroups of index } \lambda m_\Gamma \text{ in } \Gamma\,.$$

Growth behaviour and asymptotics of this function $b_\Gamma(\lambda)$ have been investigated in two previous papers [M1] and [M2]. In [M1] we also introduced an equivalence relation $\sim$ on the class $\mathcal{C}$ of all finitely generated virtually free groups by setting

$$\Gamma_1 \sim \Gamma_2 :\Leftrightarrow (m_{\Gamma_1}, b_{\Gamma_1}) = (m_{\Gamma_2}, b_{\Gamma_2})\,,$$

i.e. two groups Γ_1 and Γ_2 are identified via $\sim$ if and only if Γ_1 and Γ_2 contain the same number of free subgroups for each given finite index. We gave a characterization of this equivalence in terms of a certain structural invariant, the *type* $\tau(\Gamma)$ of Γ, and we showed that each $\sim$–class decomposes into finitely many isomorphism classes. Hence, one can say that a group Γ is "almost determined" by the combinatorial datum (m_Γ, b_Γ).

The aim of the present note is to introduce a coarser equivalence relation $\sim_a$ into this context, which classifies groups by the asymptotic behaviour of the number of free subgroups of given finite index, and to characterize this equivalence $\sim_a$ again in terms of structural invariants. We show that the $\sim_a$–class of a group $\Gamma \cong \pi_1(\Gamma(-), Y)$ is determined by the constant m_Γ, the Euler characteristic $\chi(\Gamma)$ of Γ, the rank of the fundamental group $\pi_1(Y)$ of the graph Y underlying the above Stallings decomposition of Γ and the values of certain sums formed from the type $\tau(\Gamma)$. This characterization allows us to strengthen the finiteness result [M1, Theorem 3] on $\sim$–classes by showing that even a $\sim_a$–class contains only a finite number of non–isomorphic groups and we provide examples of virtually free groups Γ_1 and Γ_2 such that $\Gamma_1 \sim_a \Gamma_2$ and $\Gamma_1 \not\sim \Gamma_2$. These results contrast in an interesting way with our recent results concerning the function $s_\Gamma(n)$ counting the number of *all* subgroups of index n in Γ.

2. The type $\tau(\Gamma)$ and the equivalence relation $\sim_a$

Recall that, as a well-known consequence of Stallings' structure theorem, a finitely generated virtually free group Γ can be presented as the fundamental group of a finite graph of groups $(\Gamma(-), Y)$ with finite vertex groups $\Gamma(v)$; cf. [KPS] or [D, IV.1.9]. The fact that, conversely, the fundamental group of a finite graph of finite groups is always virtually free of finite rank is more elementary; see [Se, II.2.6, Prop. 11]. Fix such a decomposition $\Gamma \cong \pi_1(\Gamma(-), Y)$ of Γ and denote by V, E the set of vertices and (geometric) edges of Y, respectively. The type $\tau(\Gamma)$ of Γ is defined as the tuple $(m_\Gamma; \zeta_1(\Gamma), \ldots, \zeta_\kappa(\Gamma), \ldots, \zeta_{m_\Gamma}(\Gamma))$, where the $\zeta_\kappa(\Gamma)$ are integers indexed by the divisors of m_Γ, given by

$$\zeta_\kappa(\Gamma) = \big|\{e \in E : |\Gamma(e)| \,\big|\, \kappa\}\big| - \big|\{v \in V : |\Gamma(v)| \,\big|\, \kappa\}\big|\,.$$

It can be shown that the type $\tau(\Gamma)$ is in fact an invariant of the group Γ i.e. independent of the particular decomposition of Γ in terms of a graph of groups $(\Gamma(-), Y)$; cf. [M1, Theorem 2(a)]. Note that the Euler characteristic of Γ is determined by the type $\tau(\Gamma)$ via

$$\chi(\Gamma) = -\, m_\Gamma^{-1} \sum_{\kappa | m_\Gamma} \varphi(m_\Gamma/\kappa)\, \zeta_\kappa(\Gamma)\,, \tag{1}$$

where φ is Euler's function. Indeed,

$$\begin{aligned}
&\sum_{\kappa|m_\Gamma} \varphi(m_\Gamma/\kappa)\zeta_\kappa(\Gamma) = \\
&\sum_{1\leq k\leq m_\Gamma} \Big(|\{e \in E : |\Gamma(e)| \Big| k\}| - |\{v \in V : |\Gamma(v)| \Big| k\}| \Big) = \\
&\sum_{e\in E} \frac{m_\Gamma}{|\Gamma(e)|} - \sum_{v\in V} \frac{m_\Gamma}{|\Gamma(v)|} = -m_\Gamma\, \chi(\Gamma)\,,
\end{aligned}$$

the last equality coming from the fact that the Euler characteristic $\chi(\Gamma)$ in the sense of Wall coincides with the equivariant Euler characteristic $\chi_X(\Gamma)$ relative to the tree X associated with Γ in the sense of Bass and Serre; cf. for example [B, Chap. IX, Prop. 7.3].

In [M1] the following two assertions concerning the equivalence relation $\sim$ are proved:

(2) $\Gamma_1 \sim \Gamma_2$ if and only if $\tau(\Gamma_1) = \tau(\Gamma_2)$.

(3) Each $\sim$–equivalence class of finitely generated virtually free groups contains only a finite number of non–isomorphic groups.

This is [M1, Theorem 2(b)] and [M1, Theorem 3], respectively.

It follows for example from [Se, II.2.6, Lemma 10] that $b_\Gamma(\lambda) \neq 0$ for all $\lambda \geqslant 1$, provided that Γ is infinite. We can therefore define a relation $\sim_a$ by

$$\Gamma_1 \sim_a \Gamma_2 \;:\Leftrightarrow\; m_{\Gamma_1} = m_{\Gamma_2} \;\text{ and }\; b_{\Gamma_1}(\lambda) \sim b_{\Gamma_2}(\lambda)\,(\lambda \to \infty)\,,$$

provided that both Γ_1 and Γ_2 are infinite. We extend this definition to the whole class $\mathcal{C}$ by setting

$$\Gamma_1 \sim_a \Gamma_2 :\Leftrightarrow |\Gamma_1| = |\Gamma_2|\,,$$

if one of the groups involved is finite. Clearly, this defines an equivalence relation on $\mathcal{C}$ with the property that

$$\Gamma_1 \sim \Gamma_2 \Rightarrow \Gamma_1 \sim_a \Gamma_2 \,.$$

The following result provides the desired structural characterization of the equivalence $\sim_a$.

Theorem. *Two groups $\Gamma_1, \Gamma_2 \in \mathcal{C}$ are $\sim_a$–equivalent if and only if they satisfy the following conditions:*

(i) $m_{\Gamma_1} = m_{\Gamma_2} (=: m)$.

(ii) $\chi(\Gamma_1) = \chi(\Gamma_2)$.

(iii) $\zeta_m(\Gamma_1) = \zeta_m(\Gamma_2)$.

(iv) $\sum\limits_{\mu_j=1}^{\nu_j} \zeta_{m/p_j^{\mu_j}}(\Gamma_1) = \sum\limits_{\mu_j=1}^{\nu_j} \zeta_{m/p_j^{\mu_j}}(\Gamma_2)$
for $j = 1, 2, ..., r$, *where* $m = p_1^{\nu_1} p_2^{\nu_2} ... p_r^{\nu_r}$ *is the prime decomposition of* m.

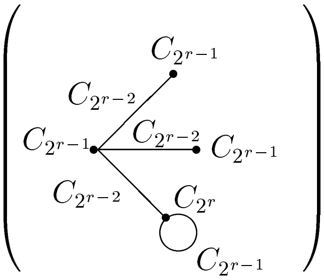
Observe that in view of (1) the conditions (i) – (iv) of the theorem are obviously implied by the condition that $\tau(\Gamma_1) = \tau(\Gamma_2)$.

Corollary. *A $\sim_a$–equivalence class of finitely generated virtually free groups contains only a finite number of non–isomorphic groups.*

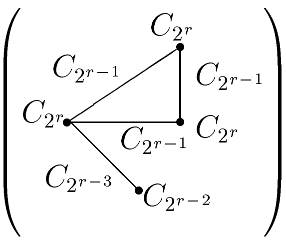
This follows from the above theorem and [M1, Prop. 4].

By definition, the type $\tau(\Gamma)$ of a virtually free group $\Gamma \cong \pi_1(\Gamma(-), Y)$ depends solely upon the graph Y and the order of the vertex and edge groups but not upon the internal structure of the $\Gamma(v)$ and $\Gamma(e)$. From (2) and this observation it is clear how to construct groups $\Gamma_1, \Gamma_2 \in \mathcal{C}$ such that $\Gamma_1 \sim \Gamma_2$ and $\Gamma_1 \ncong \Gamma_2$. Using the above theorem we can now also give examples of groups Γ_1 and Γ_2 such that $\Gamma_1 \sim_a \Gamma_2$ and $\Gamma_1 \not\sim \Gamma_2$. Fix an integer $r \geqslant 3$ and let

$\Gamma_1(r) := \pi_1$



and $\Gamma_2(r) := \pi_1$



with the natural inclusions as canonical embeddings. We have $m_{\Gamma_1(r)} = m_{\Gamma_2(r)} =: m_r = 2^r, \chi(\Gamma_1(r)) = \chi(\Gamma_2(r)) = -\frac{7}{2^r}$ and $\zeta_{m_r}(\Gamma_1(r)) = \zeta_{m_r}(\Gamma_2(r)) = 0$. Moreover, concerning the other ζ-invariants, we find the following:

$$\begin{aligned}
&\zeta_{2^s}(\Gamma_1(r)) = \zeta_{2^s}(\Gamma_2(r)) = 0 \quad \text{for} \quad 0 \leq s \leq r-4,\\
&\zeta_{2^{r-3}}(\Gamma_1(r)) = 0, \;\; \zeta_{2^{r-3}}(\Gamma_2(r)) = 1,\\
&\zeta_{2^{r-2}}(\Gamma_1(r)) = 3, \;\; \zeta_{2^{r-2}}(\Gamma_2(r)) = 0,\\
&\zeta_{2^{r-1}}(\Gamma_1(r)) = 1, \;\; \zeta_{2^{r-1}}(\Gamma_2(r)) = 3\,.
\end{aligned}$$

Hence,

$$\sum_{s=0}^{r-1} \zeta_{2^s}(\Gamma_1(r)) = 4 = \sum_{s=0}^{r-1} \zeta_{2^s}(\Gamma_2(r)),$$

i.e. condition (iv) is also satisfied, and we conclude from our theorem that for each $r \geqslant 3$ indeed $\Gamma_1(r) \sim_a \Gamma_2(r)$. Note also that m_r strictly increases to infinity as $r \to \infty$, hence (again by our theorem) pairs of groups $\{\Gamma_1(r), \Gamma_2(r)\}$ constructed to different values of r belong to different classes with respect to the equivalence relation $\sim_a$. On the other hand we have $\tau(\Gamma_1(r)) \neq \tau(\Gamma_2(r))$ and therefore $\Gamma_1(r) \not\sim \Gamma_2(r)$ by assertion (2). It follows that the three equivalence relations on the class $\mathcal{C}$ given by (i) isomorphism, (ii) the relation $\sim$ and (iii) the relation $\sim_a$ are all distinct.

It is interesting to compare these results with the situation for the function $s_\Gamma(n)$ counting the number of *all* subgroups of index n in Γ. The asymptotic behaviour of $s_\Gamma(n)$ has been analyzed in a recent paper [M3] for groups Γ of the form

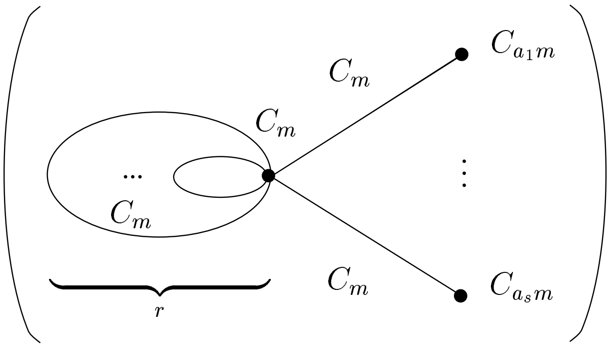
$$(\mathcal{F}) \qquad \Gamma = \mathop{*}_{\sigma=1}^{s} G_\sigma \; * F_r, \;\; 0 \leq r, s < \infty, \;\; 1 < |G_\sigma| < \infty,$$

i.e. free products of arbitrary finite groups and a free group. As a consequence we also obtain there a combinatorial classification of $\mathcal{F}$-groups via the asymptotic behaviour of their associated function $s_\Gamma(n)$ including analogues of the above theorem and its corollary; cf. [M3, §4]. It appears however that $\mathcal{F}$ is precisely the largest class of virtually free groups admitting this type of results on $s_\Gamma(n)$. Consider for instance the sequence of groups $\Gamma_m = C_{2m} \mathop{*}_{C_m} C_{3m}$, i.e. $\Gamma_1 = \Gamma \cong PSL(2,\mathbb{Z}), \Gamma_2 \cong SL(2,\mathbb{Z})$, etc. Then one can show that the relation

$$(4) \qquad \frac{s_{\Gamma_m}(n)}{s_\Gamma(n)} = 1 + \mathcal{O}(n^{-K}) \; (n \to \infty)$$

holds for all integers $m, K \geqslant 1$. Indeed, there is nothing special about the modular group here. The same result (4) holds for example, if we replace $PSL(2, \mathbb{Z})$ by any cyclic cover $\Gamma = C_{a_1} * \ldots * C_{a_s} * F_r$ in $\mathcal{F}$ such that $\chi(\Gamma) < 0$, and then lift to a sequence of groups of the form

(5) $\qquad \Gamma_m = \pi_1$



$, \qquad m = 1, 2, \ldots$.

This type of phenomenon will be pursued more systematically in a separate paper. As a consequence of (4) the asymptotic expansion provided by [M3, Theorem 1] for the function $s_\Gamma(n)$ of the base group $\Gamma = \Gamma_1$ carries over without change to the groups Γ_m in (5); in particular, the finiteness result [M3, Theorem 3(b)] breaks down if we attach one such "hair" $\{\Gamma_m\}_1^\infty$ to the class $\mathcal{F}$.

3. Proof of the theorem

Suppose that $\Gamma \in \mathcal{C}$ contains a non–abelian free subgroup and let $(\Gamma(-), Y)$ be a finite graph of finite groups (with vertex set V and edge set E) representing Γ, i. e. $\pi_1(\Gamma(-), Y) \cong \Gamma$. Then we have

$$b_\Gamma(\lambda) \sim K_\Gamma \; \varphi_\Gamma(\lambda) \; (\lambda \to \infty)\,, \tag{6}$$

where

$$K_\Gamma := m_\Gamma \left((2\pi m_\Gamma)^{\zeta_{m_\Gamma}(\Gamma)} \prod_{v \in V} |\Gamma(v)| \prod_{e \in E} |\Gamma(e)|^{-1} \right)^{1/2}$$

and

$$\varphi_\Gamma(\lambda) := \exp\left(-\, m_\Gamma \chi(\Gamma) \lambda \log(m_\Gamma \lambda) + \Big(1 + (\zeta_{m_\Gamma}(\Gamma)/2) \Big) \log(\lambda) + \; m_\Gamma \chi(\Gamma) \lambda \right).$$

This is [M1, Theorem 5]. We will also need an observation concerning real sequences. Let $\mathbb{R}^{\mathbb{N}}$ be the $\mathbb{R}$-vector space of all real sequences, $\mathfrak{N}$ the subspace consisting of those sequences $\psi(n)$ for which $\lim_{n\to\infty}\psi(n)=0$, and let $^{-}:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}^{\mathbb{N}}/\mathfrak{N}$ be the canonical projection.

Lemma. *Suppose that $\psi_1(n),\psi_2(n),...,\psi_N(n)\in\mathbb{R}^{\mathbb{N}}$ are such that*

(a) $\psi_1(n)\notin\mathfrak{N}$,

(b) $\psi_\nu(n)=o(\psi_{\nu+1}(n))$ *for all* $1\le\nu<N$.

Then $\overline{\psi_1(n)}$, $\overline{\psi_2(n)}$, ..., $\overline{\psi_N(n)}$ form a set of linearly independent vectors in $\mathbb{R}^{\mathbb{N}}/\mathfrak{N}$.

Proof. We show by induction on ν that the sets $\{\overline{\psi_1(n)},...,\overline{\psi_\nu(n)}\}\subseteq\mathbb{R}^{\mathbb{N}}/\mathfrak{N}$ are linearly independent for $\nu=1,2,...,N$. The case $\nu=1$ is just assumption (a). Assume inductively that the set $\{\overline{\psi_1(n)},...,\overline{\psi_\nu(n)}\}$ is linearly independent for some ν, $1\le\nu<N$, and suppose that $\overline{\psi_{\nu+1}(n)}\in\left\langle\overline{\psi_1(n)},...,\overline{\psi_\nu(n)}\right\rangle$, the subspace generated by $\overline{\psi_1(n)},...,\overline{\psi_\nu(n)}$. Then there exist real numbers $\alpha_1,...,\alpha_\nu$ such that

$$\psi_{\nu+1}(n)+\alpha_1\psi_1(n)+...+\alpha_\nu\psi_\nu(n)=o(1)\quad(n\to\infty),$$

and since the left-hand side equals $\psi_{\nu+1}(n)(1+o(1))$ by (b), it follows that $\psi_{\nu+1}(n)=o(1)$. Applying (b) once more yields

$$\psi_\nu(n)=\frac{\psi_\nu(n)}{\psi_{\nu+1}(n)}\psi_{\nu+1}(n)=o(1)\quad(n\to\infty),$$

which contradicts the induction hypothesis.

□

We now come to the proof of our theorem. Assume first that $\Gamma_1,\Gamma_2\in\mathcal{C}$ both contain a non-abelian free subgroup and that $\Gamma_1\sim_a\Gamma_2$. Then by (6) and the definition of the equivalence relation $\sim_a$ we have $m_{\Gamma_1}=m_{\Gamma_2}=:m$ and

$$q_\lambda=1+o(1)\,(\lambda\to\infty),\tag{7}$$

where

$$\begin{aligned}q_\lambda:=\exp\Big(&-m\,(\chi(\Gamma_1)-\chi(\Gamma_2))\,\lambda\log(m\lambda)\;+\;m\,(\chi(\Gamma_1)-\chi(\Gamma_2))\,\lambda\\&+\;\frac{\zeta_m(\Gamma_1)-\zeta_m(\Gamma_2)}{2}\log(\lambda)\;+\;\log(K_{\Gamma_1}/K_{\Gamma_2})\Big).\end{aligned}\tag{8}$$

By our lemma the functions of λ involved in the exponent on the right-hand side of (8) represent a set of linearly independent vectors in $\mathbb{R}^{\mathbb{N}}/\mathfrak{N}$. Hence (7) implies conditions (ii) and (iii) as well as $K_{\Gamma_1}=K_{\Gamma_2}$.

Next, observe that for a group $\Gamma \cong \pi_1(\Gamma(-), Y)$ in $\mathcal{C}$ the rational number $(\prod_{v\in V} |\Gamma(v)|)/(\prod_{e\in E} |\Gamma(e)|)$ is independent of the particular decomposition of Γ in terms of a graph of groups $(\Gamma(-), Y)$ and can be expressed in terms of the type $\tau(\Gamma)$ via

$$\frac{\prod\limits_{v\in V} |\Gamma(v)|}{\prod\limits_{e\in E} |\Gamma(e)|} = \prod_{\kappa|m_\Gamma} \kappa^{-\Sigma_{\nu|\kappa}\mu(\kappa/\nu)\zeta_\nu(\Gamma)}\,, \tag{9}$$

where μ is the ordinary Moebius function. To see this denote, for each positive integer κ, by v_κ the number of vertices v in Y such that $|\Gamma(v)| = \kappa$ and by e_κ the corresponding number of (geometric) edges. Then clearly

$$\sum_{\nu|\kappa}(e_\nu - v_\nu) = \zeta_\kappa(\Gamma),\ \ \kappa|m_\Gamma\,. \tag{10}$$

Using (10) to define $\zeta_\kappa(\Gamma)$ for all $\kappa \in \mathbb{N}$ and inverting we obtain

$$v_\kappa - e_\kappa = -\sum_{\nu|\kappa} \mu(\kappa/\nu)\zeta_\nu(\Gamma)$$

for all $\kappa \in \mathbb{N}$, in particular for $\kappa|m_\Gamma$, from which (9) follows.

Since we know already that $m_{\Gamma_1} = m_{\Gamma_2} = m$ and $\zeta_m(\Gamma_1) = \zeta_m(\Gamma_2)$, the fact that also $K_{\Gamma_1} = K_{\Gamma_2}$ holds implies

$$\frac{\prod\limits_{v\in V_1} |\Gamma(v)|}{\prod\limits_{e\in E_1} |\Gamma(e)|} = \frac{\prod\limits_{v\in V_2} |\Gamma(v)|}{\prod\limits_{e\in E_2} |\Gamma(e)|}\,,$$

and hence in view of (9) that

$$\prod_{\kappa|m} \kappa^{-\Sigma_{\nu|\kappa}\mu(\kappa/\nu)(\zeta_\nu(\Gamma_1)-\zeta_\nu(\Gamma_2))} = 1\,.$$

Taking log we obtain the equation

$$\sum_{\kappa|m}\sum_{\nu|\kappa} \mu(\kappa/\nu)(\zeta_\nu(\Gamma_1) - \zeta_\nu(\Gamma_2))\log(\kappa) = 0. \tag{11}$$

If, conversely, conditions (i) - (iii) plus equation (11) are satisfied for our groups Γ_1 and Γ_2 then using (9) again we find that $K_{\Gamma_1} = K_{\Gamma_2}$ and $\varphi_{\Gamma_1}(\lambda) = \varphi_{\Gamma_2}(\lambda)$ and $\Gamma_1 \sim_a \Gamma_2$ follows from the asymptotic formula (6).

We claim now that assuming (i) the conjunction of (iii) and (11) is equivalent to (iii) and (iv). Indeed, suppose that $m > 1$ (otherwise there is nothing to prove) and let $m = p_1^{\nu_1} p_2^{\nu_2} \dots p_r^{\nu_r}$ be the prime decomposition of m. Then we have

$$\sum_{\kappa|m} \sum_{\nu|\kappa} \mu(\kappa/\nu)(\zeta_\nu(\Gamma_1) - \zeta_\nu(\Gamma_2)) \log(\kappa) =$$

$$\sum_{\nu|m} \sum_{\lambda|\frac{m}{\nu}} \mu(\lambda)(\zeta_\nu(\Gamma_1) - \zeta_\nu(\Gamma_2)) \log(\nu\lambda) =$$

$$\sum_{\nu|m} (\zeta_\nu(\Gamma_1) - \zeta_\nu(\Gamma_2)) \log(\nu) \sum_{\lambda|\frac{m}{\nu}} \mu(\lambda) \; + \; \sum_{\nu|m} (\zeta_\nu(\Gamma_1) - \zeta_\nu(\Gamma_2)) \sum_{\lambda|\frac{m}{\nu}} \mu(\lambda) \log(\lambda) =$$

$$(\zeta_m(\Gamma_1) - \zeta_m(\Gamma_2)) \log(m) - \sum_{j=1}^{r} \log(p_j) \sum_{\mu_j=1}^{v_j} \left(\zeta_{m/p_j^{\mu_j}}(\Gamma_1) - \zeta_{m/p_j^{\mu_j}}(\Gamma_2) \right).$$

Here we have used the facts that

$$\sum_{\lambda|n} \mu(\lambda) = \begin{cases} 1; & n = 1 \\ 0; & n > 1 \end{cases}$$

and that

$$-\sum_{\lambda|n} \mu(\lambda) \log(\lambda) = \Lambda(n) = \begin{cases} \log(p); & n = p^\nu \text{ a prime power, } \nu > 0 \\ 0; & \text{otherwise,} \end{cases}$$

where $\Lambda(n)$ is von Mangoldt's function; cf. for example [C, Chap. VI, §5]. It follows that assuming (i) the conjunction of (iii) and (11) is equivalent to (iii) plus the condition that

$$\prod_{j=1}^{r} p_j^{\sum_{\mu_j=1}^{\nu_j} \left(\zeta_{m/p_j^{\mu_j}}(\Gamma_1) - \zeta_{m/p_j^{\mu_j}}(\Gamma_2) \right)} = 1,$$

the latter being clearly equivalent to (iv), whence our claim.

So far we have verified the theorem in the case when both groups Γ_1 and Γ_2 are large. It remains to deal with the case when one of the groups (say Γ_1) is finite or virtually infinite-cyclic. Suppose first that Γ_1 is finite and that $\Gamma_1 \sim_a \Gamma_2$. Then by the definition of $\sim_a$ the group Γ_2 is also finite and $|\Gamma_1| = |\Gamma_2| =: m$, and by definition of the type we have $\tau(\Gamma_1) = (m; 0, ..., 0, -1) = \tau(\Gamma_2)$, in particular conditions (i) - (iv) hold. If, conversely, conditions (i) - (iv) hold for the groups Γ_1 and Γ_2 and Γ_1 is finite, then by (ii), Γ_2 is also finite, and by (i) we have $|\Gamma_1| = |\Gamma_2|$ and hence $\Gamma_1 \sim_a \Gamma_2$.

Recall that a finite extension Γ of C_∞ is the fundamental group either (a) of a single loop with finite vertex group and isomorphic canonical embeddings or (b) of a segment of finite groups with canonical embeddings of index 2; see for example [St, Sect. 5.1] or [W, Lemma 4.1]. In the first case $\tau(\Gamma) = (m_\Gamma; 0, \ldots, 0)$, whereas in case (b) we have $\tau(\Gamma) = (m_\Gamma; 0, \ldots, 0, 1, -1)$. Moreover, in case (a) $b_\Gamma(\lambda) = m_\Gamma$ and in case (b) $b_\Gamma(\lambda) = m_\Gamma/2$ identically; cf. the remark preceding Corollary 6 in [M1]. Assume now that Γ_1 is virtually infinite-cyclic and that $\Gamma_1 \sim_a \Gamma_2$. Then Γ_2 must also be infinite and by definition of $\sim_a$ we have $m_{\Gamma_1} = m_{\Gamma_2}$ and $b_{\Gamma_1}(\lambda) \sim b_{\Gamma_2}(\lambda)$, in particular $b_{\Gamma_2}(\lambda)$ remains bounded as $\lambda \to \infty$. The latter fact implies via (6) that Γ_2 is also virtually infinite–cyclic and hence that $b_{\Gamma_1} = b_{\Gamma_2}$. It now follows from (2) that $\tau(\Gamma_1) = \tau(\Gamma_2)$ and in particular that conditions (i) - (iv) hold for Γ_1 and Γ_2. Conversely, assume that Γ_1 and Γ_2 satisfy conditions (i) - (iv) and that Γ_1 is virtually infinite–cyclic. Then by condition (ii) the group Γ_2 is also virtually infinite–cyclic and it follows from (iii) that Γ_1 is of the form (a) if and only if Γ_2 is of this form. The last statement together with condition (i) implies that $b_{\Gamma_1} = b_{\Gamma_2}$. We conclude that $\Gamma_1 \sim \Gamma_2$, hence a fortiori $\Gamma_1 \sim_a \Gamma_2$ and the theorem is proved.

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