Excellent algebraic groups I

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Abstract: The notion of an excellent quadratic form has been introduced by M. Knebusch [10, 11]. It has an obvious analogue for a semisimple linear algebraic group G over an arbitrary field k: the group G is excellent if, for any field extension K of k, the anisotropic kernel of the K-group $G \times_k K$ can be defined by a k-group. The aim of this paper is to investigate excellence properties of special linear and orthogonal groups.

Introduction

Not much is known about anisotropic semisimple linear algebraic groups over arbitrary fields. In order to reduce this gap of knowledge we study an important class of semisimple groups which we call *excellent* algebraic groups.

If k is any field, a semisimple k-group G is *excellent*, if, for any field extension K of k, there exists an algebraic group H over k such that $H \times_k K$ is the anisotropic kernel of the K-group $G \times_k K$.

A corresponding notion for quadratic forms was introduced by M. Knebusch [11, 7.14, p. 6]. A regular quadratic form q over k is *excellent* if, for any field extension K of k, there exists a form η over k such that $\eta \otimes_k K$ is the anisotropic kernel of the quadratic form $q_K := q \otimes_k K$.

The notion of excellence also has an obvious analogue for an Azumaya algebra A over k (cf. Def. 1.1 below). It is proved here (Th. 1.10 and Th. 1.6) that the following three statements are equivalent: i) *A is excellent.* ii) *The special linear group $G = \mathrm{SL}_1(A)$ is excellent.* iii) *The index of A is squarefree.* We like to emphasize that the squarefreeness of the index is a crucial condition for several algebraic properties of Azumaya algebras related to rationality questions discussed in the past and recently, cf. [23, 15, 19, 20, 13, 4].

Using the index reduction formula of Schofield and van den Bergh [17, Th. 1.3], we obtain in Cor. 1.7 that an Azumaya algebra is excellent if and only if its index and its exponent coincide over every field extension of k.

In section 2, the notion of excellence for the special orthogonal group of a regular quadratic form over a field k of characteristic $\neq 2$ is discussed. We prove in Th. 2.2, that the projective quadrics of two regular forms over k are isomorphic if and only if their corresponding special orthogonal groups are isomorphic as k-groups. This and the generic splitting theory [8] yield in Th. 2.5 a necessary and sufficient criterion for the special orthogonal group of a regular form q to be excellent. It shows that the class of quadratic forms which give rise to excellent groups is bigger than the class of excellent forms in the sense of Knebusch. There are non excellent quadratic forms which have excellent orthogonal groups. For example, the so called “good forms” as introduced by Fitzgerald [5] have an excellent special orthogonal group, but are not always excellent (cf. Ex. 2.7 below). On the other hand, if q is excellent, then G also is excellent. It would be interesting to get an overview about the class of quadratic forms described in Th. 2.5.

Knebusch [10] introduced a generic splitting tower $k = F_0 \subset F_1 \subset \ldots \subset F_h$ of a non-split quadratic form q over k and proved in [11, Th. 7.14], that an anisotropic quadratic form q over k is excellent if and only if there is a quadratic form η_i over k such that $\eta_i \otimes_k F_i$ is the anisotropic kernel of $q \otimes_k F_i$ for all $i = 1, \ldots, h$. We prove analogous statements for the special orthogonal group $G = \mathrm{SO}(q)$ and for the special linear group $G = \mathrm{SL}_1(A)$ in 2.4 and 1.9, 1.10 below.

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1. Azumaya algebras and excellent groups of type ${}^1\mathrm{A}_n$

An Azumaya algebra A over k is a finite dimensional central simple k-algebra, and, by Wedderburn's theorem, there is a unique integer $m \geq 1$ and a central division algebra D over k which is unique up to k-isomorphism such that $A \simeq \mathrm{M}_m(D)$, where $\mathrm{M}_m(D)$ denotes the full matrix ring of m by m matrices over D. The index of A is defined by $\mathrm{ind}\, A = \sqrt{\dim_k D}$, and the exponent $\exp A$ of A is the order of the equivalence class $[A]$ in the Brauer group $\mathrm{Br}(k)$. It is known that $\exp A$ divides $\mathrm{ind}\, A$ and that every prime divisor of $\mathrm{ind}\, A$ divides $\exp A$, cf. [16, 14.4, Prop. b, p. 261].

We write A^m for the m-fold tensor product $A \otimes_k \cdots \otimes_k A$ and set $A^0 = k$. Let A^{op} be the opposite algebra of A so that $[A^{\mathrm{op}}] = [A]^{-1}$ in $\mathrm{Br}(k)$. We write $A \sim A'$ if $[A] = [A']$ and set $K^* = K \setminus \{0\}$ for any field K.

1.1 Definition. An Azumaya algebra A over k is said to be *excellent* if for any field $K \supseteq k$, there is a k-algebra C such that $C \otimes_k K$ is the underlying central division K-algebra of $A \otimes_k K$.

If A is excellent and D is the underlying central division algebra of $A \otimes_k K$ then there is a canonical way to find a k-algebra C such that $D \simeq C \otimes_k K$ as follows from 1.5 and 1.6 below.

1.2 Lemma. *Let A be an Azumaya algebra over k, let K be a field extension of k, and let D be the underlying central division algebra of $A \otimes_k K$. Then the following two statements are equivalent.*

i) *There is a k-algebra C such that $C \otimes_k K \simeq D$.*
ii) *There is an Azumaya algebra B over k such that $B \otimes_k K$ is split and* $\mathrm{ind}\,(A \otimes_k K) = \mathrm{ind}\,(A \otimes_k B)$.

Proof. i) implies ii): Define $B := A^{\mathrm{op}} \otimes_k C$.
Then $B \otimes_k K \simeq A^{\mathrm{op}} \otimes_k C \otimes_k K \simeq (A^{\mathrm{op}} \otimes_k K) \otimes_K D \sim D^{\mathrm{op}} \otimes_K D$ is split over K, and $\mathrm{ind}\,(A \otimes_k B) = \mathrm{ind}\, C = \mathrm{ind}\, D = \mathrm{ind}\,(A \otimes_k K)$.
ii) implies i): Let C denote the underlying central division algebra of $A \otimes_k B$ over k. Then $\mathrm{ind}\, C = \mathrm{ind}\,(A \otimes_k B) = \mathrm{ind}\,(A \otimes_k K) = \mathrm{ind}\, D$. Since $B \otimes_k K$ is split, the three K-algebras $C \otimes_k K$, $A \otimes_k B \otimes_k K$ and $A \otimes_k K$ are equivalent in the Brauer group of K and represented by the central division algebra D, hence $C \otimes_k K \simeq D$. □

Let F be the function field of the Severi-Brauer variety of an Azumaya algebra B over k. In [1, 9.3], Amitsur showed that the kernel of the canonical homomorphism $\mathrm{Br}(k) \to \mathrm{Br}(F)$ is cyclic generated by the class $[B]$. The following lemma is an easy consequence of this result and the index reduction formula of Schofield/van den Bergh [17, Th. 1.3].

1.3 Lemma. *Let A be an Azumaya algebra over k of exponent e and index d with prime power decomposition $d = \prod p^{v_p(d)}$. For any divisor δ of d let $F^{(\delta)}$ denote the function field of the Severi-Brauer variety of A^δ. Then*

$$\exp(A \otimes_k F^{(\delta)}) = \gcd(e, \delta) \quad \textit{and} \quad \operatorname{ind}(A \otimes_k F^{(\delta)}) = \prod_{p|\delta} p^{v_p(d)} .$$

In particular, we have $\operatorname{ind}(A \otimes_k F^{(\delta)}) = \operatorname{ind} A$ *if and only if δ is divisible by any prime divisor of d.*

Proof. Since $F^{(\delta)} \sim A^\delta \otimes_k F^{(\delta)} \sim (A \otimes_k F^{(\delta)})^\delta$ it follows that $r := \exp(A \otimes_k F^{(\delta)})$ divides δ, and hence r divides $\delta' := \gcd(e, \delta)$. Since A^r splits over $F^{(\delta)}$ there is an integer i such that $[A^r] = [A^\delta]^i$, cf. [1, 9.3, p. 34]. Thus $r \equiv \delta i \bmod e$, and δ' divides r. Hence $\exp(A \otimes_k F^{(\delta)}) = \gcd(e, \delta)$.

Since every prime divisor of the index of a central simple algebra is also a divisor of its exponent it follows from the above that p divides $\operatorname{ind}(A \otimes_k F^{(\delta)})$ if and only if it divides $\gcd(e, \delta)$, which holds if and only if p divides δ. Hence we have to show that $p^{v_p(d)} \mid \operatorname{ind}(A \otimes_k F^{(\delta)})$ for every such p. There is a finite separable field extension L of k such that $\operatorname{ind}(A \otimes_k L) = p^{v_p(d)}$ (cf. [16, 14.4 Lemma b, p. 260]). Since k is algebraically closed in $F^{(\delta)}$ we may replace L by k and assume $\operatorname{ind} A = p^{v_p(d)}$. The index reduction formula [17, Th. 1.3, p. 733] implies that for $l = \operatorname{ind} A^\delta$

$$\operatorname{ind}(A \otimes_k F^{(\delta)}) = \min_{i=1,\dots,l}\{\operatorname{ind}(A \otimes_k A^{\delta i})\} = \min_{i=1,\dots,l}\{\operatorname{ind}(A^{1+\delta i})\}.$$

For p dividing δ the latter is equal to $\operatorname{ind} A = p^{v_p(d)}$ for every i, since $1 + \delta i$ is coprime to p for every i. □

1.4 Remark. Lemma 1.3 yields examples of so called *anisotropic splitting* of an Azumaya algebra A: Assume that $\operatorname{ind} A$ is not squarefree. Let δ denote the product of all prime divisors of $\operatorname{ind} A$, and assume that δ is a proper divisor of $\exp A$. Then $\operatorname{ind}(A \otimes_k F^{(\delta)}) = \operatorname{ind} A$ and $\exp(A \otimes_k F^{(\delta)}) = \delta$. Hence field extension by $F^{(\delta)}$ keeps the index invariant, while the exponent goes down.

For an Azumaya algebra A and any divisor δ of $\operatorname{ind} A$, there is a generic partial splitting field F_δ of A such that $\operatorname{ind}(A \otimes_k F_\delta) = \delta$ (cf. [2, Th. 3, p. 104]). Hence scalar extension by F_δ brings the index down to δ, while extension by $F^{(\delta)}$ keeps the index at its highest possible value, as follows from 1.3.

The kernel $\operatorname{Br}(F_\delta/k)$ of the homomorphism $\operatorname{Br}(k) \to \operatorname{Br}(F_\delta)$ is cyclic and generated by the class of A^δ by [2, Th. 7, p. 115].

1.5 Lemma. *Let A_ε, A_δ be central division algebras over k with coprime indices ε, δ, and let $A := A_\varepsilon \otimes_k A_\delta$. For any field extension K of k with $\delta = \operatorname{ind}(A \otimes_k K)$, the underlying central division algebra of $A \otimes_k K$ is isomorphic to $A_\delta \otimes_k K$.*
For $K = F_\delta$, the following uniqueness statement holds: Any k-algebra C satisfying $C \otimes_k F_\delta \simeq A_\delta \otimes_k F_\delta$ is isomorphic to A_δ.

Proof. We have $\delta = \operatorname{ind}(A \otimes_k K) = \operatorname{ind}(A_\varepsilon \otimes_k K) \operatorname{ind}(A_\delta \otimes_k K) = \operatorname{ind}(A_\delta \otimes_k K)$, since ε, δ are coprime, hence $A_\delta \otimes_k K$ is a division algebra. By assumption, $A_\delta \otimes_k C^{\mathrm{op}}$ is split by F_δ. Hence $A_\delta \otimes_k C^{\mathrm{op}} \sim A^{\delta j}$ for some integer j by [2, Th. 7, p. 115]. This yields $A_\delta \sim C \otimes_k A^{\delta j} \sim C \otimes_k A_\varepsilon^{\delta j} \otimes_k A_\delta^{\delta j} \sim C \otimes_k A_\varepsilon^{\delta j}$. Since $\delta = \operatorname{ind} C$ and $\operatorname{ind} A_\varepsilon^{\delta j}$ are coprime, we conclude that $A_\varepsilon^{\delta j}$ splits over k, and $A_\delta \simeq C$. □

1.6 Theorem. *An Azumaya algebra A over k is excellent if and only if its index is squarefree.*

Proof. We may assume that A is a central division algebra.

Let $i := \operatorname{ind} A$ be squarefree, and let K be a field extension of k. Set $\delta = \operatorname{ind}(A \otimes_k K)$ and $\varepsilon = i/\delta$. Then ε and δ are coprime, and $A = A_\varepsilon \otimes_k A_\delta$ (cf. [16, Primary Decomposition Th., p. 260]). Thus we can apply 1.5.

Let now i be not squarefree. Then, by 1.3, there is a field extension $F^{(\delta)}$ of k such that $\exp(A \otimes_k F^{(\delta)})$ is a proper divisor of $\operatorname{ind}(A \otimes_k F^{(\delta)})$. Since it suffices to show that $A \otimes_k F^{(\delta)}$ is not excellent we may replace k by $F^{(\delta)}$, hence $e := \exp A$ is a proper divisor of i. We have $\operatorname{ind}(A \otimes_k F_e) = e$. The kernel $\operatorname{Br}(F_e/k)$ of the homomorphism $\operatorname{Br}(k) \to \operatorname{Br}(F_e)$ is cyclic and generated by the class of A^e by [2, Th. 7, p. 115], hence $\operatorname{Br}(F_e/k)$ is trivial, which means that every Azumaya algebra over k which is split by F_e is trivial. Since $\operatorname{ind}(A \otimes_k F_e) = e \neq i$ it follows from 1.2 that A is not excellent. □

1.7 Corollary. *An Azumaya algebra over k is excellent if and only if over every field extension of k its index and its exponent are equal.*

Proof. Let A be excellent. Since every prime divisor of the index of a central simple algebra divides its exponent it follows from 1.6 that $\operatorname{ind}(A \otimes_k K) = \exp(A \otimes_k K)$ for every field extension K of k.

Let $\delta = \Pi_{p|d}\, p$ with $d = \operatorname{ind} A$. Then $\delta | e | d$. Thus 1.3 implies that $\operatorname{ind}(A \otimes_k F^{(\delta)}) = \operatorname{ind} A$ and $\exp(A \otimes_k F^{(\delta)}) = \delta$. If $\operatorname{ind}(A \otimes_k F^{(\delta)}) = \exp(A \otimes_k F^{(\delta)})$ then $\operatorname{ind} A = \delta$ is squarefree. □

We say that a field $k' \supseteq k$ is a *k-specialization* of a field $L \supseteq k$ if there is a k-place $L \to k' \cup \{\infty\}$. If $A \otimes_k L$ is excellent then so is $A \otimes_k k'$ for every k-specialization k' of L as follows from 1.6 and [8, 4.2, p. 53]. Moreover, there is a divisor δ of $\operatorname{ind} A$ such that F_δ has the following universal property.

1.8 Corollary. *Let A be an Azumaya algebra over k and let $\operatorname{ind} A = p_1^{e_1} \cdots p_r^{e_r}$ be the prime number decomposition of its index. If $\delta = p_1 \cdots p_r$ then $A \otimes_k F_\delta$ is excellent and for any field extension K of k the following statements are equivalent.*

i) *$A \otimes_k K$ is excellent.*

ii) *K is a k-specialization of F_δ.*

Proof. It follows from [8, 4.2, p. 53 and 4.8, p. 55] that $\operatorname{ind}(A \otimes_k F_\delta) = \delta$ and that every field extension K of k with $\operatorname{ind}(A \otimes_k K)$ dividing δ is a k-specialization of F_δ. Conversely, if K is a k-specialization of a field extension L of k then $\operatorname{ind}(A \otimes_k K)$ divides $\operatorname{ind}(A \otimes_k L)$ (cf. [8, 4.2]). Thus 1.8 follows from 1.6. □

1.9 Lemma. *Let A be an Azumaya algebra over k. Then A is excellent if and only if there is a k-algebra C_δ such that $C_\delta \otimes_k F_\delta$ is the underlying central division algebra of $A \otimes_k F_\delta$ for every divisor δ of $\operatorname{ind} A$.*

Proof. If A is excellent then certainly the condition stated is true by Definition 1.1. Suppose K is a field extension of k such that $\delta := \operatorname{ind}(A \otimes_k K)$ is a proper divisor of $\operatorname{ind} A$. We have to show that there is a k-algebra C and an integer $r > 0$ such that $A \otimes_k K \simeq \mathrm{M}_r(C \otimes_k K)$ and $C \otimes_k K$ is a division algebra.

It follows from [8, 4.2 and 4.8] that $\operatorname{ind}(A \otimes_k F_\delta) = \operatorname{ind}(A \otimes_k K)$ and that there is a k-place $\varphi : F_\delta \to K \cup \{\infty\}$. Set $F := F_\delta$. Let R be the valuation ring of φ. Then the canonical homomorphism $\mathrm{Br}(R) \to \mathrm{Br}(F)$, $B \mapsto B \otimes_R F$, is injective, where $\mathrm{Br}(R)$ and $\mathrm{Br}(F)$ are the Brauer groups of R and F respectively [18, Lemma 1.2]. By assumption, there is a k-algebra C and an integer $r > 0$ such that $A \otimes_k F \simeq \mathrm{M}_r(C \otimes_k F)$ and $C \otimes_k F$ is a central division algebra. Since we have canonical F-algebra isomorphisms

$$(A \otimes_k R) \otimes_R F \simeq A \otimes_k F \simeq \mathrm{M}_r(C \otimes_k F) \simeq \mathrm{M}_r(C \otimes_k R \otimes_R F) \simeq \mathrm{M}_r(C \otimes_k R) \otimes_R F\ ,$$

it follows that $A \otimes_k R$ and $\mathrm{M}_r(C \otimes_k R)$ represent the same equivalence class in $\mathrm{Br}(R)$ and that the R-ranks of $A \otimes_k R$ and $\mathrm{M}_r(C \otimes_k R)$ coincide. Since R is a local ring, equivalence in $\mathrm{Br}(R)$ means that there are integers $m, n > 0$ such that

$$(A \otimes_k R) \otimes_R \mathrm{M}_m(R) \simeq \mathrm{M}_r(C \otimes_k R) \otimes_R \mathrm{M}_n(R),$$

and for rank reasons we must have $m = n$. By the cancellation property for Azumaya algebras over semilocal rings it follows $A \otimes_k R \simeq \mathrm{M}_r(C \otimes_k R)$ (cf. [7, Chap. III, Prop. 5.2.3, p. 144]). Viewing K as an R-algebra via φ we obtain K-algebra isomorphisms

$$A \otimes_k K \simeq A \otimes_k R \otimes_R K \simeq \mathrm{M}_r(C \otimes_k R) \otimes_R K \simeq \mathrm{M}_r(C \otimes_k R \otimes_R K) \simeq \mathrm{M}_r(C \otimes_k K).$$

This implies that $C \otimes_k K$ is a division algebra since $\mathrm{ind}\,(A \otimes_k F) = \mathrm{ind}\,(A \otimes_k K)$. □

1.10 Theorem. *An Azumaya algebra A over k is excellent if and only if the special linear group $\mathrm{SL}_1(A)$ is excellent.*

Proof. Let $G = \mathrm{SL}_1(A)$. The absolute Dynkin diagram of G is given by

$$\underset{\alpha_1}{\circ}\!\!-\!\!-\!\!-\!\!\underset{\alpha_2}{\circ}\!\!-\!\!-\ \cdots\ -\!\!-\!\!\underset{\alpha_n}{\circ},$$

where $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a system of simple roots of G. Here n is such that $A \otimes_k k' \simeq \mathrm{M}_{n+1}(k')$ for a splitting field k' of A.

Let K be an arbitrary field extension of k. Then there is a central division algebra D over K such that $A \otimes_k K \simeq \mathrm{M}_{r+1}(D)$ where $d(r+1) = n+1$ for $d = \mathrm{ind}\, D$. The anisotropic kernel $(G_K)_{\mathrm{an}}$ of G_K is given by $(G_K)_{\mathrm{an}} \simeq \mathrm{SL}_1(D) \times_K \ldots \times_K \mathrm{SL}_1(D)$ with $r+1$ factors and it has an absolute Dynkin diagram

$$(*) \qquad \underset{\alpha_1}{\circ}\!\!-\ \cdots\ -\!\!\underset{\alpha_{d-1}}{\circ} \quad \underset{\alpha_{d+1}}{\circ}\!\!-\ \cdots\ -\!\!\underset{\alpha_{2d-1}}{\circ} \quad \cdots \quad \underset{\alpha_{n-d+1}}{\circ}\!\!-\ \cdots\ -\!\!\underset{\alpha_n}{\circ}.$$

If C is a k-algebra such that $D \simeq C \otimes_k K$ then $\mathrm{SL}_1(D) \simeq \mathrm{SL}_1(C) \times_k K$, hence $(G_K)_{\mathrm{an}} \simeq \big(\mathrm{SL}_1(C) \times_k \ldots \times_k \mathrm{SL}_1(C)\big) \times_k K$. It follows that G is excellent if A is so.

Conversely, assume that G is excellent. Let d be a divisor of $\mathrm{ind}\, A$ and $K = F_d$. There is a k-group H such that $(G_K)_{\mathrm{an}} \simeq H \times_k K$. Since G is of inner type, k is algebraically closed in K by [8, 3.16]. This forces that H is of inner type, since each field extension k' of k such that $H \times_k k'$ is of inner type contains a finite Galois extension k_{inn} of k by [8, 3.5 ii)]. It follows that $(G_K)_{\mathrm{an}}$ and H have the same absolute Dynkin diagram $(*)$ whose set of vertices we denote be Δ'. Let $\bar{K}$ be an algebraic closure of K and let $G' \simeq \big(\mathrm{SL}_1(D)\big)_{\bar{K}}$ be the first factor of $H_{\bar{K}}$ corresponding to $\{\alpha_1, \ldots, \alpha_{d-1}\}$. There is a parabolic subgroup P of $H_{\bar{K}}$ of type $\Delta' \setminus \{\alpha_1\}$ and a parabolic subgroup P' of G' of type $\{\alpha_2, \ldots, \alpha_{d-1}\}$ such that $P \cap G' = P'$ and $H_{\bar{K}} = G' \cdot P$. We therefore obtain an isomorphism of $\bar{K}$-varieties $H_{\bar{K}}/P \simeq G'/P'$. Since H is a k-group of inner type there is a unique k-variety V such that $H_{k'}/Q \simeq V \times_k k'$ for any field extension k' of k and any parabolic subgroup Q of $H_{k'}$ of type $\Delta' \setminus \{\alpha_1\}$ (cf. [8, 3.4, 3.7, p. 43f]), in particular, $H_{\bar{K}}/P \simeq V \times_k \bar{K}$. Similarly, since $\mathrm{SL}_1(D)$ is a K-group of inner type, there is a unique K-variety V' such that $G'/P' \simeq V' \times_K \bar{K}$. This implies that there is an isomorphism of K-varieties $V' \simeq V \times_k K$. The K-variety V' is the Severi-Brauer variety of D (cf. [8, 4.7 and 4.10, p. 55f]), hence V is a Severi-Brauer variety over k. It follows that there is an Azumaya algebra C over k such that $D \simeq C \otimes_k K$. Thus A is excellent by 1.9. □

2. Quadratic forms and excellent special orthogonal groups

Let k be an infinite field with char $k \neq 2$ and let (M, q) be a regular quadratic k-space of dimension $m \geq 3$, hence M is an m-dimensional k-vector space and q is a quadratic form with non-degenerate associated bilinear form $(\ ,\ )$ such that $q(x+y) = q(x)+q(y)+(x, y)$ holds for all $x, y \in M$. The discriminant $d(q)$ of (M, q) is defined to be the square class $(-1)^{[m/2]} \det((u_i, u_j)_{i,j=1,\ldots,m}) k^{*2} \in k^*/k^{*2}$. (Here $\{u_1, \ldots, u_m\}$ denotes a k-basis of M.) We have a *Witt decomposition* of M into mutually orthogonal subspaces

$$M = H_1 \perp \ldots \perp H_r \perp M_{\mathrm{an}}$$

where H_i is a hyperbolic plane for $i = 1, \ldots, r$ and $(M_{\mathrm{an}}, q_{\mathrm{an}})$ with $q_{\mathrm{an}} := q|_{M_{\mathrm{an}}}$ is a maximal anisotropic subspace of (M, q) which is unique up to k-isometry and is called an *anisotropic kernel* of the quadratic space (M, q). The integer $r \geq 0$ is the *Witt index* of (M, q), and we have $m = 2r + \dim_k \mathrm{M}_{\mathrm{an}}$ and $d(q) = d(q_{\mathrm{an}})$. We will write $i(q)$ for the Witt index of q and q_K for the form $q \otimes_k K$ in case K is a field extension of k.

Let $G := \mathrm{SO}(q)$ be the special orthogonal group of q. Then the k-rational points $G(k)$ of G are given by $G(k) = \{g \in \mathrm{SL}(M) \mid q(gx) = q(x)\ \forall\, x \in M\}$.

The k-rational points of $G_{\mathrm{an}} \simeq \mathrm{SO}(q_{\mathrm{an}})$ are given by

$$G_{\mathrm{an}}(k) = \{g \in G(k) \mid gM_{\mathrm{an}} \subseteq M_{\mathrm{an}} \text{ and } g|_{\mathbb{H}^r} = \mathrm{id}\}$$

where $\mathbb{H}^r = H_1 \perp \ldots \perp H_r$ is the hyperbolic part of M.

Using the definitions from the introduction we observe that excellent quadratic forms q have excellent orthogonal groups $\mathrm{SO}(q)$. But the converse is not always true. This follows from 2.7 below.

2.1 Lemma. *Let M be a k-vector space, let $q, q' : M \to k$ be regular quadratic forms with bilinear forms $(.,.)$ and $(.,.)'$ respectively. Assume that, for every quadratic field extension L of k and every $x \in M_L$, we have $q_L(x) = 0$ if and only if $q'_L(x) = 0$. Then we have $(x, y) = 0$ if and only if $(x, y)' = 0$ for all $x, y \in M$.*

The lemma says, in other words: If the notion of isotropy for q and q' is the same for all quadratic field extensions of k, then the notion of orthogonality is the same for both forms.

Proof. Let $x, y \in M$. If x, y are linearly dependent, then we may assume that $y = \lambda x$ for some $\lambda \in k$. We then have $(x, y) = 2\lambda q(x)$ and $(x, y)' = 2\lambda q'(x)$ and we are done in this case.

Hence we may assume that $W := kx \oplus ky$ is a plane. Let L be a quadratic extension of k such that $q_L|_{W_L}$ is isotropic. Let $e, f \in W_L$ be a hyperbolic pair for q_L. We then have, by our assumption, the formulas

$q_L(e) = 0 = q'_L(e)$, $q_L(f) = 0 = q'_L(f)$, $(e,f) = 1$, $(e,f)' = \gamma$ for some $\gamma \in L^*$.

If we write $x = \xi_x e + \eta_x f$ and $y = \xi_y e + \eta_y f$ with $\xi_x, \xi_y, \eta_x, \eta_y \in L$ we obtain $(x,y) = (\xi_x\eta_y + \xi_y\eta_x)(e,f) = \gamma^{-1}(\xi_x\eta_y + \xi_y\eta_x)(e,f)' = \gamma^{-1}(x,y)'$ and our claim follows. In fact, the computation shows that γ is necessarily in k^*, but we do not need that here. □

2.2 Theorem. *Let q and q' be regular quadratic forms over k of dimensions ≥ 3. Then the following statements are equivalent.*

i) *$\mathrm{SO}(q)$ and $\mathrm{SO}(q')$ are isomorphic as k-groups.*
ii) *The projective quadrics defined by q and q' are isomorphic as k-varieties.*
iii) *q and q' are similar over k, that is, there is a $\gamma \in k^*$ such that $q \simeq \gamma q'$.*

Proof. i) implies ii): Any k-isomorphism between the semisimple groups $G := \mathrm{SO}(q)$ and $G' := \mathrm{SO}(q')$ preserves the notion of parabolic subgroups and their types. The quadric of q is the *unique* (cf. [8, 3.7, p. 44]) k-variety Q_q such that, for every field extension L of k with q_L isotropic, we have $(Q_q)_L \simeq (\mathrm{SO}_q)_L/P$, where $P \subset G_L$ is the stabilizer of some isotropic line of q_L. The type of the parabolic subgroup P of G_L is defined by this description (cf. [8, §5, p. 56 ff]). If $P' \subset G'$ denotes the image of P under some k-isomorphism $G \to G'$ we obtain for the quadric $Q_{q'}$ of q' the L-isomorphism $(Q_{q'})_L \simeq G'_L/P' \simeq G_L/P$. The uniqueness statement in [8, 3.7, p. 44] asserts that $Q_q \simeq Q_{q'}$ over k.

ii) implies iii): The dimension of the quadric of q is $\dim q - 2$, hence we may assume that q and q' are defined on the same k-vector space M. Since the quadrics of q and q' are k-isomorphic there is an element $u \in \mathrm{GL}_M(k)$ such that $(q' \circ u)_L$ and q_L vanish, for every field extension L of k, on the same points $x \in M_L$.

Let $e_1, \ldots, e_m$ be an orthogonal basis of (M,q). Then, by 2.1, it is an orthogonal basis for (M,q') as well.

Let $\alpha_i := q(e_i)$ and $\alpha'_i := q'(e_i)$ and define $\gamma := \alpha_1/\alpha'_1$. Let i be maximal such that $\gamma \equiv \alpha_i/\alpha'_i \bmod k^{*2}$. If $i = m$ we are done. Assume $i < m$, and let $W := ke_i \perp ke_{i+1}$. The forms q_W and q'_W are either both isotropic or both anisotropic. By the assumption on i we have

$$(*) \qquad d := d(q|_W) = -\alpha_i\alpha_{i+1}k^{*2} \neq -\alpha'_i\alpha'_{i+1}k^{*2} = d(q'|_W) =: d'$$

hence $q|_W, q'|_W$ both have to be anisotropic, and therefore both discriminants are not trivial. But, for $L = k\sqrt{d}$, both forms $(q|_W)_L, (q'|_W)_L$ are isotropic, hence $-\alpha'_i\alpha'_{i+1} \in L^{*2}$ so that $k\sqrt{d'} = L$ and therefore $-\alpha_i\alpha_{i+1} \equiv -\alpha'_i\alpha'_{i+1} \bmod k^{*2}$ contradicting $(*)$. Hence $i = m$ and iii) follows. Trivially iii) implies i). □

Define $n := [m/2]$ where m is the dimension of M. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of vertices of the Dynkin diagram of $G_{\bar{k}}$ where $\bar{k}$ is an algebraic closure of k. Then define for $i \in \{1, \ldots, n-1\}$ the following subset Θ_i of Δ:

$$\Theta_i := \begin{cases} \Delta \setminus \{\alpha_i\} & \text{if } G \text{ is of inner type or } i \leq n-2; \\ \Delta \setminus \{\alpha_{n-1}, \alpha_n\} & \text{if } G \text{ is of outer type and } i = n-1. \end{cases}$$

For $i \in \{1, \ldots, n-1\}$ let $F_i := F_{\Theta_i}$ be the generic Θ_i-splitting field of q as introduced in [8, §§3 and 5]. Then q_{F_i} has Witt index $i(q_{F_i}) \geq i$, and every field extension L of k satisfying $i(q_L) \geq i$ is a k-specialization of F_i. Moreover, if there is a field $K \supset k$ with index $i(q_K) = i$, then $i(q_{F_i}) = i$ by [8, 5.9].

2.3 Theorem. *Let $G = \mathrm{SO}(q)$ and K be a field extension of k. Let $F := F_i$ be a generic Θ_i-splitting field of q such that $i(q_F) = i(q_K) = i$. If $(G_F)_{\mathrm{an}} \simeq H \times_k F$ for some k-group H, then there is an anisotropic quadratic form τ over k, such that $H \simeq \mathrm{SO}(\tau)$ and $(G_K)_{\mathrm{an}} \simeq H \times_k K$.*

Proof. We first show that $H \simeq \mathrm{SO}(\tau)$ for some anisotropic quadratic form τ over k. If $\dim q$ is odd, then this is clearly true since H is of Dynkin type B_n. Let now $\dim q$ be even. Let k_s be the separable closure of k contained in $\bar{k}$ and let $\Gamma = \mathrm{Gal}\,(k_s/k)$ be its Galois group. Then there is a homomorphism $\beta_G : C_s^{*\Gamma} \to \mathrm{Br}(k)$ defined by Tits [22, §3, p. 202ff] where C_s^* is the dual of the center C_s of $\widetilde{G}_{k_s}$, and where $\widetilde{G}$ denotes a simply connected covering of G. We have $\beta_G \times_k F = \beta_H \times_k F$. Since H is of Dynkin type D_n we have $H \simeq \mathrm{SU}(D, h)$ for some division k-algebra D and some hermitian form h over D [21, p. 56f]. It follows from [22, 6.2, p. 212] that $D \otimes_k F$ is trivial in $\mathrm{Br}(F)$. The kernel of $\mathrm{Br}(k) \to \mathrm{Br}(F)$ was computed by Merkurjev and Tignol [14, Cor. 2.11, p. 43f]. From this we obtain that $D = k$, hence $H \simeq \mathrm{SO}(\tau)$ for some anisotropic quadratic form τ over k.

By [8, 5.3, p. 59], there is a k-place $\varphi : F \to K \cup \{\infty\}$. Let $\mathcal{O}$ denote the valuation ring of φ with maximal ideal $\mathfrak{m}$. Since the canonical homomorphism of Witt rings $\mathrm{W}(\mathcal{O}) \to \mathrm{W}(F)$ is injective (cf. [9, p. 283]) it follows that $q_{\mathcal{O}} := q \otimes_k \mathcal{O} \simeq i \times \langle 1, -1 \rangle \perp (q_{\mathcal{O}})_{\mathrm{an}}$, hence $q_K = q_{\mathcal{O}} \otimes_{\mathcal{O}} K \simeq (i \times \langle 1, -1 \rangle \perp (q_{\mathcal{O}})_{\mathrm{an}}) \otimes_{\mathcal{O}} K$. The Witt cancellation theorem implies $(q_K)_{\mathrm{an}} \simeq (q_{\mathcal{O}})_{\mathrm{an}} \otimes_{\mathcal{O}} K$, since $i(q_K) = i(q_F)$.

We are going to show that $\mathrm{SO}((q_{\mathcal{O}})_{\mathrm{an}}) \simeq H \times_k \mathcal{O}$. Then the result will follow since $\mathrm{SO}((q_K)_{\mathrm{an}}) \simeq \mathrm{SO}((q_{\mathcal{O}})_{\mathrm{an}}) \times_{\mathcal{O}} K \simeq (H \times_k \mathcal{O}) \times_{\mathcal{O}} K \simeq H \times_k K$.

Let $V := (M \otimes_k \mathcal{O})_{\mathrm{an}}$. Then $V_F := V \otimes_{\mathcal{O}} F \simeq (M \otimes_k F)_{\mathrm{an}}$. We define

$$X := \{\, x \in V_F \mid (q_F)_{\mathrm{an}}(x) \in \mathcal{O} \,\}$$

and first show $X = V$.

We may choose a suitable orthogonal basis $e_1, \ldots, e_s$ of V such that $(q_F)_{\mathrm{an}}(e_i)$ is a unit in $\mathcal{O}$ for all $i = 1, \ldots, s$ (cf. [12, Prop. 6.4, p. 17]). Thus $X \supseteq V$.

Assume that there is an $x \in X \setminus V$. Choose $x = \sum_{i=1}^{s} \lambda_i e_i$ with $\lambda_i \in F$ such that the number t of λ_i's for which $\lambda_i \notin \mathcal{O}$ is minimal. By our assumption we have $t \geq 1$ and we may assume that $X \ni x = \sum_{i=1}^{t} \lambda_i e_i$ with $\lambda_i \in F \setminus \mathcal{O}$ for all $i = 1, \ldots, t$. Since $\mathcal{O}$ is a valuation ring we can arrange that $\lambda_1 \mathcal{O} \subseteq \lambda_2 \mathcal{O} \subseteq \ldots \subseteq \lambda_t \mathcal{O}$. Thus $\lambda_1/\lambda_i \in \mathcal{O}$ for all $i = 1, \ldots, t$. We have $\lambda_1^{-1} x = e_1 + \sum_{i=2}^{t} \lambda_i \lambda_1^{-1} e_i$ with $\lambda_1^{-1} \in \mathfrak{m}$ and, since $e_1, \ldots, e_s$ is an orthogonal basis, $(q_F)_{\mathrm{an}}(\lambda_1^{-1} x - e_1) = (q_F)_{\mathrm{an}}(\lambda_1^{-1} x) - (q_F)_{\mathrm{an}}(e_1) \in \mathcal{O}$. Thus, by minimality, $\lambda_1^{-1} x - e_1 \in V$ and $\lambda_i \lambda_1^{-1} \in \mathcal{O}$ for all i. We have shown that $\varepsilon_i := \lambda_i \lambda_1^{-1}$ is a unit in $\mathcal{O}$ for all $i = 1, \ldots, r$ and $\lambda_1^{-1} x = \sum_{i=1}^{t} \varepsilon_i e_i$. Since $(q_{\mathcal{O}})_{\mathrm{an}}(\lambda_1^{-1} x) = \lambda_1^{-2} (q_F)_{\mathrm{an}}(x) \in \mathfrak{m}$ we conclude that $\lambda_1^{-1} x$ becomes isotropic in $V \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{m}$. Since $\mathcal{O}/\mathfrak{m} \subseteq K$ we obtain $(q_K)_{\mathrm{an}}(y) = 0$ for some non-zero vector $y \in (M \otimes_k K)_{\mathrm{an}} \simeq V \otimes_{\mathcal{O}} K$ and hence a contradiction. Thus $X = V$.

Let $\sigma : V_F \to V_F$ be an isometry. Then we have $(q_F)_{\mathrm{an}}(\sigma v) = (q_F)_{\mathrm{an}}(v) \in \mathcal{O}$ for all $v \in V$. Since $X = V$ this implies that $\sigma(V) \subseteq V$, hence $\mathrm{SO}((q_{\mathcal{O}})_{\mathrm{an}})(\mathcal{O}) = \mathrm{SO}((q_{\mathcal{O}})_{\mathrm{an}})(F)$. Since $H(k) \subseteq H(\mathcal{O})$ is Zariski-dense in $H(\bar{F})$, cf. [3, 18.3, p. 220], and since $H(\mathcal{O}) \subseteq H(F) \simeq \mathrm{SO}((q_{\mathcal{O}})_{\mathrm{an}})(F) = \mathrm{SO}((q_{\mathcal{O}})_{\mathrm{an}})(\mathcal{O})$, it follows that $\mathrm{SO}((q_{\mathcal{O}})_{\mathrm{an}}) \simeq H \times_k \mathcal{O}$. □

2.4 Corollary. *Let I be the set of all integers $i \in \{1, \ldots, n-1\}$ which occur as Witt index of q_K for some field $K \supset k$. Then $G = \mathrm{SO}(q)$ is excellent if and only if there is a quadratic form τ_i over k such that $(G_{F_i})_{\mathrm{an}} \simeq \mathrm{SO}(\tau_i) \times_k F_i$ for all $i \in I$.* □

2.5 Theorem. *Let $G = \mathrm{SO}(q)$, let I be as in 2.4. Then the following statements are equivalent.*

i) *G is excellent.*

ii) *For every field extension K of k, there is an element $a \in K^*$ and a quadratic form τ over k such that $a(q_K)_{\mathrm{an}} \simeq \tau_K$.*

iii) *For each $i \in I$, there is an element $a \in F_i^*$ and a quadratic form τ over k such that $a(q_{F_i})_{\mathrm{an}} \simeq \tau_{F_i}$.*

If $\dim q$ is odd, then the elements a can be chosen in k^*, hence in this case G is excellent if and only if q is excellent.

Proof. i) implies ii): Let G be excellent and let K be a field extension of k such that $i(q_K) > 0$. By 2.3, there is an anisotropic form τ over k such that $\mathrm{SO}((q_K)_{\mathrm{an}}) \simeq \mathrm{SO}(\tau_K)$. By 2.2, we obtain $a(q_K)_{\mathrm{an}} \simeq \tau_K$ for some $a \in K^*$. This proves ii). Trivially ii) implies iii), and it follows from 2.4 that iii) implies i).

Let now $\dim q$ be odd. Then looking at the discriminants $d(\tau_K) = d(a(q_K)_{\mathrm{an}}) = a\, d(q_K)$ we observe that, up to a square, a must be in k and the last statement follows. □

2.6 Remark. If $\dim q$ is odd, we easily obtain the following uniqueness property: If $H \times_k F_i \simeq (G_{F_i})_{\mathrm{an}} \simeq H' \times_k F_i$ for some k-groups H, H' and some $i \in I$ then $H \simeq H'$.

Proof. By 2.3, there are quadratic forms τ, τ' over k such that $H \simeq \mathrm{SO}(\tau)$ and $H' \simeq \mathrm{SO}(\tau')$. Hence, by 2.2, there are $a, a' \in F_i^*$ such that $a\, \tau_{F_i} \simeq (q_{F_i})_{\mathrm{an}} \simeq a'\, \tau'_{F_i}$. If $\dim q$ is odd we find, by computing discriminants, that, up to squares, a, a' can be chosen in k^*. The collection $(F_j, j \in I)$ yields a generic splitting tower of q as introduced by Knebusch, cf. [8, 5.8, p. 61]. Proposition 7.2 in [11, p. 1] implies $a\tau \simeq a'\tau'$. This forces $H \simeq H'$. □

In [11, Ex. 9.11, p. 24], Knebusch gives an example of an anisotropic non excellent quadratic form q of height 2 with leading form defined over k. For the definition of the height and the leading form of a quadratic form we refer to [10, 5.4, p. 79 and 5.9, p. 82].

2.7 Example. *Let q be anisotropic of height 2 with leading form defined over k. Then the special orthogonal group $\mathrm{SO}(q)$ is excellent. If, in addition, $\dim q$ is a 2-power then q is not excellent.*

Proof. If m is odd then q is excellent [11, p. 27] and thus $\mathrm{SO}(q)$ is excellent. Assume that m is even and q is not excellent. Let $F = k(q)$ denote the function field of q and let τ denote a quadratic form over k such that τ_F is the leading form of q. Then there is an $a \in F^*$ such that $(q_F)_{\mathrm{an}} \simeq a\tau_F$ by the properties of the leading form. Since q has height 2, Th. 2.5 implies that G is excellent. □

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