Bounds on norms of the adjoint matrix*

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February 27, 1997

ABSTRACT

An upper bound on operator norms of the adjoint matrix is presented, and special cases of the l_1, l_2 and l_∞ norms are investigated. The results are then used to obtain lower bounds on the smallest absolute value of an eigenvalue of a nonsingular matrix.

1 Introduction

For a matrix A in the vector space C^{nn} of all $n \times n$ complex matrices we denote by $\lambda_1(A), \ldots, \lambda_n(A)$ the eigenvalues of A ordered in a non-increasing order of their moduli. Clearly, the value of $|\lambda_n(A)|$ can serve as a kind of measure for the nonsingularity of A. In this paper we find lower bounds on $|\lambda_n(A)|$.

For this purpose, we consider an absolute vector norm $||.||$ on C^n. For a matrix A in C^{nn} we denote by $||A||$ the corresponding operator norm. Also, we denote by $adj(A)$ the (classical) adjoint matrix of A, that is, the transposed matrix of cofactors. Note that the term *adjugate* is sometimes used instead of *adjoint* to avoid confusion with the Hermitian adjoint A^*.

*The research was started while the first author visited at the Technion in 1995 and completed in a visit of the second author at Universität Bielefeld in 1997. The research was partly supported by SFB 343 "Diskrete Strukturen in der Mathematik", Bielefeld.

We define a constant $\kappa(||.||)$ associated with the operator norm, and we present an upper bound on $||adj(A)||$ in terms of the constant $\kappa(||.||)$ and the vector norms of the rows of A. We apply this result to the norms l_1, l_2 and l_∞, and we use the resulting inequalities to derive our lower bounds on $|\lambda_n(A)|$.

2 Upper bounds on norms of the adjoint matrix

Let $||.||$ be an absolute vector norm on C^n, and for a matrix A in C^{nn} let $||A||$ be the corresponding operator norm. We define

$$\kappa(||.||) = max\{||adj(B)|| : B \in C^{nn},\ ||row_i(B)|| = 1,\ i = 1, \ldots, n\},$$

where $row_i(B)$ denotes the (vector in C^n which is the) transpose of the ith row of B.

The following theorem is the main tool from which we derive our results.

Theorem 2.1 *Let $A \in C^{nn}$. We have*

$$||adj(A)|| \leq \kappa(||.||) \max_k \prod_{\substack{j=1 \\ j \neq k}}^{n} ||row_j(A)||. \tag{2.2}$$

Proof. Assume first that A has no zero rows. Note that for every nonsingular matrix R we have

$$adj(R^{-1}A)adj(R) = adj(A).$$

In particular, if we choose the matrix R to be the diagonal matrix $diag(r_i)$ where $r_i = ||row_i(A)||$, $i = 1, \ldots, n$, then

$$||adj(R)|| = ||r_1 r_2 \cdots r_n diag(r_i^{-1})|| = \max_k \prod_{\substack{j=1 \\ j \neq k}}^{n} ||row_j(A)||,$$

and since $R^{-1}A$ has norm 1 rows, it now follows that

$$||adj(A)|| \leq ||adj(R^{-1}A)||\, ||adj(R)|| \leq \kappa(||.||) \max_k \prod_{\substack{j=1 \\ j \neq k}}^{n} ||row_j(A)||.$$

If A has zero rows then we apply the above procedure to the matrix $A + \epsilon I$ for ϵ sufficently small, and then use continuity arguments to prove our assertion. $\square$

In the rest of this section we evaluate $\kappa(||.||_p)$ for the l_p norm $||.||_p$, $p = 1, 2, \infty$.

Proposition 2.3 *For every $i \in \{1, \dots, n\}$ we have*

$$||row_i(adj(A)^T)||_1 \le \max_k \prod_{\substack{j=1\\ j\neq k}}^{n} ||row_j(A)||_1. \tag{2.4}$$

Furthermore, if A has at most one zero row then the following are equivalent:
(i) *Equality holds in (2.4).*
(ii) *We have*

$$||row_i(A)||_1 = \min_k ||row_k(A)||_1, \tag{2.5}$$

and the rows of A indexed by $\{1, \dots, n\} \setminus \{i\}$ have disjoint supports.

Proof. For $i, k \in \{1, \dots, n\}$ we denote by $A(i|k)$ the submatrix of A obtained by eliminating the $i-th$ row and the $k-th$ column from A. Denote by S_n the set of all permutations on $\{1, \dots, n\}$. We have

$$||row_i(adj(A)^T)||_1 = \sum_{k=1}^{n} |adj(A)_{ki}| = \sum_{k=1}^{n} |det(A(i|k))| \le \sum_{k=1}^{n} \sum_{\substack{\sigma\in S_n\\ \sigma(i)=k}} \prod_{\substack{j=1\\ j\neq i}}^{n} |a_{j\sigma(j)}| \tag{2.6}$$

$$= \sum_{\sigma\in S_n} \prod_{\substack{j=1\\ j\neq i}}^{n} |a_{j\sigma(j)}| \le \prod_{\substack{j=1\\ j\neq i}}^{n} ||row_j(A)||_1 \le \max_k \prod_{\substack{j=1\\ j\neq k}}^{n} ||row_j(A)||_1,$$

proving (2.4).

We now prove the equality case under the assumption that A has at most one zero row.

(i)$\Longrightarrow$(ii) Since A has at most one zero row it follows that equality holds in the last inequality of (2.6) if and only if (2.5) holds. If, indeed, (2.5) holds then the rows of A indexed by $\{1, \dots, n\} \setminus \{i\}$ are non-zero, and equality holds in the second last inequality of (2.6) if and only if the supports of these rows are disjoint.

(ii)$\Longrightarrow$(i) Observe that it follows from (ii) that we have equalities in the last two inequalities of (2.6). To prove that equality holds also in the first inequality of (2.6) note that, since A has at most one zero row, it also follows from (ii) that the rows of A indexed by $\{1, \dots, n\} \setminus \{i\}$ are non-zero. Since the supports of these $n-1$ rows are disjoint, it follows that $n-2$ of these rows have exactly one non-zero element (in different columns), and one of the rows have at most two non-zero elements (in other columns). It is easy to verify that in such a case, at most two of the submatrices $A(i|k)$ have non-zero diagonals, where each has just one such a diagonal. Therefore, equality holds in the first inequality of (2.6). □

Definition 2.7 A matrix A is said to be a *generalized diagonal* matrix if $A = PD$, where P is a permutation matrix and D is a diagonal matrix.

Theorem 2.8 *We have* $\kappa(||.||_1) = 1$.

Proof. Since
$$||adj(A)||_1 = \max_k ||row_k(adj(A)^T)||_1,$$
it follows immediately from Proposition 2.3 that

$$||adj(A)||_1 \leq \max_k \prod_{\substack{j=1 \\ j\neq k}}^{n} ||row_j(A)||_1, \tag{2.9}$$

implying that $\kappa(||.||_1) \leq 1$. By Proposition 2.3, a generalized diagonal matrix A satisfies

$$||adj(A)||_1 = \max_k ||row_k(adj(A)^T)||_1 = \max_k \prod_{\substack{j=1 \\ j\neq k}}^{n} ||row_j(A)||_1,$$

implying that $\kappa(||.||_1) = 1$. $\square$

Theorem 2.10 *We have* $\kappa(||.||_2) = \left(1 + \frac{1}{n-1}\right)^{\frac{n-1}{2}}$.

Proof. Let $A \in C^{nn}$ be such that $||row_i(A)||_2 = 1, \ \ i = 1, \ldots, n$. The matrix $B = AA^*$ is a positive semidefinite matrix with diagonal entries equal to 1. Let $\rho_1 \geq \rho_2 \geq \ldots \rho_n \geq 0$ be the eigenvalues of B. Note that $\rho_1 + \ldots + \rho_n = trace(B) = n$. It now follows that

$$||adj(B)||_2 = (||adj(A)||_2)^2 = \rho_1\rho_2\cdots\rho_{n-1} \leq \left(\frac{\rho_1 + \ldots + \rho_{n-1}}{n-1}\right)^{n-1}$$

$$\leq \left(\frac{n}{n-1}\right)^{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

We thus have

$$\kappa(||.||_2) \leq \left(1 + \frac{1}{n-1}\right)^{\frac{n-1}{2}}. \tag{2.11}$$

To prove that equality holds in (2.11), consider the positive semidefinite matrix $B = \left(1 + \frac{1}{n-1}\right) I - \frac{1}{n-1}J$, where I is the $n \times n$ identity matrix and J is the $n \times n$ matrix all of whose entries are equal to 1. Note that the eigenvalues of B are $1 + \frac{1}{n-1}$ with multiplicity $n - 1$ and 0 with multiplicity 1. We have

$$||adj(B)||_2 = \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

Now, let A be the positive semidefinite matrix such that $B = A^2$. Since the diagonal entries of B are all equal to 1, it follows that $||row_i(A)||_2 = 1, \ \ i = 1, \ldots, n$. Also,

$$||adj(A)||_2 = \sqrt{||adj(B)||_2} = \left(1 + \frac{1}{n-1}\right)^{\frac{n-1}{2}},$$

proving that equality holds in (2.11). $\square$

We remark that the inequality (2.11) is proven in [5, Theorem 4] as well as in [1, Lemma 2]. Our proof of this inequality is essentially the same as in [5] and [1]. The equality case is, however, not handled in these two references.

We now show that in the case of l_2 norm we cannot have equality in (2.2) for nonsingular matrices.

Proposition 2.12 *Let $A \in C^{nn}$ be a nonsingular matrix. Then*

$$||adj(A)||_2 < \left(1 + \frac{1}{n-1}\right)^{\frac{n-1}{2}} \max_k \prod_{\substack{j=1 \\ j \neq k}}^{n} ||row_j(A)||_2.$$

Proof. Since A is a nonsingular matrix, it has no zero rows. Let R be the diagonal matrix $diag(r_i)$ where $r_i = ||row_i(A)||_2$, $i = 1, \ldots, n$. Note that the l_2 norms of the rows of the matrix $C = R^{-1}A$ are all equal to 1. Since C is nonsingular as well, it follows that the matrix $B = CC^*$ is a positive definite Hermitian matrix with diagonal elements equal to 1. Let $\rho_1 \geq \rho_2 \geq \ldots \rho_n > 0$ be the eigenvalues of B. Note that $\rho_1 + \ldots + \rho_{n-1} < trace(B) = n$. It now follows that

$$(2.13) \qquad ||adj(C)||_2 = \sqrt{||adj(B)||_2} = \sqrt{\rho_1 \rho_2 \cdots \rho_{n-1}} \leq \left(\frac{\rho_1 + \ldots + \rho_{n-1}}{n-1}\right)^{\frac{n-1}{2}}$$

$$< \left(\frac{n}{n-1}\right)^{\frac{n-1}{2}} = \left(1 + \frac{1}{n-1}\right)^{\frac{n-1}{2}}.$$

Since

$$||adj(A)||_2 \leq ||adj(R^{-1}A)||_2 \, ||adj(R)||_2 = ||adj(C)||_2 \max_k \prod_{\substack{j=1 \\ j \neq k}}^{n} ||row_j(A)||_2,$$

our claim now follows from (2.13). $\square$

Theorem 2.14 *We have $\kappa(||.||_\infty) = \sqrt{n}^{\,n}$.*

Proof. Let $A \in C^{nn}$ be such that $||row_i(A)||_\infty = 1$, $\imath = 1, \ldots, n$. Let x be a vector in C^n and let $y = adj(A)x$. For every $i \in \{1, \ldots n\}$ we have

$$y_i = \sum_{j=1}^{n} det(A(j|i)) x_j (-1)^{i+j} = det \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1,i-1} & x_1 & a_{1,i+1} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2,i-1} & x_2 & a_{2,i+1} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{n,i-1} & x_n & a_{n,i+1} & \ldots & a_{nn} \end{pmatrix}.$$

Thus, if we choose x such that $||x||_\infty = 1$ then the above matrix has entries of modulus less than or eqal to 1, and by the Hadamard determinant theorem, e.g. [2, p.114, Theorem 4.1.7] it follows that $|y_i| \leq \sqrt{n}^n$. Hence, we have $||adj(A)||_\infty \leq \sqrt{n}^n$, implying that

$$\kappa(||.||_\infty) \leq \sqrt{n}^n. \tag{2.15}$$

To prove that equality holds in (2.15), consider a general complex $n \times n$ Hadamard matrix H_n, that is, a matrix all of whose entries have modulus 1 and $H_n H_n^* = nI_n$. Note that $H_n^{-1} = \frac{1}{n} H_n^T$, and so

$$adj(H_n) = det(H_n) H_n^{-1} = \frac{\sqrt{n}^n}{n} H_n^T.$$

It now follows that $||adj(H_n)||_\infty = \sqrt{n}^n$, proving that equality holds in (2.15). □

Remark 2.16 Note that one can define $\kappa(||.||)$ using columns instead of rows and obtain similar results where columns replace rows all over. This remark refers to the results of the next section as well.

3 Lower bounds on the smallest eigenvalue

For an $n \times n$ complex matrix A we denote by $\lambda_1(A), \ldots, \lambda_n(A)$ the eigenvalues of A ordered in a non-increasing order of their moduli. In this section we find a lower bound on $|\lambda_n(A)|$. Clearly, the interesting case is where A is nonsingular.

Our results follow from the following corollary of Theorem 2.1.

Theorem 3.1 *Let $A \in C^{nn}$ be a nonsingular matrix and let $||.||$ be an absolute vector norm on C^n. Then*

$$|\lambda_n(A)| \geq \frac{det(A)}{\kappa(||.||) \max_k \prod_{\substack{j=1 \\ j \neq k}}^n ||row_j(A)||}. \tag{3.2}$$

Proof. As is well known, the spectral radius $\rho(adj(A))$ of $adj(A)$ satisfies

$$\rho(adj(A)) = \left| \frac{det(A)}{\lambda_n(A)} \right|. \tag{3.3}$$

Since every operator norm satisfies

$$\rho(adj(A)) \leq ||adj(A)||, \tag{3.4}$$

the inequality (3.2) follows from (3.3), (3.4) and (2.2). □

By applying Theorem 3.1 to the norms l_1, l_2 and l_∞, using the results of the previous section, we obtain

Theorem 3.5 *Let $A \in C^{nn}$ be a nonsingular matrix. Then*

$$|\lambda_n(A)| \geq \frac{det(A)}{\max_k \prod_{\substack{j=1\\ j\neq k}}^n ||row_j(A)||_1}, \tag{3.6}$$

$$|\lambda_n(A)| > \frac{det(A)}{\left(1+\frac{1}{n-1}\right)^{\frac{n-1}{2}} \max_k \prod_{\substack{j=1\\ j\neq k}}^n ||row_j(A)||_2}, \tag{3.7}$$

and

$$|\lambda_n(A)| \geq \frac{det(A)}{\sqrt{n}^{\,n} \max_k \prod_{\substack{j=1\\ j\neq k}}^n ||row_j(A)||_\infty}. \tag{3.8}$$

Proof. Note that all we have to prove is that the inequality (3.7) is strict. Since, by Proposition 2.12, nonsingular matrices A satisfy

$$\left|\frac{det(A)}{\lambda_n(A)}\right| = \rho(adj(A)) \leq ||adj(A)||_2 < \left(1+\frac{1}{n-1}\right)^{\frac{n-1}{2}} \max_k \prod_{\substack{j=1\\ j\neq k}}^n ||row_j(A)||_2,$$

our claim follows. $\square$

Remark 3.9 The weak form of (3.7), that is, the inequality

$$|\lambda_n(A)| \geq \frac{det(A)}{\left(1+\frac{1}{n-1}\right)^{\frac{n-1}{2}} \max_k \prod_{\substack{j=1\\ j\neq k}}^n ||row_j(A)||_2},$$

follows from [1, Theorem 1]. The inequality (3.6) was already proven in Theorem 1 of [3], see also [2, p.145, Theorem 1.7]. Both papers [1] and [3] employ essentially the same techniques we do. The inequality (3.6) was also proven in Theorem 8 of [4]. Our inequality (3.7) is better than the inequality

$$|\lambda_n(A)| \geq \frac{det(A)}{n \max_k \prod_{\substack{j=1\\ j\neq k}}^n ||row_j(A)||_2}$$

proven in [4], since

$$\left(1+\frac{1}{n-1}\right)^{\frac{n-1}{2}} < \sqrt{e} < n \qquad (n > 1).$$

Remark 3.10 The lower bounds on $\lambda_n(A)$ given by (3.6) and by (3.7) are not comparable. The bound given by (3.6) is better, for example, in the case of a generalized diagonal matrix A, since in such a case the l_1 norm and the l_2 norm of the rows of A are the same.

On the other hand, if A is a Hadamard matrix then the l_1 norm of any row of A is equal to n, while the l_2 norm of any row of A is equal to $\sqrt{n}$. Therefore, since for $n = 2$ we have

$$\left(1 + \frac{1}{n-1}\right)^{\frac{n-1}{2}} = \sqrt{n}$$

and for $n \geq 3$ we have

$$\left(1 + \frac{1}{n-1}\right)^{\frac{n-1}{2}} < \sqrt{e} < \sqrt{n},$$

it follows that the bound given by (3.6) is better whenever A is a Hadamard matrix.

Remark 3.11 The lower bounds on $\lambda_n(A)$ given by (3.8) is weaker than the bound given by (3.7). To see this observe that for every row $row_i(A)$ of an $n \times n$ matrix A we have

$$||row_i(A)||_2 \leq \sqrt{n\,||row_i(A)||_\infty^2} = \sqrt{n}\,||row_i(A)||_\infty.$$

Therefore, we have

$$\left(1 + \frac{1}{n-1}\right)^{\frac{n-1}{2}} \max_k \prod_{\substack{j=1 \\ j\neq k}}^{n} ||row_j(A)||_2 \leq \left(1 + \frac{1}{n-1}\right)^{\frac{n-1}{2}} \sqrt{n}^{\,n-1} \max_k \prod_{\substack{j=1 \\ j\neq k}}^{n} ||row_j(A)||_\infty$$

$$\leq \sqrt{n}^{\,n} \max_k \prod_{\substack{j=1 \\ j\neq k}}^{n} ||row_j(A)||_\infty,$$

implying our claim.

The lower bounds on $\lambda_n(A)$ given by (3.8) is still not comparable to the bound given by (3.6). As was shown above, for generalized diagonal matrices the bound given by (3.6) is better. However, for a Hadamard matrix A the l_1 norm of any row of A is equal to n while the l_∞ norm of any row of A is equal to 1, and since $\sqrt{n}^{\,n} \leq n^{n-1}$ it follows that in this case the bound given by (3.8) is better.

Remark 3.12 The lower bounds on $\lambda_n(A)$ given by (3.6) is sharp in the sense that one cannot find a constant $C,\ C < 1$ such that

$$|\lambda_n(A)| \geq \frac{det(A)}{C \max_k \prod_{\substack{j=1 \\ j\neq k}}^{n} ||row_j(A)||_1}.$$

In fact, note that for every nonsingular generalized diagonal matrix A we have equality in (3.6).

Remark 3.13 The lower bounds on $\lambda_n(A)$ given by (3.7) is sharp in the sense that one cannot find a constant C, $C < \left(1+\frac{1}{n-1}\right)^{\frac{n-1}{2}}$ such that

$$|\lambda_n(A)| \geq \frac{det(A)}{C \max_k \prod_{\substack{j=1 \\ j\neq k}}^{n} ||row_j(A)||_2}.$$

To see this we equivalently prove that one cannot find a constant C, $C < \left(1+\frac{1}{n-1}\right)^{\frac{n-1}{2}}$ such that

$$C \geq \frac{det(A)}{|\lambda_n(A)| \max_k \prod_{\substack{j=1 \\ j\neq k}}^{n} ||row_j(A)||_\infty}. \tag{3.14}$$

As in the proof of Theorem 2.10, let B be the positive semidefinite matrix $\left(1+\frac{1}{n-1}\right)I - \frac{1}{n-1}J$, where I is the $n \times n$ identity matrix and J is the $n \times n$ matrix all of whose entries are equal to 1, and let A be the positive semidefinite matrix such that $B = A^2$. We now define the matrix $A(\epsilon) = A + \epsilon I$ where ϵ is a positive number. The eigenvalues of $A(\epsilon)$ are $1 + \frac{1}{n-1} + \epsilon$ with multiplicity $n-1$ and ϵ with multiplicity 1. For ϵ sufficiently small we have $\lambda_n(A(\epsilon)) = \epsilon$, and then

$$\frac{det(A(\epsilon))}{|\lambda_n(A(\epsilon))| \max_k \prod_{\substack{j=1 \\ j\neq k}}^{n} ||row_j(A(\epsilon))||_\infty} = \frac{\left(1+\frac{1}{n-1}+\epsilon\right)^{n-1}}{\max_k \prod_{\substack{j=1 \\ j\neq k}}^{n} ||row_j(A(\epsilon))||_\infty}. \tag{3.15}$$

If we let ϵ approach 0 then the right hand side of (3.15) approaches $\left(1+\frac{1}{n-1}\right)^{\frac{n-1}{2}}$. Hence, there exists no constant C, $C < \left(1+\frac{1}{n-1}\right)^{\frac{n-1}{2}}$ such that (3.14) holds.

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