ASYMPTOTIC DISTRIBUTION OF QUADRATIC FORMS I

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Abstract. We consider quadratic forms

$$Q_n = \sum_{1\le j<k\le n} a_{jk} X_j X_k,$$

where X_j are i.i.d. random variables with finite third moment. We obtain optimal bounds for the Kolmogorov distance between the distribution of Q_n and the distribution of the same quadratic forms with X_j replaced by corresponding Gaussian random variables.

1. Introduction and results

Let $X_1, X_2, \dots$ denote independent random variables (r.v.) such that $\mathbf{E}\, X_j = 0$ and $\mathbf{E}\, X_j^2 = 1$, $j = 1, 2, \dots$. Let $A = A^{(n)} = \{a_{njk}\}_{j,k=1}^n$ denote a symmetric matrix with eigenvalues $\lambda_{n1}, \dots, \lambda_{nn}$ ordered to be non increasing in absolute value. Throughout we shall assume that

$$\sum_{j,k=1}^n a_{njk}^2 = \sum_{j=1}^n \lambda_{nj}^2 = 1. \tag{1.1}$$

Consider the quadratic forms (q.f.)

$$Q_n = \sum_{j,k=1}^n a_{njk} X_j X_k \qquad \text{and} \qquad G_n = \sum_{j,k=1}^n a_{njk} Y_j Y_k,$$

where $Y_1, Y_2, \dots$ are orthonormal Gaussian r.v.'s.

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There exists an extensive literature on different probability problems for quadratic forms. One of the most important problems is the investigation of the asymptotic of distributions of quadratic forms G_n and Q_n. Sevastyanov (1961) described the class of distributions which are limits of G_n as $n \to \infty$. Whittle (1960, 1964) obtained inequalities for moments of Q_n similar to the Rosenthal inequalities for sums of independent r.v.'s and proved a Central Limit Theorem (CLT) under the additional condition that the matrices $A^{(n)}$ have a special quasi-diagonal structure. Varberg (1966) obtained conditions of convergence of Q_n in quadratic mean and almost surely and investigated weak convergence of G_n too. There is a number of papers on the convergence of quadratic forms Q_n and especially of G_n to various limit distributions. See, for example, de Jong (1987) (CLT for quadratic forms), Fox and Taqqu (1985, 1987) (limit theorems for quadratic forms G_n of Gaussian r.v.'s with long-range dependence), Mikosch (1990, 1991) (functional CLT for quadratic forms, law of iterated logarithm). De Wet and Venter (1973) gave conditions under which the distribution of a quadratic form Q_n converges to that of $\sum_m \alpha_m (Y_m^2 - 1)$, where $\{\alpha_m\}$ is a sequence of real numbers.

Rotar' (1973) proved that under sufficiently weak conditions on the matrix A and for large n the distribution of Q_n is close to the distribution of G_n. Rotar' and Gamkrelidze (1977) obtained bounds for the error of this approximation which were improved by Rotar' and Shervashidze (1985). Below we formulate their result. We shall use the following notation:

$$F_j(x) = \mathbf{P}\{X_j \le x\}, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-u^2/2\}\, du,$$

$$\nu_j = 3 \int_{-\infty}^{\infty} x^2 \, |F_j(x) - \Phi(x)| \, dx, \qquad s_j^2 = \sum_{k=1}^{n} a_{njk}^2,$$

$$L = \sum_{j=1}^{n} \nu_j s_j^3 + \sum_{j,k=1}^{n} \nu_j \nu_k \big| a_{njk} \big|^3, \qquad \Delta = \sum_{j=1}^{n} \lambda_{nj}^4.$$

Here and in what follows we shall consider matrices A with zero diagonal only, that is, $a_{njj} = 0$ for $1 \le j \le n$.

Theorem (Rotar', Shervashidze (1985)). *Let $\Delta < 1/2$. Then*

$$\sup_x \Big| \mathbf{P}\big\{ Q_n \le x \big\} - \mathbf{P}\big\{ G_n \le x \big\} \Big| \le C \big(1 - \log(1 - 2\Delta)\big)^{3/4} L^{1/4},$$

where C is an absolute constant.

The aim of our paper is to improve this bound for independent identically distributed (i.i.d.) $X_1, X_2, \dots, X_n$ under various assumptions on the spectra of the matrices A. We apply these results in particular to investigate the rate of convergence of the distributions of Q_n to Gaussian distributions.

Write

$$\delta_n(A, F) = \sup_x \Big| \mathbf{P}\big\{ Q_n \le x \big\} - \mathbf{P}\big\{ G_n \le x \big\} \Big|,$$

and

$$\Delta_n(A,F) = \sup_x \Big|\mathbf{P}\left\{Q_n \le x\right\} - \Phi(x)\Big|,$$

where F denotes the distribution function (d.f.) $F(x) = \mathbf{P}\{X_1 \le x\}$, and A the symmetric matrix of coefficients of Q_n. We shall assume that the matrices $A = A^{(n)}$ satisfy the same conditions as in the paper of Rotar' and Shervashidze (1985). That is, we assume (1.1) and

$$a_{njj} = 0, \qquad j = 1, \dots, n. \tag{1.2}$$

Throughout the paper we shall suppose that $X, X_1, X_2, \dots$ are i.i.d and

$$\mathbf{E}\,X = 0, \qquad \mathbf{E}\,X^2 = 1, \quad \beta_3 = \mathbf{E}\,\big|X\big|^3 < \infty. \tag{1.3}$$

By C (with an index or without it) we shall denote absolute constants, whereas $C(\cdot,\cdot)$ will denote positive constants depending on arguments.

Theorem 1. *Assume the conditions* (1.1)–(1.3) *hold. Then there exists an absolute constant C such that*

$$\max\big\{\,\delta_n(A,F),\, \Delta_n(A,F)\,\big\} \le C\beta_3\,\big|\lambda_{n1}\big|.$$

Write

$$\mathcal{L}_n^2 = \max_{1\le j\le n}\sum_{k=1}^n a_{njk}^2, \quad \text{and } \Gamma_n = \max_{1\le j\le n}\sum_{k=1}^n \big|a_{njk}\big|.$$

It is well-known that

$$\mathcal{L}_n \le \big|\lambda_{n1}\big| \le \Gamma_n$$

(see, for example, Lancaster (1969, pp. 208–215)). Theorem 1 immediately implies the following result.

Corollary 2. *Assume that the conditions* (1.1)–(1.3) *hold. Then there exist absolute positive constants C_1, C_2 such that*

$$\max\big\{\,\delta_n(A,F),\, \Delta_n(A,F)\big\} \le C_1\,\beta_3\,\Gamma_n.$$

For $x > 0$ define the function $\log^+ x = \max\big\{\,\frac{1}{2}\,, |\log x|\,\big\}$.

Theorem 3. *Assume that the conditions* (1.1)–(1.3) *hold and that there exists a positive constant b_0 such that for some $q \ge 2$*

$$\big|\lambda_{nq}\big| \ge b_0. \tag{1.4}$$

Then there exists a constant $C(b_0)$ such that

$$\delta_n(A,F) \le \begin{cases} C(b_0)\,\beta_3\,\mathcal{L}_n^{4/(16-q)}, & \text{if } \ 2 \le q \le 11, \\ C(b_0)\,\beta_3\,\mathcal{L}_n \log^+ \mathcal{L}_n, & \text{if } q = 12, \\ C(b_0)\,\beta_3\,\mathcal{L}_n, & \text{if } q \ge 13. \end{cases}$$

Note that, if $\mathcal{L}_n \ge e^{-1/2}$ the results of Theorem 3 are trivial. We shall assume that $\mathcal{L}_n \le e^{-1/2}$, and therefore, $\log^+ \mathcal{L}_n = -\log \mathcal{L}_n$.

Remark 1. Note that the bounds in Theorem 1 have an optimal dependence on λ_{n1} and on $\mathcal{L}_n$ respectively. Indeed, consider the following q.f.

$$Q_n = n^{-1/2} \sum_{k=1}^{n} X_{2k-1} X_{2k}.$$

Here $|\lambda_{n1}| = 1/\sqrt{n}$, and if $X_j = \pm 1$ are i.i.d. Rademacher r.v., then

$$\min\{\, \delta_n(A, F),\, \Delta_n(A, F)\,\} \ge C/\sqrt{n} = C|\lambda_{n1}|.$$

Remark 2. Note for comparison that the best available result so far (by Rotar') yields

$$\Delta_n(A, F) \le C\mathcal{L}^{1/4}.$$

2. Applications

Toeplitz matrices. Let $b_0, b_1, \dots$ be a sequence of real numbers such that $b_0 = 0$. Assume

$$B = \sum_{j=1}^{\infty} b_j^2 < \infty. \tag{2.1}$$

Introduce the quantity $D_n^2 = \sum\limits_{j=1}^{n} (n-j) b_j^2$, and define the matrix A with the elements

$$a_{jk} = \frac{1}{D_n}\, b_{|j-k|}. \tag{2.2}$$

By Whittle (1964) proved $\delta_n(A, F) \to 0$, as $n \to \infty$. We have following result.

Proposition 2.1. *Assume* (2.1) *holds. Then there exists an absolute constant C such that*

$$\delta_n(A, F) \le C\beta_3 n^{-1/2} \sum_{j=1}^{n} |b_j|.$$

Proof. From Corollary 2 it follows that

$$\delta(A, F) \le C\beta_3 \Gamma_n \tag{2.3}$$

By (2.2)

$$\Gamma_n \le \frac{2}{D_n} \sum_{j=1}^{n} |b_j|. \tag{2.4}$$

Without loss of generality we may assume that $\sum_{j \ge n/2} b_j^2 \le B/2$. Then, it is easy to check that

$$D_n^2 \ge Bn/4. \tag{2.5}$$

The inequalities (2.3)–(2.5) together prove the Proposition. $\square$

Random matrices. Let $B = \Big(b_{jk}\Big)$ be symmetric $n \times n$ matrix such that the entries for $k > j$ are independent r.v.'s independent of $X, X_1, \dots$. Assume that there exists a constant C_0 such that

$$|b_{jk}| \le C_0, \quad \mathbf{E}\, b_{jk} = 0, \quad \text{and} \quad \mathbf{E}\, b_{jk}^2 = \sigma^2 \quad \text{for } j > k. \tag{2.6}$$

Denote by D_n the quantity $D_n^2 = 2 \sum\limits_{1 \le j < k \le n} b_{jk}^2$. Let $A = \dfrac{1}{D_n} B$. Then we have the following result.

Proposition 2.2. *Assume* (2.6) *holds. Then*

$$\mathbf{P}\left\{ \sup_x \Big| \mathbf{P}\left\{ Q_n \le x \Big| a_{jk},\ j > k \right\} - \mathbf{P}\left\{ G_n \le x \Big| a_{jk},\ j > k \right\} \Big| \le C\beta_3 n^{-1/2} \right\} \to 1,$$

as $n \to \infty$.

Proof. By Theorem 1 we have

$$\sup_x \Big| \mathbf{P}\left\{ Q_n \le x \big| a_{jk},\ j > k \right\} - \mathbf{P}\left\{ G_n \le x \big| a_{jk},\ j > k \right\} \Big| \le C\beta_3 |\lambda_1|. \tag{2.7}$$

By Theorem 2 of Füredi and Komlos (1980)

$$\mathbf{P}\left\{ |\lambda_1| \le C n^{1/2} / D_n \right\} \to 1, \tag{2.8}$$

as $n \to \infty$. It is easy to check that

$$\mathbf{E}\, D_n^2 = n(n-1)\sigma^2, \quad \mathbf{E}\, (D_n^2 - \mathbf{E}\, D_n^2)^2 \le n(n-1) C_0^4. \tag{2.9}$$

By (2.9)

$$\mathbf{P}\left\{ D_n \ge Cn \right\} \le C n^{-2}. \tag{2.10}$$

The inequalities (2.7), (2.8) and (2.10) together complete the proof. □

Nearly non stationary AR(1) processes. Assume that $Z_j(n)$, $j = 1, \dots, n$, $n = 1, 2, \dots$ is a the first order autoregressive process with unknown parameter $\rho_n = 1 - \dfrac{\alpha_n}{n}$, where $\alpha_n > 0$,

$$Z_j(n) = \rho_n Z_{j-1}(n) + X_j, \quad Z_0(n) = 0, \quad \text{for all } n. \tag{2.11}$$

Here $X_1, X_2, \dots$ are i.i.d. r.v.'s satisfying condition (1.3). The least-squares estimate of the parameter ρ_n based on the observations $Z_1(n), \dots, Z_n(n)$ is given by

$$\widehat{\rho}_n = \sum_{j=1}^n Z_j(n) Z_{j-1}(n) \Big(\sum_{k=1}^n Z_{k-1}^2(n) \Big)^{-1}. \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$W_n = (\widehat{\rho}_n - \rho_n) \sum_{k=1}^n Z_{k-1}^2(n) = \sum_{j=1}^n \sum_{k=1}^{j-1} \rho_n^{j-k} X_j X_k.$$

Such processes described for instance in Chan and Wei (1987) and Rachkauskas (1996). It is well-known that, if $\alpha_n \to \infty$ for $n \to \infty$,

$$\delta_n = \sup_x \Big| \mathbf{P}\left\{ W_n / \sqrt{Var(W_n)} \le x \right\} - \Phi(x) \Big| \to 0,$$

as $n \to \infty$. Our results yield the following rates of convergence.

Proposition 2.3. *Assume that* (1.3) *holds. Then there exists an absolute constant* C *such that*

$$\delta_n = \sup_x \Big| \mathbf{P}\Big\{ W_n/\sqrt{\mathrm{Var}\{W_n\}} \le x \Big\} - \Phi(x) \Big| \le C\beta_3 \alpha_n^{-1/2}.$$

Proof. We can represent the statistic $W_n/\sqrt{\mathrm{Var}\,(W_n)}$ in the form of a q.f. Q_n with matrix A defined by

$$a_{jk} = \frac{\rho_n^{|j-k|}}{2\sqrt{Var(W_n)}} \,. \tag{2.13}$$

By Theorem 1 we have $\delta_n \le C\beta_3|\lambda_1|$. It is easy to see that $\mathrm{Var}(W_n) = \sum_{j=1}^n (n-j)\rho_n^{2j}$. A simple calculation yields

$$C_1 n^2 \alpha_n^{-1} \le \mathrm{Var}(W_n) \le C_2 n^2 \alpha_n^{-1}. \tag{2.14}$$

From (2.14) we derive

$$\Gamma_n \le C n^{-1} \alpha_n^{1/2} \sum_{j=1}^n \rho_n^j \le C\alpha_n^{-1/2}. \tag{2.15}$$

Corollary 2 and inequality (2.15) together complete the proof. □

Consider now the case

$$\alpha_n \to \alpha > 0, \quad n \to \infty. \tag{2.16}$$

In this case we obtain the following result.

Proposition 2.4. *Assume that* (1.3) *and* (2.16) *hold. Then there exists a constant* $C(\alpha)$ *such that*

$$\sup_{x\in\mathbb{R}^1} \Big| \mathbf{P}\Big\{ W_n \le x \Big\} - \mathbf{P}\Big\{ G_n \le x \Big\} \Big| \le C(\alpha)\beta_3 n^{-1/2},$$

where G_n *is the q.f of Gaussian r.v.'s and the matrix* A *is defined in* (2.13).

Proof. Using (2.16) we obtain that

$$\mathcal{L}_n^2 \le \frac{2}{\mathrm{Var}(W_n)} \sum_{k=1}^n \rho_n^{2k} \le C n^{-1} \frac{\alpha}{\exp\{-2\alpha\} - 1 + 2\alpha} \,. \tag{2.17}$$

It finally remains to check the condition (1.4) and to apply Theorem 3. The spectrum of the matrix A satisfies for any $q \ge 1$

$$v_\alpha \frac{2\alpha}{\alpha^2 + \pi^2 (q + \frac{1}{2})^2} \le \lim_{n\to\infty} |\lambda_q(n)| \le v_\alpha \frac{2\alpha}{\alpha^2 + \pi^2 q^2} \,, \tag{2.18}$$

where $v_\alpha = \Big(\frac{4\alpha^2}{\exp\{-2\alpha\} - 1 + 2\alpha} \Big)^{1/2}$. The proof of relation (2.18) will be deferred to the Appendix. Combining Theorem 3, relations (2.17) and (2.18) we obtain the result of Proposition 2.4. □

3. The method of symmetrization

Let $\overline{X}$ denote a copy of r.v. X which is independent of all other r.v.'s. Throughout $\widetilde{X}$ denote a symmetrization of X, i.e. $\widetilde{X} = X - \overline{X}$. By $\mathbf{N}$ we denote the set of integer numbers $\{1, \dots, n\}$. Let $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ be a random vector with zero-one coordinates, which is independent of $X_1, \dots, X_n$. Write $\mathcal{X} = (X_1, \dots, X_n)$ and $\xi_j = (0, \dots, 0, X_j, 0, \dots, 0)$, $j \in \mathbf{N}$. By $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ we shall denote the Euclidean scalar product and the Euclidean norm respectively. For simplicity of notation, the index n will be omitted. In this notation we have $Q = \langle A\mathcal{X}, \mathcal{X} \rangle$, and $f(t) = \mathbf{E} \exp\{itQ\}$.

There is an extensive literature on the distribution of quadratic forms of the type $R = \|Z_1 + \cdots + Z_n\|^2$ where Z_j, $j \in \mathbf{N}$, denotes a sequence of i.i.d. random vectors with values in $\mathbb{R}^q$ assuming, non-degenerate covariance operators and finite third absolute moments. Here the error of approximation by replacing Z_j by Gaussian random vectors is of order $n^{-1/2}$ and depends on the smallest eigenvalue of the covariance of Z_1. The first result of this type is due to Götze (1979). The crucial step of the proofs is an estimate of the characteristic function (ch.f.) of R using a symmetrization inequality reducing the quadratic form to a conditional linear form (see Götze (1979)). In our case this procedure yields the following result.

Lemma 3.1. *For all $t \in \mathbb{R}$, we have*

$$\big| f(t) \big|^4 \le \mathbf{E} \exp\Big\{ 2it \sum_{j,k \in \mathbf{N}} a_{jk} \varepsilon_j (1 - \varepsilon_k) \widetilde{X}_j \widetilde{X}_k \Big\}. \tag{3.1}$$

Proof. Assume that $U, V, V_1, V_2 \in \mathbb{R}^n$ are independent random vectors such that V, V_1, V_2 are i.i.d. Let $h(u, v)$ denote an arbitrary measurable real valued function defined on $\mathbb{R}^n \times \mathbb{R}^n$. Then

$$\big| \mathbf{E} \exp\big\{ it\, h(U, V) \big\} \big| \le \mathbf{E} \exp\Big\{ it\, \Delta_2(V_1 - V_2)\, h(U, V_2) \Big\}, \tag{3.2}$$

where the difference operator $\Delta_2(s) h(u, v) = h(u, v+s) - h(u, v)$ acts on the second variable of h. Indeed, applying Hölder's inequality and using that V, V_1, V_2 are i.i.d., we have

$$\begin{aligned} \big| \mathbf{E} \exp\big\{ ith(U, V) \big\} \big| &\le \mathbf{E} \Big| \mathbf{E} \left(\exp\{ith(U, V)\} \big| U \right) \Big|^2 \\ &= \mathbf{E} \exp\{ith(U, V_1)\} \exp\{-ith(U, V_2)\}, \end{aligned}$$

which coincides with (3.2) (cf. Götze (1984)). For the proof of (3.1) we fix $\boldsymbol{\varepsilon}$ and apply the inequality (3.2) twice with

$$U = (\varepsilon_1 X_1, \dots, \varepsilon_n X_n), \qquad V = \big((1 - \varepsilon_1) X_1, \dots, (1 - \varepsilon_n) X_n \big) \tag{3.3}$$

and $h(U, V) = \langle A(U + V), (U + V) \rangle$. Note that

$$\Delta_1(s_1) \Delta_2(s_2) h(U, V) = 2\langle AU, V \rangle,$$

where the difference operator Δ_1 acts on the first arguments of h. This obviously yields the result.□

Denote by $\widetilde{X}^{(H)}$ the truncation of $\widetilde{X}$, i.e.

$$\widetilde{X}^{(H)} = \widetilde{X}\,\mathbf{I}\{|\widetilde{X}| \le H\},$$

where $\mathbf{I}\{B\}$ denotes the indicator of an event B. Introduce the notation

$$\sigma^2(H) = \mathbf{E}\,(\widetilde{X}^{(H)})^2, \quad \mu_4(H) = \mathbf{E}\,(\widetilde{X}^{(H)})^4.$$

Lemma 3.2. *Let $\varepsilon_1, \dots, \varepsilon_n$ denote i.i.d. Bernoulli r.v.'s with*

$$\mathbf{P}\,\{\varepsilon_1 = 1\} = 1 - \mathbf{P}\,\{\varepsilon_1 = 0\} = p.$$

Then, for all $t \ge 0$, $K > 0$,

$$\begin{aligned} \big|f(t)\big|^4 \le \mathbf{E}\,\exp\Big\{\,\frac{1}{4}\,i(t\wedge T_0)\sigma(H)\sum_{j,k\in\mathbf{N}} a_{jk}\varepsilon_k(1-\varepsilon_j)Y_jY_k\Big\} + \mathbf{I}\{|t| > \gamma\mathcal{L}^{-1}\} \\ + \mathbf{P}\,\Big\{\max_{k\in\mathbf{N}}\Big|\sum_{j\in\mathbf{N}} a_{jk}(1-\varepsilon_j)Y_j\Big| \ge K\mathcal{L}\Big|\ln\mathcal{L}\Big|^{1/2}\Big\} \\ + \mathbf{P}\,\Big\{\max_{j\in\mathbf{N}}\Big|\sum_{k\in\mathbf{N}} a_{jk}\varepsilon_k\widetilde{X}_l^{(H)}\Big| > K\mathcal{L}\Big\}, \end{aligned} \tag{3.4}$$

where $T_0 = \gamma\mathcal{L}^{-1}(\log^+\mathcal{L})^{-1/2}$, $\quad t\wedge T_0 = \min\{t, T_0\}$ and $\gamma = \dfrac{\sigma(H)}{2\sqrt{K\mu_4(H)}}$.

The corresponding estimates for negative t is similar since $|f(t)| = |f(-t)|$.

Proof. By Lemma 3.1

$$\big|f(t)\big|^4 \le \mathbf{E}\,\exp\Big\{\,2it\sum_{j,k\in\mathbf{N}} a_{jk}\varepsilon_k(1-\varepsilon_j)\widetilde{X}_j\widetilde{X}_k\Big\}.$$

Since $\widetilde{X}$ has a symmetric distribution and

$$\mathbf{E}\,\cos(a\widetilde{X})\mathbf{I}\{|\widetilde{X}| \ge H\} \le \mathbf{E}\,\cos(a\cdot 0)\mathbf{I}\{|\widetilde{X}| \ge H\},$$

we can easily show that

$$\big|f(t)\big|^4 \le \mathbf{E}\,\exp\Big\{2it\sum_{j,k\in\mathbf{N}} a_{jk}\varepsilon_j(1-\varepsilon_k)\widetilde{X}_j^{(H)}\widetilde{X}_k^{(H)}\Big\}. \tag{3.5}$$

With U, V defined by (3.3) write

$$\widetilde{U} = (\varepsilon_1\widetilde{X}_1, \dots, \varepsilon_n\widetilde{X}_n), \quad \widetilde{V} = ((1-\varepsilon_1)\widetilde{X}_1, \dots, (1-\varepsilon_n)\widetilde{X}_n).$$

Write

$$S_j(\boldsymbol{\varepsilon}) = \sum_{k\in\mathbf{N}} a_{jk}\varepsilon_k\widetilde{X}_k^{(H)}.$$

Since the coordinates of $\widetilde{V}$ are conditionally independent given $\varepsilon_1, \dots, \varepsilon_n$ and $\widetilde{U}$, we have

$$\big|f(t)\big|^4 \le \mathbf{E} \prod_{j \in \mathbf{N}} \mathbf{E}\Big(\exp\big\{2it(1-\varepsilon_j)\widetilde{X}_j^{(H)} S_j(\boldsymbol{\varepsilon})\big\} \Big| S_j(\boldsymbol{\varepsilon})\Big). \tag{3.6}$$

Note that

$$\max_{j \in \mathbf{N}} \big| S_j(\boldsymbol{\varepsilon})\big| \le K\mathcal{L}, \tag{3.7}$$

implies

$$\begin{aligned}
&\Big| \mathbf{E}\Big(\exp\Big\{ 2it(1-\varepsilon_j)\widetilde{X}_j^{(H)} S_j(\boldsymbol{\varepsilon})\Big\} \Big| S_j(\boldsymbol{\varepsilon})\Big)\Big| \\
&\le 1 - \frac{t^2}{2}(1-\varepsilon_j)\sigma^2(H) S_j^2(\boldsymbol{\varepsilon}) + t^4(1-\varepsilon_j)\mu_4(H) S_j^4(\boldsymbol{\varepsilon}) \\
&\le \max\Big\{ \mathbf{I}\{t > \gamma\mathcal{L}^{-1}\},\ \exp\Big\{ -\frac{1}{4} t^2 \sigma^2(H) S_j^2(\boldsymbol{\varepsilon})(1-\varepsilon_j)\Big\}\Big\}.
\end{aligned} \tag{3.8}$$

The inequalities (3.6)–(3.8) together imply

$$\begin{aligned}
\big|f(t)\big|^4 &\le \mathbf{E} \exp\Big\{ \frac{1}{2} it\sigma(H) \sum_{j,k \in \mathbf{N}} a_{jk}\varepsilon_k(1-\varepsilon_j) Y_j \widetilde{X}_k^{(H)} \Big\} + \mathbf{I}\{t > \gamma\mathcal{L}^{-1}\} \\
&\quad + \mathbf{P}\Big\{ \max_{j \in \mathbf{N}} \big| S_j(\boldsymbol{\varepsilon})\big| > K\mathcal{L}\Big\}.
\end{aligned} \tag{3.9}$$

Repeating the arguments in (3.6)–(3.9) and replacing $\mathcal{L}$ in (3.7) by $\mathcal{L}\Big(\log^+ \mathcal{L}\Big)^{1/2}$, we obtain

$$\begin{aligned}
\big|f(t)\big|^4 &\le \mathbf{E} \exp\Big\{ \frac{1}{4} i(t \wedge T_0)\sigma(H) \sum_{j,k \in \mathbf{N}} a_{jk}\varepsilon_k(1-\varepsilon_j) Y_j Y_k \Big\} + \mathbf{I}\{t > \gamma\mathcal{L}^{-1}\} \\
&\quad + \mathbf{P}\Big\{ \max_{k \in \mathbf{N}} \Big| \sum_{j \in \mathbf{N}} a_{jk}(1-\varepsilon_j) Y_j \Big| \ge K\mathcal{L}\big(\log^+ \mathcal{L}\big)^{1/2}\Big\} \\
&\quad + \mathbf{P}\Big\{ \max_{j \in \mathbf{N}} \big| S_j(\boldsymbol{\varepsilon})\big| > K\mathcal{L}\Big\}.
\end{aligned} \tag{3.10}$$

□

To investigate the ch.f. in a neighborhood of zero we shall use Lemma 3.3 below which is a generalization of Lemma 3.1 in Bentkus and Götze (1994). Let J_0, J_1 denote J_0, J_1 denote an arbitrary non-intersecting splitting of $\mathbf{N}$ and let J_{il} be denote an arbitrary non-intersecting splitting of J_i, $i = 0, 1$, respectively. Introduce the corresponding vectors

$$U_i = \sum_{j \in J_l} \xi_j, \quad U_{im} = \sum_{j \in J_{im}} \xi_j, \quad i = 0, 1, \quad m = 1, \dots, 5. \tag{3.11}$$

Note that,

$$\mathcal{X} = U_0 + U_1, \quad U_i = \sum_{m=1}^{5} U_{im}, \quad i = 0, 1.$$

Lemma 3.3. *For any $\eta \in \mathbb{R}^n$, $t \in \mathbb{R}$ we have*

$$\Big| \mathbf{E}\,\big\langle A\eta,\, \mathcal{X}\big\rangle^3 \exp\Big\{ it\big\langle A\eta,\, \mathcal{X}\big\rangle + it\,Q\Big\}\Big|^2 \le C \max_{\substack{i=0,1\\ 1\le l\le 5}} \mathbf{E}^{\,2}\big|\big\langle A\eta, U_{i,l}\big\rangle\big|^3$$

$$\times \Big(\max_{1\le j\le 5} \mathbf{E}\,\exp\Big\{2it\big\langle \widetilde{U}_0, A\widetilde{U}_{1,j}\big\rangle\Big\} + \max_{1\le j,l\le 5} \mathbf{E}\,\exp\Big\{2it\big\langle A\widetilde{U}_{1j}, \widetilde{U}_{0l}\big\rangle\Big\}\Big), \quad (3.12)$$

Proof. Write

$$\mathbf{E}\,\big\langle A\eta,\, \mathcal{X}\big\rangle^3 \exp\Big\{ it\big\langle A\eta,\, \mathcal{X}\big\rangle + it\,Q_n\Big\} = \sum_{\alpha_0+\alpha_{1,1}+\alpha_{1,2}+\ldots+\alpha_{1,5}=3} H(\alpha_0, \alpha_{1,1}, \ldots, \alpha_{1,5}), \quad (3.13)$$

where

$$H(\alpha_0, \alpha_{1,1}, \ldots, \alpha_{1,5}) = \mathbf{E}\,\big\langle A\eta, , U_0\big\rangle^{\alpha_0} \prod_{i=1}^{5} \big\langle A\eta, U_{1,i}\big\rangle^{\alpha_{1,i}} \exp\Big\{ it\big\langle A\eta,\, \mathcal{X}\big\rangle + it\,Q\Big\}.$$

If $\alpha_0 = 0$, there are at least two r.v.'s U_{1,j_1}, U_{1,j_2} which are not involved in the product used in the definition of the function $H(\alpha_0, \ldots, \alpha_{1,5})$. Arguing in the same way as in proof of Lemma 3.1 in Bentkus V., Götze F. (1996) we obtain the estimate

$$\big| H(0, \alpha_{1,1}, \ldots, \alpha_{1,5})\big| \le \max_{i=0,1} \max_{1\le j\le 5} \mathbf{E}\,\big|\big\langle A\eta, U_{i,j}\big\rangle\big|^3 \max_{1\le j\le 5} \mathbf{E}\,\exp\big\{2it\big\langle A\widetilde{U}_0, \widetilde{U}_{1j}\big\rangle\big\}.$$

In the case $\alpha_0 \ne 0$ we fix a set J_{1,j_0} which is not contained in the product used in the definition of the function $H(\alpha_0, \alpha_{1,1}, \ldots, \alpha_{1,5})$ (in this case we have at least three similar sets). We represent U_0 as $\sum_{j=1}^{5} U_{0,j}$. Raising $\langle A\eta, U_0\rangle$ to the power α_0 we get a similar sum as in equality (3.13). For each term in this equality there exist at least two sets J_{0,l_1}, J_{0,l_2} which are not involved in the product used in the definition of this term. In the same way as above we obtain the estimate

$$\big| H(1, \alpha_{1,1}, \ldots, \alpha_{1,5})\big| \le C \max_{\substack{i=0,1;\\ 1\le j\le 5}} \mathbf{E}\,\big|\big\langle A\xi_k, Z_{i,j}\big\rangle\big|^3 \times \max_{1\le j,l\le 5} \mathbf{E}^{\,1/2} \exp\big\{2it\big\langle \widetilde{U}_{0,j}, A\widetilde{U}_{1,l}\big\rangle\big\}. \quad (3.15)$$

The conclusion of Lemma 3.3 follows immediately from the inequalities (3.14) and (3.15). □

Remark. The conclusions of Lemma 3.3 still valid, if the splitting J_{0j}, $j = 1, \ldots, 5$, will dependent on J_{1l} for any fixed $l = 1, \ldots, 5$.

Denote by $g_{l,m}(t)$, $l, m, = 1, \ldots, 5$ the function

$$g_{l,m}(t) = \mathbf{E}\,\exp\Big\{\,2\,i\,t\,\big\langle A\widetilde{U}_{0l}, \widetilde{U}_{1m}\big\rangle\Big\}, \quad (3.16)$$

where U_{0l}, U_{1m} was defined in (3.11). Then we have

Lemma 3.4. *For all $l, m = 1, \dots, 5$ and for all $t \ge 0$, $K > 0$ we have*

$$\begin{aligned} g_{l,m}(t) \le \mathbf{E} \exp\Big\{ \frac{1}{4}\, it\sigma(H) \sum_{j\in J_{0l}, k\in J_{1m}} a_{jk} Y_j Y_k \Big\} + \mathbf{I}\Big\{ t > \gamma \mathcal{L}^{-1} \big(\log^+ \mathcal{L}\big)^{-1} \Big\}, \\ + \mathbf{P}\Big\{ \max_{j\in J_{0l}} \Big| \sum_{k\in J_{1m}} a_{jk} X_k \Big| \ge K\mathcal{L} \log^+ \mathcal{L} \Big\} \\ + \mathbf{P}\Big\{ \max_{k\in J_{1m}} \Big| \sum_{j\in J_{0l}} a_{jk} \widetilde{Y}_j^{(H)} \Big| \ge K\mathcal{L} \log^+ \mathcal{L} \Big\} \end{aligned} \tag{3.17}$$

where $\gamma = \dfrac{\sigma(H)}{2\sqrt{K\mu_4(H)}}$.

Proof. Similar to (3.6) we have

$$g_{lm}(t) \le \mathbf{E} \prod_{j\in J_{0l}} \mathbf{E} \Big(\exp\Big\{ 2it(1-\varepsilon_j) \widetilde{X}_j^{(H)} \sum_{k\in J_m} a_{jk}\varepsilon_k \widetilde{X}_k^{(H)} \Big\} \Big| \widetilde{U}_m \Big).$$

Using instead of (3.7) the inequality

$$\max_{j\in\mathbf{N}} \Big| \sum_{k\in\mathbf{N}} a_{jk}\varepsilon_k \widetilde{X}_k^{(H)} \Big| \le K\mathcal{L}\log^+\mathcal{L},$$

we obtain

$$\begin{aligned} \mathbf{E}\Big(\exp\Big\{ 2it(1-\varepsilon_j) \widetilde{X}_j^{(H)} \sum_{k\in J_m} a_{jk}\varepsilon_k \widetilde{X}_k^{(H)} \Big\} \Big| \widetilde{U}_m \Big) \le \mathbf{I}\Big\{ t > \gamma\mathcal{L}^{-1}(\log^+\mathcal{L})^{-1} \Big\} \\ + \exp\Big\{ -\frac{1}{4}\, t^2 \sigma^2(H)(1-\varepsilon_j) \Big(\sum_{k\in\mathbf{N}} a_{jk}\varepsilon_k \widetilde{X}_k^{(H)} \Big)^2 \Big\}. \end{aligned}$$

Repeating the arguments for Gaussian r.v.'s we obtain the result of the lemma.□

Remark. The results of Section 2 may be used for non-i.i.d. r.v.'s $X_1, \dots, X_n$. In this case one should replace $\sigma^2(H)$ by $\sigma_1^2(H) = \min_{j\in\mathbf{N}} \mathbf{E}\,|X_j^{(H)}|^2$ and $\mu_4(H)$ by $\mu_4'(H) = \max_{j\in\mathbf{N}} \mathbf{E}\,|X_j^{(H)}|^4$.

4. Spectral properties of sub-matrices

In order to estimate the right hand side of (3.4) we need lower bounds for sufficiently many eigenvalues of the matrix

$$A_{\boldsymbol{\varepsilon}} = \big(a_{\boldsymbol{\varepsilon}jk}\big)_{j,k\in\mathbf{N}} \quad \text{with} \quad a_{\boldsymbol{\varepsilon}jk} = (\varepsilon_j(1-\varepsilon_k) + \varepsilon_k(1-\varepsilon_j))a_{jk},$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ is a random vector with zero-one coordinates, which is independent of $X_1, \dots, X_n$. Introduce the diagonal matrix $\mathcal{E}$ with $\mathcal{E}(j,j) = \varepsilon_j$. Then we have

$$A_{\boldsymbol{\varepsilon}} = \mathcal{E}\, A\, (I - \mathcal{E}) + (I - \mathcal{E})\, A\, \mathcal{E}.$$

Define for any $J \subset \mathbf{N}$ the diagonal matrix $\mathcal{E}_J$ with $\mathcal{E}_J(j,j) = 1$, if $j \in J$ and 0 otherwise. Let

$$A_J = \mathcal{E}_J\, A\, (I - \mathcal{E}_J) + (I - \mathcal{E}_J)\, A\, \mathcal{E}_J.$$

By $\lambda_k(J),\ k \in \mathbf{N}$, we denote the eigenvalues of the matrix A_J, ordered to be non-increasing in absolute value.

Unfortunately, our assumptions are expressed in terms of spectral properties of A. Hence we need to find a splitting $\boldsymbol{\varepsilon}$ (respectively J) such that $A_{\boldsymbol{\varepsilon}}$ (respectively A_J) satisfies all the necessary spectral conditions. To this end we will investigate the spectral properties of $A_{\boldsymbol{\varepsilon}}$ and A_J in Lemmas 3.1–3.3.

Let $\Lambda^q(\mathbb{R}^n)$ denote the q-th exterior product of $\mathbb{R}^n$ endowed with the Euclidean norm

$$\|x\|^2 = \sum_{1 \le i_1 < \cdots < i_q \le n} x^2_{i_1 \cdots i_q}, \quad x = \sum_{1 \le i_1 < \cdots < i_q \le n} x_{i_1 \cdots i_q} e_{i_1} \wedge \cdots \wedge e_{i_q},$$

where $e_1, \dots, e_n$ denotes the canonical basis of $\mathbb{R}^n$.

In what follows let $A \wedge B$ denote the exterior product of the matrices A and B and let $A^{\wedge q} = A \wedge \cdots \wedge A$ (q times) denote the exterior power of A, which maps $\Lambda^q(\mathbb{R}^n)$ into itself. Let $j^r = (j_1, j_2, \dots, j_r)$ be a multi-index such that $1 \le j_1 < \cdots < j_r \le n,\ r = 1, 2, \dots, q$. The symbol $A_{j^r k^r}$ will denote the minor of the matrix A with rows $1 \le j_1 < \cdots < j_r \le n$ and columns $1 \leqslant k_1 < \cdots < k_r \le n$. It is well-known that the j^q, k^q entry of $A^{\wedge q}$, say $A^{\wedge q}(j^q, k^q)$ is given by $A_{j^q k^q}$. Furthermore, eigenvalues of the matrix $A^{\wedge q}$ are $\lambda^*_{j^q} = \prod_{\nu=1}^{q} \lambda_{j_\nu}$, and the eigenvalue with largest absolute value is given by

$$\lambda_{*q} = \prod_{\nu=1}^{q} \lambda_\nu.$$

By $\lambda^*_{j^q}(J),\ \lambda_{*q}(J),\ q \in \mathbf{N}$, we shall denote the corresponding eigenvalues of the matrix $A_J^{\wedge q}$. Note that $A^{\wedge 1} = A$ and $A^{\wedge 0} = I$.

Below we shall prove some results about the spectrum of the matrix A_J.

Lemma 4.1. *Assume that the conditions* (1.1) *and* (1.2) *hold. For any J there exist sets J, J_1 such that, for $q \in \mathbf{N}$,*

$$\big|\lambda_q(J)\big| \ge \frac{1}{2^q}\, |\lambda_{*q}| - 16\mathcal{L}\, \mathbf{I}\{q \ne 1\}, \tag{4.4}$$

and

$$\sum_{k \in \mathbf{N}} \lambda_k^2(J_1) \ge \frac{1}{2}, \quad |\lambda_1(J_1)| \le |\lambda_1|. \tag{4.5}$$

Let $J_1 \subset J \subset \mathbf{N}$. Denote by A_{JJ_1} and by A'_{JJ_1} the matrices defined by

$$A_{JJ_1} = \mathcal{E}_{J_1} A_J + A_J \mathcal{E}_{J_1}, \quad A'_{JJ_1} = (\mathcal{E} - \mathcal{E}_{J_1}) A_J + A_J (\mathcal{E} - \mathcal{E}_{J_1}).$$

By $\lambda_k(JJ_1)$, respectively $\lambda'_k(JJ_1)$ $k \in \mathbf{N}$ we denote the eigenvalues of these matrices. Note that, in general, $A_{JJ_1} \ne A_{J_1}$.

Lemma 4.2. *Assume that* (1.1) *and* (1.2) *are fulfilled. Then, for any $J \subset \mathbf{N}$ there exists set J_1 such that $J_1 \subset J$ and*

$$\min\{\, |\lambda_q(JJ_1)|, |\lambda_q(JJ_1^{(c)}|\,\} \geq \frac{1}{2^{q+1}}\, |\lambda_{*q}(J)| - 19\,\mathcal{L}\,\mathbf{I}\{q \neq 1\}, \tag{4.6}$$

and

$$\min\Big\{ \sum_{k\in\mathbf{N}} \lambda_k^2(JJ_1), \sum_{k\in\mathbf{N}} \lambda_k^2(JJ_1^{(c)}) \Big\} \geq \frac{1}{4} \sum_{k\in\mathbf{N}} \lambda_k^2(J) - \sqrt{2}\,\mathcal{L}, \tag{4.7}$$

where $J_1^{(c)} = J \setminus J_1$.

Remark. The inequalities (4.6) and (4.7) are still valid $\lambda_q(JJ_1)$ replaced by $\lambda'_q(JJ_1)$.

Lemma 4.3. *Let $\varepsilon_1, \dots, \varepsilon_n$ be i.i.d. Bernoulli r.v.'s with*

$$\mathbf{P}\{\varepsilon_1 = 1\} = 1 - \mathbf{P}\{\varepsilon_1 = 0\} = p.$$

Then, for any $0 \leq \alpha < 2 - 2p$,

$$\mathbf{P}\Big\{ \sum_{j\in\mathbf{N}} |\lambda_{\boldsymbol{\varepsilon} j}|^2 \leq \alpha p \Big\} \leq C\,(2 - 2p - \alpha)^{-6} \mathcal{L}^6 p^{-6}. \tag{4.8}$$

Let $\mathcal{L} \leq 2^{-5} B(q,p) |\lambda_{*q}|^2$, $B(q,p) = 2^{2q-1}(1-p)^{2q}$. Then, for any $q \geq 1$, there exists an absolute constant C such that, for $0 \leq \alpha^2 \leq B(q,p)$, we have

$$\mathbf{P}\Big\{\, |\lambda_{*\boldsymbol{\varepsilon} q}| \leq \alpha p^q |\lambda_{*q}|\,\Big\} \leq C(B(q,p) - \alpha^2)^{-6} |\lambda_{*q}|^{-12q} \mathcal{L}^6 p^{-12q}. \tag{4.9}$$

Proof of Lemma 4.1. Introduce a random vector $\boldsymbol{\varepsilon}$ with i.i.d. Bernoulli coordinates $\varepsilon_j, j \in \mathbf{N}$ such that $\mathbf{P}\{\varepsilon_j = 1\} = p$. Denote by A_{*q} and $A_{*\boldsymbol{\varepsilon} q}$ the q-th exterior powers of the matrices A and $A_{\boldsymbol{\varepsilon}}$ respectively. The proof is based on the following relations:

$$\mathbf{E}\,\big|\lambda_{*\boldsymbol{\varepsilon} q}\big| \geq \|(\,\mathbf{E}\, A_{*\boldsymbol{\varepsilon} q})x\,\|, \quad \text{for any } x \in \Lambda^q(\mathbb{R}^n), \text{ such that } \|x\| = 1, \tag{4.10}$$

$$\mathbf{E}\, A_{\boldsymbol{\varepsilon} j^q k^q} = (2p(1-p))^q A_{j^q k^q} \quad \text{for any } j^q, k^q \text{ such that } j^q \cap k^q = \varnothing, \tag{4.11}$$

and

$$\sum_{j^q, k^q : j^q \cap k^q \neq \varnothing} |A_{j^q k^q}|^2 \leq 256\,\mathcal{L}^2\ \mathbf{I}\{q \neq 1\}. \tag{4.12}$$

Let $q \geq 2$. Then, by (4.11) and (4.12),

$$\|\,\mathbf{E}\, A_{*\boldsymbol{\varepsilon} q} - (2p(1-p))^q A_{*q}\|_2 \leq 16\mathcal{L}. \tag{4.13}$$

Here $\|\cdot\|_2$ denotes the Frobenius-norm of matrix, i.e., $\|A\|_2^2 = \sum_{j,k\in\mathbf{N}} a_{jk}^2$. Since $\|A\| \leq \|A\|_2$ we obtain by (4.13)

$$\|\,\mathbf{E}\, A_{*\boldsymbol{\varepsilon} q} x\| \geq (2p(1-p))^q \|A_{*q} x\| - 16\,\mathcal{L}. \tag{4.14}$$

Inequalities (4.10) and (4.14) imply

$$\mathbf{E}\,|\lambda_{*\boldsymbol{\varepsilon}q}| \geq (2p(1-p))^q |A_{*q}x| - 16\,\mathcal{L}. \tag{4.15}$$

Choosing here x as the eigenvector of A_{*q} corresponding to the eigenvalue λ_{*q}, $\|A_{*q}x\| = |\lambda_{*q}|$, we obtain

$$\mathbf{E}\,|\lambda_{*\boldsymbol{\varepsilon}q}| \geq (2p(1-p))^q |\lambda_{*q}| - 16\mathcal{L}. \tag{4.16}$$

Let $\lambda_{\boldsymbol{\varepsilon}k}$ denote the k-th eigenvalue of $A_{\boldsymbol{\varepsilon}}$ (ordered in absolute value). Since $\lambda_{\boldsymbol{\varepsilon}k} \leq 1$ for all $k \in \mathbf{N}$ we have

$$|\lambda_{\boldsymbol{\varepsilon}q}| \geq \prod_{\nu=1}^{q} |\lambda_{\boldsymbol{\varepsilon}\nu}| = |\lambda_{*\boldsymbol{\varepsilon}q}|. \tag{4.17}$$

Inequality (4.4) for $q \geq 2$ follows from (4.16) and (4.17) with $p = \dfrac{1}{2}$. In the case $q = 1$ we have $\mathbf{E}\,a_{\boldsymbol{\varepsilon}jk} = p(1-p)a_{jk}$, and (4.10) implies (4.4).

It is easy to see that for any $J \subset \mathbf{N}$

$$|\lambda_1(J)| \leq |\lambda_1|.$$

Furthermore,

$$\sum_{k \in \mathbf{N}} \lambda_{\boldsymbol{\varepsilon}k}^2 = \|A_{\boldsymbol{\varepsilon}}\|_2^2. \tag{4.18}$$

The relations (4.11), (4.12) with $q = 1$ and (4.18) together imply

$$\mathbf{E} \sum_{k \in \mathbf{N}} \lambda_{\boldsymbol{\varepsilon}k}^2 = 2p(1-p)\,\|A\|_2^2. \tag{4.19}$$

This implies for $p = \dfrac{1}{2}$ inequality (4.5). It remains to prove the assertions (4.10)–(4.12). Since the matrix $A_{\boldsymbol{\varepsilon}}$ is symmetric, we have

$$|\lambda_{*\boldsymbol{\varepsilon}q}| = \sup_{x \in \Lambda^q(\mathbb{R}^n):\|x\|=1} \|A_{*\boldsymbol{\varepsilon}q}x\|.$$

This equality implies (4.9). Let $j^q \cap k^q = \varnothing$. Note that in this case $\varepsilon_{j_1}, \dots, \varepsilon_{j_q}$ and $\varepsilon_{k_1}, \dots, \varepsilon_{k_q}$ are independent. It is easy to see that

$$\varepsilon_j(1-\varepsilon_k) + \varepsilon_k(1-\varepsilon_j) = (\varepsilon_j - \varepsilon_k)^2. \tag{4.20}$$

Hence, for any permutation π of $\{1, \dots, q\}$

$$\begin{aligned}\mathbf{E} \prod_{\nu=1}^{q} (\varepsilon_{j_\nu} - \varepsilon_{k_{\pi(\nu)}})^2 &= \mathbf{E} \prod_{\nu=1}^{q} \mathbf{E}\left((\varepsilon_{j_\nu} - \varepsilon_{k_{\pi(\nu)}})^2 \middle| \varepsilon_{j_\nu}\right) \\ &= \prod_{\nu=1}^{q} \mathbf{E}\, p^{(1-\varepsilon_{j_\nu})}(1-p)^{\varepsilon_{j_\nu}} = (2p(1-p))^q. \end{aligned} \tag{4.21}$$

Expanding the minor $A_{\boldsymbol{\varepsilon} j^q k^q}$, we get

$$A_{\boldsymbol{\varepsilon} j^q k^q} = \sum_\pi (-1)^\pi \prod_{\nu=1}^q a_{\boldsymbol{\varepsilon} j_\nu k_{\pi(\nu)}} = \sum_\pi (-1)^\pi \prod_{\nu=1}^q (\varepsilon_{j_\nu} - \varepsilon_{k_{\pi(\nu)}})^2 \prod_{\nu=1}^q a_{j_\nu k_{\pi(\nu)}}. \tag{4.22}$$

Averaging (4.22) over all $\boldsymbol{\varepsilon}$ and applying (4.21), we obtain

$$\mathbf{E}\, A_{\boldsymbol{\varepsilon} j^q k^q} = (2p(1-p))^q \sum_\pi (-1)^\pi \prod_{\nu=1}^q a_{j_\nu k_{\pi(\nu)}} = (2p(1-p))^q A_{j^q k^q}.$$

This proves (4.11).

Finally let $j^q \cap k^q \neq \varnothing$. The assumption (1.2) yields (4.12) for $q=1$. If $q=2$, we note that $|A_{\boldsymbol{\varepsilon} j^q k^q}| = |a_{\boldsymbol{\varepsilon} j k_1} a_{\boldsymbol{\varepsilon} j_2 j}|$ for $j \in j^q \cap k^q$. This implies for $q=2$

$$\sum_{j^q,\, k^q:\, j^q \cap k^q \neq \varnothing} |A_{\boldsymbol{\varepsilon} j^q k^q}|^2 \leq \sum_{j,k,l \in \mathbf{N}} |a_{jk}|^2 |a_{lj}|^2 \leq \mathcal{L}^2 \sum_{j,k \in \mathbf{N}} a_{jk}^2 = \mathcal{L}^2, \tag{4.23}$$

thus proving the inequality (4.12) for $q=2$.

Let $q \geq 3$ and $j \in j^q \cap k^q$. Expanding $A_{\boldsymbol{\varepsilon} j^q k^q}$ first in the j-th row and then in the j-th column, we obtain

$$\sum_{j^q,\, k^q:\, j^q \cap k^q \neq \varnothing} |A_{\boldsymbol{\varepsilon} j^q k^q}|^2 \leq 2\, q^4 \Big(\sum_{j,k,l \in \mathbf{N}} |a_{jk}|^2 |a_{lj}|^2 \Big) \Big(\sum_{j^{(q-2)},\, k^{(q-2)}} |A_{\boldsymbol{\varepsilon} j^{(q-2)} k^{(q-2)}}|^2 \Big). \tag{4.24}$$

Note that, for any $r \in \mathbf{N}$,

$$\begin{aligned} \sum_{j^r,\, k^r} |A_{\boldsymbol{\varepsilon} j^r k^r}|^2 &= \sum_{j^r} |\lambda^*_{\boldsymbol{\varepsilon} j^r}|^2 = \sum_{1 \leq j_1 < \ldots < j_r \leq n} \prod_{\nu=1}^r |\lambda_{\boldsymbol{\varepsilon} j_\nu}|^2 \\ &\leq \frac{1}{r!} \big(\sum_{k \in \mathbf{N}} |\lambda_{\boldsymbol{\varepsilon} k}|^2 \big)^r \leq \frac{1}{r!} \Big(\sum_{j,k \in \mathbf{N}} a_{jk}^2 \Big)^r = \frac{1}{r!}. \end{aligned} \tag{4.25}$$

It is easy to show that $2q^4/(q-2)! \leq 256$. The inequalities (4.24) and (4.25) together imply (4.12) in the general case. □

Proof of Lemma 4.2. Consider the sets L_k, $k = 0, 1, \ldots, K = |J|$, consisting of the first k elements of J, i.e. $L_0 = \varnothing$ and $L_K = J$. Let $\mathcal{E}'_k$ be the diagonal matrix with $\mathcal{E}'_k(j,j) = \varepsilon_j^{(k)}$, where $\varepsilon_j^{(k)} = 1$, if $i \in L_k$, and 0, otherwise. Let

$$\mathcal{C}_k = \mathcal{E}'_k\, A\, (I - \mathcal{E}) + (I - \mathcal{E})\, A\, \mathcal{E}'_k, \quad \mathcal{C}'_k = (\mathcal{E} - \mathcal{E}'_k)\, A\, (I - \mathcal{E}) + (I - \mathcal{E})\, A\, (\mathcal{E} - \mathcal{E}'_k),$$

and let $\lambda_{*q}(k)$ and $\lambda'_{*q}(k)$ denote the eigenvalues of the q-th exterior power of matrices $\mathcal{C}_k$ and $\mathcal{C}'_k$ respectively of largest absolute values. Note that $|\lambda_{*q}(A)|$ is the operator-norm in the space of q-th exterior power of symmetric matrices, which is continuous with respect to the Frobenius-norm of matrices. It is well-known that

$$\big|\, |\lambda_{*q}(A)| - |\lambda_{*q}(B)| \,\big| \le \big|\, \lambda_{*q}(A-B) \,\big| \le \|A_{*q} - B_{*q}\|_2$$

for any symmetric matrices A and B (see Lancaster (1969, Ch. 6)). Assume that, for any $q \in \mathbf{N}$,

$$\|\mathcal{C}_k^{\wedge q} - \mathcal{C}_{k+1}^{\wedge q}\|_2^2 \le 2\,\frac{q^2}{(q-1)!}\,\mathcal{L}^2. \tag{4.26}$$

Then

$$\max_{1\leqslant k\leqslant K-1} \big|\, |\lambda_1(k+1)| - |\lambda_1(k)| \,\big| \le \sqrt{2\,\frac{q^2}{(q-1)!}}\,\mathcal{L}. \tag{4.27}$$

We note that $\dfrac{2q^2}{(q-1)!} \le 9$ for $q \ge 1$. Since $\lambda_{*q}(0) = 0$, (4.27) implies that one can find a number k_0 such that

$$\frac{1}{2}\,|\lambda_{*q}(K)| - 3\,\mathcal{L} \le |\lambda_{*q}(k_0)| \leqslant \frac{1}{2}\,|\lambda_{*q}(K)| + 3\mathcal{L}. \tag{4.28}$$

Put $J_1 = L_{k_0}$. For $J_{k_0}^{(c)} = J \setminus J_{k_0}$ we have

$$|\lambda'_{*q}(k_0)| \ge |\lambda_{*q}(K)| - |\lambda_{*q}(k_0) \ge \frac{1}{2}\,|\lambda_{*q}(K)| - 3\,\mathcal{L}. \tag{4.29}$$

The inequalities (4.28) and (4.29) together imply (4.6). The same proof yields (4.7). Thus it remains to prove (4.26) only. If $q = 1$, we have

$$\|\mathcal{C}_{k+1} - \mathcal{C}_k\|_2^2 \le 2\,\mathcal{L}^2$$

and (4.26) holds. The proof of (4.26) in the case $q \ge 2$ is similar to the proof of (4.12). Indeed, let l_k be the k-th element of J. Then

$$\|\mathcal{C}_k^{\wedge q} - \mathcal{C}_{k+1}^{\wedge q}\|_2^2 \le 2 \sum_{l^q, j^q : l_{k+1} \in l^q} |A_{\boldsymbol{\varepsilon} l^q j^q}|^2.$$

Expanding the minor $A_{\boldsymbol{\varepsilon} l^q j^q}$ in the l_{k+1}-th row, we obtain

$$\sum_{l^q,\, j^q : l_{k+1} \in l^q} |A_{\boldsymbol{\varepsilon} l^q j^q}|^2 \le q^2 \Big(\sum_{j \in \mathbf{N}} a^2_{l_{k+1} j} \Big) \sum_{l^{q-1},\, j^{q-1}} |A_{* l^{q-1} j^{q-1}}|^2. \tag{4.30}$$

The inequalities (4.30) and (4.25) together imply (4.26). $\square$

Proof of Lemma 4.3. We shall prove the second inequality only, because the first one can be proved similarly. First note that

$$|\lambda_{*\boldsymbol{\varepsilon} q}|^2 = \sup_{x \in \Lambda^q(\mathbb{R}^n) : \|x\|^2 = 1} \|A_{*\boldsymbol{\varepsilon} q}\, x\|^2.$$

Hence, for any $x \in \Lambda^q(\mathbb{R}^n)$ such that $\|x\|^2 = 1$, we have

$$\mathbf{P}\left\{\,|\lambda_{*\boldsymbol{\varepsilon} q}|^2 \le \alpha^2 p^{2q} |\lambda_{*q}|^2\,\right\} \le \mathbf{P}\left\{\,\|A_{*\boldsymbol{\varepsilon} q}\, x\|^2 \le \alpha^2 p^{2q} |\lambda_{*q}|^2\,\right\}.$$

Let x be the eigenvector of matrix $A^{\wedge q}$ corresponding to the eigenvalue λ_{*q}. For the proof of the lemma it is sufficient to estimate the quantity

$$D_n = \mathbf{P}\left\{\,\|A_{*\boldsymbol{\varepsilon} q}\, x\|^2 \le \alpha^2 p^{2q} |\lambda_{*q}|^2\,\right\}.$$

Note that, for $0 \le u \le 1$,

$$|u - v|^2 \ge u^2 - 2v.$$

Lyapunov's inequality, (4.14) and the last inequality together imply

$$\begin{aligned}\mathbf{E}\,\|A_{*\boldsymbol{\varepsilon} q}\, x\|^2 &\ge 2^{2q} p^{2q} (1-p)^{2q} \|A_{*q} x\|^2 - 2^5\,\mathcal{L} \\ &= 2^{2q} p^{2q} (1-p)^{2q} |\lambda_{*q}|^2 - 2^5\,\mathcal{L} \ge B(q,p) p^{2q} |\lambda_{*q}|^2. \end{aligned} \tag{4.31}$$

Write $\zeta = \|A_{*\boldsymbol{\varepsilon} q}\, x\|^2$. If we show

$$\mu_6 = \mathbf{E}\,|\zeta - \mathbf{E}\,\zeta|^6 \le C\mathcal{L}^6, \tag{4.32}$$

the lemma follows. Indeed, (4.31) implies

$$\begin{aligned} D_n &= \mathbf{P}\left\{\,\|A_{*\boldsymbol{\varepsilon} q}\, x\|^2 - \mathbf{E}\,\|A_{*\boldsymbol{\varepsilon} q}\, x\|^2 \le \alpha^2 p^{2q} |\lambda_{*q}|^2 - \mathbf{E}\,\|A_{*\boldsymbol{\varepsilon} q}\, x\|^2\,\right\} \\ &\le \mathbf{P}\left\{\,\|A_{*\boldsymbol{\varepsilon} q}\, x\|^2 - \mathbf{E}\,\|A_{*\boldsymbol{\varepsilon} q}\, x\|^2 \le -(B(p,q) - \alpha^2) p^{2q} |\lambda_{*q}|^2\,\right\} \\ &\le \mathbf{P}\left\{\,|\zeta - \mathbf{E}\,\zeta| \ge (B(p,q) - \alpha^2) p^{2q} |\lambda_{*q}|^2\,\right\} \end{aligned}$$

Applying Chebyshev's inequality and inequality (4.32) we obtain (4.8). It remains to prove (4.32) only. Write μ_6 in the form

$$\mu_6 = \sum_{i_1^q} \cdots \sum_{i_6^q} \mathbf{E} \prod_{k=1}^{6} \big(\zeta(i_k^q) - \mathbf{E}\,\zeta(i_k^q)\big), \quad \text{where} \quad \zeta(i_k^q) = \Big(\sum_{l^q} A_{\boldsymbol{\varepsilon} i_k^q, l^q} x_{l^q}\Big)^2.$$

Introducing for $k = 1, 2, \ldots, 6$ the notation

$$\zeta(i_k^q, l_1^q, l_2^q) = \Big(A_{\boldsymbol{\varepsilon} i_k^q, l_1^q} A_{\boldsymbol{\varepsilon} i_k^q, l_2^q} - \mathbf{E}\, A_{\boldsymbol{\varepsilon} i_k^q, l_1^q} A_{\boldsymbol{\varepsilon} i_k^q, l_2^q}\Big) x_{l_1^q} x_{l_2^q},$$

we can rewrite the equality (4.31) in the form

$$\mu_6 = \sum_{i_1^q, l_1^q, j_1^q} \cdots \sum_{i_6^q, l_6^q, j_6^q} \mathbf{E}\, \zeta(i_1^q, l_1^q, j_1^q) \cdots \zeta(i_6^q, l_6^q, j_6^q). \tag{4.33}$$

Without loss of generality we may assume that, for $k = 1, \ldots, 6$, the multi-indices i_k^q are disjoint to the multi-indices l_k^q and j_k^q. Write

$$J_k = \{i_k^{(1)}, \ldots, i_k^{(q)}\} \cup \{l_k^{(1)}, \ldots, l_k^{(q)}\} \cup \{j_k^{(1)}, \ldots, j_k^{(q)}\}.$$

If for some fixed k_0 and a multi-index i^q_k, l^q_k, j^q_k, $k = 1, 2, \dots, 6$,

$$J_{k_0} \cap \Big(\cup_{k \neq k_0} J_k \Big) = \varnothing,$$

then

$$\mathbf{E}\, \zeta(i^q_1, l^q_1, j^q_1) \zeta(i^q_1, l^q_2, j^q_2) \cdots \zeta(i^q_6, l^q_6, j^q_6) = 0.$$

This implies that the summands in (4.33) are non-zero in three cases only:

1) the set of indexes $J_1, \dots, J_6$ can be decomposed in three non-intersecting pairs of subsets $\big(J_{i_1}, J_{i_2}\big)$, $\big(J_{i_3}, J_{i_4}\big)$, $\big(J_{i_5}, J_{i_6}\big)$, such that any pair has at least one common index;

2) the set of indices $J_1, \dots, J_6$ can be decomposed in two non-intersecting triples of subsets $\big(J_{i_1}, J_{i_2}, J_{i_3}\big)$, $\big(J_{i_4}, J_{i_5}, J_{i_6}\big)$, such that any triple has at least one common index;

3) the set of indices $J_1, \dots, J_6$ can be decomposed in two non-intersecting subsets: a quadruple $\big(J_{i_1}, J_{i_2}, J_{i_3}, J_{i_4}\big)$ and a pair $\big(J_{i_5}, J_{i_6}\big)$, such that both systems have at least one common index. We show that in all these cases the following estimate holds

$$\Big| \sum_{i^q_1, l^q_1, j^q_1} \cdots \sum_{i^q_6, l^q_6, j^q_6} \mathbf{E}\, \zeta(i^q_1, l^q_1, j^q_1) \cdots \zeta(i^q_6, l^q_6, j^q_6) \Big| \le C \mathcal{L}^6. \tag{4.34}$$

Note, for instance, that in the first case the sum on the right hand side of (4.37) can be decomposed in a product of the following sums

$$\mathcal{S}_{\nu,\mu} = \sum_{i^q_\nu, l^q_\nu, j^q_\nu} \sum_{i^q_\mu, l^q_\mu, j^q_\mu} \zeta(i^q_\nu, l^q_\nu, j^q_\nu) \zeta(i^q_\mu, l^q_\mu, j^q_\mu),$$

where the sets J_ν and J_μ have non-empty intersection, $1 \le \nu \neq \mu \le 6$.

Let i_0 be the common index of J_ν and J_μ. For simplicity we can assume that i_0 is a common index for l^q_ν and j^q_μ. Expanding the determinants $A_{\boldsymbol{\varepsilon} i^q_\nu, l^q_\nu}$ and $\big(A(\varepsilon)\big)^{\wedge q}\big(i^q_\mu, j^q_\mu\big)$ in the common column, and the determinants $A_{\boldsymbol{\varepsilon} i^q_\nu, j^q_\nu}$, and $A_{\boldsymbol{\varepsilon} i^q_\mu, l^q_\mu}$ in the common row and using $a_{nii} = 0$, we can obtain the following estimate

$$\begin{aligned} \mathcal{S}_{\nu,\mu} \le q^2 &\sum_{i_0, j_1, l_1, j_2, l_2 \in \mathbf{N}} |a_{nj_1,i_0}| |a_{nl_1,i_0}| |a_{nl_1,l_2}| |a_{nj_1,j_2}| |x_{i_0}|^2 \, |x_{l_2}| |x_{j_2}| \\ &\times \sum_{i^{q-2}, j^{q-2}} \big| A_{i^{q-2} j^{q-2}} \big| |x_{i^{q-2}}| |x_{j^{q-2}}|. \end{aligned}$$

Hölder's inequality, (4.23) and $\sum_i x_i^2 = 1$ together imply

$$\begin{aligned} \mathcal{S}_{\nu,\mu} &\le q^2/(q-2)! \sum_{i_0, l_1, j_1 \in \mathbf{N}} \Big(\sum_{l_2 \in \mathbf{N}} a^2_{l_1, l_2} \Big)^{1/2} \Big(\sum_{j_2 \in \mathbf{N}} a^2_{l_1, l_2} \Big)^{1/2} |a_{nj_1,i_0}| \, |a_{nl_1,i_0}| x^2_{i_0} \\ &\le q^2/(q-2)! \sum_{i_0 \in \mathbf{N}} \Big(\sum_{l_1 \in \mathbf{N}} a^2_{l_1, i_0} \Big)^{1/2} \Big(\sum_{j_1 \in \mathbf{N}} a^2_{j_1, i_0} \Big)^{1/2} x^2_{i_0} \le C \mathcal{L}^2. \end{aligned}$$

The last inequality proves(4.34) for the case 1). The proof of inequality (4.34) in the cases 2) and 3) is similar. Inequality (4.34) finally implies (4.32). □

5. Some bounds for distributions of randomized linear forms

In this section we obtain the bounds for the last two terms in (4.4). Let

$$\mathcal{L}_j^2 = \sum_{k\in\mathbf{N}} a_{jk}^2.$$

Without loss of generality we shall assume that $\mathcal{L}_j \neq 0$. Let $\rho_j = \mathcal{L}\mathcal{L}_J^{-1}(\log^+\mathcal{L}_j)^{-1}$ and $\varkappa = \frac{e}{H^2}\,\log^+\mathcal{L}$.

Lemma 5.1. *Let $\varepsilon_1, \dots, \varepsilon_n$ be i.i.d. Bernoulli r.v.'s with*

$$\mathbf{P}\left\{\varepsilon_1 = 1\right\} = 1 - \mathbf{P}\left\{\varepsilon_1 =\right\} = p.$$

Then for $\mathcal{L} \leq e^{-1/2}$

$$\mathbf{P}\Big\{\max_{j\in\mathbf{N}}\Big|\sum_{\substack{k\in\mathbf{N}:\\ |a_{jk}|\leq\rho_j\mathcal{L}_j}} a_{jk}\varepsilon_k\widetilde{X}_k^{(H)}\Big| > K\mathcal{L}\Big\} \leq \mathcal{L}^{K/H-2-p\varkappa} \tag{5.1}$$

Proof. If $p\varkappa \leq K/H-2$ the inequality (5.1) is trivial. Assume that $p\varkappa > K/H-2$. Denote by Σ_j the quantity

$$\Sigma_j = \Big|\sum_{\substack{k\in\mathbf{N}:\\ |a_{jk}|\leq\rho_j\mathcal{L}_j}} a_{kj}\varepsilon_k\widetilde{X}_l^{(H)}\Big|, \tag{5.2}$$

and by $\sum^*$, $\prod^*$ denote the sum and the product over $k \in \mathbf{N}:\quad |a_jk| \leq \rho_j\mathcal{L}_j$. In this notation we have

$$\mathbf{P}\Big\{\max_{j\in\mathbf{N}}\Sigma_j > K\mathcal{L}\Big\} \leq \sum_{j\in\mathbf{N}}\mathbf{P}\Big\{\big|\Sigma_j\big| > K\mathcal{L}\Big\}.$$

By Chebyshev's inequality

$$\begin{aligned}\mathbf{P}\Big\{\Sigma_j > K\mathcal{L}\Big\} &\leq 2\exp\{-h_jK\mathcal{L}\}\,\mathbf{E}\,\exp\{h_j\Sigma_j\}\\ &= 2\exp\{-h_jK\mathcal{L}\}\textstyle\prod^*\Big(1+p\big(\mathbf{E}\,\exp\{h_ja_{jk}\widetilde{X}_k^{(H)}\}-1\big)\Big)\\ &\leq 2\exp\{-h_jK\mathcal{L}\}\exp\Big\{p\textstyle\sum^*\big(\mathbf{E}\,\exp\{h_ja_{jk}\widetilde{X}_k^{(H)}\}-1\big)\Big\}.\end{aligned} \tag{5.3}$$

Furthermore, we have

$$\begin{aligned}\mathbf{E}\,\exp\{h_ja_{jk}\widetilde{X}_k^{(H)}\}-1 &\leq \sigma^2(H)h_j^2|a_{jk}|^2\exp\{h_jH|a_{jk}|\}\\ &\leq h_j^2|a_{jk}|^2\exp\{h_jH|a_{jk}|\}.\end{aligned} \tag{5.4}$$

By (5.3) and (5.4) it follows that

$$\sum_{j\in\mathbf{N}}\mathbf{P}\Big\{\Sigma_j > K\mathcal{L}\Big\} \leq 2\sum_{j\in\mathbf{N}}\exp\{-K\,h_j\mathcal{L}_j + ph_j^2\mathcal{L}_j^2\exp\{h_jH\rho_j\mathcal{L}_j\}\}. \tag{5.5}$$

Choosing $h_j = H^{-1}\mathcal{L}_j^{-1}\rho_j^{-1} = H^{-1}\mathcal{L}^{-1}(\log^+ \mathcal{L}_j)^{-1}$, we get

$$\sum_{j\in\mathbf{N}} \mathbf{P}\left\{|\Sigma_j| > K\mathcal{L}\right\} \leq 2\sum_{j\in\mathbf{N}} \exp\Big\{ - KH^{-1}\big|\log^+ \mathcal{L}_j\big| + pH^{-2}\left(\log^+ \mathcal{L}_j\right)^2\mathcal{L}_j^2\mathcal{L}^{-2}e\Big\}. \quad (5.6)$$

Note that

$$p(\log^+ \mathcal{L}_j)\mathcal{L}_j^2\mathcal{L}^{-2} \leq p\log^+ \mathcal{L}, \quad (5.7)$$

since the function $x^2\log^+ x$ is increasing, for $0 \leq x \leq e^{-1/2}$. Inequalities (5.6) and (5.7)together finally imply

$$\mathbf{P}\left\{|\Sigma_j| > K\mathcal{L}\right\} \leq 2\sum_{j\in\mathbf{N}} \mathcal{L}_j^{KH^{-1}-p\varkappa} \leq \mathcal{L}^{KH^{-1}-2-p\varkappa} \quad (5.8)$$

□

Lemma 5.2. *Let $\varepsilon_1, \dots, \varepsilon_n$ be i.i.d. Bernoulli r.v.'s with*

$$\mathbf{P}\{\varepsilon_1 = 1\} = 1 - \mathbf{P}\{\varepsilon_1 = 0\} = p.$$

Then, for any $K, H > 0$ such that $\frac{K}{H} \geq 3$

$$\mathbf{P}\Big\{\max_{j\in\mathbf{N}}\Big|\sum_{\substack{k\in\mathbf{N}:\\ |a_{jk}|>\rho_j\mathcal{L}_j}} a_{jk}\varepsilon_k\widetilde{X}_k\Big| > K\mathcal{L}\Big\} \leq p^{K/H}\mathcal{L}^{-2}(\log^+ \mathcal{L})^{K/H}.$$

Proof. Denote by Σ_j' the quantity

$$\Sigma_j' = \sum_{\substack{k\in\mathbf{N}:\\ |a_{jk}|>\rho_j\mathcal{L}_j}} a_{jk}\varepsilon_k\widetilde{X}_k,$$

and by $\sum'$, $\prod'$ denote the sum and the product over $k \in \mathbf{N}: \ |a_jk| > \rho_j\mathcal{L}_j$.The inequality

$$|\Sigma_j'| \leq H\max_{l\in\mathbf{N}}|a_{jl}|\textstyle\sum'\varepsilon_k \leq H\mathcal{L}_j\sum'\varepsilon_k,$$

implies

$$\mathbf{P}\Big\{\max_{1\leq j\leq n}|\Sigma_j'| > K\mathcal{L}\Big\} \leq \sum_{j\in\mathbf{N}}\mathbf{P}\left\{|\Sigma_j'| > K\mathcal{L}_j\right\} \leq \sum_{j\in\mathbf{N}}\mathbf{P}\left\{\textstyle\sum'\varepsilon_k \geq \frac{K}{H}\right\}. \quad (5.9)$$

Let n_j^* the number of elements a_{lj} such that $k \in \mathbf{N} : |a_{kj}| > \rho_j\mathcal{L}_j$. Note that $n_j^* \leq 1/\rho_j^2$, $j = 1, 2, \dots, n$. Let $\mu_n = \sum_1^n \varepsilon_j$. The μ_n has binomial distribution with parameters n, p. Assuming

$$\mathbf{P}\{\mu_n > C\} \leq (np)^C, \quad C \geq 3 \quad (5.10)$$

we obtain

$$\mathbf{P}\,\{\textstyle\sum' \varepsilon_k > \dfrac{K}{H}\,\} \le (n_j^* p)^{K/H} \le (\rho_j^{-2} p)^{K/H}. \tag{5.11}$$

Note that,

$$\sum_{j\in\mathbf{N}} (\rho_j^{-2} p) \le p^{K/H} \mathcal{L}^{-2} (\log^+ \mathcal{L})^{K/H}.$$

This implies the result of Lemma. It remains to prove (5.10). If $np \ge 1$, the inequality (5.10) is trivial. If $np < 1$, assume for simplicity that C is an integer. Then,

$$\mathbf{P}\,\{\mu_n > C\} = \sum_{j>C} C_n^k p^k (1-p)^k \le (np)^C \,\frac{1}{C!} \sum_{k\ge 0} \frac{(np)^k}{k!} \le (np)^C \,\frac{e}{C!} \le (np)^C.$$

□

Remark 5.3. *For* $\mathcal{L} \le e^{-\frac{1}{2}}$ *we have*

$$\mathbf{P}\Big\{ \max_{1\le j\le n} \Big| \sum_{k\in\mathbf{N}} a_{kj} \varepsilon_k \widetilde{X}_k^{(H)} \Big| > 2\,K\,\mathcal{L} \Big\} \le \mathcal{L}^{KH^{-1}-2-p\varkappa} + p^{K/H} \mathcal{L}^{-2} (\log^+ \mathcal{L})^{-2K/H}. \tag{5.12}$$

Lemma 5.4. *Let* $\varepsilon_1, \ldots, \varepsilon_n$ *be i.i.d. Bernoulli r.v.'s with*

$$\mathbf{P}\,\{\varepsilon_1 = 1\} = 1 - \mathbf{P}\,\{\varepsilon_1 = 0\} = \frac{1}{2}\,.$$

Then

$$\mathbf{P}\Big\{ \max_{1\le j\le n} \Big| \sum_{k\in\mathbf{N}} a_{kj} \varepsilon_k \widetilde{X}_k^{(H)} \Big| > 2\,K\,\mathcal{L}\,|\log^+ \mathcal{L}| \Big\} \le \exp\big\{ e/2H \big\} \mathcal{L}^{K/H}. \tag{5.13}$$

Proof. Note that $\mathcal{L}_j \log^+ \mathcal{L}_j \le \mathcal{L} \log^+ \mathcal{L}$. Therefore,

$$\mathbf{P}\Big\{ \max_{1\le j\le n} \Big| \sum_{k\in\mathbf{N}} a_{kj} \varepsilon_k \widetilde{X}_k^{(H)} \Big| > 2\,K\,\mathcal{L}\,\log^+ \mathcal{L} \Big\} \\ \le \sum_{j\in\mathbf{N}} \mathbf{P}\Big\{ \Big| \sum_{k\in\mathbf{N}} a_{kj} \varepsilon_k \widetilde{X}_k^{(H)} \Big| > 2\,K\,\mathcal{L}_j\,\log^+ \mathcal{L}_j \Big\}.$$

Similarly to (5.5) we obtain that

$$\mathbf{P}\Big\{ \max_{1\le j\le n} \Big| \sum_{k\in\mathbf{N}} a_{kj} \varepsilon_k \widetilde{X}_k^{(H)} \Big| > 2\,K\,\mathcal{L}\,\log^+ \mathcal{L} \Big\} \\ \le \sum_{j\in\mathbf{N}} \exp\big\{ -K\,h_j \mathcal{L}_j \log^+ \mathcal{L}_j + \frac{1}{2}\,h_j^2 \mathcal{L}_j^2 \exp\{ h_j H \rho_j \mathcal{L}_j \} \big\}. \tag{5.14}$$

By (5.14) with $h_j = H^{-1} \mathcal{L}_j^{-1}$ (5.13) holds.

Lemma 5.5. *The inequality*

$$\mathbf{P}\Big\{\max_{1\le j\le n}\Big|\sum_{k\in\mathbf{N}} a_{jk}(1-\varepsilon_k)Y_k\Big| > K\mathcal{L}\big(\log^+\mathcal{L}\big)^{1/2}\Big\} \le \mathcal{L}^{\frac{K^2}{2}-2}$$

holds.

Proof. Note that $\sum\limits_{k\in\mathbf{N}} a_{jk}(1-\varepsilon_k)Y_k$ has normal distribution with variance $\mathcal{L}_j^2(\varepsilon)=\sum\limits_{k\in\mathbf{N}} a_{jk}^2(1-\varepsilon_k)$. Thus

$$\begin{aligned}
&\mathbf{P}\Big\{\max_{j\in\mathbf{N}}\Big|\sum_{k\in\mathbf{N}} a_{jk}(1-\varepsilon_k)Y_k\Big| > K\mathcal{L}(\log^+\mathcal{L})^{1/2}\Big\}\\
&\le \sum_{j\in\mathbf{N}}\mathbf{P}\Big\{\Big|\sum_{k\in\mathbf{N}} a_{jk}(1-\varepsilon_k)Y_k\Big| > K\mathcal{L}(\log^+\mathcal{L})^{1/2}\Big\}\\
&\le \sum_{j\in\mathbf{N}}\exp\Big\{-\frac{K^2\mathcal{L}^2\log^+\mathcal{L}_j}{\mathcal{L}_j^2}\Big\} \le \sum_{j\in\mathbf{N}}\exp\{-\tfrac{1}{2}K^2\log^+\mathcal{L}_j\} \le \mathcal{L}^{\left(\frac{K^2}{2}-2\right)}.
\end{aligned}$$

□

Remark. Note that the conclusions of this section still hold for non-i.i.d. r.v.'s $X_1,\dots,X_n$.

6. The proofs of Theorems 1 and 3

Let $f(t)$ and $g_n(t)$ the ch.f.'s of Q and G respectively. Under the conditions of Theorems 1 and 3 the distribution of G has a bounded density. Therefore, by Esseen's inequality, we have, for any $T>0$

$$\Delta(A,F)\le C_1\int_0^T \frac{|f(t)-g(t)|}{t}\,dt+\frac{C_2}{T}. \tag{6.1}$$

See, for instance, Petrov (1975). To bound the right hand side of (6.1) we need more notation. Denote by $\zeta_j,\ j=1,2,\dots,n$, the i.i.d. Gaussian r.v.'s with covariance operator coinciding with that of ξ_k. Introduce the random variables $\mathcal{X}_k=\sum\limits_{j=1}^{k}\xi_j+\sum\limits_{j=k+1}^{n}\zeta_j,\ k=1,\dots,n-1$ and $\mathcal{X}_k'=\sum\limits_{j=1}^{k-1}\xi_j+\sum\limits_{j=k+1}^{n}\zeta_j$. For $k=n$ we define $\mathcal{X}_n=\mathcal{X}$ and for $k=0$ respectively $\mathcal{X}_0=\sum\limits_{j\in\mathbf{N}}\zeta_j$. Let $Q_k'=\langle A\mathcal{X}_k',\mathcal{X}_k'\rangle$.

Introduce the quadratic forms $Q_k=\langle A\mathcal{X}_k,\mathcal{X}_k\rangle$ with characteristic functions $f_k(t)=\mathbf{E}\exp\{itQ_k\}$. We have

$$g(t)-f(t)=\sum_{k\in\mathbf{N}}\big(f_{k-1}(t)-f_k(t)\big). \tag{6.2}$$

Consider now the difference $h_k(t) = \big(f_{k-1}(t) - f_k(t)\big)$. It is easy to see that

$$Q_k - Q_{k-1} = \langle A\mathcal{X}_k,\, \mathcal{X}_k\rangle - \langle A\mathcal{X}_{k-1},\, \mathcal{X}_{k-1}\rangle = \langle A\xi_k, \mathcal{X}'_{k-1}\rangle - \langle A\zeta_k, \mathcal{X}'_{k-1}\rangle. \tag{6.3}$$

Equality (6.3) implies that

$$h_k(t) = \mathbf{E}\,\Big(\exp\big\{2it\langle A\xi_k,\, \mathcal{X}'_{k-1}\rangle\big\} - 1\Big)\exp\big\{it Q'_{k-1}\big\} \\ - \mathbf{E}\,\Big(\exp\big\{2it\langle A\zeta_k,\, \mathcal{X}'_{k-1}\rangle\big\} - 1\Big)\exp\big\{it Q'_{k-1}\big\}. \tag{6.4}$$

Since ξ_k and ζ_k are independent of $\mathcal{X}'_{k-1}$ and, hence, of Q'_{k-1} as well, we can rewrite the equality (6.4) in the form

$$h_k(t) = h_k(t,\xi_k) - h_k(t,\zeta_k), \tag{6.5}$$

where

$$h_k(t,\eta) = \mathbf{E}\,\Big(\exp\big\{2it\langle A\eta,\, \mathcal{X}'_{k-1}\rangle\big\} - 1 - 2it\langle A\eta,\, \mathcal{X}'_{k-1}\rangle - \frac{1}{2}\,\big(2it\langle A\eta,\, \mathcal{X}'_{k-1}\rangle\big)^2\Big) \\ \times \exp\big\{it Q'_{k-1}\big\}.$$

In the last formula we have used $\mathbf{E}\,\zeta_k^2 = \mathbf{E}\,\xi_k^2$. Now we shall derive a bound for $h_k(t,\eta)$. A bound for the second term can be proved in the same way. Let τ be a uniformly distributed r.v., on the unit interval, which is independent of all other r.v.'s. By Tailor expansion with integral remainder we have

$$\exp\{iu\} - 1 - iu - \frac{1}{2}\,(iu)^2 = \frac{1}{6}\,(iu)^3\,\mathbf{E}\,(1-\tau)^2\exp\{i\tau u\}.$$

Using this formula for $h_k(t,\eta)$, we obtain that

$$h_k(t,\eta) = \frac{1}{6}\,(it)^3\,\mathbf{E}\,(1-\tau)^2\langle A\eta,\, \mathcal{X}'_{k-1}\rangle^3\exp\big\{i\tau t\,\langle A\eta,\, \mathcal{X}_{k-1}\rangle + it Q'_{k-1}\big\}. \tag{6.6}$$

From Rosental's inequality it follows that, for $\eta = \xi_k$ or $\eta = \zeta_k$,

$$\max_{\substack{i=0,1\\ l=1,\dots,5}}\ \mathbf{E}\,|\langle A\eta, U_{il}\rangle|^3 \le C\beta_3\big(\sum_{j\in\mathbf{N}} a_{jk}^2\big)^{3/2} = C\beta_3\mathcal{L}_k^3 \tag{6.7}$$

Proof of Theorem 1. By Lemmas 4.1 and 4.2 we can choose a subset $J_0 \subset \mathbf{N}$ and non-intersecting subsets J_{1l}, $l = 1\dots,5$, such that $J_{1l} \subset J_1 = \mathbf{N}\setminus J_0$ and

$$\min_{1\le l\le 5}\Big\{\sum_{k\in\mathbf{N}}\lambda_k^2(J_0 J_{1l})\Big\} \ge \frac{1}{16} - 5\sqrt{2}\,\mathcal{L}. \tag{6.8}$$

For any fix J_{1l} we can choose the non-intersecting splitting J_{0j}, $j = 1,\dots,5$, such that

$$\min_{1\le m\le 5}\Big\{\sum_{k\in\mathbf{N}}\lambda_k^2(J_{1l} J_{0m})\Big\} \ge \frac{1}{128} - 5\sqrt{2}\,\mathcal{L}. \tag{6.9}$$

Let $B = (b_{jk})_{j,k\in\mathbf{N}}$ denote a symmetric matrix with eigenvalues μ_k, $k = 1, \dots, n$, which are nondecreasing in absolute value. Consider the quadratic form $G_b = \sum_{j,k\in\mathbf{N}} b_{jk}Y_jY_k$. For its ch.f. $\phi_b(t) = \mathbf{E}\exp\{itG_b\}$ we have the representation

$$\phi_b(t) = \exp\Big\{ i \int_0^t \Big(\sum_{k\in\mathbf{N}} \frac{\mu_k}{1-2iu\mu_k} \Big)\, du \Big\}. \tag{6.10}$$

From this formula it immediately follows that

$$|\phi_b(t)| \le \exp\Big\{ -\frac{3}{4}\, t^2 \sum_{k\in\mathbf{N}} \mu_k^2 \} + \mathbf{I}\{t > 2|\mu_1|^{-1}\Big\}.$$

Relations (6.7)–(6.9), Lemma 3.1, Lemma 3.4, Lemma 4.4 and the last inequality together imply that

$$|h_k(t,\eta)| \le Ct^3 \max_{\substack{i=0,1\\ l=1,\dots,5}} \mathbf{E}\,|\langle A\eta, U_{il}\rangle|^3 r(t), \tag{6.11}$$

where

$$r(t) = \exp\{-1/256\,\sigma^2(H)t^2\} + \mathbf{I}\{t > \gamma\mathcal{L}^{-1}(\log^+\mathcal{L})^{-1}\} + \theta_1 \exp\{\frac{e}{4H}\}\mathcal{L}^{K/2H} + \theta_2\mathcal{L}^{\frac{K^2}{4}-1}.$$

Note that,

$$\sigma^2(H) \ge 1 - \beta_3/H, \quad \mu_4(H) \le H\beta_3. \tag{6.12}$$

Choose $H = 2\beta_3$ and $K = 16H$. Then, $\gamma \ge (2\beta_3)^{-1}$, we obtain from (6.7) and (6.12)

$$|h_k(t)| \le Ct^3\beta_3\mathcal{L}_k^3\Big(\exp\{-1/512\,t^2\} + \mathcal{L}^6 + \mathbf{I}\{t \ge (\beta_3\mathcal{L})^{-1}(\log^+\mathcal{L})^{-1}\}\Big). \tag{6.13}$$

Applying Lemmas 3.2, 4.3 and inequalities (6.8) and (4.12) we obtain for $t \ge 0$

$$|f(t)| \le \exp\{-\frac{1}{8}\,\alpha p(t\wedge T_0)^2\} + \mathbf{I}\{t > \gamma\mathcal{L}^{-1}\} + C(2-2p-\alpha)^{-3/2}p^{-3/2}\mathcal{L}^{-3/2} + \mathcal{L}^{\frac{K}{4H}-2-p\varkappa} + p^{K/H}\mathcal{L}^{-2}(\log^+\mathcal{L})^{-2K/H}.$$

Let $p = \mathcal{L}^{1/4}$ and $\alpha = 1$. Then,

$$|f(t)| \le \exp\{-\frac{1}{8}\,(t\wedge T_0)^2\mathcal{L}^{1/4}\} + \mathbf{I}\{t > \frac{1}{8}\,\beta_3^{-1}\mathcal{L}^{-1}\} + C\mathcal{L}^{-9/8} + \mathbf{I}\{t > |\lambda_1|^{-1}\}, \tag{6.14}$$

and

$$|g(t)| \le \exp\{-\frac{1}{2}\,t^2\} + \mathbf{I}\{t > 2|\lambda_1|^{-1}\}. \tag{6.15}$$

Applying inequality (6.13) for $0 \le t \le (2\beta_3\mathcal{L})^{-1}(\log^+\mathcal{L})^{-1}$, inequality (6.15) for $(2\beta_3\mathcal{L})^{-1}(\log^+\mathcal{L})^{-1} \le t \le (2\beta_3\mathcal{L})^{-1}$, inequality (6.15) and (6.1) together completes the proof of Theorem 1. $\square$

Proof of Theorem 3. Similarly as in (6.8) and (6.9), by Lemmas 3.1 and 3.2 we can choose a subset $J_0 \subset \mathbf{N}$ and non-intersecting subsets J_{1l}, $l = 1 \dots, 5$, such that $J_{1l} \subset J_1 = \mathbf{N} \setminus J_0$ and

$$\min_{1 \le l \le 5} \Big\{ |\lambda_q(J_0 J_{1l}|) \Big\} \ge \frac{1}{2^{3(q+1)}} - 5\sqrt{2}\mathcal{L}. \tag{6.16}$$

For any fixed J_{1l} we can choose a non-intersecting splitting J_{0j}, $j = 1, \dots, 5$, such that

$$\min_{1 \le m \le 5} \Big\{ |\lambda_q(J_{1l} J_{0m})| \Big\} \ge \frac{1}{2^{3(q+1)}} - 5\sqrt{2}\mathcal{L}. \tag{6.17}$$

It is well-known that

$$|\phi_b(t)| \le \min\Big\{ \frac{1}{(1+2\mu_q^2 t^2)^{q/4}}, \frac{1}{|\mu_{*q}| t^{q/2}} \Big\}, \tag{6.18}$$

where $\mu_{*q} = \prod_{l=1}^{q} \mu_l$; see, for instance, Götze (1984). Arguing similar to (6.13), we obtain

$$|h_k(t)| \le C\mathcal{L}_k^3 t^3 \Big(\frac{1}{(1+2\delta t^2)^{q/4}} + \mathcal{L}^6 \Big). \tag{6.19}$$

If $2 \le q \le 12$ the inequalities (6.19) and (6.1) with $T = C\beta_3^{-1}\mathcal{L}^{-1}(\log^+ \mathcal{L})^{-1}$ imply the required results. In the case $q \le 13$ we shall apply for $C\beta_3^{-1}\mathcal{L}^{-1}(\log^+ \mathcal{L})^{-1} \le t \le C\beta_3^{-1}\mathcal{L}^{-1}$ the following inequality. Choose $K = 16qH$ and $p = \mathcal{L}^{1/4q}$. Then, Lemmas 3.2, 4.2, 4.3, and 5.1 and inequalities (6.17) and (6.18) together imply that

$$|f(t)| \le \frac{C}{(t\mathcal{L}^{1/4q})^{q/8}} + \mathcal{L}^3 \le C\mathcal{L}^{51/32}. \tag{6.20}$$

Note that

$$\int_0^t u^2 (1+\alpha u)^{-13/4} du \le C, \tag{6.21}$$

and

$$\sum_{k \in \mathbf{N}} \mathcal{L}_k^3 \le \mathcal{L}. \tag{6.22}$$

The inequalities (6.19) and (6.20) complete the proof. □

7. Appendix

In this Section we prove relation (2.18). Consider the matrix A with entries

$$a_{jk} = \frac{1}{B_n}\rho_n^2, \qquad B_n^2 = 2\sum_{k=1}^{n-1}(n-k)\rho_n^{2k}. \tag{7.1}$$

Assume that $\rho_n = 1 - \frac{\alpha_n}{n}$, $\alpha_n \to \alpha > 0$ as $n \to \infty$. Denote by $\widetilde{A}$ the matrix $B_n A$ and by $\widetilde{\lambda}_q(n)$, $q = 1, \dots, n$ its eigenvalues . First we note that $\frac{B_n^2}{n^2} \to v_\alpha^2 = \frac{\exp\{-2\alpha\} - 1 + 2\alpha}{4\alpha^2}$, which follows from

$$B_n^2 = n\rho_n^2 \frac{1-\rho_n^{2(n-1)}}{1-\rho_n^2} + \rho_n^2 \frac{(n-1)\rho_n^{2(n-2)}(1-\rho_n^2) + (1-\rho_n^{2(n-1)})}{(1-\rho_n^2)^2}.$$

We shall write the eigenvalues of the matrix $\widetilde{A}$ in the form $\widetilde{\lambda}_q(n) = nr_q(n)$. In this notation we have $\lim\limits_{n\to\infty} \lambda_q(n) = \lim\limits_{n\to\infty} \dfrac{1}{B_n} \widetilde{\lambda}_q(n) = v_\alpha^{-1/2} \lim\limits_{n\to\infty} r_q(n) = v_\alpha^{-1/2} r_q$.

Denote by $U_n(\lambda)$ the characteristic polynomial of matrix $\widetilde{A}$. Reducing the matrix $\widetilde{A}-\lambda I$ to the three-diagonal form and and expanding the corresponding determinant in the last row, we obtain that

$$U_n(\lambda) = -\lambda V_{n-1}(\lambda) - \rho_n^2(1+\lambda)^2 V_{n-2}(\lambda), \tag{7.2}$$

where $V_n(\lambda)$ is the solution of the difference equation

$$V_n(\lambda) = -b_1 V_{n-1}(\lambda) - b_2^2 V_{n-2}(\lambda), \quad V_0(\lambda) = 1, \quad V_1(\lambda) = -b_1. \tag{7.3}$$

Here $b_1 = \lambda(1+\rho_n^2) + 2\rho_n^2$, $\quad b_2^2 = \rho_n^2(1+\lambda)^2$. Denote by s_1 and s_2 the characteristic roots of the difference equation (7.3),

$$s_1 = -\frac{b_1}{2} + \frac{\sqrt{D}}{2}, \quad s_2 = -\frac{b_1}{2} - \frac{\sqrt{D}}{2}, \quad D = b_1^2 - 4b_2^2.$$

From (7.2)–(7.3) it follows that the eigenvalues of matrix $\widetilde{A}$ satisfy the following equation

$$\frac{s_1^n}{s_2^n} = \frac{\lambda + s_1}{\lambda + s_2} \tag{7.4}$$

If $D > 0$, then (7.4) yields that there exists an eigenvalue $\widetilde{\lambda} = -nr(n)$ with $r(n) > 0$ such that $r = \lim\limits_{n\to\infty} r(n)$ satisfies the following equation.

$$\exp\{-2\sqrt{(\alpha r + 2)\alpha r^{-1}}\} = \Big(1 - \alpha r - \sqrt{(\alpha r + 2)\alpha r}\Big(1 - \alpha r + \sqrt{(\alpha r + 2)\alpha r}\Big)^{-1}$$

If $D < 0$, then then for any $q \ge 1$ $\widetilde{\lambda}_q(n) = nr_q(n) > 0$, and $r_q = \lim\limits_{n\to\infty} r_q(n)$ satisfies the equation

$$\sqrt{\frac{\alpha(2-\alpha r_q)}{r_q}} = \arcsin\Big(\sqrt{(2-\alpha r_q)\alpha r_q}\Big) + q\pi. \tag{7.5}$$

It is easy to see that for any $q \ge 1$ exists exactly one root of the equation (7.5), and

$$\frac{2\alpha}{\alpha^2 + q^2\pi^2} \le r_q \le \frac{2\alpha}{\alpha^2 + \big(q + 1/2\big)^2\pi^2}.$$

This inequality completes the proof of (2.18).

References

1. V. Bentkus, F. Götze, *Optimal rates of convergence in the CLT for quadratic forms*, Annals of Probab. **24** (1996), 466-490.
2. N. H. Chan, C. Z. Wei, *Asymptotic inference for nearly nonstationary AR(1) processes*, Annals of Statist. **15** (1987), 1050–1063.

3. N. G. Gamkrelidze, V. I. Rotar', *On the rate of convergence in the limit theorem for quadratic forms*, Theory Prob. Appl., **22** (1977), 394–397.
4. F. Götze, *Asymptotic expansions for bivariate von Mises functionals*, Z. Wahrsch. verw. Geb., **50** (1979), 333–355.
5. F. Götze, *Expansions for von Mises functionals*, Z. Wahrsch. verw. Geb., **65** (1984), 599–625.
6. F. Götze, *Approximations for multivariate U-statistics*, J.Multivar.Anal., **22** (1987), 212–229.
7. J. Gurland, *Distribution of quadratic forms and ratios of quadratic forms*, Ann. Math. Stat., **24** (1953), 416–427.
8. J. Gurland, *Distribution of definite and indefinite quadratic forms*, Ann. Math. Stat., **26** (1955), 122–127.
9. R. Fox, M.S.Taqqu, *Central limit theorems for quadratic forms in random variables having long-range dependence*, Prob. Th. Rel. Fields, **74** (1987), 213–240.
10. Z. Füredi, J. Komlos, *The eigenvalues of random symmetric matrices*, Combinatorica, **1** (1981), 233–241.
11. P. de Jong, *A central limit theorem for generalized quadratic forms*, Prob.Theory Rel.Fields, **75** (1987), 261-277.
12. P. Lancaster, *Theory of matrices*, Academic Press, New York and London W.1, 1969.
13. T. Mikosch, *Functional limit theorems for quadratic forms*, Stochastic. Process. Appl. **37** (1991), 81–98.
14. T. Mikosch, *On the lower estimate in the law of the iterated logarithm for Gaussian quadratic forms*, Theory Prob. Appl. **35** (1990), 363–367.
15. J. Pachares, *A new result on the distribution of quadratic forms*, Ann. Math. Stat., **26** (1955), 128-131.
16. V. V. Petrov, *Sums of Independent Random Variables*, Springer-Verlag, Berlin Heidelberg New York, 1975.
17. A. Rachkauskas, *Asymptotic accuracy of the least-squares estimates in nearly nonstationary autoregressive models*, Lith. Math. Jour., **36** (1996), 92–103.
18. V. I. Rotar', *Some limit theorems for polynomials of second degree*, Theory Prob. Appl., **18** (1973), 499–507.
19. V. I. Rotar', T. L. Shervashidze, *Some estimates of distributions of quadratic forms*, Theory Prob. Appl., **30** (1985), 585–591.
20. H. Ruben, *A new result on the distribution of quadratic forms*, Ann. Math. Stat., **34** (1963), 1582–1584.
21. B. A. Sevast'yanov, *A class of limit distribution for quadratic forms of normal stochastic variables*, Theory Prob. Appl., **6** (1961), 337–340.
22. D. E. Varberg, *Convergence of quadratic forms in independent random variables*, Ann. Math. Stat., **37** (1966), 567–575.
23. J. H. Venter, T. de Wet, *Asymptotic distribution for quadratic forms with applications to test of fit*, Ann. Stat., **1** (1973), 380-387.
24. P. Whittle, *On the convergence to normality of quadratic forms in independent variables*, Theory Prob. Appl., **9,** (1964), 103–109.
25. P. Whittle, *Bounds for the moments of linear and quadratic forms in independent variables*, Theory Prob. Appl., **5,** (1960), 303–305.

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