SIMPLICIAL DEGREES OF FUNCTORS

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Dedicated to the memory of Professor Roin Nadiradze

1.INTRODUCTION

The main result of this paper is to show that if G_* is a simplicial group of finite length, then $H_n G_*$ also has finite length. Here the length of a simplicial group means the length of the corresponding Moore normalization and $H_n G_*$ is a simplicial abelian group given by $[k] \mapsto H_n G_k$. A similar fact is true if we replace G_* by a simplicial ring and we take the algebraic K-functors instead of group homology.

The origin of such results goes back to the classical paper of Dold and Puppe (see Hilfsatz 4.23 of [DP]), where the following was proved: let $T : \mathbf{A} \to \mathbf{B}$ be a functor between abelian categories of degree d, meaning that the $(d+1)$-st cross-effect functor in the sense of Eilenberg and Mac Lane [EM] vanishes. Then $l(TX_*) \le dl(X_*)$ for any simplicial object X_* in $\mathbf{A}$. Here $l(X_*)$ denotes the length of X_*. Actually, this property characterizes the degree of functors in the abelian case (see Lemma 3.6).

One can modify the notion of the cross-effects in the nonabelian framework (see [BaP]) and define the notion of degree of functors. One can also use the above inequality to define the simplicial degree in the nonabelian set-up. However in this way we get generally different invariants and the aim of this paper is to establish the relationship between the different notions. We show that many classical functors arising in homological algebra and K-theory have finite simplicial degree. Our Conjecture 4.7 claims that this should be always the case.

In the second section we deal with simplicial groups of finite length, where we essentially use the results of Conduché [C]. Based on [BaP], in the next section, we define the cross-effects and the notion of degree of functors in the nonabelian framework. In the section 4, we introduce simplicial degrees and give some inequalities relating all kinds of degrees. In section 5 we show our main result, that simplicial degree of n-dimensional group homology is n. In the last section we consider the algebraic K-functors and André-Quillen homology.

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2. SIMPLICIAL GROUPS OF FINITE LENGTH

Here we give the definition and some elementary facts about simplicial groups of finite lengths. We use the standard notation and terminology on simplicial objects (see [M]). Let $\mathcal{S}$ be the category of simplicial groups. We recall that for a simplicial group G_*, the *Moore normalization* NG_* is a chain complex of (nonabelian) groups defined by

$$N_nG_* = \bigcap_{0\le i<n} Ker\ \partial_i,$$

whose boundary operator is the restriction of ∂_n. According to Lemma 5 and Proposition 2 of section 3 of ChII of [QHA], the functor

$$N_n : \mathcal{S} \to Groups$$

is exact, for any $n \ge 0$.

2.1. Definition. *We say that length of G_* is n if $N_iG_* = 0$, for $i > n$ and $N_nG_* \neq 0$.*

Let $\mathcal{S}^{(k)}$ be the category of such simplicial groups whose length is less or equal to k. We denote the length of G_* by $l(G_*)$. Obviously $l(G'_*) \le l(G_*)$ if $G'_* \subset G_*$. Moreover, for any short exact sequence of simplicial groups

$$0 \longrightarrow G'_* \longrightarrow G_* \longrightarrow G''_* \longrightarrow 0$$

one has

$$l(G_*) = max\{l(G^{'}_*), l(G^{''}_*)\}. \tag{2.1.1}$$

This is a consequence of the exactness of the functor N_n. We shall use (2.1.1) very extensively in the following situations. Assuming one has a spectral sequence in the category of simplicial objects: $E^r_{pq} \Longrightarrow A_{p+q}$. Then (2.1.1) has following consequences: a) if $l(E^r_{p,n-p}) \le d$ for any p, then $l(A_n) \le d$ and b) if $l(A_n) < \infty, n \ge 0$ and $l(E^r_{pq}) < \infty$ for $q \neq k$, then $l(E^r_{pk}) < \infty$ as well.

2.2. Examples and notations. i) For any group G, one denotes by $K(G,0)$ the constant simplicial group whose value is G. Then $l(K(G,0)) = 0$. For any abelian group A, one denotes by $K(A,n)$ the simplicial abelian group, whose normalization is A in the dimension n and 0 in other dimension. Thanks to the Dold-Puppe equivalence, such object is unique up to isomorphism. By definition $l(K(G,n)) = n$.

ii) The category of simplicial groups of length one is isomorphic to the category of *crossed modules* (see [L]). Similarly the category of simplicial groups of length two is isomorphic to the category of *2-crossed modules* in the sense of Conduché (see [C]).
iii) Let $f : G \to G^{''}$ be an epimorphism of groups. Define a simplicial group $C_*(f)$ by

$$C_n(f) = \{(x_0, ..., x_n) \in G^{n+1} \mid f(x_0) = ... = f(x_n)\},$$

$$\partial_i(x_0, ..., x_n) = (x_0, ..., \widehat{x}_i, ..., x_n),$$

$$s_i(x_0, ..., x_n) = (x_0, ..., x_i, x_i, ..., x_n).$$

Clearly $l(C_*(f)) = 1$. It is also clear that $BC_*(f) \to B(G^{''})$ is a weak equivalence. Therefore one has a spectral sequence

$$E^1_{pq} = H_q(C_p(f), M) \Longrightarrow H_{p+q}(G^{''}, M),$$

where M is a $G^{''}$-module. For more about this spectral sequence see Corollary 5.6, Proposition 5.7 and [P].
iv) Let G be a group. Then for each $n \geq 0$ there exists unique (up to isomorphism) simplicial group $T(G, n)$, whose normalization is trivial in all dimensions except n and $n + 1$,

$$N_nT(G, n) = G, \ N_{n+1}T(G, n) = G$$

and where the boundary map $d : N_{n+1}T(G, n) \to N_nT(G, n)$ is the identity map. This follows from the Theorems 2.6 and 3.4 of [C]. Here is the explicit construction of $T(G, n)$. As usual we set $[n] = \{0, ..., n\}$. Let $I(n, k)$ be the set of all order-preserving injections from $[n]$ to $[k]$. So $I(n, k)$ is a subset of $\Delta(n, k)$. Varying k, we get a well-known cosimplicial set $\Delta(n, -)$. However $I(n, -)$ is not a subcosimplicial set, because it is stable only with respect to face maps. If we add one more additional point $+$ in each dimension, one can obtains the pointed cosimplicial set $I_+(n, -)$ with base point $+$. The face operators act on points from $I(n, -)$ in the same way as it was in $\Delta(n, -)$. The action of a degeneracy operator on points from $I(n, -)$ is the same as it was in $\Delta(n, -)$, if we still land in $I(n, -)$ and is $+$ otherwise. One readily checks that we get a well-defined pointed cosimplicial set. Now we put

$$T(G, n) = Sets_*(I_+(n, -), G) = G^{I(n,-)}.$$

Here $Sets_*$ denotes the category of pointed sets. Thus for face operators we have

$$(\partial_i f)(\alpha) = f(\delta^i\alpha),$$

where $f : I(n, k) \to G$ is a map and $\alpha \in I(n, k - 1)$. For degenerecy operators we have: $(s_i f)(\beta) = 0$ if $i, i + 1 \in Im\beta$ and $(s_i f)(\beta) = f(\sigma^i\beta)$

otherwise, where $\beta \in I(n, k+1)$. One easily shows that this simplicial group has the desired normalisation. Hence $l(T(G, n)) = n + 1$.

2.3. Lemma. *Let A_* and B_* be two simplicial abelian groups and $D : Ab \times Ab \to Ab$ be an additive bifunctor. Then*

$$l(D(A_*, B_*)) \leq l(A_*) + l(B_*).$$

Proof. Let $S(n, k)$ be the set of all order-preserving surjections from $[n]$ to $[k]$. Then the Dold-Puppe correspondence yields

$$A_n = \bigoplus_{s \in S(n,k)} s(N_k A_*).$$

We remark also that in the abelian case the inequality $l(A_*) \leq k$ holds iff A_n is generated by degenerate elements, $n > k$. Therefore the Lemma is a consequence of the following elementary fact: if $f \in S(n, k)$ and $g \in S(n, l)$ with $n > k + l$, then there exist $i \in [n-1]$ such that $f(i) = f(i+1)$ and $g(i) = g(i+1)$.

□

Now we recall the following result of Conduché (see Theorem 1.5 of [C]).

2.4. Lemma. *Let $\mathcal{S}_k$ be the category of such k-truncated simplicial objects for which the following equations hold*

$$[\bigcap_{i \in I} Ker\partial_i, \bigcap_{j \in J} Ker\partial_j] = 0$$

for any nonempty decomposition $I \bigcup J = [k]$. Then truncation yields an isomorphism of categories

$$\mathcal{S}^{(k)} \cong \mathcal{S}_k.$$

□

2.5. Lemma. *The inclusion $\mathcal{S}^{(k)} \subset \mathcal{S}$ has the left adjoint functor*

$$\tau_{\leq k} : \mathcal{S} \to \mathcal{S}^{(k)}.$$

Moreover, the natural homomorphism

$$G_* \to \tau_{\leq k} G_*$$

is an isomorphism in dimensions $< k$ and is an epimorphism in dimensions $\geq k$.

Proof. For $x \in G_k$ and $y \in N_{k+1}G_*$ one has $z = (s_k x)y(s_k x^{-1}) \in N_{k+1}G_*$. Since $dz = xd(y)x^{-1}$, we can conclude that $d(N_{k+1}G_*)$ is a normal subgroup of G_k. Morover $\partial_i(N_{k+1}G_*) = 0$ for $i \in [k]$, because of

$$\partial_i dy = \partial_i \partial_{k+1} y = \partial_k \partial_i y = 0.$$

Hence, there exists a well-defined k-truncated simplicial group for which

$$(\tau_{\leq k}G_*)_i = G_i, \ i < k,$$

$$(\tau_{\leq k}G_*)_k = G_k/dN_{k+1}G_* \ .$$

By Lemma 2.4 one needs only to show that this truncated object belongs to $\mathcal{S}_k$. Let $I \bigcup J = [k]$ be a nonempty decomposition and $x \in \bigcap_{i \in I} Ker\partial_i, y \in \bigcap_{j \in J} Ker\partial_j$. Without loss of generality we may assume existence of such $r \in J$ that $r - 1 \notin J$. Then one checks $\partial_i[s_r x, s_{r-1}y] = 0, \ i \neq r$ and $\partial_r[s_r x, s_{r-1}y] = [x, y]$. Now according to Lemma 1.6 of [C] there exists an element $z \in G_{k+1}$ such that $\partial_i z = 0$ for $i < k+1$ and $dz = [x, y]$. This shows that $\tau_{\leq k}G_* \in \mathcal{S}_k$. □

2.6. Corollary. *Let Φ be a full subcategory of $\mathcal{S}^{(k)}$. Consider the following properties of Φ:*

i) *$K(G, 0) \in \Phi$ and $T(G, n) \in \Phi$ for any group G and $0 \leq n < k$,*

ii) *if*

$$0 \longrightarrow G'_* \longrightarrow G_* \longrightarrow G''_* \longrightarrow 0$$

is a short exact sequence of simplicial groups and $G'_*, G''_* \in \Phi$, then $G_* \in \Phi$,

iii) *if*

$$0 \longrightarrow G'_* \longrightarrow G_* \longrightarrow G''_* \longrightarrow 0$$

is a short exact sequence of simplicial groups with $l(G_*) = l(G'')$ and $G'_*, G_* \in \Phi$ then $G''_* \in \Phi$.

Assuming Φ satisfies i)-iii)*, then $\Phi = \mathcal{S}^{(k)}$.*

Proof. For any $G_* \in \mathcal{S}^{(t)}$ the normalization of the kernel of the map $G_* \to \tau_{\leq t-1}G_*$ is nontrivial only in dimensions t and $t-1$, thanks to Lemma 2.5. So, by induction it is enough to show that $X_* \in \Phi$, assuming $N_i X_* = 0$ for $i \neq n, n+1$, where $n < k$. In order to show the last statement we remark that there exist following short exact sequences:

$$0 \longrightarrow K(\pi_{n+1}X_*, n+1) \longrightarrow X_* \longrightarrow Y_* \longrightarrow 0,$$

$$0 \longrightarrow T(Im\ d, n) \longrightarrow Y_* \longrightarrow K(\pi_n X_*, n) \longrightarrow 0,$$

$$0 \longrightarrow K(\pi_n X_*, n) \longrightarrow T(A, n) \longrightarrow K(\pi_{n+1}X_*, n+1) \longrightarrow 0.$$

Here d is a boundary homomorphism $d : N_{n+1}X_* \to N_n X_*$. The simplicial group Y_* is defined via first short exact sequence. The existence of the desired morphisms in the first and second sequence is a consequence of Theorems 2.6 and 3.4 of [C].

$\square$

3. CROSS-EFFECTS OF FUNCTORS

Here we recall some ideas and facts from [BaP] where the notion of the cross-effects of functors was introduced in the nonabelian case. Let $\mathcal{C}$ be a category with a zero object and finite sums. Denote the sum of objects X and Y by $X \vee Y$. Let $F : \mathcal{C} \to Groups$ be a functor with $F(0) = 0$. We recall that the *second cross-effect* of F is given by

$$F(X \mid Y) = ker(F(X \vee Y) \to F(X) \times F(Y)).$$

If the second cross-effect vanishes, then F is called *linear* and F is called *quadratic* if $F(X \mid Y)$ is linear on X and Y. More generally the n-th cross-effect of F can be defined by

$$F(X_1 \mid ... \mid X_n) = ker(F(X_1 \vee ... \vee X_n) \to F(X_2 \vee ... \vee X_n) \times ... \times F(X_1 \vee ... \vee X_{n-1}))$$

We remark that, in the case when $\mathcal{C}$ is an additive category and values of F are abelian groups, then this is a classical notion of Eilenberg and Mac Lane (see [EM]). If the $(n+1)$-st cross-effect vanishes and n-th cross-effect is nonzero, then F is called *functor of degree* n. In this case we write $deg\ F = n$. Obviously $deg(F') \leq deg(F)$ if $F' \subset F$. Moreover, for any short exact sequence of functors

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

one has

$$deg(F) = max\{deg(F^{'}), deg(F^{''})\}. \tag{3.1}$$

This is a consequence of the fact that the cross-effect sends exact sequences to exact sequences. The next fact for $n = 2$ was proved in [BaP]. The same argument works in the general case also.

3.2. Proposition. *Assume $deg\ F = n$. Then the values of F lie in the category of nilpotent groups of degree n. Moreover the image of the last nontrivial cross-effect under the composition*

$$F(X \mid ... \mid X) \subset F(X \vee ... \vee X) \to F(X) \tag{3.2.1}$$

is a central subgroup and the cokernel of the same map defines functor whose degree is $< n$. The last map in (3.2.1) is induced by

$$(1_X, ..., 1_X) : X \vee ... \vee X \to X.$$

$\square$

In the sequel we are mostly interested in the case when $\mathcal{C}$ is a *variety of groups with operators*. This means that the objects are groups together with some additional operations satisfying some identities. Examples are the category of groups, nilpotent, or solvable groups of given degree, as well as rings, Lie algebras and their subcategories of nilpotent or solvable objects. We call $\mathcal{C}$ *pointed* if it has a zero object. If $\mathcal{C}$ is a such category, then we define the *degree* of $\mathcal{C}$ as the degree of the forgetful functor $\mathcal{C} \to$ *Groups*. For example, the degree of the category of modules over some ring is one, while the degree of the category of all groups, or all (Lie, associative,...) algebras is infinity.

3.3. Lemma. *Let $\mathcal{C}$ be a variety of groups. Then $deg(\mathcal{C}) \leq n$ iff $\mathcal{C}$ is a subcategory of the category of nilpotent groups of degree n.*

Proof. By Proposition 3.2, if $deg(\mathcal{C}) \leq n$ then $\mathcal{C}$ is a subcategory of the category of nilpotent groups of degree n. Assume now that $\mathcal{C}$ is a subcategory of the category of nilpotent groups of degree n. For a given group G, denote by $\Gamma_n G$ the n-th level of the lower central series of abelianization$G$. It is well-known that abelianization of groups sends finite sums to products, so it is a degree one functor. Furthermore, for any i and any group G there exists an epimorphism

$$(G_{ab})^i \longrightarrow \Gamma_i G/\Gamma_{i+1} G$$

and by lemma 2.3 we have $deg(\Gamma_i/\Gamma_{i+1}) \leq i$. Therefore one has

$$deg(\mathcal{C}) = deg(\Gamma_1) = max_{1 \leq i \leq n} deg(\Gamma_i/\Gamma_{i+1}) = n.$$

$\square$

A similar proposition is true for the varieties of Lie, associative or other sorts of algebras.

Let $\mathcal{C}$ be a pointed variety of groups with operators. Denote by $\mathcal{P}_n(\mathcal{C})$ the category of such functors

$$F : \mathcal{C} \longrightarrow Groups$$

which preserve filtered colimits, coequalizers and have degree less or equal to n. We remark that this category is equivalent to the category of less than or equal to n degree functors from the category of finitely generated free objects in $\mathcal{C}$ to the category of groups. Hence a sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

in $\mathcal{P}_n(\mathcal{C})$ is exact iff the sequence of groups

$$0 \longrightarrow F'(X) \longrightarrow F(X) \longrightarrow F''(X) \longrightarrow 0$$

is exact for every finitely generated free object X.

3.4. Examples. i) For an object $X \in \mathcal{C}$, one denotes by X_{ab} the abelianization of X. In the case of groups, and Lie algebras it is just quotient by the commutator, and for rings it is X/X^2. In general it can be described as the left adjoint to the inclusion of the category of abelian group objects in the whole category. It is a standard fact of category theory that there exists an unique (up to isomorphism) ring R for which the category of left R-modules is isomorphic to the category of abelian group objects in $\mathcal{C}$. Since abelianization carries finite sums to products, it has degree one. More generally, consider the ring $R \sim \Sigma_n$, which is $R^{\otimes n}[\Sigma_n]$ as an abelian group, but with the following multiplication:

$$(r_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes \sigma)(s_1 \otimes s_2 \otimes \cdots \otimes s_n \otimes \tau) = ((r_1 s_{\sigma^{-1}1} \otimes r_2 s_{\sigma^{-1}2} \otimes \cdots \otimes r_n s_{\sigma^{-1}n}) \otimes \sigma\tau$$

and for any right $R \sim \Sigma_n$-module M, consider the functor

$$M^\flat : \mathcal{C} \to Ab$$

given by $M^\flat(X) = M \bigotimes_{R \sim \Sigma_n} (X_{ab})^{\otimes n}$. Then $M^\flat \in \mathcal{P}_n(\mathcal{C})$.
ii) We recall by [BaP] that a *square group*

$$M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e$$

is given by a group M_e and an abelian group M_{ee}. Both groups are written additively. Moreover P is a homomorphism and H is a quadratic function, that is the cross-effect

$$(a, b)_H = H(a + b) - H(a) - H(b)$$

is linear in $a, b \in M_e$. In addition, the following properties are satisfied ($x, y \in M_{ee}$):

$$(Px, b)_H = 0, \quad \text{and} \quad (a, Py)_H = 0,$$

$$P(a, b)_H = a + b - a - b,$$

$$PHP(x) = P(x) + P(x).$$

We also recall that [BaP], for a group G and a square group M, the group $G \otimes M$ is defined by the generators $g \otimes a$ and $[g, h] \otimes x$ with $g, h \in G$, $a \in M_e$ and $x \in M_{ee}$, subject to the relations

$$(g + h) \otimes a = g \otimes a + h \otimes a + [g, h] \otimes H(a),$$

$$[g,g]\otimes x \;=\; g\otimes P(x),$$

where $g\otimes a$ is linear in a and where $[g,h]\otimes x$ is central and linear in each variable g,h and x. As follows from [BaP], the degree of the functor $-\otimes M : Groups \to Groups$ is less or equal to 2 (it is exactly 2 iff $M_{ee}\neq 0$) and in this way we get an equivalence between the category of square groups and $\mathcal{P}_2(Groups)$.
iii) Let $T\in\mathcal{P}_n(Groups)$. By Proposition 3.2 values of T lie in the category Nil_n of nilpotent groups of class less or equal to n. As was shown in [BaP] T factors through $Id/\Gamma_{n+1} : Groups \to Nil_n$. Therefore there exists an isomorphism of categories $\mathcal{P}_n(Groups)\cong\mathcal{P}_n(Nil_n)$.

3.5. Proposition. *Let $\mathcal{C}$ be a pointed variety of groups with operators and let $\mathcal{P}$ be a full subcategory of the category $\mathcal{P}_n(\mathcal{C})$. Consider the following properties of $\mathcal{P}$:*

i) *For any $k\leq n$ and any right $R\sim\Sigma_n$-module M, the functor $M^\flat\in\mathcal{P}$.*

ii) *If*

$$0\longrightarrow F'\longrightarrow F\longrightarrow F''\longrightarrow 0$$

is a central extension in $\mathcal{P}_n(\mathcal{C})$ with $F', F''\in\mathcal{P}$, then $F\in\mathcal{P}$.

iii) *If*

$$0\longrightarrow F'\longrightarrow F\longrightarrow F''\longrightarrow 0$$

is a central extension in $\mathcal{P}_n(\mathcal{C})$ with $F', F\in\mathcal{P}$, then $F''\in\mathcal{P}$.

If $\mathcal{P}$ satisfies properties i) - iii) *then $\mathcal{P}=\mathcal{P}_n(\mathcal{C})$.*

Proof. Let F be a functor of degree $\leq n$. The group Σ_n acts on the n-th cross-effect of F. Moreover the map (3.2.1) is equivariant and it yields a central homomorphism:

$$F(X\mid\ldots\mid X)_{\Sigma_n}\to F(X).$$

One readily checks that this is an isomorphism on n-th cross-effects. So the kernel and cokernel have smaller degrees and the proof can be finished by remarking that by [BaP] we know that $F(X\mid\ldots\mid X)_{\Sigma_n}$ is of the form $M^\flat$ for

$$M=F(<1>\mid\ldots\mid<1>),$$

where $<1>$ is a free object in $\mathcal{C}$ with one generator.

□

3.6. Lemma. *For a functor $T:\mathcal{A}\to Ab$, from an abelian category $\mathcal{A}$ to the category of abelian groups, the following properties are equivalent to each other*:

i) *$deg(T)\leq d$,*

ii) *for any simplicial object X_*, one has: $l(TX_*)\leq d\ l(X_*)$,*

iii) *for any simplicial object X_* of length one, one has: $l(TX_*) \leq d$.*

Proof. By the Hilfsatz 4.23 of [DP] one knows that i) $\Longrightarrow$ ii). Obviously ii) $\Longrightarrow$ iii). In order to show iii) $\Longrightarrow$ i) it is enough to take $X_* = K(A, 1)$.

□

4. SIMPLICIAL DEGREES

In this section we introduce two versions of simplicial degrees and we will establish the relationship with degree in the sense of previous section. According to Lemma 3.6 in abelian case all approaches give the same result. Let $\mathcal{C}$ be a variety of groups with operators. We let $\mathcal{S}(\mathcal{A})$ denote the category of simplicial objects in $\mathcal{C}$, while $\mathcal{S}^{(n)}(\mathcal{A})$ denotes the category of simplicial objects in $\mathcal{C}$ whose length is $\leq n$.

4.1. Definition. *We say that the simplicial degree of a functor*

$$F : \mathcal{C} \to Groups$$

with $F(0) = 0$ is less or equal to d, if for any $n \geq 0$ one has the inclusion

$$F(\mathcal{S}^{(n)}(\mathcal{A})) \subset \mathcal{S}^{(dn)}.$$

In other words for any $n \geq 0$ and any simplicial object X_* with $l(X_*) \leq n$ one has $l(F(X_*)) \leq dn$. In that case we write: $sdeg(F) \leq d$.

4.2. Definition. *We say that the strong simplicial degree of a functor $F : \mathcal{C} \to Groups$ with $F(0) = 0$ is less or equal to d, if for any $n \geq 0$ and any simplicial object X_* with $l(X_{*ab}) \leq n$ one has*

$$l(F(X_*)) \leq dn.$$

In that case we write $Sdeg(F) \leq d$.

The following proposition is a direct consequence of the definitions.

4.3. Proposition. i) *If $F' \subset F$, then $sdeg(F') \leq sdeg(F)$ as well as $Sdeg(F') \leq Sdeg(F)$.*

ii) *If*

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

is a short exact sequence of functors, then

$$sdeg(F) = max\{sdeg(F^{'}), sdeg(F^{''})\},$$

$$Sdeg(F) = max\{Sdeg(F^{'}), Sdeg(F^{''})\}.$$

iii) *Let*

$$\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{T} Groups$$

be a diagram of functors, where $\mathcal{C}'$ is also a variety of groups with operators. Then

$$sdeg(T \circ F) \leq (sdegT)(sdegF) \text{ and } Sdeg(T \circ F) \leq (sdegT)(SdegF).$$

□

4.4. Proposition. $sdeg(F) \leq Sdeg(F) \leq (deg\mathcal{C})sdeg(F)$.

Proof. Thanks to 2.1.1 one has $l(X_{ab}) \leq l(X)$ and first inequality follows from this. In order to prove second one, assume $deg(\mathcal{C}) = d < \infty$. Let $l(X_{*ab}) \leq n$. Using the same argument as in the proof of Lemma 3.3 we get

$$l(X_*) \leq max_{1 \leq i \leq d}\{ni\} = dn.$$

□

4.5. Proposition. *If $\mathcal{C}$ is a pointed variety of groups with operators, then $deg(F) \leq Sdeg(F)$. Moreover, if $F \in \mathcal{P}_n(\mathcal{C})$, then $deg(F) = Sdeg(F)$.*

Proof. Let $G \in \mathcal{C}$. Consider $S^1 \otimes G$, where S^1 is a pointed simplicial circle, with $n+1$ points in degree n and tensor product is pointed version of [QCR]. Clearly $(S^1 \otimes G)_{ab} = K(G_{ab}, 1)$. So $l(S^1 \otimes G_{ab}) = 1$ and hence $l(F(S^1 \otimes G)) \leq SdegF$. By definition

$$N_i F(S^1 \otimes G) = F(G_1 \mid .... \mid G_i)$$

with $G_1 = ... = G_i = G$. Hence, $l(F(S^1 \otimes G)) = degF$. Comparing the two calculations we obtain first inequality. If $F \in \mathcal{P}_n(\mathcal{C})$, with minimal n, then $n = deg(F)$. By Proposition 3.5 and 4.3 one can assume $F = M^\flat$, where M is a representation of Σ_i with $i \leq n$ and then one can use Lemma 2.3 to finish the proof.

□

4.6. Examples. i) If $\mathcal{C}$ is an abelian category, then $deg = sdeg = Sdeg$. Simple examples show that if $\mathcal{C}$ is not abelian, then $deg, sdeg$ and $Sdeg$ do not coincide in general. For example, if F is the forgetful functor in groups, then $sdegF = 1$. Moreover $degF = 1$ iff $\mathcal{C}$ is an abelian category and $SdegF = \infty$ when $\mathcal{C}$ is the category of all groups, or all rings.
ii) Since the simplicial degree of the forgetful functor from the category of Lie algebras to the category of groups is one, one easily checks that

$X \mapsto \Lambda^n X$ has simplicial degree n. Since for Lie algebras these are chains we obtain that the simplicial degree of the functor

$$H_n : Lie\ k-algebras \to k-mod$$

is n. For the same reasons the simplicial degree of the functors

$$HH_n,\ HC_n : Associative\ k-algebras\ \to\ k-mod$$

are $n+1$. The same is true for Harrison homology (= André-Quillen homology in characteristic 0). More generally, the simplicial degree of the n-dimensional homology of $\mathcal{P}$-algebras is n, where $\mathcal{P}$ is Koszul operad over characteristic zero field (see [GK]). Less trivial situations arise, when one considers algebraic K-functors, the homology of groups or André-Quillen homology in positive charactreristic, because in the last cases, the chains have infinite degree (see sections 5 and 6).
iii) We consider the case, when $\mathcal{C} = Nil_2$. Let M be a square group and $T = -\otimes : Nil_2 \to Nil_2$ be corresponding functor of degree 2 (see example 3.4 ii)). Recall that, T preserves finite products iff the following homomorpfism $\Delta : M_e \to M_{ee}$ given by $\Delta(a) = HPH(a) + H(a+a) - 4H(a)$ is surjective. Moreover in that case T is also right exact (see section 6 of [BaP]). According to Lemma 3.3 we have $deg(Nil) = 2$. So $sdeg(T) = 1$ or $sdeg(T) = 2$ by Proposition 4.4. We show that $sdeg(T) = 1$ iff Δ is surjective. In order to show this fact, observe that any functor of simplicial degree one preserves finite products. Conversely, if T is of such type, then it is right exact and so we have following exact sequence

$$T([G_*, G_*]) \to T(G_*) \to T(G_{*ab}) \to 0,$$

which gives

$$l(T(G_*)) \leq max\{l(TG_{*ab}), l(T([G_*, G_*]))\}.$$

Here G_* is a simplicial group in Nil. On the other hand, since T preserves finite products, it sends abelian groups to abelian groups and yields an additive functor $Ab \to Ab$. Since G_{*ab} and $[G_*, G_*]$ are abelian, we have $l(T(G_{*ab})) = l(G_{*ab})$ and $l(T([G_*, G_*])) = l([G_*, G_*])$ and we obtain:

$$l(T(G_*)) \leq max\{l(G_{*ab}), l(([G_*, G_*])\} = l(G_*).$$

Thus $sdeg(T) = 1$.

Now we formulate a conjecture.

4.7. Conjecture. Let $\mathbf{C}$ be a variety of groups with operators and $T : \mathbf{C} \to Groups$ be a functor of finite simplicial degree. Then the left simplicial derived functors of T have finite simplicial degree.

This conjecture might be too optimistic. The following is much weaker:

4.8. Conjecture (weak). Let $\mathbf{C}$ be a pointed variety of groups with operators and $T \in \mathcal{P}_n(\mathbf{C})$. Then the left simplicial derived functors of T have finite simplicial degree.

4.9. Remark. The crucial part of Conjecture 4.8 is the case when T is the abelianization. Simplicial derived functors of the abelianization are known as Quillen homology. Based on Proposition 3.5 and Proposition 6.5, it is not hard to show that if the Conjecture 4.8 is true, when T is abelianization then it is true for arbitrary $T \in \mathcal{P}_n(\mathcal{C})$. According to sections 5 and 6, Conjecture 4.8 is true when $\mathcal{C}$ is the one of the following categories: the category of groups, the category of associative algebras or the category of commutative algebras over any commutative ring, the category of $\mathcal{P}$-algebras, where $\mathcal{P}$ is a Koszul operad over characteristic zero field [GK]. It is also relatively easy to show that Conjecture 4.8 is true when $\mathcal{C}$ has finite degree.

4.10. Variations. The property iii) from Lemma 3.6 suggests two more notions: $sd\ F$ and $Sd\ F$. We say that $sd(F) \leq d$ (resp. $Sd(F) \leq d$) if for any simplicial object X_* with $l(X) \leq 1$ (resp. $l(X_{*ab}) \leq 1$) one has $l(F(X_*)) \leq d$. Obviously $sd \leq sdeg$ (resp. $Sd \leq Sdeg$) and $sd \leq Sd$. In abelian case all notions coincide. However in nonabelian case we do not know whether sd and $sdeg$ are really different or not. We recall that in [P] we introduced still different notion, which is defined as follows. Let $\mathcal{C}$ be a category and $\mathcal{E}$ be a class of morphisms in $\mathcal{C}$ satisying the following properties i) if f is isomorphism, then $f \in \mathcal{E}$, ii) any arrows from $\mathcal{E}$ has a section, iii) if $f : B \to A \in \mathcal{E}$ and $g : C \to A$ is an arbitrary morphism in $\mathcal{E}$, then the pull-back $B \times_A C$ exists and the projection $B \times_A C \to C \in \mathcal{E}$. For a functor $F : \mathcal{C} \to Groups$, we define the relation $deg^{\mathcal{E}} F \leq d$ by induction. By definition, $deg^{\mathcal{E}} F = -1$ iff $F = 0$, and $deg^{\mathcal{E}} F \leq d$ iff $deg^{\mathcal{E}'} F' \leq d-1$. Here $\mathcal{C}'$ is the category whose objects are $\mathcal{E}$ and whose morphisms are pull-backs in $\mathcal{C}$. The functor $F' : \mathcal{C}' \to Groups$ is given by $F'(f : B \to A) := Ker(f_* : F(B) \to F(A))$. Here $\mathcal{E}'$ consists of such split morphisms in $\mathcal{C}'$, whose arrows are in $\mathcal{E}$. Assuming $\mathcal{E}$ is the class of split morphisms, this notion has the following properties (see [P]): a) when $\mathcal{C}$ is an abelian category, then $deg^{\mathcal{E}} = deg$, b) $deg^{\mathcal{E}} F = 1$ iff F preserves all pull-becks with split arrows, c) $sdF \leq deg^{\mathcal{E}} F$, d) when $\mathcal{C}$ is the category of groups and $F = H_n$ is the n-dimensional group homology, then $deg^{\mathcal{E}} H_n = n$. However we do not know of any nontrivial relationship between $sdeg$ and $deg^{\mathcal{E}}$.

5. HOMOLOGY OF GROUPS

In this section we are going to consider the homology of groups

$$H_n : Groups \to Ab$$

from the point of view of our theory. Since homology sends finite free products to direct sums we see that $degH_n = 1$. However $Sdeg(H_n) = \infty$, $n > 1$, because of the existence of such simplicial groups of finite length whose components are perfect groups. Such objects can be constructed by means of the degreewise action of the functor

$$E = [GL, GL] : Rings\ with\ unit \to Groups$$

on a simplicial ring of finite length, because $sdeg(E) = 1$ (see the next section). The main result of this paper shows that $sdeg(H_n) = n$. Since $sdeg(H_n) \geq n$ (because the restriction of H_n on the category of free abelian groups is Λ^n), this fact is a particular case of the following

5.1. Theorem. *Let G_* be a simplicial group, and M_* be a simplicial module over G_*. Then*

$$l(H_t(G_*, M_*)) \leq tl(G_*) + l(M_*),$$

where $H_t(G_*, M_*)$ denotes the simplicial group whose n-th component is $H_t(G_n, M_n)$.

5.2. Lemma. i) *If*

$$0 \longrightarrow G'_* \longrightarrow G_* \longrightarrow G''_* \longrightarrow 0$$

is a short exact sequence of simplicial groups and it is known that Theorem 5.1 is true for $G'_*, G^{''}_*$, then Theorem 5.1 is true for G_* also.

ii) *If*

$$0 \longrightarrow G'_* \longrightarrow G_* \longrightarrow G''_* \longrightarrow 0$$

is a short exact sequence of simplicial groups with $l(G_*) = l(G^{''})$ and it is known that Theorem 5.1 is true for G'_*, G_*, then Theorem 5.1 is true for $G^{''}_*$ also.

Proof. i) For each $n \geq 0$ one has the Hochschild-Serre spectral sequence:

$$E^2_{pq}(n) = H_p(G^{''}_n, H_q(G'_n, M_n)) \Longrightarrow H_{p+q}(G_n, M_n)$$

Varying n, we get a spectral sequence in the category of simplicial objects. By (2.1.1) it is enough to show that length of the simplicial object $[n] \mapsto$

$E^2_{pq}(n)$ is less or equal to $(p+q)l(G_*)+l(M_*)$. By the assumption, Theorem 5.1 is true for G'_*, so one has

$$l(H_q(G'_*,M_*)) \leq ql(G'_*)+l(M_*) \leq ql(G_*)+l(M_*).$$

Again by assumption, Theorem 5.1 is true for G_*''. Using this fact for the coefficients $H_q(G'_*,M_*)$ we finally get $l(E^2_{pq}) \leq pl(G_*'')+ql(G_*)+l(M_*) \leq pl(G_*)+ql(G_*)+l(M_*)$

ii) Applying the spectral sequence from 2.2 ii) degreewise, we still have a spectral sequence in the category of simplicial objects. So it is enough to show that $l(E^1_{pq}) \leq (p+q)k+l(M_*)$, where $k = l(G''_*) = l(G_*)$. Consider the face homomorphism $\partial_0 : C_{p+1}(f) \to C_p(f)$. The kernel of this epimorphism is isomorphic to G'_*.We remark also that $C_0(f) = G_*$. Hence by induction, based on 2.1.1 we obtain $l(C_p(f)) = k$ for any $p \geq 0$. Again by induction, based on Lemma 5.2 i) we may conclude that Theorem 5.1 is true for $C_p(f), p \geq 0$. Therefore $l(E^1_{pq}) \leq qk + l(M_*)$.

$\square$

5.3. Lemma. *Theorem 5.1 is true for* $G_* = K(G,0)$.

Proof. Since $T = H_n(G,-)$ is an additive functor defined on abelian category, we have $sdeg(T) = 1$.

$\square$

5.4. Proof of Theorem 5.1. Without loss of generality one can assume that all modules are defined over a field. Thanks to Corollary 2.6 and the above lemmata one needs only to consider the case when $G_* = T(G,k-1)$. Let G_*-*mod* be the category of simplicial modules over G_* and $(G_*-mod)^{(k)}$ be the full subcategory of G_*-*mod*, whose objects have length $\leq s$. Then the inclusion

$$(G_* - mod)^{(s)} \subset G_* - mod$$

has a left adjoint functor τ_s. If M_* is a G_*-module, then $\tau_s M_* = M_*/M'_*$ where M'_* is the submodule generated by $N_i M_*$, for $i > s$. The natural epimorphism $M_* \to \tau_s M_*$ is an isomorphism in dimensions $< s$. Using the same argument as in the proof of Corollary 2.6, we conclude that in order to prove theorem, it is enough to consider the cases $M_* = K(M,0)$ and $M_* = T(M,m-1)$. Obviously $K(M,0) \in G_*$-*mod* iff $M \in \pi_0 G_*$-*mod* and the G_*-module structure on $T(M,m-1)$ yields a G_{m-1}-module structure on M. Knowledge of the letter determines the whole action of G_* on $T(M,m-1)$:

$$(xf)(\alpha) = \alpha_*(x)f(\alpha)$$

where $x \in G_n$, $\alpha \in I(m-1,n)$, $\alpha_* : G_n \to G_m$ is an operator in G_* and $f : I(m-1,n) \to G$ is a map. So one needs to verify theorem in the following cases:

i) $G_* = T(G, k-1)$ and $M_* = K(M, 0)$ with trivial action.

ii) $G_* = T(G, k-1)$ and $M_* = T(M, m-1)$, where $m \geq k$ and $M \in G^{I(k-1,m-1)}$-*mod*.

Case i). One can assume that M is the base field. Moreover H_* will denote the homology with coefficients in this field. Using Künneth formula, we have

$$H_p(G_n) = H_p(G^X) \cong \bigoplus_{|h|=p} \bigotimes_{x \in X} H_{h(x)} G,$$

where $X = I(k-1, n), h : X \to \mathbf{N}$ runs over all maps with the property

$$| h | := \sum_{x \in X} h(x) = p.$$

The restriction of the face operator ∂_i on the summand h is nontrivial iff $| h | = | h\delta^i |$, and sends this summand isomorphically to the summand corresponding to $h\delta^i$. We remark that $| h | = | h\delta^i |$ iff

$$(h(x) \neq 0) \Longrightarrow i \notin Imx.$$

One needs to show that, for $n > pk$ and nontrivial $a \in H_p(G_n)$ there exist $i \leq n-1$ such that $\partial_i a \neq 0$. Since $h \neq g$ yields $h\delta^i \neq g\delta^i$, one can assume that a is homogeneous, corresponding say to a map h. Denote by $x_1, ..., x_p$ all points (not necessarilly different) in $I(k-1, n)$ where h is nonzero. Since

$$n > pk \geq Card(Imx_1 \cup ... \cup Imx_p),$$

there exists $i \notin Imx_1 \cup ... \cup Imx_p$, which has the desired property.

Case ii). Let M be a $G^{I(k-1,m-1)}$-module. As was mentioned before $T(M, m-1)$ is a $T(G, k-1)$-module via

$$(xf)(\alpha) = \alpha_*(x) f(\alpha),$$

where $x : I(k-1, n) \to G$ and $f : I(m-1, n) \to M$ are maps and $\alpha \in I(m-1, n)$, while $\alpha_*(x) \in G^{I(k-1,m-1)}$ is given by: $\alpha_*(x)(\beta) = x(\alpha\beta)$, where $\beta \in I(k-1, m-1)$. Denote by $\alpha_* M$ a $G^{I(k-1,n)}$-module whose underlying abelian group is M and the action is given via $\alpha_* : G^{I(k-1,n)} \to G^{I(k-1,m-1)}$. Then one has an isomorphism of $G^{I(k-1,n)}$-modules:

$$M^{I(m-1,n)} \cong \bigoplus_{\alpha \in I(m-1,n)} \alpha_* M.$$

Thus,

$$H_p(G_n, M_n) = H_p(G^{I(k-1,n)}, M^{I(m-1,n)}) \cong$$

$$\cong \bigoplus_{\alpha\in I(m-1,n)} H_p(G^{I(k-1,n)}, \alpha_* M).$$

Let

$$\alpha^* : I(k-1, m-1) \longrightarrow I(k-1,n)$$

be the map given by $\beta \mapsto \alpha\beta$. We set $X = I(k-1,n)$ and

$$X_\alpha = X - \alpha^* I(k-1, m-1).$$

Then $H_p(G^X, \alpha_* M) = H_p(G^{X_\alpha} \times G^{I(k-1,m-1)}, \alpha_* M)$. Since the action of G^{X_α} on $\alpha_* M$ is trivial, one has

$$H_p(G^X, \alpha_* M) \cong \bigoplus_{i+j=p} A_i \otimes H_j(G^{X_\alpha}),$$

where $A_i = H_i(G^{I(k-1,m-1)}, M)$. Hence

$$H_p(G_n, M_n) \cong \bigoplus_{\alpha\in I(m-1,n)} \bigoplus_{i+j=p} \bigoplus_{|h|=j} A_i \otimes \bigotimes_{x\in X_\alpha} H_{h(x)} G,$$

where $h : X_\alpha \to \mathbf{N}$ runs over all maps with the property that $\mid h \mid = j$.

The face operator ∂_i acts on this decomposition as follows. If $i \in Im\alpha$, then ∂_i is trivial on the summand α. For $i \notin Im\alpha$ one has $\alpha = \delta^i \alpha'$ and the summand corresponding to α goes to the summand corresponding to α'. If $h : X_\alpha \to \mathbf{N}$ is given, then the face operator ∂_i sends the summand corresponding to h to the summand corresponding to $h\delta^i$ and the corresponding homomorphism between these summands is either an isomorphism or zero, depending on whetter $\mid h\delta^i \mid$ equals to $\mid h \mid$ or not. We remark that $\mid h \mid = \mid h\delta^i \mid$ iff

$$(h(x) \neq 0) \Longrightarrow i \notin Imx.$$

One needs to show that, for $n > pk + m$ and nontrivial $a \in H_p(G_n)$ there exist $i \leq n-1$ such that $\partial_i a \neq 0$. As in the case i), without loss of generality one can assume that a is homogeneous, corresponding say to $\alpha \in I(M-1,n)$ and $h : X_\alpha \to \mathbf{N}$, $\mid h \mid = j$.

Denote by $x_1, ..., x_p$ all points (not necessarilly different) in $I(k-1,n)$ where h is nonzero. Since

$$n > pk + m \geq Card(Im\alpha \cup Imx_1 \cup ... \cup Imx_p),$$

there exists $i \notin Im\alpha \cup Imx_1 \cup ... \cup Imx_p$, which has the desired property. $\square$

5.6. Corollary. *For the spectral sequence* 2.2 iii) *one has*

$$E^2_{pq} = 0, \ \ when\ p > q.$$

Proof. Since $l(C_*(f)) = 1$, we have $l(H_qC_*(f)) \leq q$ and thus $\pi_pH_qC_*(f)) = 0$, for $p > q$.

□

Corrollary 5.6 was established earlier in [P]. It is well-known that the first quadrant spectral sequence gives rise to five-term exact sequence in low dimensions. Thanks to Corrollary 5.6 for the spectral sequence 2.2 iii) one has a slightly longer exact sequence. One can check (see *loc. cit.*) that in this way we obtain an exact sequence, which for central extensions, gives the well-known exact sequence from [EHS]. Proposition 5.7 was also established in [P], but we reproduce it here because of the beautiful consequence: *the functor H_2 : Groups $\to$ Ab determines all higher homology.* In order to give a more precise meening of this statement we introduce some auxiliary notation. For a functor

$$T : Groups \to Ab$$

one defines new functors $S_nT : Groups \to Ab$, $n \geq 0$, as follows. For a group G one can choose an epimorphism $f : F \to G$, where F is a free group. We consider the simplicial abelian group $TC_*(f)$ obtained by degreewise action of T on the simplicial group $C_*(f)$ (see 2.2 iii). It is well-known that $\pi_*TC_*(f)$ is isomorphic to the homology of the chain complex $(TC_*(f), d)$, where $d = \Sigma(-1)^i\partial_i$. We set

$$S_nT(G) := \pi_nT(C_*(f)).$$

One easily shows that this definition does not depend on the choise of f and we get the well-defined functors S_nT, $n \geq 0$.

5.7. Proposition. *For $n \geq 1$, one has isomorphisms*

$$H_{2n} \cong S_nH_n \text{ and } H_{2n+1} \cong S_nH_{n+1}.$$

Proof. Comparing with notations we have:

$$E^2_{pq} = S_pH_q.$$

By Corollary 5.6, we know that $E^2_{pq} = 0$ for $p > q$. By definition $C_n(f)$ is a subgroup of F^{n+1}, thus homological dimension of it is less than or equal to $n+1$. Hence $E^1_{pq} = 0$ for $p < q - 1$. Now we see that in the line $p+q = n$ there is only one nontrivial E^2-term and the proposition proved. Moreover, from our calculation it follows that

$$H_{2n+1} \cong S_nH_{n+1} = Coker(d : H_{n+1}(C_{n+1}(f)) \to H_{n+1}(C_n(f))).$$

□

5.8. Remarks. i) It is classical fact that, the n-th left simplicial derived functor of the functor $H_1 : Groups \to Ab$ is isomorphic to (n+1)-st homology. So the *functor* $H_1 : Groups \to Ab$ *determines all higher homology* too. We do not know whether the functor $H_k : Groups \to Ab, k > 2$ determines or not all $H_n : Groups \to Ab, n > k$.
ii) Ellis considered homology of crossed modules, i.e. simplicial groups of length one (see [E]). Our Theorem 5.1 shows that for the spectral sequence of Corollary 3 of *loc. cit.*, one has $E^2_{pq} = 0$ for $p > q$. Using this fact T. Datuashvili in [D] extended some results of Ellis.

6. MORE EXAMPLES

6.1. Algebraic K-functors. Main observation of this section is the fact that algebraic K-functors, when considered as functors from the category of rings to the category of abelian groups, have finite simplicial degree. Indeed Since GL is left exact it preserves the normalization, so $sdeg(GL) = 1$. Rationally $K_*(R)$ is the primitive part of $H_*GL(R)$. Hence Theorem 5.1 and Proposition 4.3 yield $sdeg(K_n \otimes \mathbf{Q}) \leq n$. Based on Goodwillie's theorem on relative K-theory [Go] one easily shows that

$$sdeg(K_n \otimes \mathbf{Q}) = n, \ \ n \geq 1.$$

Now we return to the integral case. Since $sdeg(GL) = 1$ we have $sdeg(E) = 1$ and $sdeg(K_1) = 1$ and the fundamental theorem of algebraic K-theory shows that $sdeg(K_n) = 1$, for $n \leq 1$. On the other hand, using Theorem 5.1 and Proposition 4.3 iii) we get $sdeg(K_2) = 2$, because $K_2 = H_2 \circ E$. Again Proposition 4.3 ii) shows that $sdeg(St) = 2$. Since $K_3 = H_3 \circ St$ we may conclude that $sdeg(K_3) \leq 6$. However we do not know what is the exact value of $sdeg(K_3)$. In order to get the information for higher K-theory we use a trick of Dror Farjoun (see Proposition 2.20 of [Ge]). Let X_3 be the space as defined in *loc. cit.* Recall that it fits in a fibration

$$X_3 \hookrightarrow B_{StR} \longrightarrow K(K_3R, 3)$$

and $H_4X_3 = K_4(R)$. Using Serre spectral sequence we get an exact sequence

$$\mathbf{Z}/2\mathbf{Z} \otimes K_3 \longrightarrow K_4 \longrightarrow H_4St \longrightarrow 0.$$

Thus $sdeg(K_4) \leq 8$. It is now evident, that this argument works as well for $X_n, n > 3$, and we get

$$sdeg(K_n) < \infty.$$

Example 6.2. Let $f : R \to S$ be a surjective homomorphism of rings. In this case $C_*(f)$ is a simplicial ring of length one. Thus length of $StC_*(f)$ is two and $\pi_i StC_*(f) = 0$ for $i > 2$. This fact using direct calculation with Steinberg relations was established before by Keune (see Lemma 5.3 of [K]).

6.3. Shukla homology and Mac Lane homology. We recall that homology is a kind of derived Hochschild homology (see [QCR]). More precisely. Let k be a commutative ring with unit and A be an associative k-algebra. We let A_* denote a free simplicial resolution of A in the category of associative k-algebras. Then by definition:

$$Shukla_*(A) := HH_*(A_*).$$

We claim that $sdeg\ (Shukla_n) = n+1$. In order to show this statement, we need more notation. We let $Mult_n tor_*(M_1, \cdots, M_n)$ denote the value of the left derived functor of $(X_1, \cdots, X_n) \mapsto X_1 \otimes \cdots \otimes X_n$ on $(M_1, \cdots, M_n)$. Since $Mult_n tor_*(M_1, \cdots, M_n)$ is a multilinear functor, we see that $M \mapsto Mult_n tor_*(M, \cdots, M)$ has $\leq n$ degree. The Hochschild chains of A_* form a bicomplex, which gives the spectral sequence

$$E^1_{pq} = Mult_{p+1} tor_q(A, \cdots, A) \Longrightarrow Shukla_{p+q}(A).$$

Hence, $sdeg\ Shukla_n = max_{0 \leq p \leq n}\ sdeg Mult_{p+1} tor_{n-p} = n+1$ and the claim is proved. According to the spectral sequence 4.2(4) of [PW], the same relation is true for Mac Lane homology.

6.4. André-Quillen homology. In what follows k denotes a commutative ring with unit. For a commutative k-algebra A, one can define André-Quillen homology of A [QCR] by

$$D_n(A) = \pi_n(\mathbf{L}_{A/k}),$$

where $\mathbf{L}_{A/k}$ is the *cotangent complex* of A over k (see *loc. cit.*). Using the cotriple resolution, we may assume that

$$\mathbf{L}_{-/k} : Comm.\ k-alg. \to free\ simplicial\ k-modules$$

is a functor. By Theorem 8.1 of *loc. cit.* one has a spectral sequence

$$E^2_{pq} = \pi_p(\Lambda^A_q \mathbf{L}_{A/k}) \Longrightarrow Shukla_{p+q}(A).$$

We use this spectral sequence together with Proposition 6.5 to show that André-Quillen homology has finite simplicial degree. Indeed, since $E^2_{p0} = 0$ for $p > 0$, we have an isomorphism $E^2_{01} = D_0 \cong Shukla_1 = \Omega^1_{A/k}$ and an epimorphism $Shukla_2 \to D_1 \to 0$. Hence $sdeg\ D_0 = 2$ and $sdeg\ D_1 \leq 3$. Assume we have already proved that $sdeg\ D_q < \infty$, for any $q < n$. By Proposition 6.5, we obtain $sdeg\ E^2_{pq} < \infty$ for any $q < n$ and $p \geq 0$.

Since the abutment has finite degree (thanks to 6.3), we conclude that $sdeg\ D_n = sdeg\ E^2_{n1} < \infty$.

6.5. Proposition. *Let $\mathcal{C}$ be a variety of groups with operators, and let*

$$T : \mathcal{C} \to \text{free simplicial } k-\text{modules}$$

be a functor and $F \in \mathcal{P}_d(k-mod)$. Then

$$sdeg(\pi_n(FT)) \le d\ max_{0\le i\le n}\{sdeg\ \pi_i T\}.$$

6.6. Lemma. i) *If*

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0$$

is a short exact sequence in $\mathcal{P}_d(k-mod)$ and it is known that 6.5 is true for F_1 and F, then 6.5 is true for F_2 also.

ii) *If*

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0$$

is a short exact sequence in $\mathcal{P}_d(k-mod)$ and it is known that 6.5 is true for F_1 and F_2, then 6.5 is true for F also.

Proof. Since the values of T are free k-modules, for any $c \in \mathcal{C}$, we have a short exact sequence of simplicial k-modules

$$0 \longrightarrow F_1T(c) \longrightarrow FT(c) \longrightarrow F_2T(c) \longrightarrow 0.$$

According to Proposition 4.3 ii) our Lemma is a consequence of the long exact sequence for homotopy groups. $\square$

Proof of Proposition 6.5. We use the same method as was used in [BP], by induction on d. If $d = 1$, then F is additive functor. For an object $c \in \mathcal{C}$, one puts $X_* = T(c)$. One can use the universal coefficient theorem for the functor F and for the chain complex associated to X_*, to get the spectral sequence

$$E_2^{pq} = L_pF(\pi_q X_*) \Longrightarrow \pi_{p+q}(FX_*).$$

Here L_*F denotes the left derived functor of F. Since left derived functors of an additive functor are still additive, varying c, we get

$$sdeg\ (E_2^{pq}) \le sdeg\ \pi_q T.$$

Hence, $sdeg\ (\pi_n(FT)) \le max_{0\le i\le n} sdeg\ (\pi_i T)$. Now we assume that Proposition 6.5 is proved for all functors from $\mathcal{P}_{d-1}(k-mod)$ and consider

$F \in \mathcal{P}_d(k - mod)$. According to Proposition 3.5 and Lemma 6.6, it is sufficient to consider the case $F = M^\flat$, for some right $k \sim \Sigma_d$-module M. We remark that when M is free $k \sim \Sigma_d$-module with one generators, then $F(X) = X^{\otimes d}$ and the result is a simple consequence of the Künneth theorem (see also the argument in 6.3). For arbitrary M, we consider a free resolution $P_* \to M$ in the category of right $k \sim \Sigma_d$-modules. We let A_* denote the homology of the total complex of the bicomplex $P_* \otimes_{k \sim \Sigma_d} X_*^{\otimes d}$. We have the following spectral sequences

$$E^2_{pq} = \pi_p Tor_q^{k\sim\Sigma_d}(M, X_*^{\otimes d}) \Longrightarrow A_{p+q},$$

$$'E^2_{pq} = Tor_p^{k\sim\Sigma_d}(M, \pi_q X_*^{\otimes d}) \Longrightarrow A_{p+q}.$$

Varying c, we obtain the spectral sequences in the category of functors from $\mathcal{C}$ to k-mod. Since $Tor_p^{k\sim\Sigma_d}(M, -)$ is additive, we have $sdeg\ ('E^2_{pq}) \leq d(sdeg\ \pi_q T)$. Hence,

$$sdeg\ A_n \leq d\ max_{0\leq q\leq n} sdeg \pi_q T. \tag{6.7}$$

We remark that for $q > 0$, the degree of the functor $X \mapsto Tor_q^{k\sim\Sigma_d}(M, X_*^{\otimes d})$ is less or equal to $d-1$, because the d-th cross-effect of $X \mapsto X^{\otimes d}$ is $X^{\otimes d}[\Sigma_d]$. Hence, by the induction assumption we have

$$sdeg\ E^2_{pq} \leq (d-1)\ max_{0\leq q\leq n} sdeg \pi_q T,\ \ q > 0.$$

Comparing the last inequality with (6.7), we get

$$sdeg\ E^2_{n0} \leq d\ max_{0\leq i\leq n} sdeg \pi_i T.$$

□

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