Atomic compactness and reflexive graphs.

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Abstract

A first order structure $\mathfrak{M}$ with universe M is atomic compact if every system of atomic formulas with parameters in M is satisfiable in $\mathfrak{M}$ provided each of its finite subsystems is. The definition of atomic compactness goes back to Weglorz [Weg]. It extends to general first order structures the notion of equational compactness, first considered by I. Kaplansky and J. Løs for Abelian groups, and later by J. Mycielsky for general algebras (see [Myc]).

We present here miscellaneous results about atomic compactness of reflexive graphs.

Contents

1 Preliminaries.

Let us recall some basic definitions and general facts about atomic compactness ; they appear in [Myc], [Weg] and [Myc,Ry-Na], see also [Weg] for details and examples.

Given a structure $\mathfrak{M} = (M; ...)$ with *universe* M over a first order language $\mathcal{L}$, an *atomic formula with parameters* in M is any formula $t_1 \equiv t_2$ (*equality formula*) or $r(t_1, \cdots t_n)$, where r is a relation symbol and $t_1, \cdots, t_n$ are terms of the language $\mathcal{L}_M$, obtained from $\mathcal{L}$ by the adjunction of the elements of M as new constant symbols ; *positive formulas* are built up from atomic ones using the connectives $\wedge$, $\vee$ and the quantifiers $\exists$, $\forall$; $(\exists, \wedge)$-*formulas* are obtained using only $\wedge$ and $\exists$. A *sentence* is a formula with no free variables.

A *system* of $\mathfrak{M}$ with *unknowns* in X is any set $\Phi(X)$ of formulas with parameters in M and with free variables in X ; the set of unknowns may be uncountably infinite. The system Φ is an *atomic*, resp. a *positive*, resp. a $(\exists, \wedge)$-*system* if its elements are respectivly atomic, positive or $(\exists, \wedge)$-formulas ; Φ^{at} and Φ^{+} will denote respectively the set of atomic and positive elements of Φ. Given cardinals κ and λ, Φ is a κ-*system* (resp. a κ^{-}-*system*) if it has at most (resp. strictly less than) κ elements, it is a $[\lambda]$-*system* (resp. a $[\lambda^{-}]$-*system*) if it has at most (resp. strictly less than) λ unknowns. We shall freely combine these various prefixes.

Given a structure $\mathfrak{N} = (N; ...)$ over $\mathcal{L}_M$, a *solution* of Φ in $\mathfrak{N}$ is any family $(c_x : x \in X)$ of elements of N such that $\mathfrak{N}$ satisfies the set of sentences $\Phi[c_x : x \in X]$. The system Φ is *solvable* if it has a solution in $\mathfrak{M}$, it is *finitely solvable* if each of its finite subsets is solvable.

The structure $\mathfrak{M}$ is *atomic compact* if all its finitely solvable atomic systems are solvable. One similarly defines $\mathfrak{M}$ being compact with respect to any class of systems (e.g. $\mathfrak{M}$ is saturated if and only if it is $[1]$-card$(M)^{-}$-compact.)

The atomic compactness of $\mathfrak{M}$ can be checked by considering only "small" systems. Indeed, given any infinite cardinal κ greater or equal to the cardinality of $\mathcal{L}_M$, if $\mathfrak{M}$ is κ-atomic compact, then it is $[1]$-$(\exists, \wedge)$-compact, but then, any finitely solvable atomic system can be solved "one unknown after the other" (see [Myc,Ry-Na]).

Given an extension $\mathfrak{N} = (N; ...)$ of $\mathfrak{M} = (M; ...)$, recall that $\mathfrak{M}$ is a *retract* of $\mathfrak{N}$, if there exists a *retraction* from $\mathfrak{N}$ to $\mathfrak{M}$ (i.e. an $\mathcal{L}_M$-homomorphism, that is an $\mathcal{L}$-homomorphism fixing M pointwise) ; $\mathfrak{N}$ is an *elementary* (resp. a *pure*) *extension* of $\mathfrak{M}$, if $\mathfrak{M}$ and $\mathfrak{N}$ satisfy exactly the same sentences (resp. the same $(\exists, \wedge)$-sentences) with parameters in M.

Observe that if $\mathfrak{M}$ is a retract of $\mathfrak{N}$, then any positive system of $\mathfrak{M}$ with a solution in $\mathfrak{N}$ has a solution in $\mathfrak{M}$. Besides (by the compactness theorem) the finitely solvable systems of $\mathfrak{M}$ are exactly the systems which have a solution in an elementary extension of $\mathfrak{M}$ (they are the so called incomplete X-types of $\mathfrak{M}$ over M). But for each extension $\mathfrak{N}$ of $\mathfrak{M}$, the set of $\mathcal{L}_N$-sentences true in $\mathfrak{N}$ can be viewed as a system $\Phi(N - M)$ of $\mathfrak{M}$, with the property that any solution of Φ^{at} defines a retraction of $\mathfrak{N}$ to $\mathfrak{M}$; moreover, Φ (resp. Φ^{at}) is finitely solvable precisely when $\mathfrak{N}$ is an elementary (resp. a pure) extension of $\mathfrak{M}$.

From these observations, together with the fact that homomorphisms onto "preserve" positive formulas (note incidentally that each retract of an atomic compact structure is atomic compact as well) one gets : If $\mathfrak{M}$ is atomic compact, then it is a retract of all its pure extensions, hence it is a retract of all its elementary extensions, and then it is positively compact ; thus these four properties are mutually equivalent, and even equivalent to $\mathfrak{M}$ being a retract of all its reduced powers (resp. of all its ultrapowers) [Weg], as for any finitely solvable atomic system Φ of $\mathfrak{M}$, the system Φ^{at} has a solution in the reduced power $\mathfrak{M}^{[\Phi]}/\mathcal{F}$, where $[\Phi]$ denotes the set of finite subsets of Φ, and $\mathcal{F}$ is any filter (resp. any ultrafilter) on $[\Phi]$ containing $\{v \in [\Phi] : u \subset v\}$ for all u in $[\Phi]$; conversely, recall that any reduced power is a pure extension.

A topology $\mathcal{T}$ on M is *compatible* with $\mathfrak{M}$ if the interpretations of the relation and function symbols are respectively closed and continuous (the powers of M involved being endowed with the product topology) ; when in addition $\mathcal{T}$ is Hausdorff (in which case the equality is closed too), for each atomic system $\Phi(X)$ of $\mathfrak{M}$ and each φ in Φ, the set of solutions of φ in $\mathfrak{M}$ defines a closed subset of E^X. Hence, any topologically compact structure (admitting a Hausdorff-compact compatible topology), and in particular any finite structure is atomic compact [Myc]. But the converse need not hold (consider for instance $(\mathbb{Q}, +)$) ; in fact, there are some atomic compact structures which are retract of no topologically compact ones [Tay].

Given a *relational* structure $\mathfrak{R} = (E; \mathcal{R}_i : i \in I)$, let $\check{\mathfrak{R}} = (\check{E}; \overline{\mathcal{R}_i} : i \in I)$ denote the structure with universe the Stone-Čech compactification of the discrete space E, each $\overline{\mathcal{R}_i}$ being the closure of $\mathcal{R}_i$ in $\check{E}^{n_i}$ (n_i denoting the arity of $\mathcal{R}_i$). The relational structure $\check{\mathfrak{R}}$ is a topologically compact extension of $\mathfrak{R}$ with the property that every homomorpism from $\mathfrak{R}$ into a topologically compact structure has a (unique) extension to $\check{\mathfrak{R}}$; thus if $\mathfrak{R}$ is a retract of some topologically compact structure, then it is a retract of $\check{\mathfrak{R}}$. We shall consider $\check{E}$ as the set of ultrafilters on E (hence $(\mathcal{U}_1, \cdots, \mathcal{U}_{n_i}) \in \overline{\mathcal{R}_i}$ if and only if for all $U_1 \in \mathcal{U}_1, \cdots, U_{n_i} \in \mathcal{U}_{n_i}$, there are $a_1 \in U_1, \cdots, a_{n_i} \in U_{n_i}$ such that $(a_1, \cdots, a_{n_i}) \in \mathcal{R}_i$.)

2 Graphs.

2.1 Conventions.

The *graphs* that we consider are the binary relations $\mathfrak{G} = (E; \mathcal{G})$ which are reflexive and symmetric. For any subset F of E, the graph *induced* on F by $\mathfrak{G}$ is $\mathfrak{G}(F) = (F; \mathcal{G} \cap F \times F)$. In handling systems Φ of $\mathfrak{G}$, we shall adopt the convention to denote by letters x, y, z, ... the free variables (unknowns), by letters a, b, c, ... the vertices (parameters) and by letters u, v, w, ... indiscriminately the unknowns or parameters ; the symbol of the binary relation will be $\sim$. So the general forms of atomic formulas with parameters are $u \equiv v$ and $u \sim v$. The notations $u \sim_\Phi v$ and $u \sim_\mathfrak{G} v$ will stand respectively for $u \sim v \in \Phi$ and $(u, v) \in \mathcal{G}$. The *neighbourhood* of a vertex a is $\mathcal{G}_a = \{b \in E : a \sim_\mathfrak{G} b\}$ and its *degree* is $\mathrm{card}(\mathcal{G}_a \backslash \{a\})$; the vertex a is a (weak) *neighbour* of the subset A of E (write $a \sim_\mathfrak{G} A$) if $A \cap \mathcal{G}_a \neq \varnothing$; the set $\mathcal{G}_A = \{a \in E : a \sim_\mathfrak{G} A\}$ (which is also $\cup\{\mathcal{G}_a : a \in A\}$) is the *neighbourhood* of A. In the same way, we shall write $x \sim_\Phi A$ for : "there is an a in A such that $x \sim_\Phi a$". The graph $\mathfrak{G}$ is *locally finite* (resp. *almost locally finite*) if all its vertices (resp. all but finitely many of its vertices) have a finite degree ; an *essential degree* of $\mathfrak{G}$ is any cardinal which bounds the degree of all but finitely many of its vertices.

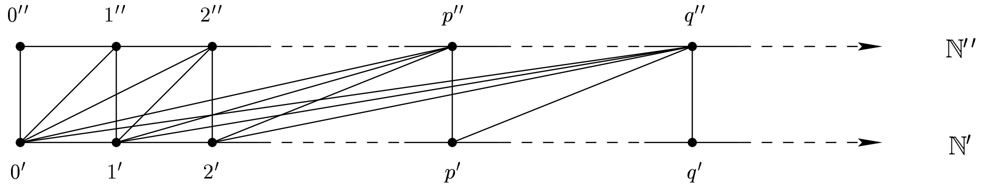
2.2 Examples.

All complete graphs and more generally all graphs with a central vertex a (i.e. for which $\mathcal{G}_a = E$) are atomic compact (any atomic system $\Phi(X)$ admits the constant family $(a : x \in X)$ as a solution) ; they are actually topologically compact (consider the Alexandroff compactification on a of the discrete space $E \backslash \{a\}$).

A forest (graph with no cycle) is atomic compact, as each of its elementary extensions is a forest in which there is no path between two vertices previously in distinct connected components.

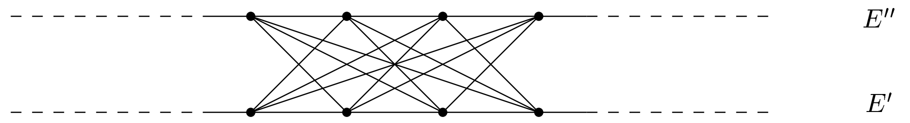
The simplest examples of non-atomic compact graphs are those for which there is a set of vertices A with no neighbour while each of its finite subsets has a neighbour (consider the [1]-atomic system $\Phi(x) = \{x \sim a : a \in A\}$.) In the first example below, no infinite subset of $\mathbb{N}'$ has a neighbour ; in the second one, every proper subset of E' has a neighbour.

$\star$**Example 1.** The vetex set of the *half-graph* $\mathfrak{H}_{\mathbb{N}}$ is the disjoint union of two copies $\mathbb{N}' = \{0', 1', 2', \cdots\}$ and $\mathbb{N}'' = \{0'', 1'', 2'', \cdots\}$ of the set of non-negative integers ; any two distinct vertices are linked if one is a p', the other is a q'' and $p \leq q$. (One defines similarly $\mathfrak{H}_{\mathbf{P}}$ for any poset $\mathbf{P}$.)



$\star$

$\star$**Example 2.** The vertex set of $\mathfrak{D}_E$ is the disjoint union of two copies $E' = \{a', b', c', \cdots\}$ and $E'' = \{a'', b'', c'', \cdots\}$ of the same infinite set E ; any two distinct vertices are linked if they lie in distinct copies and corrsespond to two different points of E.



$\star$

Note that each elementary extension of $\mathfrak{D}_E$ is some $\mathfrak{D}_F$, hence it is never atomic compact. The elementary extensions of $\mathfrak{H}_{\mathbb{N}}$ are not atomic compact either, since they are exactly the graphs $\mathfrak{H}_{\mathbf{C}}$ where $\mathbf{C}$ is any elementary extension of $\mathbb{N}$ (but such a $\mathbf{C}$ has no upper bound while each of its finite subsets has one ...).

Remark. (c.f. [Del]) Given any infinite cardinal κ, the successor cardinal κ_+ endowed with its order topology has the following property : for any $f : \kappa \to \kappa_+$ and any ultrafilter $\mathcal{F}$ on κ, the image ultrafilter $f(\mathcal{F})$ is convergent in κ_+ (observe that κ_+ having a cofinality strictly greater than κ, $f(\mathcal{F})$ has bounded elements, hence it is "essentially" a ultrafilter on some compact subspace of κ_+.)

It follows that the graph $\mathfrak{C}_{\kappa_+}$ is κ-atomic compact but not κ_+-atomic compact (it is obviously not [1]-κ_+-atomic compact). Indeed, the disjoint union topology on $K = \kappa'_+ \cup \kappa''_+$ being compatible, for any ultrafilter $\mathcal{F}$ on κ, $\lim_{\mathcal{F}} : K^\kappa \to K$ yields a retraction from $\mathfrak{C}^\kappa_{\kappa_+}/\mathcal{F}$ on $\mathfrak{C}_{\kappa_+}$.

3 Atomic compactness and almost locally finite graphs.

We already know that finite graphs are topologically compact ; more generally, it is the case for locally finite ones, and even for those with no more than two vertices of infinite degree ; as for general almost locally finite graphs, they are $[\omega^-]$-atomic compact.

3.1 Topologically compact graphs.

Lemma 3.1 *If $\mathfrak{G}$ has a retract $\mathfrak{G}(F)$ which is topologically compact and contains all vertices of infinite degree, then it is topologically compact.*

Proof. Let $\mathcal{T}$ be a Hausdorff-compact topology on F compatible with $\mathfrak{G}(F)$, and $r : E \to F$ a retraction. One easily checks that the following set $\mathcal{B}$ is a base for a Hausdorff-compact topology $\widetilde{\mathcal{T}}$ on E.

$$\mathcal{B} = \Big\{ \{a\} : a \in E \backslash F \Big\} \cup \Big\{ r^{-1}(U) \backslash A : U \in \mathcal{T},\ A \text{ finite} \Big\}$$

Now, given $(a, b) \notin \mathcal{G}$,
— either $\{a, b\} \subset F$, and then for any $\mathcal{T}$-open neighbourhood $U \times V$ of (a, b) disjoint from $\mathcal{G} \cap F \times F$, $r^{-1}(U) \times r^{-1}(V)$ is a $\widetilde{\mathcal{T}}$-neighbourhood of (a, b) which is disjoint from $\mathcal{G}$.
— or $a \notin F$ for example, and then $\mathcal{G}_a$ is finite (by hypothesis) hence closed, and a is isolated (by construction) ; thus $\{a\} \times (E \backslash \mathcal{G}_a)$ is a neighbourhood (a, b) disjoint from $\mathcal{G}$. □

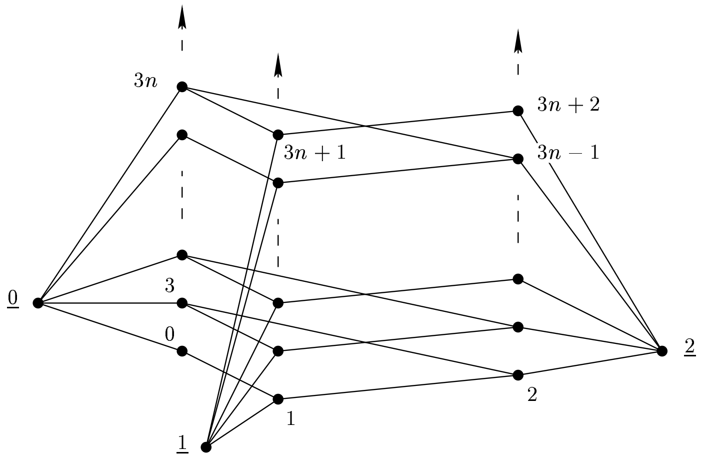
Note that, if in the above lemma, we allow vertices of infinite degree outside F, provided that such vertices are linked to their respective image under r, then the same construction yields the topological compactness of $\mathfrak{G}$. From this follows for instance, that every equivalence relation is a topologically compact graph (take F containing exactly one point in each class ...)

Corollary 3.1 *If the graph $\mathfrak{G}$ has at most two vertices of infinite degree, then it is topologically compact.*

Proof. Suppose that every vertex distinct from a and b has a finite degree and consider a path $a = c_0 \sim_{\mathfrak{G}} c_1 \sim_{\mathfrak{G}} \cdots \sim_{\mathfrak{G}} c_n = b$ of minimal length between a and b ; the expression $r(x) = c_{\min\{d_{\mathfrak{G}}(a,x), n\}}$ defines a retraction from $\mathfrak{G}$ onto $\mathfrak{G}(\{c_0, \cdots, c_n\})$, where $d_{\mathfrak{G}}$ denotes the distance of the "shortest path" in $\mathfrak{G}$. □

A graph with exactly 3 vertices of infinite degree may not even be atomic compact, as shown by the next example ; note however that it is $[\omega^-]$-atomic compact, according to the results of the following section.

$\star$**Example 3.** Consider the graph $\mathfrak{C}_{\mathbb{N}}$ with vertex set the disjoint union of $\mathbb{N}$ and $\mathbb{Z}_{/3\mathbb{Z}}$ (respectively the set of non-negative integers and the set of integers modulo 3), and for which any two distinct vertices are linked if and only if one is an integer n and the other is either its successor $n+1$ in $\mathbb{N}$ or its class $\underline{n}$ in $\mathbb{Z}_{/3\mathbb{Z}}$.



Consider the system $\Phi(x_k : k \in \mathbb{N}) = \{x_k \sim x_{k+1} : k \in \mathbb{N}\} \cup \{x_k \sim -\underline{k} : k \in \mathbb{N}\}$. This system can be represented in the following way

$$\left\{ \begin{array}{ccc} & \vdots & \\ -\underline{k+1} \sim x_{k+1} & \sim & x_k \sim -\underline{k} \\ & \vdots & \end{array} \right. \qquad (\mathcal{F}_k)$$

Now, if $(a_k : k \in \mathbb{N})$ is a solution of $\mathcal{F}_k$ in $\mathfrak{C}_{\mathbb{N}}$, then a_{k+1} and a_k must be two elements of $\mathbb{N}$ such that $a_{k+1} = a_k - 1$. Then any solution of Φ in $\mathfrak{C}_{\mathbb{N}}$ would be a descending sequence in $\mathbb{N}$. Thus Φ is not solvable ; it is however finitely solvable (take a_0 large enough in $\underline{0}$ ...) $\star$

3.2 Almost locally finite graphs.

3.2.1 Reduced atomic systems.

Let us describe a process for reducing the finitely solvable atomic systems of a structure $\mathfrak{M}$. That will enable us to consider only certain kinds of "maximal" systems with no "non-trivial" equality relation.

Given a system $\Phi(X)$, maximal for inclusion with respect to the property of being a finitely solvable atomic system of $\mathfrak{M} = (M, ...)$ with unknowns in X, consider an elementary extension $\mathfrak{N} = (N, ...)$ of $\mathfrak{M}$ in which Φ has a solution $(c_x : x \in X)$. Consider the set of atomic $\mathcal{L}_N$-sentences true in $\mathfrak{N}$ as a system $\widetilde{\Phi}(Y)$ with unknown set $Y = N - M$. Note, by maximality of Φ, that $\widetilde{\Phi}(Y)$ is nothing else than $\Phi[c_x : x \in X]$, which itself is a maximal atomic finitely solvable system of $\mathfrak{M}$ that contains no $u \equiv v$ for distinct u and v in $M \cup Y$. Call any such $\widetilde{\Phi}(Y)$ a *reduced* atomic system of $\mathfrak{M}$. Now observe that for every subset Ψ of Φ, any solution $(e_y : y \in Y)$ of $\Psi[c_x : x \in X]$ in an extension of $\mathfrak{M}$ yields a solution $(e_{c_x} : x \in X)$ of

Ψ in the same extension. Hence, one can consider only reduced atomic systems, in checking the atomic compactness of $\mathfrak{M}$ (and even its $[\kappa]$ or $[\kappa^-]$-atomic compactness, as the cardinality of Y does not exceed the cardinality of X).

Lemma 3.2 *Given a reduced atomic system $\Phi(X)$ of $\mathfrak{M}$, for every x in X, and $a_1 \cdots, a_n$ in M, there is a finite subset Ψ of Φ such that no solution of Ψ in $\mathfrak{M}$ maps x to any of $a_1 \cdots, a_n$.*

Proof. As Φ is maximal finitely solvable and contains no $x \sim a$, there are finite subsets Ψ_k of Φ, $k \in \{1, \cdots, n\}$, such that no $\Psi_k \cup \{x \equiv a_k\}$ is solvable ; consider then $\Psi = \Psi_1 \cup \cdots \cup \Psi_n$.□

Corollary 3.2 *Given a reduced atomic system Φ and a vertex a of a graph $\mathfrak{G}$, if for some x in X, $x \sim a$ is in Φ, then a has an infinite degree.*

Proof. If $\mathcal{G}_a = \{a_1 \cdots, a_n\}$, then the above lemma gives a finite subset Ψ of Φ such that $\Psi \cup \{x \sim a\}$ is not solvable ; hence, as it is finitely solvable, Φ cannot contain $x \sim a$. □

Corollary 3.3 *Almost locally finite graphs are $[\omega^-]$-atomic compact.*

Proof. According to the corollary above, any reduced atomic system $\Phi(x_1 \cdots, x_n)$ of such a graph is finite, hence solvable. □

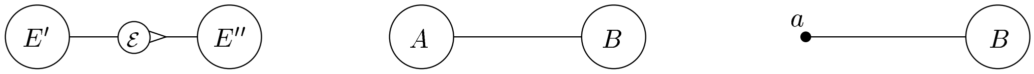
4 $[n]$-atomic compact graphs.

Most of the graphs that we give as examples share some common features. They are built according to patterns that we describe before proceeding further.

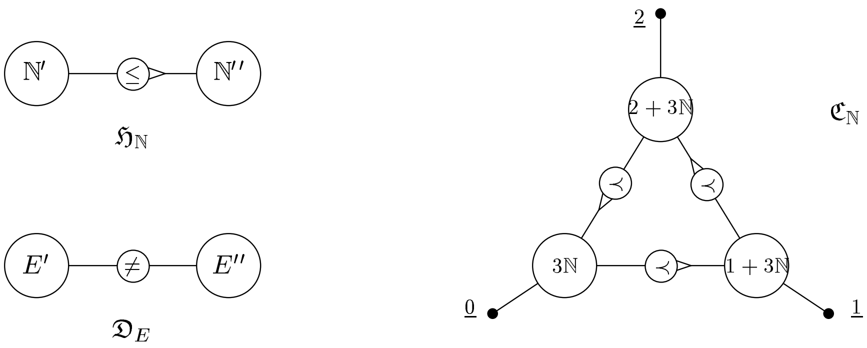
4.1 Conventions.

Given copies E', E_1, E'_1, ... of some set E and an element a of E, we shall denote by a', a_1, a'_1, ... the elements corresponding to a in each of these copies.

We shall define graphs by diagrams from which one should be able to recover all the information required. Diagrams are made up of blocks interconnected by labelled lines. To each block is assigned a set of vertices ; the names of these vertices shall be clear whenever needed (e.g. $\mathbb{N}'_2 = \{0'_2, 1'_2, \cdots\}$). Each vertex has a loop (unless specified) and vertices in distinct blocks are linked according to the label of the line between these blocks. The generic primitive patterns correspond to the diagrams below. In the first pattern, E' and E'' are two disjoint subsets of copies of a set E and $\mathcal{E}$ is some subset of $E \times E$; an element a' of E' and an element b'' of E'' are linked when $(a, b) \in \mathcal{E}$. In the second pattern, every element of A is linked to every element of B. In the third one, the vertex a is linked to every element of B.



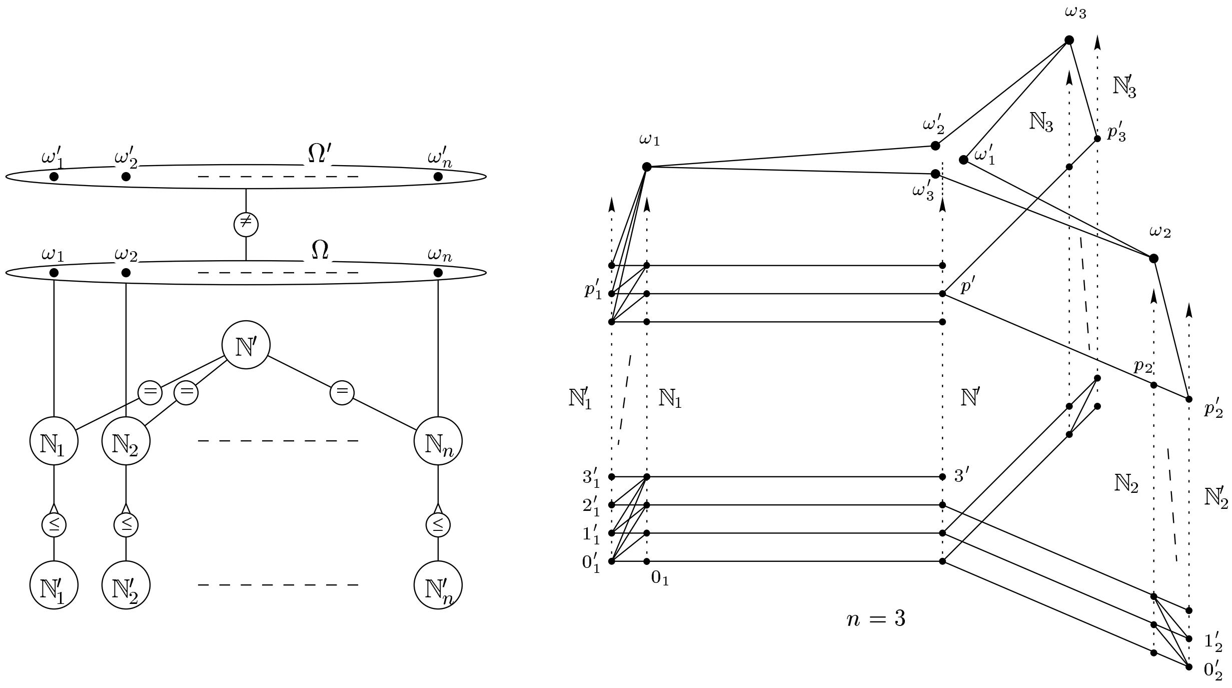
For instance, the three examples above are represented by the following diagrams ($\prec$ denotes the successor relation corresponding to the natural ordering).



4.2 Generalized half-graphs.

Let us show that for any n, there are $[n]$-atomic compact graphs which are not $[n+1]$-atomic compact (for $n = 1$ see the first two examples.)

$\star$**Example 4.** Consider an integer $n \geq 2$ and the graph $\mathfrak{H}_n$ described by the figure below (c.f. example 1).



The following system is easily seen to be finitely solvable but not solvable :

$$\begin{aligned}
&\Phi(x, x_1, \cdots, x_n) = \Phi_1(x_1, \cdots, x_n) \cup \Phi_2(x, x_1, \cdots, x_n) \quad \text{with} \\
&\Phi_1(x_1, \cdots, x_n) = \Big\{ x_k \sim p'_k : k \in \{1, \cdots, n\}, p \in \mathbb{N} \Big\} \quad \text{and} \\
&\Phi_2(x, x_1, \cdots, x_n) = \Big\{ x_k \sim x : k \in \{1, \cdots, n\} \Big\}.
\end{aligned}$$

Any finite subset of Φ has a solution $(p', p_1, \cdots, p_n)$ for p large enough ; but $(\omega_1, \cdots, \omega_n)$ is the only solution of Φ_1 and cannot be extended to any solution of Φ_2.

Now, let us prove that $\mathfrak{H}_n$ is not $[n]$-atomic compact. So let us consider a reduced atomic system $\Phi(x_1, \cdots, x_m)$ with $m \leq n$, and let us show that it is solvable (then a posteriori, m has to be null).

For each $k \in \{1, \cdots, n\}$, define $\widetilde{\mathbb{N}}_k = \mathbb{N}'_k \cup \{\omega_k\}$. Recall (corollary 3.3) that given any $x \in \{x_1, \cdots, x_m\}$, $x \sim d$ can be in Φ only for some d's of infinite degree, hence for d in $\widetilde{\mathbb{N}}_1 \cup \cdots \cup \widetilde{\mathbb{N}}_n$.

First observe, given distinct k and ℓ in $\{1, \cdots, n\}$, that any vertices a and b such that $a \sim_{\mathfrak{H}} \widetilde{\mathbb{N}}_k$, $b \sim_{\mathfrak{H}} \widetilde{\mathbb{N}}_\ell$ and $a \sim_{\mathfrak{H}} b$ must lie in the finite set $\Omega \cup \Omega'$; hence, according to corollary 3.2, for any x and y such that $x \sim_\Phi \widetilde{\mathbb{N}}_k$ and $y \sim_\Phi \widetilde{\mathbb{N}}_\ell$, Φ cannot contain $x \sim y$ and moreover, x and y have to be distinct.

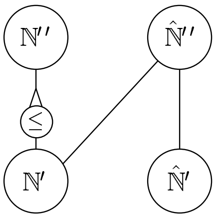
Next note that, as $n \geq m$, if there happens to be some x for which no $x \sim_\Phi \widetilde{\mathbb{N}}_k$ does hold, then there must be some k for which no $x \sim_\Phi \widetilde{\mathbb{N}}_k$ does hold ; in that case, consider k_0 such a k.

Now we can define a family $(d_x : x \in \{x_1, \cdots, x_m\})$ in the following way :
— $d_x = \omega_k$ if and only if $x \sim_\Phi \widetilde{\mathbb{N}}_k$,
— $d_x = \omega_{k_0}$ if and only if for no k does $x \sim_\Phi \widetilde{\mathbb{N}}_k$ hold.
From the four facts below, it follows that $(d_x : x \in \{x_1, \cdots, x_m\})$ is a solution of Φ :
— for each k, $\omega_k \sim_{\mathfrak{H}} \widetilde{\mathbb{N}}_k$,
— for any distinct k and ℓ, $x \sim_\Phi \widetilde{\mathbb{N}}_k$ and $y \sim_\Phi \widetilde{\mathbb{N}}_\ell$ is incompatible with $x \sim_\Phi y$,
— for each k distinct from k_0, $\omega'_{k_0} \sim_{\mathfrak{H}} \omega_k$,
— for no distinct x and y does $x \equiv y$ lie in Φ. ⋆

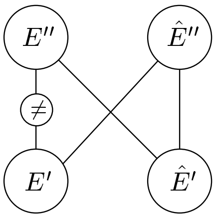
5 Purity.

It is known [Tay] that an atomic compact structure need not be a retract of any topologically compact one. In fact, the first example, given by Taylor, is an atomic compact "antireflexive graph" (locally finite and with finitely many cycles of any given size) which has an infinite chromatic number : For any binary relation $\mathfrak{R}$ with infinite chromatic number, $\check{\mathfrak{R}}$ has loops, hence it is not a pure extension of $\mathfrak{R}$ when this latter is moreover antireflexive. In this section, we focus on the property for a (reflexive) graph, of being a pure restriction of its Stone-Čech compactification. Given a set E, we shall denote by $\hat{E} = \check{E} \backslash E$, the set of non-principal ultrafilters on E. A topologically compact graph $\mathfrak{G}$ is obviously pure in $\check{\mathfrak{G}}$; but a graph $\mathfrak{G}$ may be pure in $\check{\mathfrak{G}}$ without even being [1]-atomic compact :

⋆**Example 5.** The graphs $\check{\mathfrak{H}}$ and $\check{\mathfrak{D}}$ (see the first two examples) correspond to the following diagrams :



$\mathfrak{H}_{\mathbb{N}}$

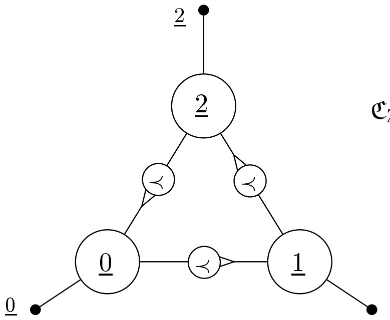


$\mathfrak{D}_E$

⋆

A graph can be atomic compact and pure in its Stone-Čech compactification (i.e. retract of a topologically compact relation) without being topologically compact :

⋆**Example 6.** Consider the graph $\mathfrak{C}_{\mathbb{Z}}$ (c.f. example 3) defined by the diagram below (recall that for each $n \in \mathbb{Z}$, $\underline{n}$ denotes the class of n modulo 3, that is $n + 3\mathbb{Z}$).



$\mathfrak{C}_{\mathbb{Z}}$

The graph $\mathfrak{C}_{\mathbb{Z}}$ satisfying the following two sentences with parameters in $\mathbb{Z}_{/3\mathbb{Z}}$, each of its elementary extensions satisfies them as well.

— $\alpha/$ each vertex is linked to one and only one element of $\mathbb{Z}_{/3\mathbb{Z}}$;

— $\beta/$ given $\underline{n} \in \mathbb{Z}_{/3\mathbb{Z}}$, every vertex linked to $\underline{n}$ and different from $\underline{n}$ has exactly two other neighbours, one linked to $\underline{n-1}$ and the other linked to $\underline{n+1}$.

But any such extension $\mathfrak{G} = (E, \mathcal{G})$ has a retraction on $\mathfrak{C}_{\mathbb{Z}}$ (for each connected component C of $\mathfrak{G}(E \backslash \mathbb{Z}_{/3\mathbb{Z}})$, there is an isomorphism i_C from $\mathfrak{G}(C \cup \mathbb{Z}_{/3\mathbb{Z}})$ onto $\mathfrak{C}_{\mathbb{Z}}$; the union of these i_C is a retraction of $\mathfrak{G}$ onto $\mathfrak{C}_{\mathbb{Z}}$.) Hence $\mathfrak{C}_{\mathbb{Z}}$ is atomic compact, but it is not topologically compact : Suppose that $\mathcal{T}$ is a Hausdorff-compact compatible topology and consider σ the bijection mapping each n to $n+1$ and each $\underline{n}$ to $\underline{n+1}$. For each $\underline{n}$, $\mathcal{G}_{\underline{n}}$ is closed, so for any $\mathcal{U} \in \check{\underline{n}}$, $\lim_{\mathcal{T}} \mathcal{U} \in \mathcal{G}_{\underline{n}} = \underline{n} \cup \{\underline{n}\}$; so, as $\sigma(\mathcal{U}) \in \check{\underline{n+1}}$, $\lim_{\mathcal{T}} \sigma(\mathcal{U}) \in \underline{n+1} \cup \{\underline{n+1}\}$; but moreover, for all $p \in \mathcal{U}$, $\sigma(p) \sim_{\mathfrak{C}_{\mathbb{Z}}} p$, hence $\lim_{\mathcal{T}} \sigma(\mathcal{U}) \sim_{\mathfrak{C}_{\mathbb{Z}}} \lim_{\mathcal{T}} \mathcal{U}$, which now imposes $\lim_{\mathcal{T}} \mathcal{U} \in \underline{n}$ and $\lim_{\mathcal{T}} \sigma(\mathcal{U}) = \sigma(\lim_{\mathcal{T}} \mathcal{U})$. Now take any $\mathcal{U} \in \hat{\underline{0}}$; as for each $p \in \mathbb{Z}$, $\lim_{\mathcal{T}} \sigma^p(\mathcal{U}) = \sigma^p(\lim_{\mathcal{T}} \mathcal{U})$, each element of $\mathbb{Z}$ is the $\mathcal{T}$-limit of a non-principal ultrafilter, hence is a $\mathcal{T}$-accumulation point ; but by the Baire category theorem, a countable Hausdorff-compact space must have at least one, and then infinitely many isolated points.

It is an easy task to check directly that $\mathfrak{C}_{\mathbb{Z}}$ is a retract of $\check{\mathfrak{C}}_{\mathbb{Z}}$ ($\check{\mathfrak{C}}_{\mathbb{Z}}$ satisfies the properties α and β above, as any ultrafilter $\mathcal{U}$ on $\underline{n}$ is linked exactly to itself, to $\underline{n}$, to $\sigma(\mathcal{U})$ and to $\sigma^{-1}(\mathcal{U})$). This is also a consequence of the following general proposition. ⋆

5.1 Graphs with a finite essential degree.

Given $\mathcal{I}$ a set of selfmappings of some set E, let, for every a in F, $\mathcal{I}(a)$ and $\mathcal{I}^{-1}(a)$ denote respectively $\{f(a) : f \in \mathcal{I}\}$ and $\cup\big\{f^{-1}(\{a\}) : f \in \mathcal{I}\big\}$.

Proposition 5.1 *Every graph with a finite essential degree is pure in its Stone-Čech compactification.*

Lemma 5.1 *Consider a graph $\mathfrak{G} = (E, \mathcal{G})$, $\mathcal{U}$ a ultrafilter on E, H an element of $\mathcal{U}$, $\mathcal{I} = \{f_0, \cdots, f_d\}$ a finite set of selfmappings of E and K a subset of E. If $\mathcal{I}(a) = \mathcal{G}_a \cap K$ for all a in H, then $\mathcal{I}(\mathcal{U}) = \overline{\mathcal{G}}_{\mathcal{U}} \cap \check{K}$ (i.e. $\{f_0(\mathcal{U}), \cdots, f_d(\mathcal{U})\}$ is the set of the elements of $\overline{\mathcal{G}}_{\mathcal{U}}$ which contain K.)*

Proof of the lemma. As obviously $\mathcal{I}(\mathcal{U}) \subset \check{K}$, we have to check that for all $\mathcal{V}$ in $\check{K}$, $\mathcal{V} \in \overline{\mathcal{G}}_{\mathcal{U}}$ if and only if $\mathcal{V} \in \mathcal{I}(\mathcal{U})$. So let $\mathcal{V}$ be a ultrafilter on E which contains K :

$$\begin{aligned}
\mathcal{V} \sim_{\check{\mathfrak{G}}} \mathcal{U} \quad &\Leftrightarrow \quad \forall U \in \mathcal{U}, \forall V \in \mathcal{V}, \quad V \cap \mathcal{G}_U \neq \varnothing \\
&\Leftrightarrow \quad \forall U \in \mathcal{U}, \forall V \in \mathcal{V}, \quad (V \cap K) \cap \mathcal{G}_{U \cap H} \neq \varnothing \\
&\Leftrightarrow \quad \forall U \in \mathcal{U}, \forall V \in \mathcal{V}, \quad V \cap (K \cap \mathcal{G}_{U \cap H}) \neq \varnothing \\
&\Leftrightarrow \quad \forall U \in \mathcal{U}, \forall V \in \mathcal{V}, \quad V \cap \{f_0(U \cap H), \cdots, f_d(U \cap H)\} \neq \varnothing \\
&\Leftrightarrow \quad \forall U \in \mathcal{U}, \forall V \in \mathcal{V}, \quad V \cap \{f_0(U), \cdots, f_d(U)\} \neq \varnothing,
\end{aligned}$$

which is easily checked to precisely characterize (the ultrafilter) $\mathcal{V}$ belonging to (the finite set of ultrafilters) $\{f_0(\mathcal{U}), \cdots, f_d(\mathcal{U})\}$. □

Lemma 5.2 *Given $\mathcal{U}_0$ a non-principal ultrafilter on some set F, $\mathcal{I}$ a finite set of selfmappings of F and ℓ an integer, let $\mathcal{I}_\ell$ be the set of all products (under composition) of at most ℓ factors from $\mathcal{I}$. Suppose that, for some integer γ, the cardinalities of the sets $\mathcal{I}^{-1}(a)$ are all bounded above by γ. Then there is an element U of $\mathcal{U}_0$ such that any two elements of $\mathcal{I}_\ell$ either coincide on U or map it on two disjoint sets.*

Proof of the lemma. As $\mathcal{I}_\ell$ satisfies the hypothesis of the lemma (with γ^ℓ replacing γ), the general statement follows from the case $\ell = 1$, which we handle now.

First consider $U_0 \in \mathcal{U}_0$ such that any two elements of $\mathcal{I}$ coincide, either on every point or on no point of U_0. Let us define two sequences $(U_0, U_1, \cdots)$ and $(V_0, V_1, \cdots)$ satisfying, for each integer p :
— U_0 is the disjoint union $V_0 \sqcup \cdots \sqcup V_{p-1} \sqcup U_p$,
— any two elements of $\mathcal{I}$ either coincide on V_p or map it on two disjoint sets,
— for each b in U_p and $q < p$, there is an a in V_q such that $\mathcal{I}(b) \cap \mathcal{I}(a) \neq \varnothing$ (i.e. $a \in \mathcal{I}^{-1}(b)$).

Before describing the construction of these sequences, let us conclude : The set $U_{1+\gamma\times\mathrm{card}(\mathcal{I})}$ is empty, as for any p and any element b of U_p, $\mathcal{I}^{-1}(b)$ must meet each of the pairwise disjoint $V_0, \cdots, V_{p-1}$; hence one V_p among $V_0, \cdots, V_{\gamma\times\mathrm{card}(\mathcal{I})}$ suits for U. (Remark : each element of $\mathcal{I}$ is one to one on U.)

Let us now define our sequences by induction. Given $(U_0, \cdots, U_p)$ and $(V_0, \cdots, V_{p-1})$, define V_p and U_{p+1} in the following way : The set U_p is well ordered and its elements are erased at successive stages indexed by ordinals. At the α-th stage, consider a_α, the smallest element of U_p which is bigger than the previous a_β's and has not been erased so far (if there is any) ;

then, among the following elements of U_p, erase all the b's for which $\mathcal{I}(b) \cap \mathcal{I}(a_\alpha) \neq \varnothing$. When the process stops (i.e. when each element of U_p, either has been erased or is an a_α), put the a_α's in V_p and the erased elements in U_{p+1}. □

Proof of the proposition. Let $c_1, \cdots, c_e$ be the vertices of infinite degree of $\mathfrak{G}$ and F the set $E \backslash \{c_1, \cdots, c_e\}$. Consider $\mathcal{I}$ any finite set of selfmappings of F such that, for every $a \in F$, $\mathcal{I}(a) = \mathcal{G}_a \cap F$; from lemma 5.2 follows that, for each $\mathcal{U}$ in $\hat{E}$, $\overline{\mathcal{G}}_\mathcal{U} \cap \hat{E} = \mathcal{I}(\mathcal{U})$ (take H and K both equal to F.)
Now consider $A \subset E$ and $B = \{\mathcal{U}_0, \cdots, \mathcal{U}_q\} \subset \hat{E}$, two finite sets. We must prove that there is a family $(b_\mathcal{U})_{\mathcal{U} \in B}$ in E^B such that,

— for any $\mathcal{U}$ and $\mathcal{V}$ in B, $\mathcal{U} \sim_{\check{\mathfrak{G}}} \mathcal{V} \Rightarrow b_\mathcal{U} \sim_\mathfrak{G} b_\mathcal{V}$,

— for any a in A and $\mathcal{U}$ in B, $a \sim_{\check{\mathfrak{G}}} \mathcal{U} \Rightarrow a \sim_\mathfrak{G} b_\mathcal{U}$.

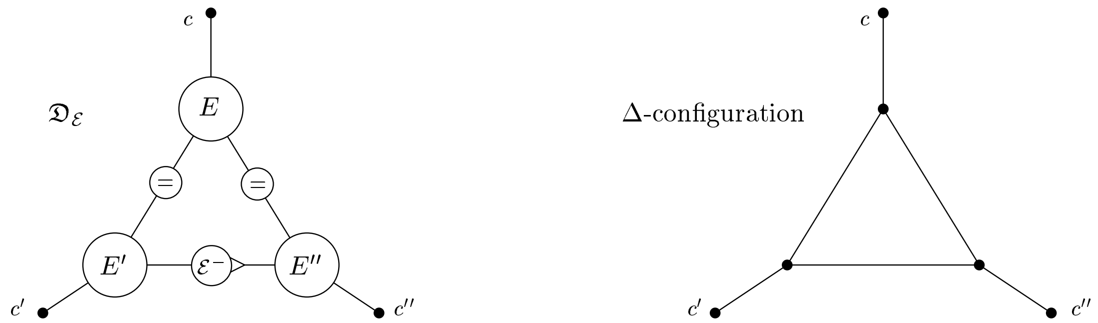
We can easily reduce to the case where $\check{\mathfrak{G}}(B)$ is connected ; then for each $\mathcal{U} \in B$, consider a $h_\mathcal{U} \in \mathcal{I}_q$ such that $\mathcal{U} = h_\mathcal{U}(\mathcal{U}_0)$. Lemma 5.2, invoked with $\ell = q+1$, provides us with some U that we restrict in such a way that, for each $a \in A$ and $\mathcal{U} \in B$, if $\mathcal{G}_a \in \mathcal{U}$ (i.e. $a \sim_{\check{\mathfrak{G}}} \mathcal{U}$), then $h_\mathcal{U}(U) \subset \mathcal{G}_a$.

We claim that for any a in U, the family $(h_\mathcal{U}(a))_{\mathcal{U} \in B}$ suits : For any $\mathcal{U}$ and $\mathcal{V}$ in B, if $\mathcal{U} \sim_{\check{\mathfrak{G}}} \mathcal{V}$, then there is an h in $\mathcal{I}$ for which $\mathcal{U} = h(\mathcal{V})$, i.e. for which $h_\mathcal{U}(\mathcal{U}_0) = h \circ h_\mathcal{V}(\mathcal{U}_0)$, hence $h_\mathcal{U}(U) \cap h \circ h_\mathcal{V}(U) \neq \varnothing$; but as $h_\mathcal{U}$ and $h \circ h_\mathcal{V}$ both lie in $\mathcal{I}_\ell$, they coincide on U and finally $h_\mathcal{U}(a) = h \circ h_\mathcal{V}(a) \sim_\mathfrak{G} h_\mathcal{V}(a)$. □

Note that the conclusion of that proposition may fail if $\mathfrak{G}$ is simply an almost locally finite graph (see example 8 below).

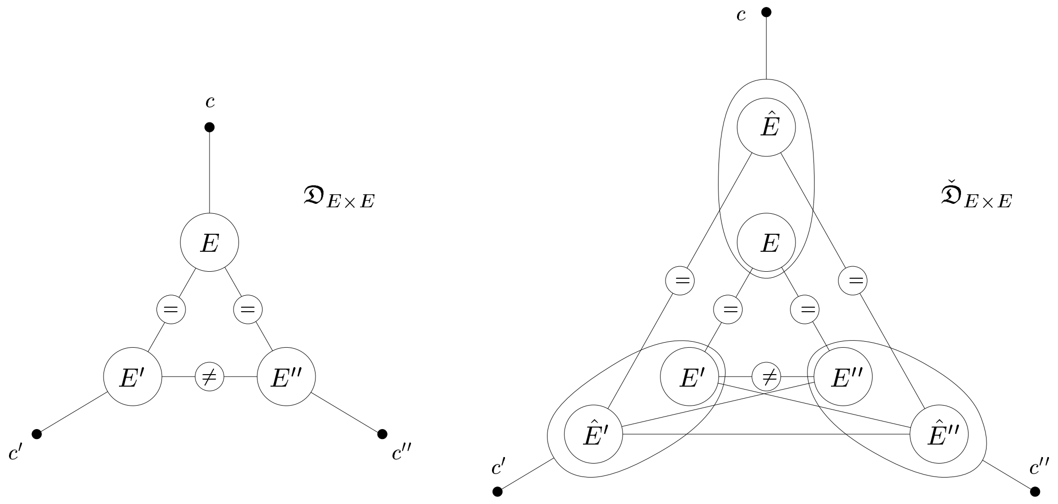
5.2 Atomic compact graphs which are retract of no topologically compact relation.

Given a set E and a subset $\mathcal{E}$ of $E \times E$, consider the graph $\mathfrak{D}_\mathcal{E}$ corresponding to the diagram on the left hand side of the figure below, with $\mathcal{E}^-$ denoting the set $\mathcal{E} \backslash \{(x,x) : x \in E\}$.



The Δ-configuration (right hand side of the above figure) cannot be mapped in $\mathfrak{D}_\mathcal{E}$ (by an homomorphism fixing c, c' and c'') ; however, it may occur in $\check{\mathfrak{D}}_\mathcal{E}$ (as an induced subgraph) : in that case, $\check{\mathfrak{D}}_\mathcal{E}$ is not a pure extension of $\mathfrak{D}_\mathcal{E}$.

⋆**Example 7.** $\mathfrak{D}_{E\times E}$ and $\check{\mathfrak{D}}_{E\times E}$ correspond to the following diagrams.



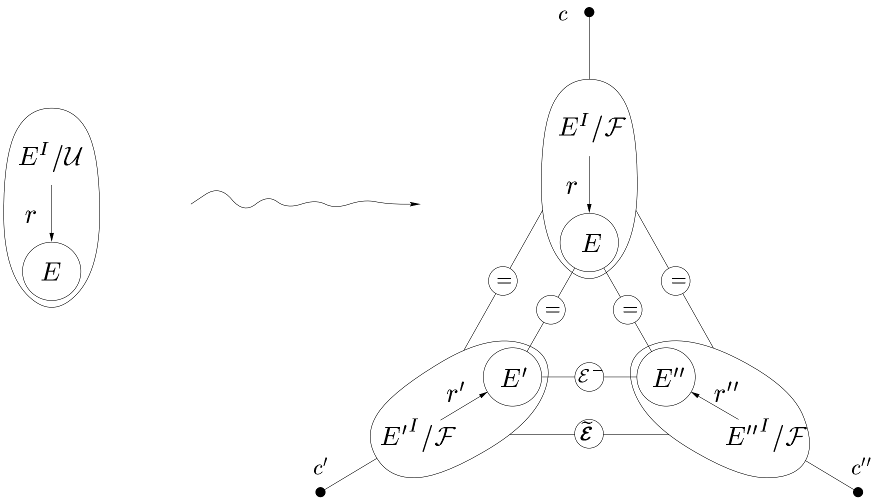
$\star$

The Δ-configuration occurs in $\check{\mathfrak{D}}_{\mathcal{E}}$, every time there is a $\mathcal{U}$ in $\check{E}$ each element of which contains an element of $\mathcal{E}^-$: $\mathcal{U}'$ and $\mathcal{U}''$ are then linked in $\check{\mathfrak{D}}_{\mathcal{E}}$ (besides being both linked to $\mathcal{U}$) ; that happens precisely when the chromatic number χ of the binary relation $(E, \mathcal{E}^-)$ is infinite, see [Tay] (when χ is infinite, the ideal of subsets of E with a finite chromatic number is proper, and when χ is finite, every ultrafilter contains an independent set.)

$\star$**Example 8.** Consider $\big(\mathbb{N}, \{(p,q) : p < q \leq 2p\}\big)$. $\star$

Now note that, when the binary relation $\mathfrak{R} = (E, \mathcal{E}^-)$ is symmetric and atomic compact, $\mathfrak{D}_{\mathcal{E}}$ is atomic compact as well. Indeed, given any ultrafilter $\mathcal{F}$ on some set I, every retraction $r : \mathfrak{R}^I/\mathcal{F} \to \mathfrak{R}$ gives rise to a retraction $s : \mathfrak{D}_{\mathcal{E}}^I/\mathcal{F} \to \mathfrak{D}_{\mathcal{E}}^I$ as follows : Note that $\mathfrak{D}_{\mathcal{E}}^I/\mathcal{F}$ canonically identifies with $\mathfrak{D}_{\widetilde{\boldsymbol{\mathcal{E}}}}$ (where $\mathfrak{R}^I/\mathcal{F} = (E^I/\mathcal{F}, \widetilde{\boldsymbol{\mathcal{E}}})$) and consider for s the extension of r, r' and r'', with $r' : E'^I/\mathcal{F} \to E'$ and $r'' : E''^I/\mathcal{F} \to E''$ corresponding to r (for every $\mathbf{a}$ in E^I, $\widetilde{\mathbf{a}}$, $\widetilde{\mathbf{a}}'$ and $\widetilde{\mathbf{a}}''$ denoting respectively the class of $\mathbf{a}$ in $E^I/\mathcal{F}$ and the elements corresponding to $\widetilde{\mathbf{a}}$ in $E'^I/\mathcal{F}$ and $E''^I/\mathcal{F}$, $r'(\widetilde{\mathbf{a}}') = \big(r(\widetilde{\mathbf{a}})\big)'$ and $r''(\widetilde{\mathbf{a}}'') = \big(r(\widetilde{\mathbf{a}})\big)''$.) — see the figure below— s is a retraction as :

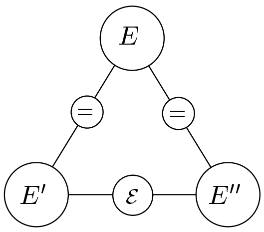
$$\left\{\begin{array}{llll}
\widetilde{\mathbf{a}} \sim_{\mathfrak{D}_{\mathcal{E}}^I/\mathcal{F}} \widetilde{\mathbf{b}}' & \Leftrightarrow \widetilde{\mathbf{a}} = \widetilde{\mathbf{b}} \Rightarrow r(\widetilde{\mathbf{a}}) = r(\widetilde{\mathbf{b}}) \Leftrightarrow & r(\widetilde{\mathbf{a}}) \sim_{\mathfrak{D}_{\mathcal{E}}} r'(\widetilde{\mathbf{b}}') & \\
\widetilde{\mathbf{a}} \sim_{\mathfrak{D}_{\mathcal{E}}^I/\mathcal{F}} \widetilde{\mathbf{b}}'' & \qquad\cdots \Rightarrow \cdots & r(\widetilde{\mathbf{a}}) \sim_{\mathfrak{D}_{\mathcal{E}}} r''(\widetilde{\mathbf{b}}'') & \\
\widetilde{\mathbf{a}}' \sim_{\mathfrak{D}_{\mathcal{E}}^I/\mathcal{F}} \widetilde{\mathbf{b}}'' & \Leftrightarrow \widetilde{\mathbf{a}} \sim_{\mathfrak{R}^I/\mathcal{F}} \widetilde{\mathbf{b}} \Rightarrow r(\widetilde{\mathbf{a}}) \sim_{\mathfrak{R}} r(\widetilde{\mathbf{b}}) \Leftrightarrow & \underbrace{\big(r(\widetilde{\mathbf{a}})\big)'}_{r'(\widetilde{\mathbf{a}}')} \sim_{\mathfrak{D}_{\mathcal{E}}} \underbrace{\big(r(\widetilde{\mathbf{b}})\big)''}_{r''(\widetilde{\mathbf{b}}'')} &
\end{array}\right.$$



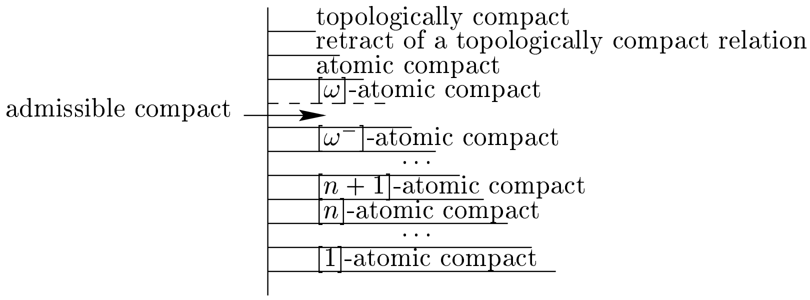
In particular, the atomic compact "antireflexive graphs", retract of no topologically compact binary relation given in [Tay] have a finite chromatic number, and then yield atomic compact (reflexive) graphs which are retract of no topologically compact binary relation. Now, we can state :

Proposition 5.2 *There exists atomic compact (reflexive) graphs which are retract of no topologically compact relation and have a finite chromatic number.* □

Remark. Given $(E, \mathcal{E})$ an atomic compact antireflexive symmetric binary relation with an infinite chromatic number, consider the antireflexive symmetric binary relation $\mathfrak{R}$ corresponding to the following diagram. It is atomic compact, has a finite chromatic number (it is 3-partite) and is retract of no topologically compact relation ($\breve{\mathfrak{R}}$ contains triangles).



6 Admissible compactness.



Up to now, we know that the classes of graphs listed on the right hand side of the above diagram are pairwise distinct ; but we shall need to distinguish among the $[\omega^-]$-atomic compact ones which are not $[\omega]$-atomic compact : Given a graph $\mathfrak{G}$ and an atomic $[\omega]$-system Φ of $\mathfrak{G}$, let Φ_P denote the set of the elements of Φ which do have a parameter ; Φ is *admissible* if for each finite subset Ψ of Φ_P, $\Phi \backslash \Phi_P \cup \Psi$ is solvable (in $\mathfrak{G}$) ; the graph $\mathfrak{G}$ is *admissible compact* if all its admissible systems are solvable. (In the case of a general structure $\mathfrak{M} = (M, ...)$, Φ_P is the set of formulas $\varphi(x_1, \cdots, x_p, a_1, \cdots, a_q)$ from Φ satisfying : for all $(b_1, \cdots, b_p) \in M^p$, there is $(c_1, \cdots, c_q) \in M^q$ such that $\mathfrak{M} \models \varphi(b_1, \cdots, b_p, c_1, \cdots, c_q)$.)

Our reason for introducing such a distinction is the following : Given a cardinal κ, a structure $\mathfrak{M} = (M, ...)$ has the *κ-projection property* (κ-PP for short) if the only retractions of $\mathfrak{M}^\kappa$ on $\mathfrak{M}$ (diagonally embedded) are the projections. (Note that, being precisely retract of its ultrapowers, an atomic compact structure has the κ-PP for no infinite κ.) In [Del], we showed that a structure with the finite projection property (n-PP for every integer n) has the ω-PP provided it is not admissible compact and satisfies some simple additional neccessary conditions ; for instance, a graph has the ω-PP provided it has the 2-PP, has a finite diameter and is not admissible compact.

So far, all our examples of non ω-atomic compact graphs possess non-solvable admissible systems ; our question is then :

Question. Is every admissible compact graph $[\omega]$-atomic compact as well ?

In case of a positive answer, a countable graph would have the ω-PP if and only if it has the 2-PP, has a finite diameter and is not atomic compact.

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